# Streaming Graph Mining

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# Large-Scale Dynamic Data

Examples:

- The Web, with approximately 5 billion Web Pages;
- Models of the Human brain, approx. $10^{10}$ neurons;
- Twitter, highly dynamic, 6000 tweets per second;
- Facebook: 1.86 billion monthly active users.

To process the sheer amount of data, several models of computations have emerged...

# Models of Computations

- *streaming algorithms* The input is defined by a stream of data (e.g. edges/nodes in a graph). Algorithms in this model must process the input stream in the order it arrives (sequential access) while using only limited amount of memory. Semi-streaming algorithms are allowed to make multiple passes over the data. See [4, 5, 6].
- *dynamic algorithms* The main goal is to support query and update operations (e.g. remove an edge or update its weight) as quickly as possible, in particular, much faster than recomputing from scratch. Amortized analysis is used to analyze an algorithm: total worst case time / number of update operations. The input might or might not fit into memory. See [8, 7].
- *MapReduce Model* The input is partitioned into different machines, with each machine processing its chunk in parallel. The results are then aggregated. This can be iterated multiple times, typically constant or logarithmic in the size of the input. See [9, 10].

# Streaming Algorithm for Densest Subgraph

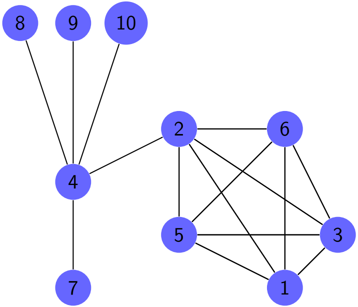
**Require**: an undirected graph $G$, a value $\epsilon > 0$

$H = G$;
while ($G$ contains at least one edge)

- rem. all nodes $v$ (and their edges) with $\delta_G(v) \leq 2(1+\epsilon)\rho(G)$ from $G$.
- if $\rho(G) > \rho(H)$ then $H \leftarrow G$;

return $H$;

# Faster algorithm: Example

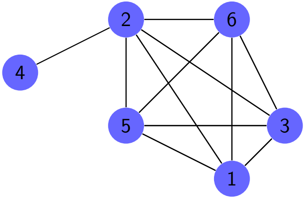



$\epsilon = 0.1$

**Iteration 1:**

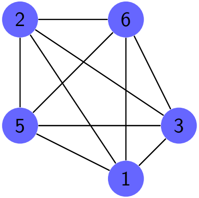
$\rho(G) = \frac{16}{10}$, remove nodes with degree $\leq 2 * (1.1) * 1.6 = 3.52$.

# Faster algorithm: Example




$\epsilon = 0.1$
**Iteration 2:**
$\rho(G) = \frac{11}{6}$, remove nodes with degree $\leq 2 * (1.1) * \frac{11}{6} = 3.45$.

# Faster algorithm: Example




$\epsilon = 0.1$

**Iteration 3:**

$\rho(G) = \frac{10}{5}$, remove nodes with degree $\leq 2 * (1.1) * 2 = 4.4$.

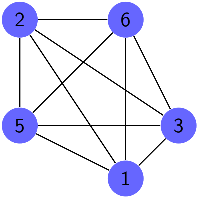
# Faster algorithm: Example

$\epsilon = 0.1$

**Iteration 4:**

Empty Graph.

# Faster algorithm: Example

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$2(1+\epsilon)-$**Approx. Densest Subgraph!**

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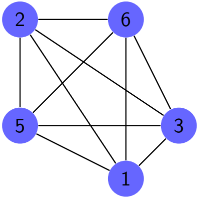


$\epsilon = 0.1$
$2(1+\epsilon)-$**Approx. Densest Subgraph!**
What if $\epsilon$ is large? (say $\epsilon = 0.5$)

# Approx. guarantee of the fast algo

## Theorem 1

*Let $O = (V_O, E_O)$ be a densest subgraph and let $H = (V_H, E_H)$ be the subgraph found by our algo, with parameter $\epsilon > 0$. Then, $\rho(H) \geq \frac{\rho(O)}{2(1+\epsilon)}$.*

## Proof.

Let $O = (V_O, E_O)$ be a densest subgraph. Consider the first step $t$ in the algo such that we remove a node $v \in V_O$ from the current graph $G_t$ (there must be such a step). From Lemma **??**, $\delta_{G_t}(v) \geq \delta_O(v) \geq \rho(O)$. Hence,

$$\rho(O) \leq \delta_{G_t}(v)$$

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$$\begin{aligned}\rho(O) &\leq \delta_{G_t}(v) \\ &\leq 2(1+\epsilon)\rho(G_t) \\ &\leq 2(1+\epsilon)\rho(H)\end{aligned}$$

# Running time of the fast algo

## Theorem 2

*The number of iterations of the fast algo with input* $G = (V_G, E_G)$ *and* $\epsilon > 0$ *is at most* $\lceil \log_{1+\epsilon}(|V_G|) \rceil$.

## Proof.

Consider any step $t$ of the algo and let $G_t = (V_{G_t}, E_{G_t})$ be the subgraph at the beginning of that step. Let $R_t$ be the set of nodes removed at the end of such step, i.e. the degree of any node in $R_t$ is $\leq 2(1+\epsilon)\rho(G_t)$. Then,

$$2|E_{G_t}| = \sum_{v \in R_t} \delta_{G_t}(v) + \sum_{v \in V_{G_t} \setminus R_t} \delta_{G_t}(v)$$

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$$\begin{aligned} 2|E_{G_t}| &= \sum_{v \in R_t} \delta_{G_t}(v) + \sum_{v \in V_{G_t} \setminus R_t} \delta_{G_t}(v) \\ &> 2(1+\epsilon)(|V_{G_t}| - |R_t|)\rho(G_t) \end{aligned}$$

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$$
\begin{aligned}
2|E_{G_t}| &= \sum_{v \in R_t} \delta_{G_t}(v) + \sum_{v \in V_{G_t} \setminus R_t} \delta_{G_t}(v) \\
&> 2(1+\epsilon)(|V_{G_t}| - |R_t|)\rho(G_t) \\
&= 2(1+\epsilon)(|V_{G_t}| - |R_t|)\frac{|E_{G_t}|}{|V_{G_t}|}.
\end{aligned}
$$

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$$2|E_{G_t}| > 2(1+\epsilon)(|V_{G_t}| - |R_t|)\frac{|E_{G_t}|}{|V_{G_t}|}, \qquad \Leftrightarrow$$

□

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Then,

$$
\begin{aligned}
2|E_{G_t}| > 2(1+\epsilon)(|V_{G_t}| - |R_t|)\frac{|E_{G_t}|}{|V_{G_t}|}, &\qquad \Leftrightarrow \\
|V_{G_t}| > (1+\epsilon)(|V_{G_t}| - |R_t|), &\qquad \Leftrightarrow
\end{aligned}
$$

$\square$

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$$
\begin{aligned}
2|E_{G_t}| > 2(1+\epsilon)(|V_{G_t}| - |R_t|)\frac{|E_{G_t}|}{|V_{G_t}|}, &\qquad \Leftrightarrow \\
|V_{G_t}| > (1+\epsilon)(|V_{G_t}| - |R_t|), &\qquad \Leftrightarrow \\
|V_{G_{t+1}}| = |V_{G_t}| - |R_t| < \frac{|V_{G_t}|}{1+\epsilon}. &
\end{aligned}
$$

□

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## Proof.

Then,

$$2|E_{G_t}| > 2(1+\epsilon)(|V_{G_t}| - |R_t|)\frac{|E_{G_t}|}{|V_{G_t}|}, \qquad \Leftrightarrow$$

$$|V_{G_t}| > (1+\epsilon)(|V_{G_t}| - |R_t|), \qquad \Leftrightarrow$$

$$|V_{G_{t+1}}| = |V_{G_t}| - |R_t| < \frac{|V_{G_t}|}{1+\epsilon}.$$

Therefore $|V_{G_t}| \leq 1$ in $\leq t$ steps for any $t$ such that $\frac{|V_G|}{(1+\epsilon)^t} \leq 1$, in particular when $t = \lceil \log_{1+\epsilon} |V_G| \rceil$. $\square$

# Streaming Algorithm for Densest Subgraph

The algorithm makes multiple passes over the data. During each pass:

- computes the current degree of each node as well as the current density;
- produces a new graph containing only the nodes with sufficiently large degree.
- maintains the current densest subgraph

**Theorem 3 (From [4])**

*There is a (semi-) streaming algorithm that for any $\epsilon > 0$, computes a $2(1+\epsilon)$-approximation of the densest subgraph while making $\lceil \log_{1+\epsilon}(|V_G|) \rceil$ passes over the data and requiring $O(n \log n)$ total memory.*

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