Controlling a (Highly Simplified) Surface-to-Air Missile

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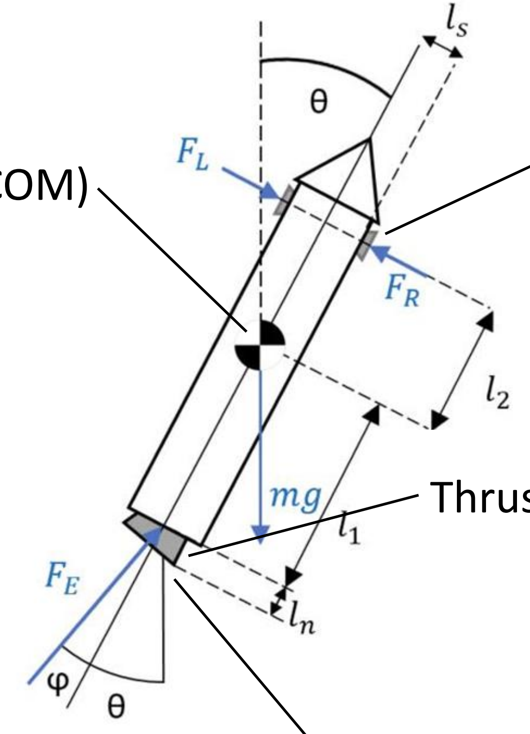
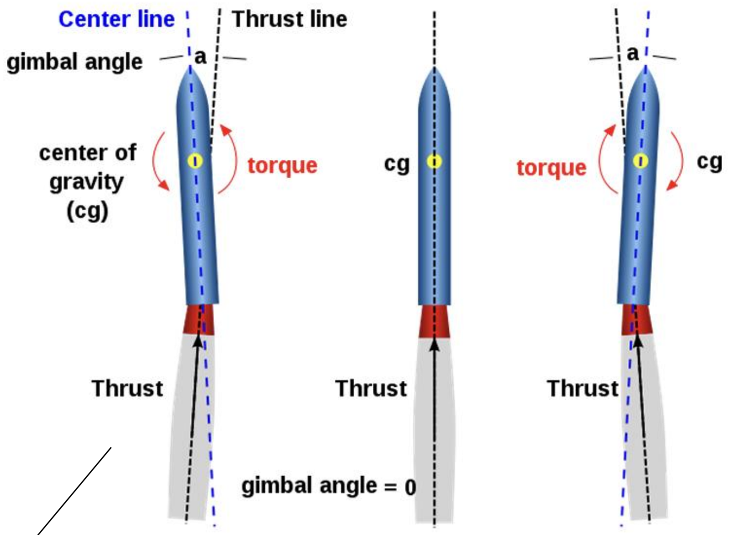
The Missile Model: A First Glance

Center of mass (COM)



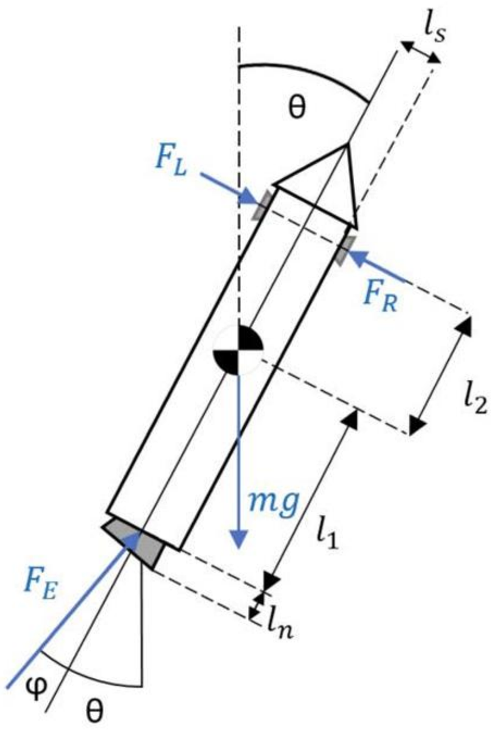
Lateral thrusters for
small, quick adjustments

Thrust generated by rocket engine



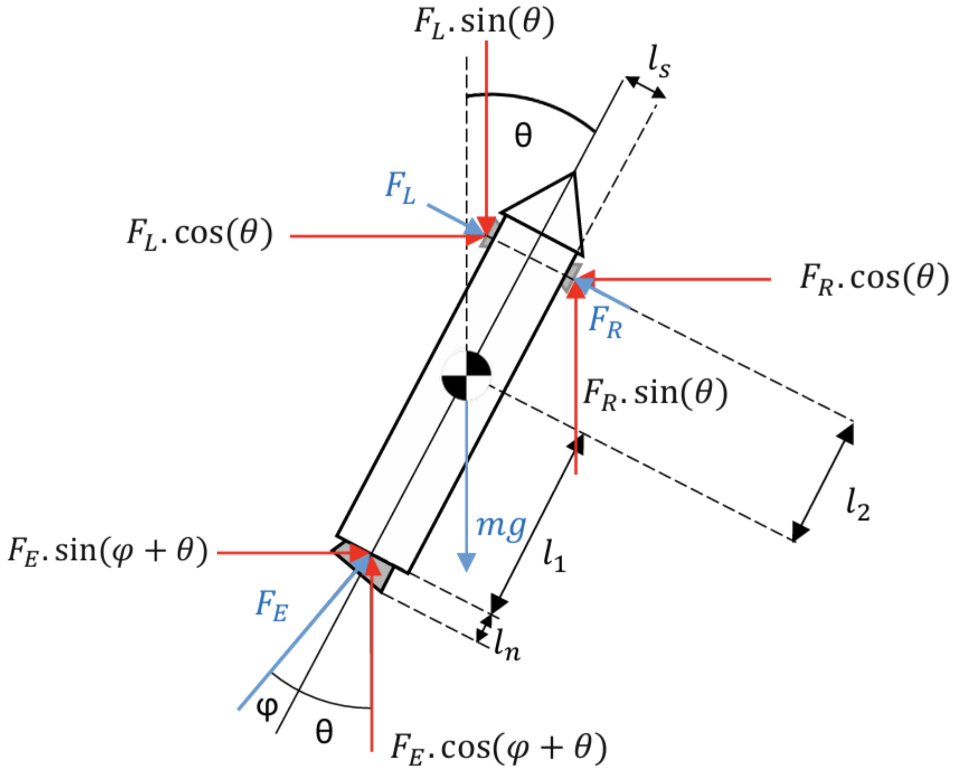
Exhaust nozzle can be angled to pitch the
missile upwards or downwards
(Thrust Vector Control)

State and Input Variables



Variable	Description	Units	Constraints
x	Position of COM along X axis	Meters (m)	NA
$\dot{x}$	Time derivative of x	Meters per second (m/s)	NA
z	Position of COM along Z axis	Meters (m)	$[0, \infty]$
$\dot{z}$	Time derivative of z	Meters per second (m/s)	NA
θ	Angle between missile's longitudinal axis and positive Z axis	Radians	NA
$\dot{\theta}$	Time derivative of θ (i.e. angular velocity)	Radians per second	NA
m	Mass of missile (decreasing overtime)	Kilograms (kg)	$[M_{dry}, M_0]$
F_E	Force from rocket engine	Newtons (N)	$[0, 5000]$
F_S	Net force from side thrusters: $F_L - F_R$	Newtons (N)	$[0, 10]$
ϕ	Nozzle angle	Radians	$[-\pi/4, \pi/4]$

Equations of Motion



- Using mechanics, derive the equations

$$\frac{d}{dt}(m\dot{x}) = F_E \sin(\phi+\theta) + F_s \cos(\theta)$$

$$\frac{d}{dt}(m\dot{z}) = F_E \cos(\phi+\theta) - F_S \sin(\theta) - mg$$

$$\underbrace{\frac{d}{dt}(J\dot{\theta})}_{\text{LHS}} = \underbrace{l_2 F_S - (l_1 + \cos(\phi)\, l_n) F_E \sin(\phi)}_{\text{RHS}}$$

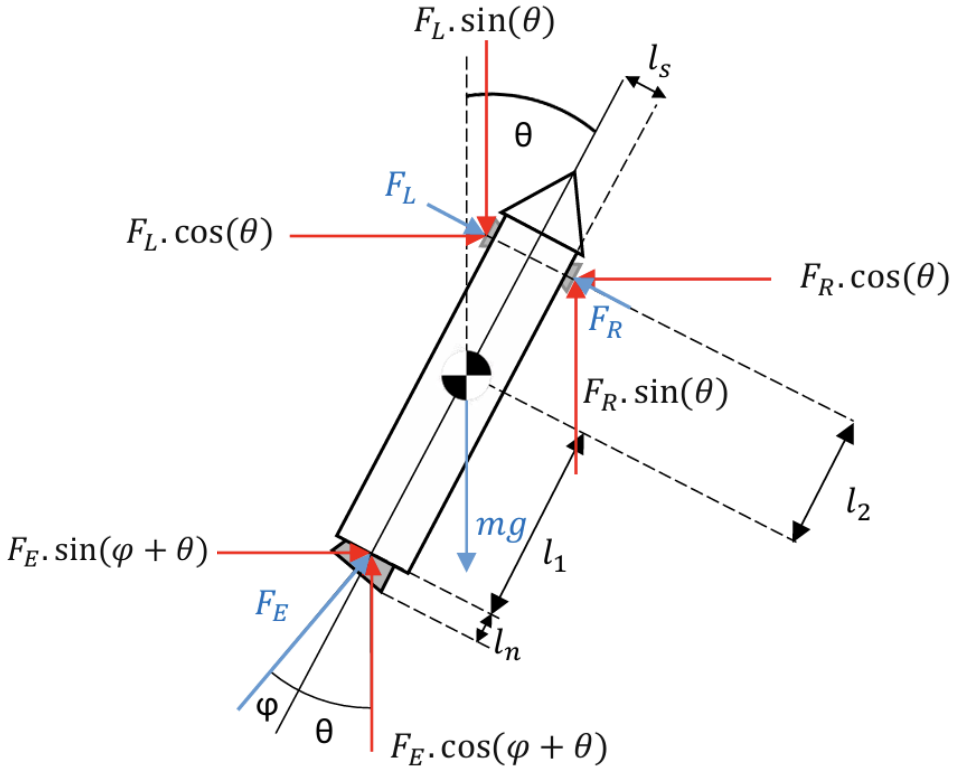
LHS: Time derivative of linear/angular momentum

RHS: Net force / torque

- Assume the rocket loses mass proportional to the thrust of its engine:

$$\dot{m} = -\alpha F_E - \beta |F_s|$$

Dynamics Model



- The dynamics model is therefore

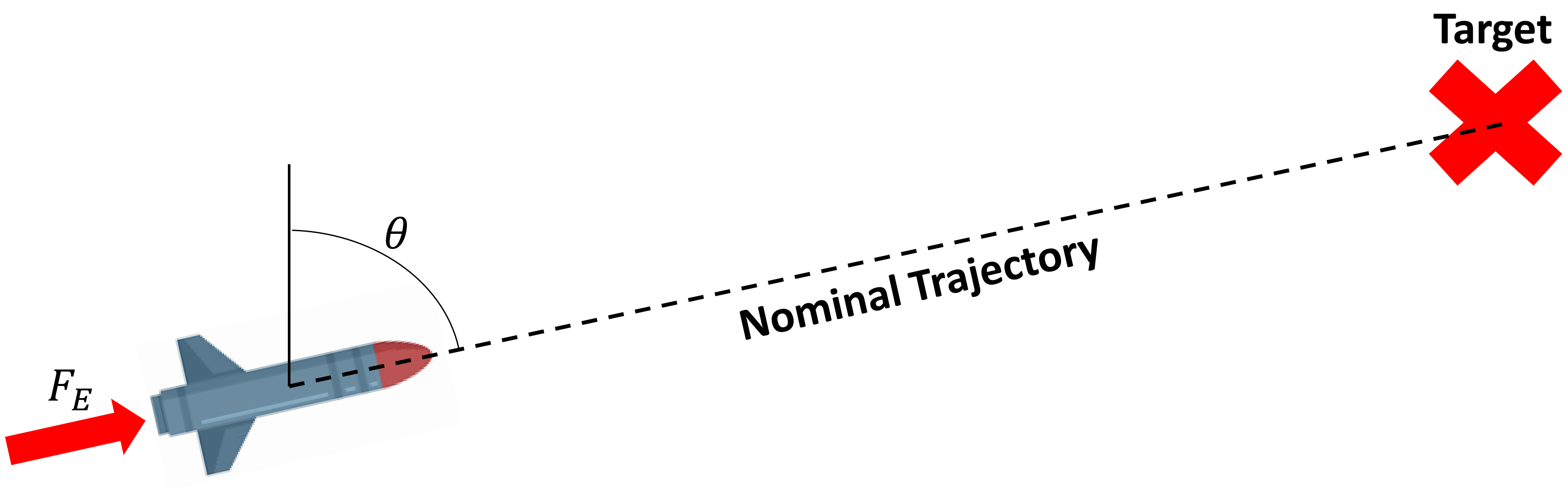
$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{z} \\ \ddot{z} \\ \dot{\theta} \\ \ddot{\theta} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \frac{1}{m}[F_E \sin(\phi+\theta) + F_S \cos(\theta) + (\alpha F_E + \beta|F_S|)\dot{x}] \\ \dot{z} \\ \frac{1}{m}[F_E \cos(\phi+\theta) - F_S \sin(\theta) - mg + (\alpha F_E + \beta|F_S|)\dot{z}] \\ \dot{\theta} \\ \frac{1}{J}[l_2 F_S - (l_1 + l_n \cos\phi) F_E \sin(\phi) + C(\alpha F_E + \beta|F_S|)\dot{\theta}] \\ -\alpha F_E - \beta|F_S| \end{bmatrix},$$

which we express as $\dot{\boldsymbol{s}} = \boldsymbol{f}(\boldsymbol{s}, \boldsymbol{u})$ where

$$s = \begin{bmatrix} x \\ \dot{x} \\ z \\ \dot{z} \\ \theta \\ \dot{\theta} \\ m \end{bmatrix} \text{ and } u = \begin{bmatrix} F_E \\ F_S \\ \phi \end{bmatrix}.$$

Nominal Trajectory

- The ultimate goal is to guide the missile towards an aerial target using LQR.
- We need a nominal trajectory to linearize around!



- Set $F_S = \phi = 0$, and find a constant F_E, θ such that the missile arrives at a **fixed** target.
- F_E and θ are constant throughout the entire flight path.

Constant Mass Case: Nominal Trajectory

- Suppose for a moment that the mass of the missile is constant (i.e., $\alpha = \beta = 0$).
- Along a nominal trajectory with fixed F_E and θ, the equations of motion are easy:

$$\ddot{x}_o(t) = \tfrac{1}{m} F_E \sin(\theta)$$
$$\ddot{z}_o(t) = \tfrac{1}{m} F_E \cos(\theta) - g$$

Integrate 2x

$$x_o(t) = \tfrac{1}{2m} F_E \sin(\theta)\, t^2 + c_x t + d_x$$
$$z_o(t) = \tfrac{1}{2}\left(\tfrac{1}{m} F_E \cos(\theta) - g\right) t^2 + c_z t + d_z$$

- Constants are chosen using initial conditions, i.e.

$$d_x = x_o(0) \triangleq x(0) \qquad c_x = \dot{x}_o(0) \triangleq \dot{x}(0)$$
$$d_z = z_o(0) \triangleq z(0) \qquad d_z = \dot{z}_o(0) \triangleq \dot{z}(0)$$

- Solve for F_E, θ to hit a target at position (x_d, z_d) at time $T > 0$ using equations

$$x_o(T) = \tfrac{1}{2m} F_E \sin(\theta)\, T^2 + c_x T + d_x = x_d$$
$$z_o(T) = \tfrac{1}{2}\left(\tfrac{1}{m} F_E \cos(\theta) - g\right) T^2 + c_z T + d_z = z_d$$

Two equations, two unknowns

Constant Mass Case: Linearization

- We have our nominal trajectory $s_o(t) = [x_o(t)\ \ \dot{x}_o(t)\ \ z_o(t)\ \ \dot{z}_o(t)\ \ \theta\ \ 0]^T,\ u_o(t) = [F_E\ \ 0\ \ 0]^T$.
- Linearizing the dynamics function about $(s_o(t), u_o(t))$ we get

Sanity check: (A, B) is controllable, and (A, C) is observable if $C = I$.

$$A = \left.\frac{\partial f}{\partial s}\right|_{s_o(t),u_o(t)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{F_E}{m}\cos(\theta) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{F_E}{m}\sin(\theta) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Because true dynamics $\boldsymbol{f}$ is nonlinear, this doesn't guarantee system is controllable.

$$B = \left.\frac{\partial f}{\partial u}\right|_{s_o(t),u_o(t)} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{m}\sin(\theta) & \frac{1}{m}\cos(\theta) & \frac{F_E}{m}\cos(\theta) \\ 0 & 0 & 0 \\ \frac{1}{m}\cos(\theta) & -\frac{1}{m}\sin(\theta) & -\frac{F_E}{m}\sin(\theta) \\ 0 & 0 & 0 \\ 0 & \frac{l_2}{J} & -\frac{F_E}{J}(l_1+l_n) \end{pmatrix}$$

Constant Mass Case: LQR Control

- **Goal**: control $\frac{d}{dt}(\delta s) = A\,\delta s + B\,\delta u$, where $\delta s(t) = s(t) - s_o(t)$ and $\delta u(t) = u(t) - u_o(t)$.
- Because nominal trajectory is guaranteed to hit target, **we want $\boldsymbol{\delta s(t) \to 0}$ as $\boldsymbol{t \to T}$**.
- Find feedback controller $u^*(t) = -K\delta s(t) + u_o(t)$ by minimizing

$$J = \int_0^\infty (\delta s)^T Q(\delta s) + (\delta u)^T R(\delta u)\, dt$$

- We have $K = -R^{-1}B^T P$ where P solves the **Algebraic Riccati Equation (ARE)**:

$$0 = PA + A^T P - PBR^{-1}B^T P + Q$$

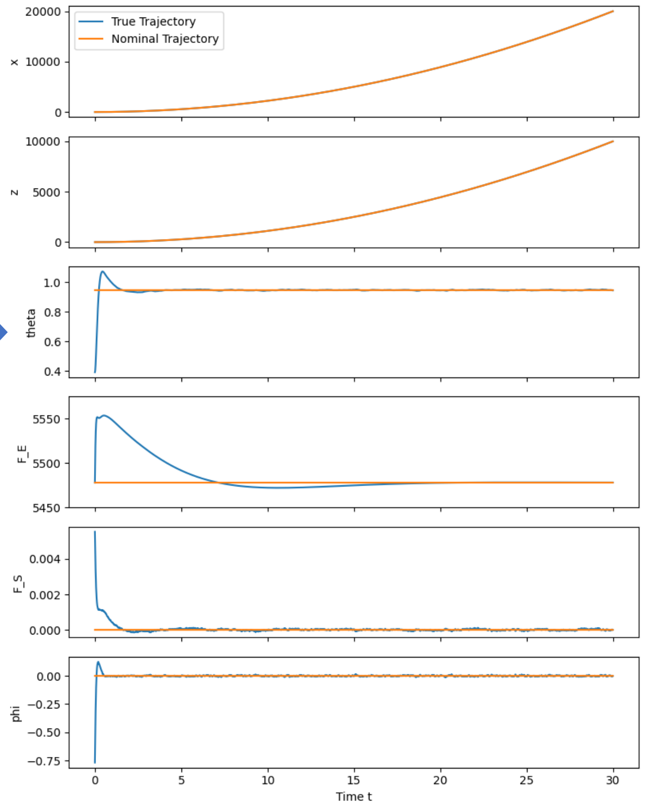
- Choose Q and R according to Bryson's Rule:

$$Q_{ii} = \frac{1}{(maximum\ acceptable\ value\ of\ \delta s_i)^2} \qquad R_{ii} = \frac{1}{(maximum\ acceptable\ value\ of\ \delta u_i)^2}$$

Constant Mass Case: LQR Control & Simulation

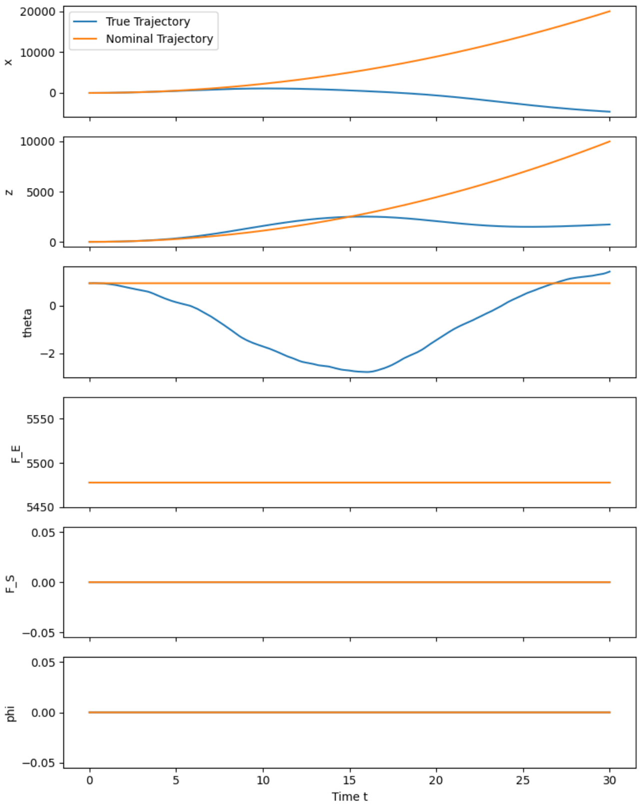
- Simulated $ds = f(s, u(s))dt + [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ \sigma]^T dW$ ($\dot{\theta}$ subject to continuous noise).
- **Goal:** with a missile beginning from rest at origin, hit a target at $(x, z) = (20{,}000, 10{,}000)$ within 30 seconds.

Missile initially pointing in the wrong direction but then converges



LQR control: $u^*(t) = -K\delta s(t) + u_o(t)$

Missile initially pointing in the right direction but then diverges



Open loop control: $u(t) = u_o(t) = [F_E\ \ 0\ \ 0]^T$

Decaying Mass Case: Nominal Trajectory

- Now suppose mass $m(t)$ is no longer constant.
- Along a nominal trajectory with fixed F_E and θ, the equations of motion are now more difficult:

$$m(t)\ddot{x}_o(t) = F_E \sin(\theta) + \alpha F_E \dot{x}_o(t)$$

$$m(t)\ddot{z}_o(t) = F_E \cos(\theta) - m(t)g + \alpha F_E \dot{z}_o(t)$$

Solve ODEs

$$x_o(t) = \frac{\sin(\theta)}{\alpha^2 F_E} m(t) + c_x \log\big(m(t)\big) + d_x$$

$$z_o(t) = \frac{\cos(\theta)}{\alpha^2 F_E} m(t) - \frac{g}{4\alpha^2 F_E^2} m(t)^2 + c_z \log\big(m(t)\big) + d_z$$

- Once again choose constants c_x, d_x, c_z, d_z so that initial conditions hold: $x_o(0) = x(0)$, $z_o(0) = z(0)$,… etc.

- Once again use Newton's method to find F_E, θ such that

$$x_o(T; F_E, \theta) = x_d$$

$$z_o(T; F_E, \theta) = z_d$$

Two equations, two unknowns

Decaying Mass Case: Linearization

- We again have nominal trajectory $s_o(t) = [x_o(t)\ \ \dot{x}_o(t)\ \ z_o(t)\ \ \dot{z}_o(t)\ \ \theta\ \ 0\ \ m_o(t)]^T,\ u_o(t) = [F_E\ \ 0\ \ 0]^T$.
- Linearizing the dynamics function about $(s_o(t), u_o(t))$ we get

Note:
We assume $\beta|F_S| \approx 0$ to ensure $A(t)$ and $B(t)$ are continuous.

$$A(t) = \left.\frac{\partial f}{\partial s}\right|_{s_o(t),u_o(t)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha F_E}{m} & 0 & 0 & \frac{F_E}{m}\cos(\theta) & 0 & -\frac{F_E}{m^2}(\sin(\theta)+\alpha\dot{x}) \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha F_E}{m} & -\frac{F_E}{m}\sin(\theta) & 0 & -\frac{F_E}{m^2}(\cos(\theta)+\alpha\dot{z}) \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\alpha F_E}{m} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B(t) = \left.\frac{\partial f}{\partial u}\right|_{s_o(t),u_o(t)} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{m}\sin(\theta) + \frac{\alpha}{m}\dot{x} & \frac{1}{m}\cos(\theta) & \frac{F_E}{m}\cos(\theta) \\ 0 & 0 & 0 \\ \frac{1}{m}\cos(\theta)+\frac{\alpha}{m}\dot{z} & -\frac{1}{m}\sin(\theta) & -\frac{F_E}{m}\sin(\theta) \\ 0 & 0 & 0 \\ 0 & \frac{l_2}{J} & -\frac{F_E}{J}(l_1+l_n) \\ -\alpha & 0 & 0 \end{pmatrix}$$

Decaying Mass Case: LQR Control

- **Goal**: control $\frac{d}{dt}(\delta s)=A(t)\,\delta s+B(t)\,\delta u$, where $\delta s(t)=s(t)-s_o(t)$ and $\delta u(t)=u(t)-u_o(t)$.
- Because nominal trajectory is guaranteed to hit target, **we want $\boldsymbol{\delta s(t)\to 0}$ as $\boldsymbol{t\to T}$**.
- Find feedback controller $u^*(t)=-K(t)\delta s(t)+u_o(t)$ by minimizing

$$J=\int_0^T(\delta s)^TQ(\delta s)+(\delta u)^TR(\delta u)\,dt+\delta s(T)Q_f\delta s(T)$$

Now using **finite-horizon**!

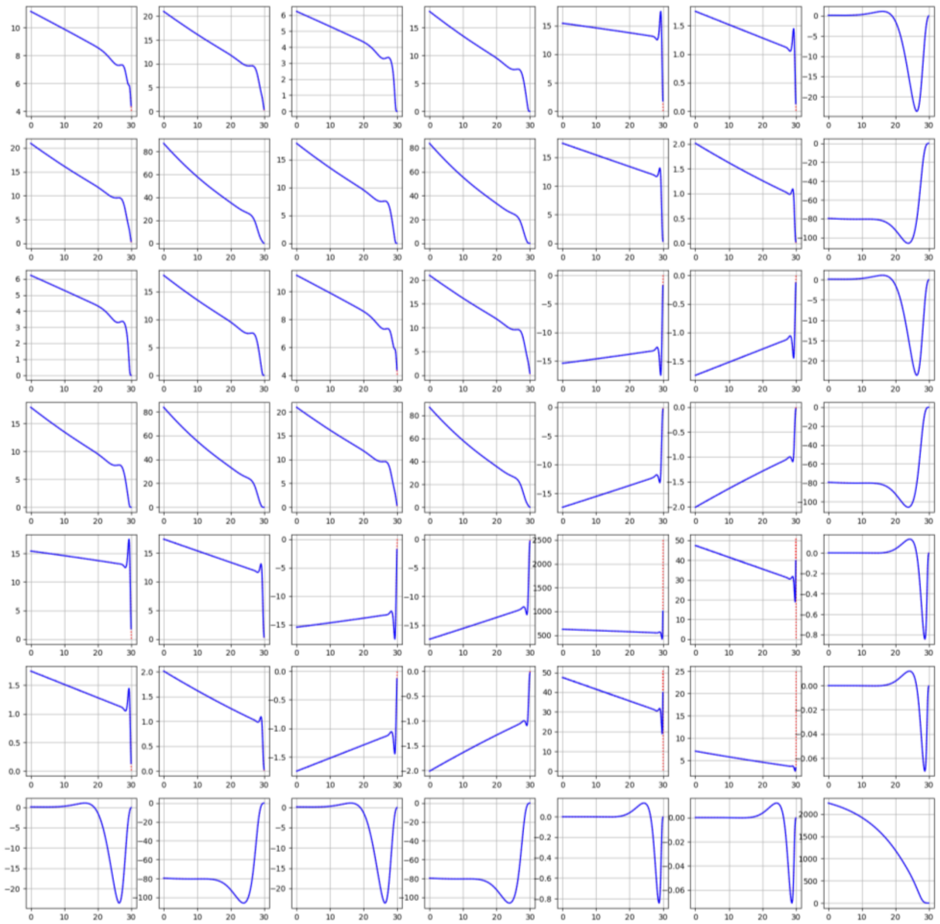
- We have $K(t)=-R^{-1}B(t)^TP(t)$ where $P(t)$ solves the **Differential Riccati Equation**:

$$\begin{cases}-\dot{P}(t)=P(t)A(t)+A(t)^TP(t)\ -P(t)B(t)R^{-1}B(t)^TP(t)\ +Q\\ P(T)=Q_f\end{cases}$$

- The above system has no closed-form solution. **Approximate $\boldsymbol{P(t)}$ by using an ODE solver + cubic splines**.
- Choose Q and R same as before.

Decaying Mass Case: Visualizing $P(t)$

Note:
Each cell is a graph of $P_{ij}(t)$ for $t \in [0, T]$.

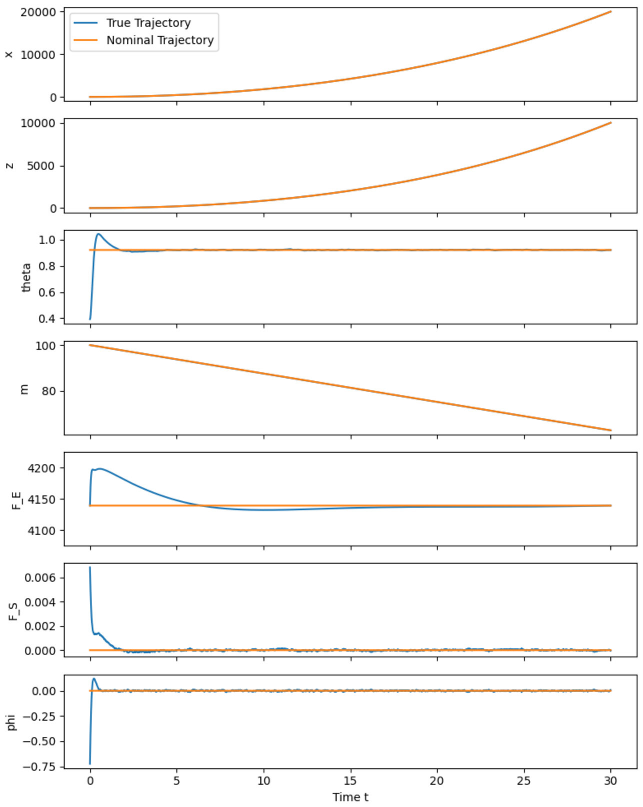


- $P(t)$ solves the **Differential Riccati Equation**: $\begin{cases} -\dot{P}(t) = P(t)A(t) + A(t)^T P(t) - P(t)B(t)R^{-1}B(t)^T P(t) + Q \\ P(T) = Q_f \end{cases}$

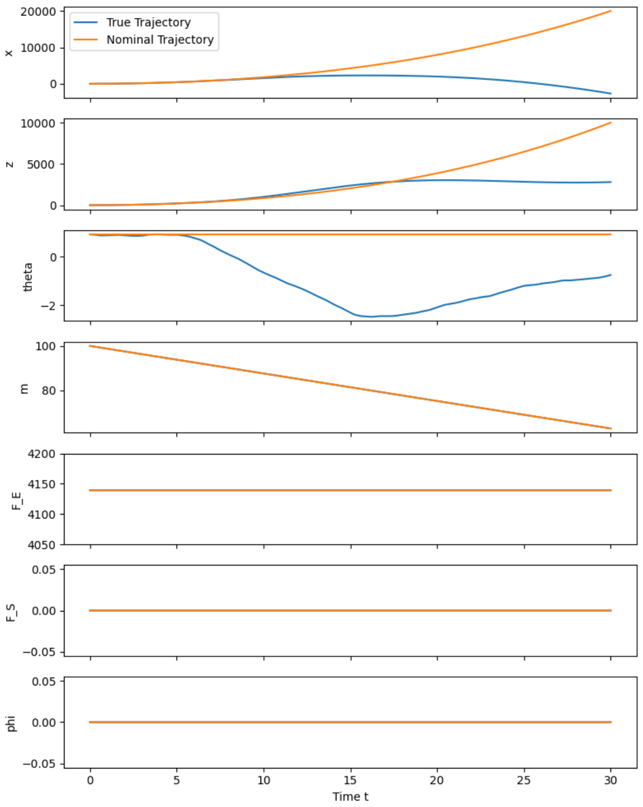
Decaying Mass Case: LQR Control & Simulation

- Simulated $ds = f(s, u(s))dt + [0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \sigma \;\; 0]^T dW$ ($\dot{\theta}$ subject to continuous noise).
- **Goal:** with a missile beginning from rest at origin, hit a target at $(x, z) = (20{,}000, 10{,}000)$ within 30 seconds.

Missile initially pointing in the wrong direction but then converges



LQR control: $u^*(t) = -K(t)\delta s(t) + u_o(t)$

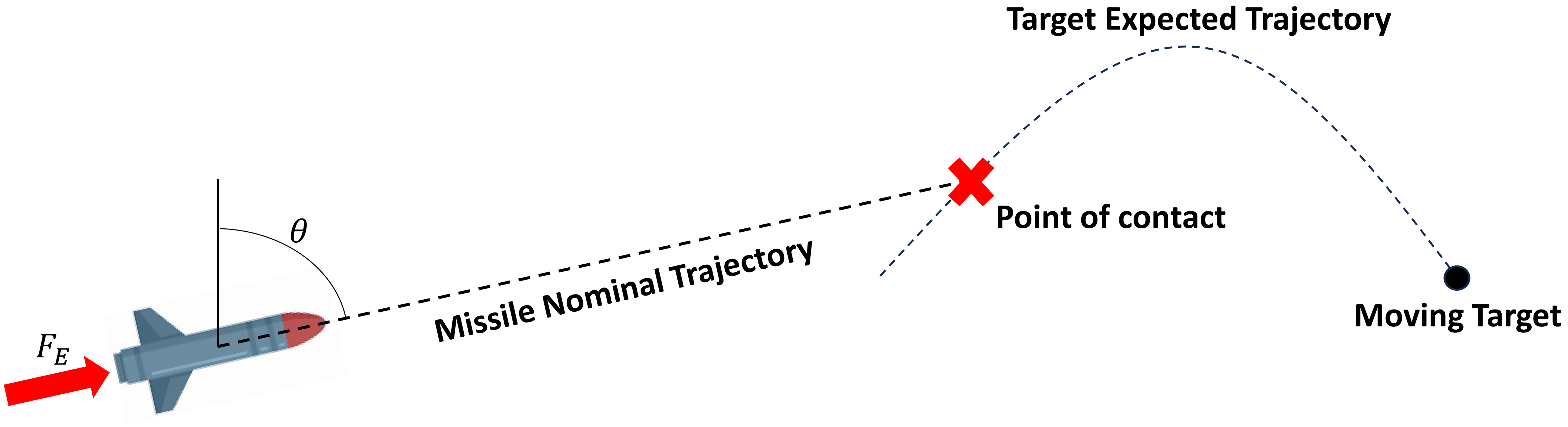


Missile initially pointing in the right direction but then diverges

Open loop control: $u(t) = u_o(t) = [F_E \;\; 0 \;\; 0]^T$

Task: Intercept a Moving Target

- Suppose we want to intercept a moving target along its expected path.
- We need to point the missile to where the moving target **will** be.



- **Nominal Trajectory:** Set $F_S = \phi = 0$, and find fixed constants F_E, θ, T such that missile intercepts target at time T.

Task: Intercept a Moving Target

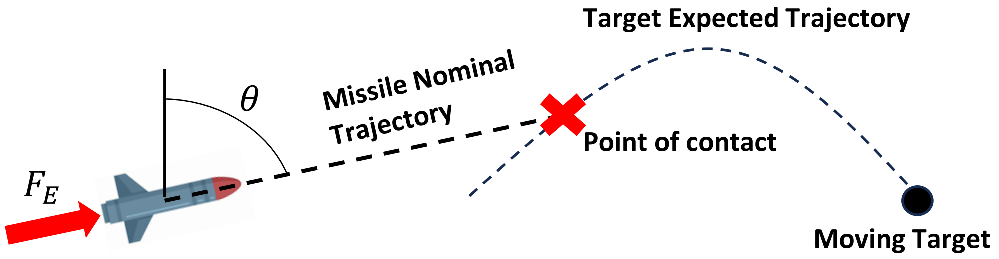
- Mathematically, the nominal trajectory is found by solving

$$\min_{T, F_E, \theta} T \ \ s.t. \quad \begin{aligned} & x_o(T; F_E, \theta) = x_{target}(T) \\ & z_o(T; F_E, \theta) = z_{target}(T) \\ & m(T) = m_o - \alpha F_E T \geq m_{dry} \\ & F_E \leq F_{max} \end{aligned}$$

Shoot down target as quickly as possible

Hit target along its trajectory

Make sure there's enough fuel (m_{dry} is mass of missile when no fuel left)



Task: Intercept a Moving Target

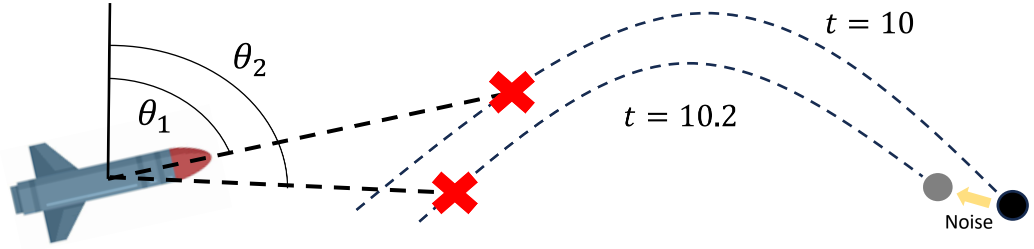
- For simplicity, assume target follows a parabolic trajectory and is subject to noise:

$$x_{target}(t) = p_x t + q_x$$
$$z_{target}(t) = -\frac{1}{2}gt^2 + p_z t + q_z$$

Turn into SDE

$$ds = A^{targ} s dt + [0 \;\; 0 \;\; \sigma_x \;\; 0]^T dW, \text{ where}$$
$$s = \left[x_{target}(t) \;\; z_{target}(t) \;\; \dot{x}_{target}(t) \;\; \dot{z}_{target}(t)\right]^T$$

- Because target has a noisy trajectory, need to recompute nominal trajectory of missile every Δt seconds.



- Using LQR, missile will smoothly track shifting nominal trajectories.

Task: Intercept a Moving Target

- I will now show **three different simulations** animated **on a single plot**.
- Each simulation is unique as the missile tracks the target which follows a random path.
- We assume the **missile is launched only 10 seconds after** the target is launched.

Summary

What I was able to accomplish:

1. **Defined a simple model** of a missile in 2D space.
2. **Found a nominal trajectory** to guide the missile from rest to a fixed aerial target.
3. **Linearized the nonlinear dynamics model** around the nominal trajectory.
4. **Controlled the missile using LQR** in the presence of noise and suboptimal initial conditions.
5. Managed to find a control strategy to **intercept moving targets** that follow random trajectories.

Shortcomings:

1. Model was confined to 2D.
2. Model did not take drag, lift, and other realistic aerodynamic factors into consideration.
3. Assumed control system had perfect information with regards to state information (did not consider estimation problem).

GitHub repository: https://github.com/Thiagodcv/lqr-missile-guidance

Borrowed diagrams and missile model from

Ferrante, Reuben. "A robust control approach for rocket landing." Theses Univ. Edinburgh (2017).