Low rank tensor approximation in OpenTURNS

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Plan

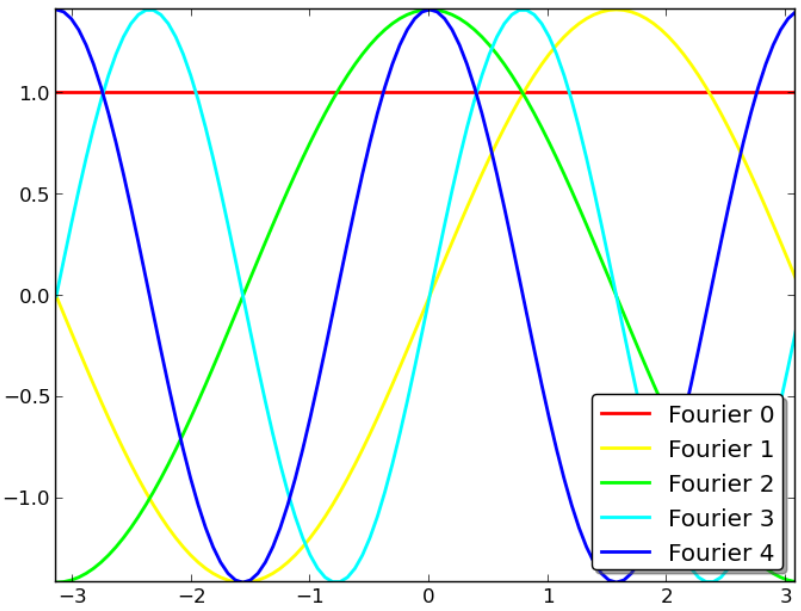
Non-polynomial basis

Canonical tensor

Fourier series

Fourier series

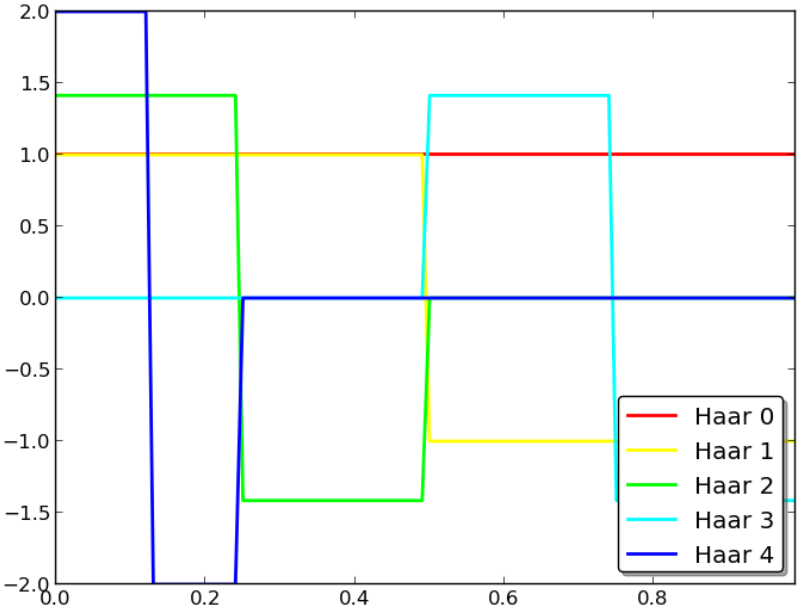
$$
\begin{aligned}
\psi_0(x) &= 1 \\
\psi_{2k+1}(x) &= \sqrt{2}\sin(kx) \\
\psi_{2k+2}(x) &= \sqrt{2}\cos(kx)
\end{aligned}
$$



Haar wavelets

Haar wavelets

$$
\begin{aligned}
\psi_0(x) &= \mathbb{1}_{[0,1]}(x) \\
\psi_n(x) &= \frac{1}{2^{j/2}}\left[\mathbb{1}_{[\frac{k}{2^j},\frac{k+1/2}{2^j}]}(x) - \mathbb{1}_{[\frac{k+1/2}{2^j},\frac{k+1}{2^j}]}(x)\right]
\end{aligned}
$$



Usage

In Python...

```
# as a regular function
family = ot.FourierSeriesFactory()
family = ot.HaarWaveletFactory()

for i in range(5):
    f = family.build(i)
    d = f.draw(xmin, xmax, 100)
```

Functional chaos tensorization

Tensorization

Functional chaos decomposition

$$Y \equiv h(\underline{X}) = \sum_{j=0}^{\infty} a_j \, \psi_j(\underline{X})$$

on univariate orthogonal function families

$$\phi_1^{(j)}, ..., \phi_M^{(j)} \qquad \forall j \in [1, d]$$

upon tensorized basis

$$\psi_{\underline{\alpha}}(\underline{x}) \equiv \phi_{\alpha_1}^{(1)}(x_1) \times \cdots \times \phi_{\alpha_d}^{(d)}(x_d)$$

Functional chaos tensorization

Tensorization

upon tensorized basis

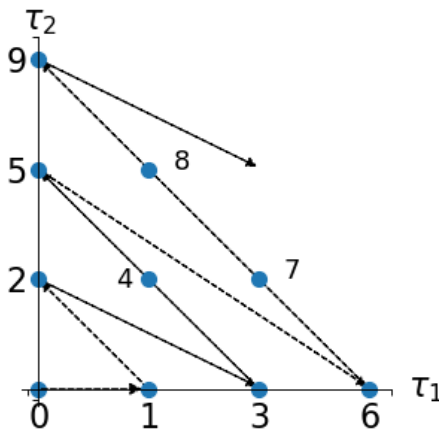
$$\psi_{\underline{\alpha}}(\underline{x}) \equiv \phi^{(1)}_{\alpha_1}(x_1) \times \cdots \times \phi^{(d)}_{\alpha_d}(x_d)$$

multi-indices notation

$$\alpha \equiv \{\alpha_1, \ldots, \alpha_d\}$$

curse of dimensionality

$$P = C_M^{d+M}$$



Usage

In Python...

```
# polynomial basis
ef = ot.LinearEnumerateFunction(dim)
factC = [LegendreFactory()] * dim
prod = ot.OrthogonalProductPolynomialFactory(factC, ef)

# non-polynomial basis
factC = [ot.FourierSeriesFactory()] * dim
prod = ot.OrthogonalProductFunctionFactory(factC)

algo = ot.FunctionalChaosAlgorithm(...)
```

Rank one tensor

Rank one tensor

$$f(x_1, \ldots, x_d) = \prod_{i=1}^{d} v_i(x_i)$$

with

$$v_i = \sum_{j=1}^{n_i} \alpha_j^{(i)} \phi_j(x_i)$$

expanding to

$$\begin{aligned} f(x_1, \ldots, x_d) = & \left(\alpha_1^{(1)} \phi_1(x_1) + \cdots + \alpha_{n_1}^{(1)} \phi_{n_1}(x_1)\right) \\ & \times \ldots \\ & \times \left(\alpha_1^{(d)} \phi_1(x_d) + \cdots + \alpha_{n_d}^{(d)} \phi_{n_d}(x_d)\right) \end{aligned}$$

Canonical tensor format

Available representation

$$f(x_1, \ldots, x_d) = \sum_{k=1}^{r} \prod_{i=1}^{d} v_i^{(k)}(x_i)$$

with

$$v_i = \sum_{j=1}^{n_i^{(k)}} \alpha_j^{(i,k)} \phi_j(x_i)$$

linear number of terms wrt dimension

$$P = r \sum_{i=1}^{d} n_i$$

Alternating Least Squares

Alternating Least Squares algorithm

Allows to learn a rank-one tensor.

Algorithm 1 ALS

```
Initialize v_i(x_i) = 1
while v does not converge do
    for i = 1 to d do
        [Ψ^i(x)]_j = ∏_{u=1≠i}^d v_u(x_u) φ_j^i(x_i)
        Solve argmin_{β_i} ||y − Ψ^i(x)^t β_i||_2^2
    end for
end while
```

Greedy rank-one approximation

Alternating Least Squares algorithm

Allows to learn a rank-r tensor.

Algorithm 2 Greedy rank-one

1: Rank-1 approximation $\prod_{i=1}^{d} v_i^{(1)}(x_i)$
2: **for** $r = 2$ to r_{max} **do**
3: Rank-1 approximation $\prod_{i=1}^{d} v_i^{(r)}(x_i)$
4: $y^m = y - \sum_{k=1}^{r} \alpha_k \prod_{i=1}^{d} v_i^{(r)}(x_i)$
5: Update α to minimize error (least-squares)
6: **end for**

Usage

In Python...

```
import openturns as ot

def model(X):
    ...
    return Y

r = 4
dim = 5
nk = [10] * dim
factC = [ot.FourierSeriesFactory()] * dim
prod = ot.OrthogonalProductFunctionFactory(factC)
X = dist.getSample(size)
Y = model(X)
algo = ot.TensorApproximationAlgorithm(X, Y, dist, factC, nk, r)
```

Conclusion

Conclusions

1. Greedy rank-1
2. Regularized greedy rank-one

Perspectives

- Rank-M