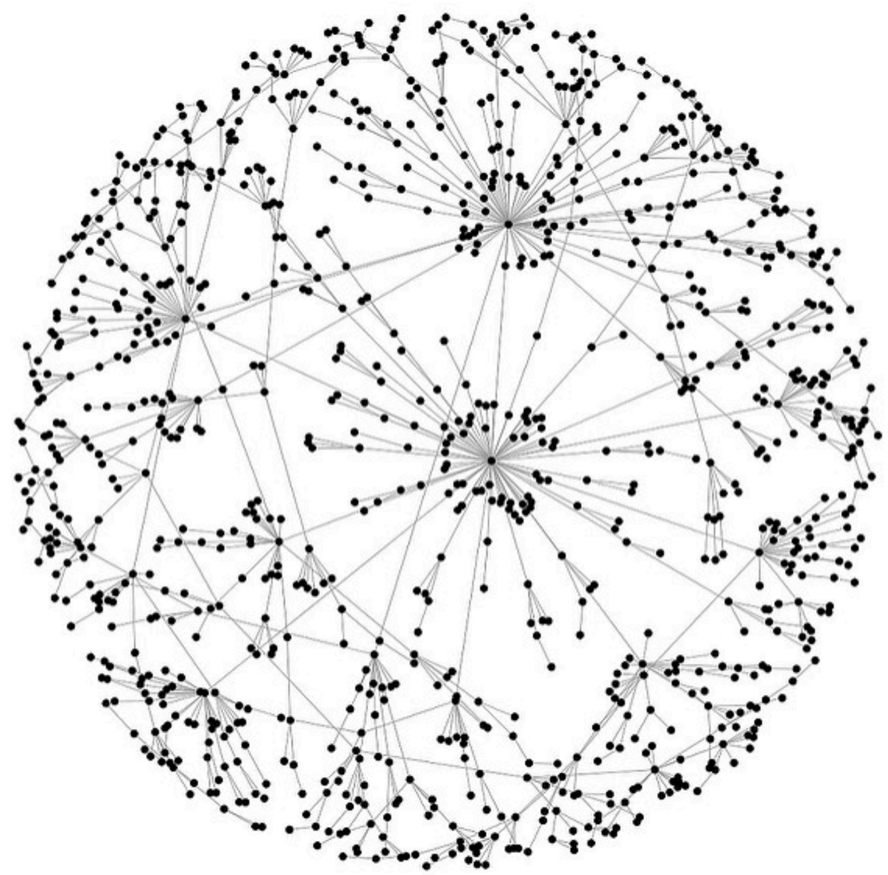
Complex Networks

Scale-free networks

Class 7



Dr. Márton Karsai

ENS Lyon 2016

Todays schedule

1. Scale-free properties and observations
2. Evolving networks and preferential attachment
3. BA network model
4. Alternative models

Slides are available:

http://perso.ens-lyon.fr/marton.karsai/protected/

login: complexnet

password: cnet123

Scale-free networks

Scale-free networks - first observations

Networks of scientific papers **Derek J. de Solla Price**, Science (1965)

- Nodes: scientific papers, Links: citations between them
- Number of citations to scientific papers shows a heavy-tailed distribution
- It can be characterised as a Pareto distribution or power-law distribution

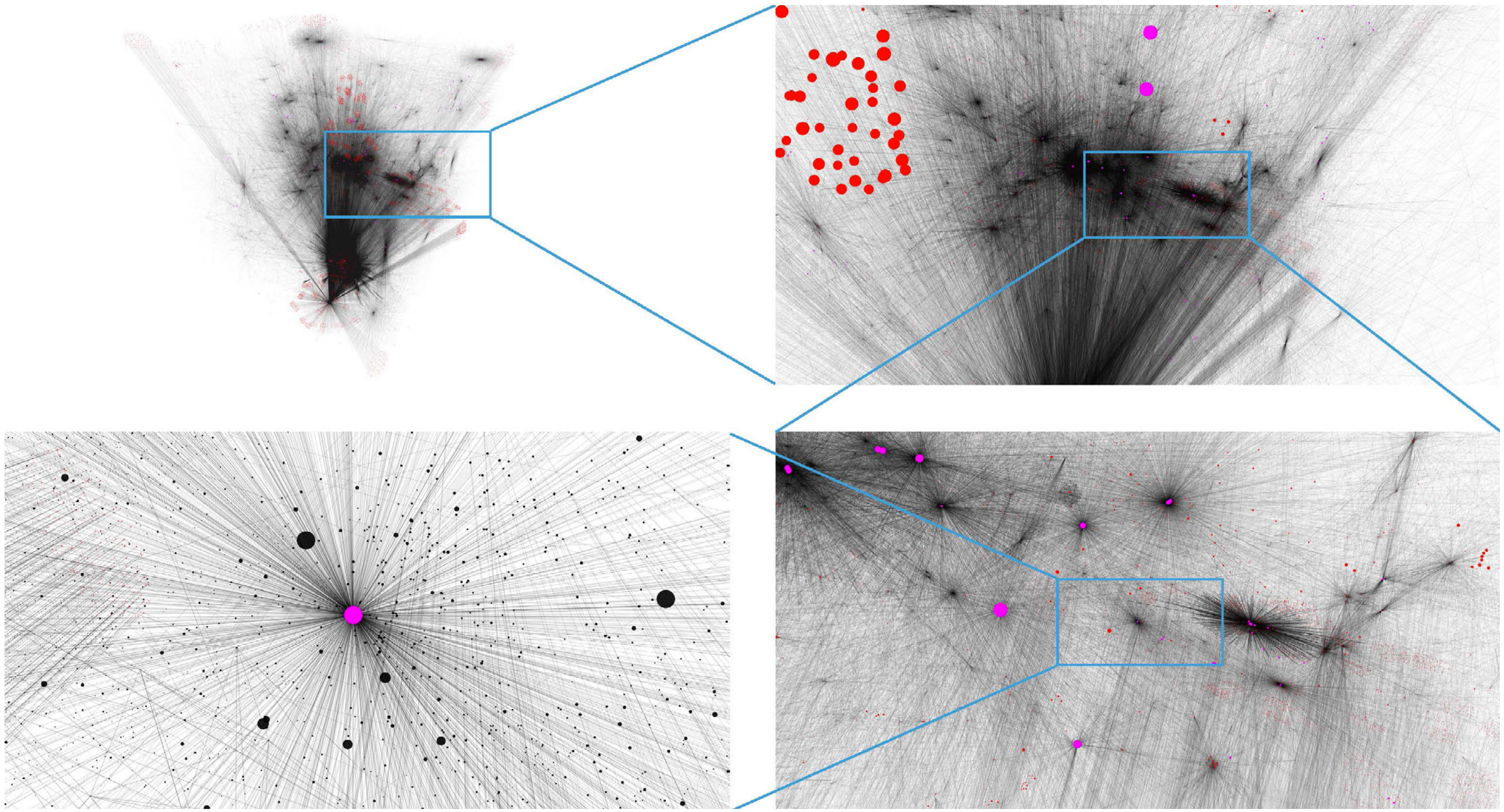
Structure of the WWW **R. Albert, H. Jeong, A-L Barabási**, Nature (1999)

- Nodes: WWW documents, Links: URL links
- More than 3 billions of documents
- Collection by a robot which explores all URL links in a document (web site) and follow them recursively
- They found a heavy-tailed degree distribution which could be well approximated with a power-law function

$$P(k) \sim k^{-\gamma}$$

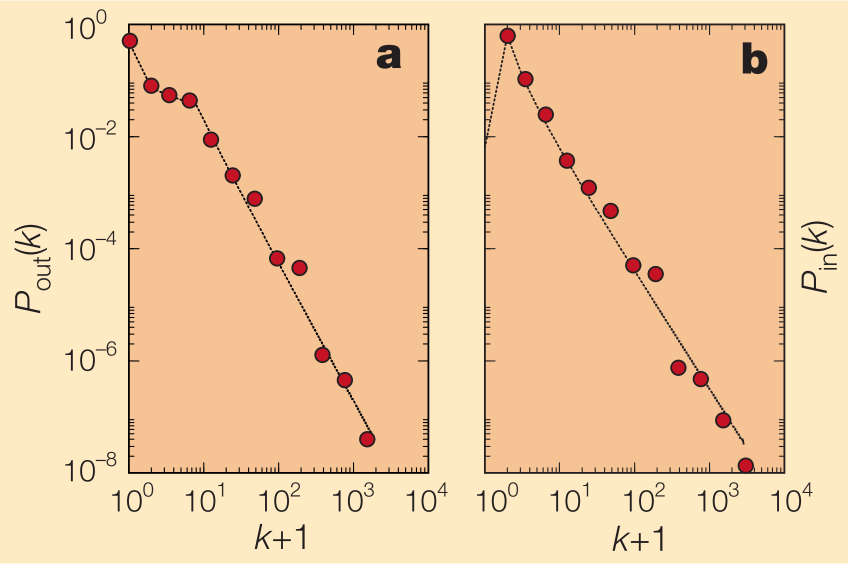
- It is a scale-free network

Scale-free networks - first observations



AL Barabási, Network Science Book (2013)

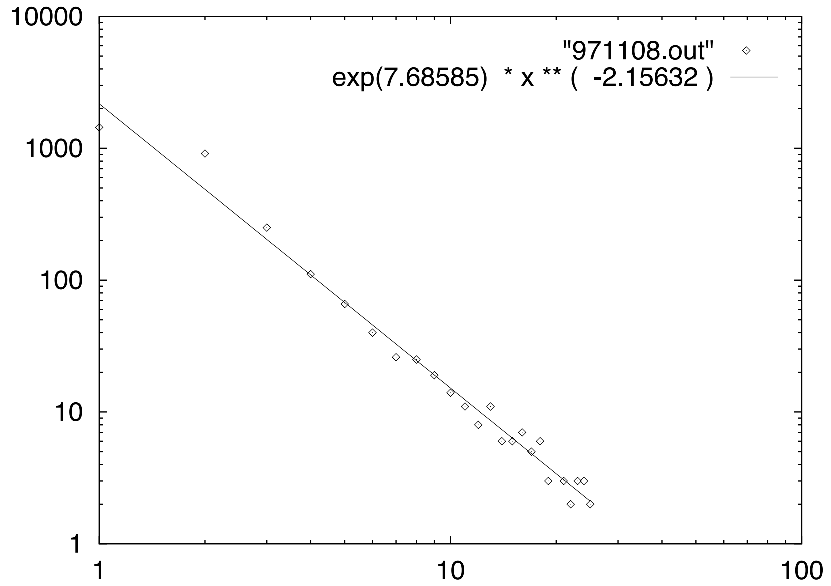
R. Albert, H. Jeong, A-L Barabási, Nature (1999)



Scale-free networks - other examples

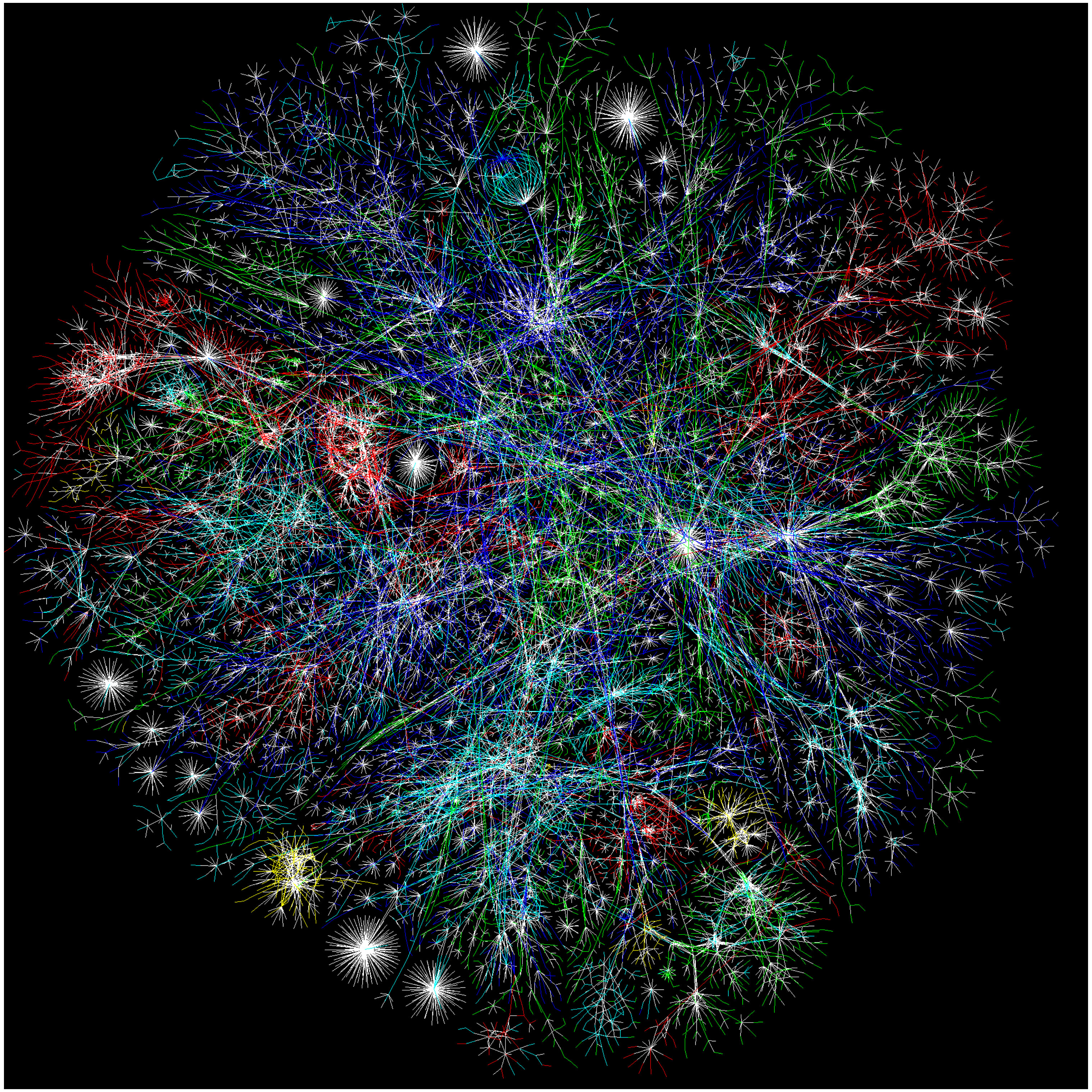
The internet

- Nodes: routers
- Links: Physical wires



(a) Int-11-97

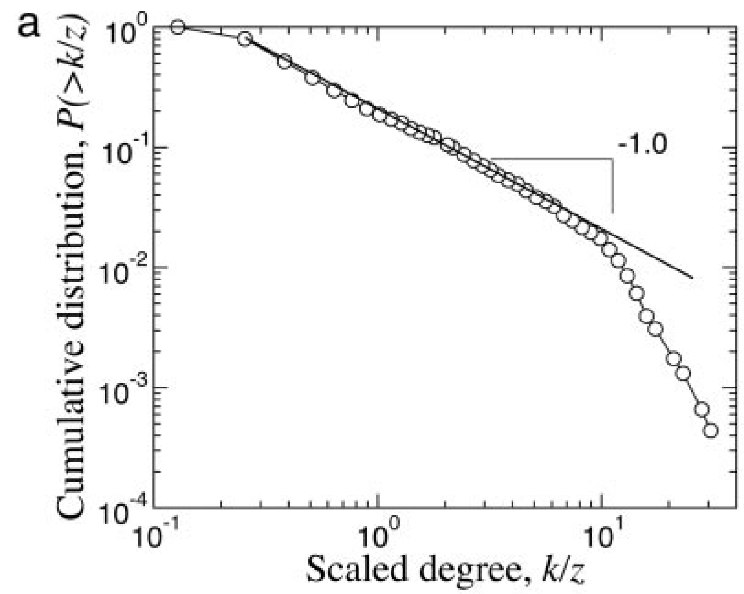
Faloutsos, Faloutsos and Faloutsos (1999)



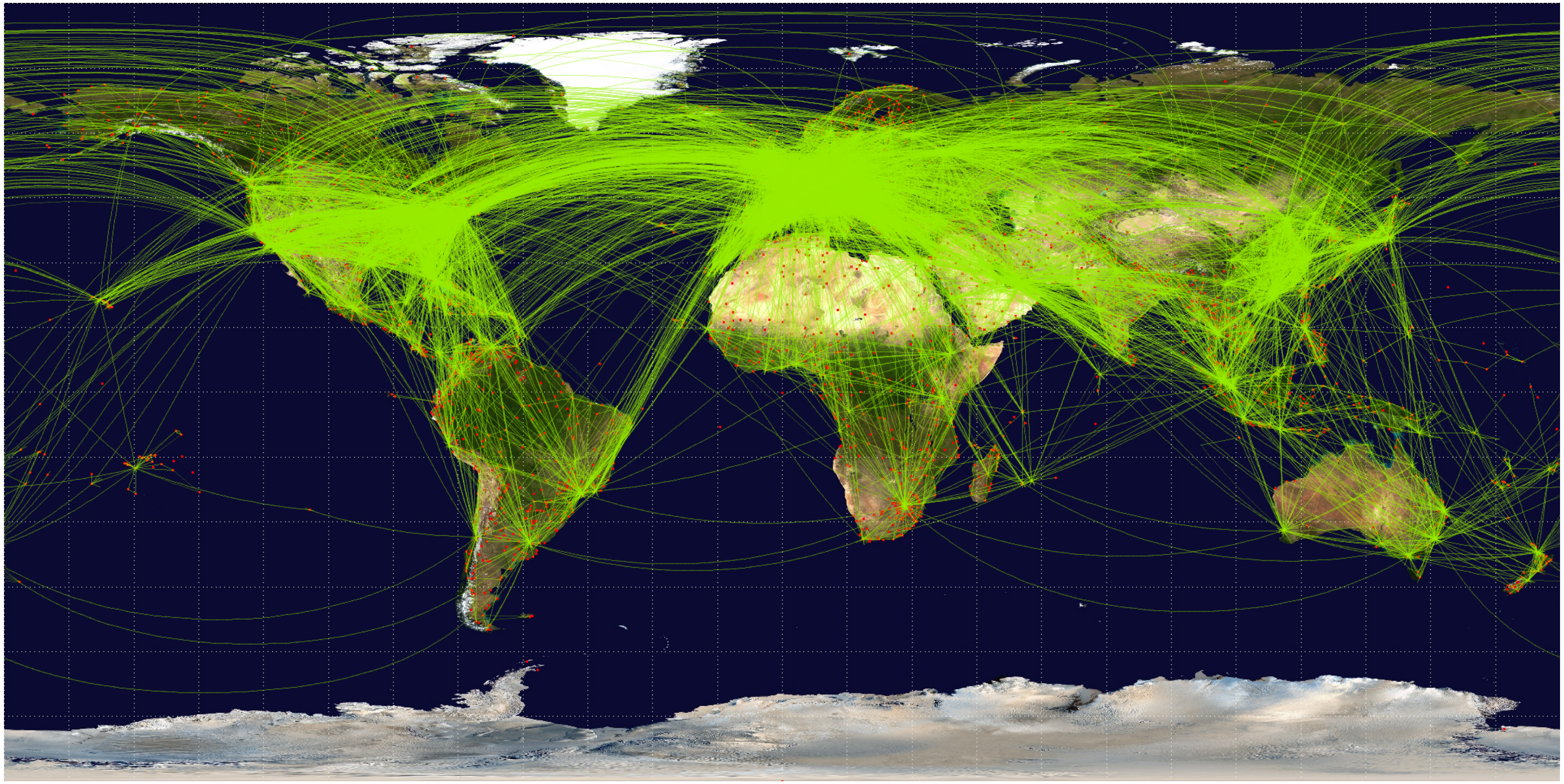
Scale-free networks - other examples

Airline route map network

- Nodes: airports
- Links: airplane connections



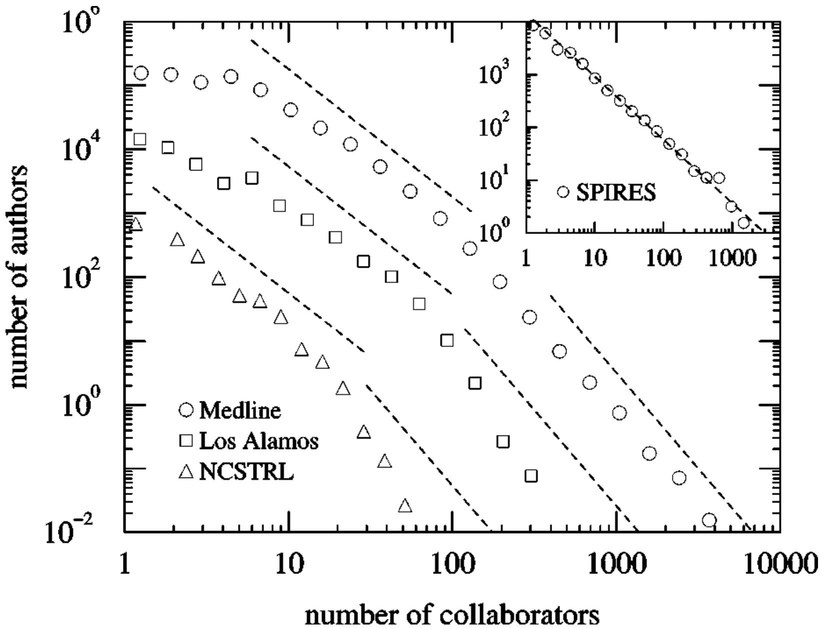
Guimera et.al. (2004)



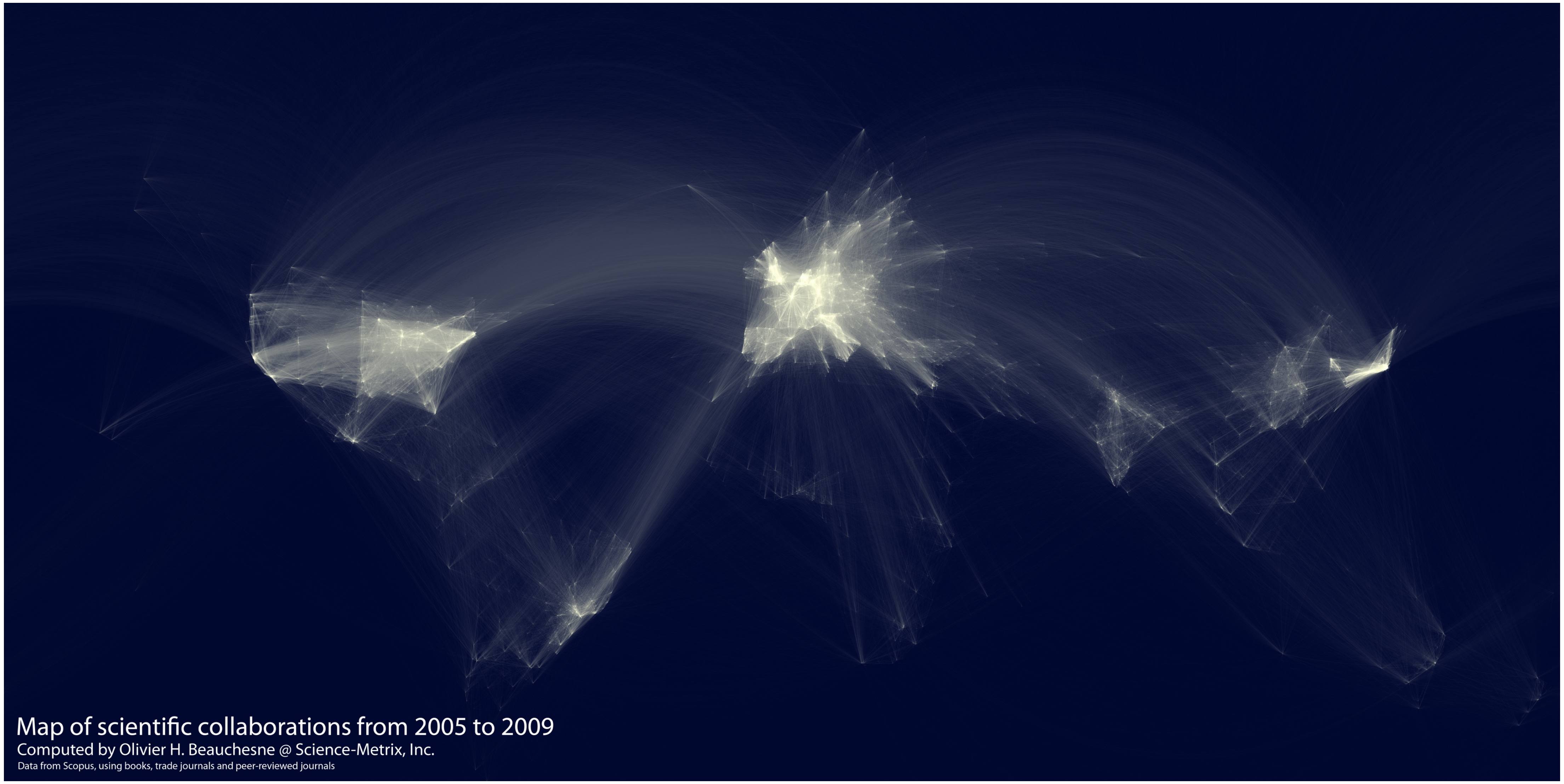
Scale-free networks - other examples

Scientific collaborations

- Nodes: scientists (here geo-localised)
- Links: common papers



Newman (2001)

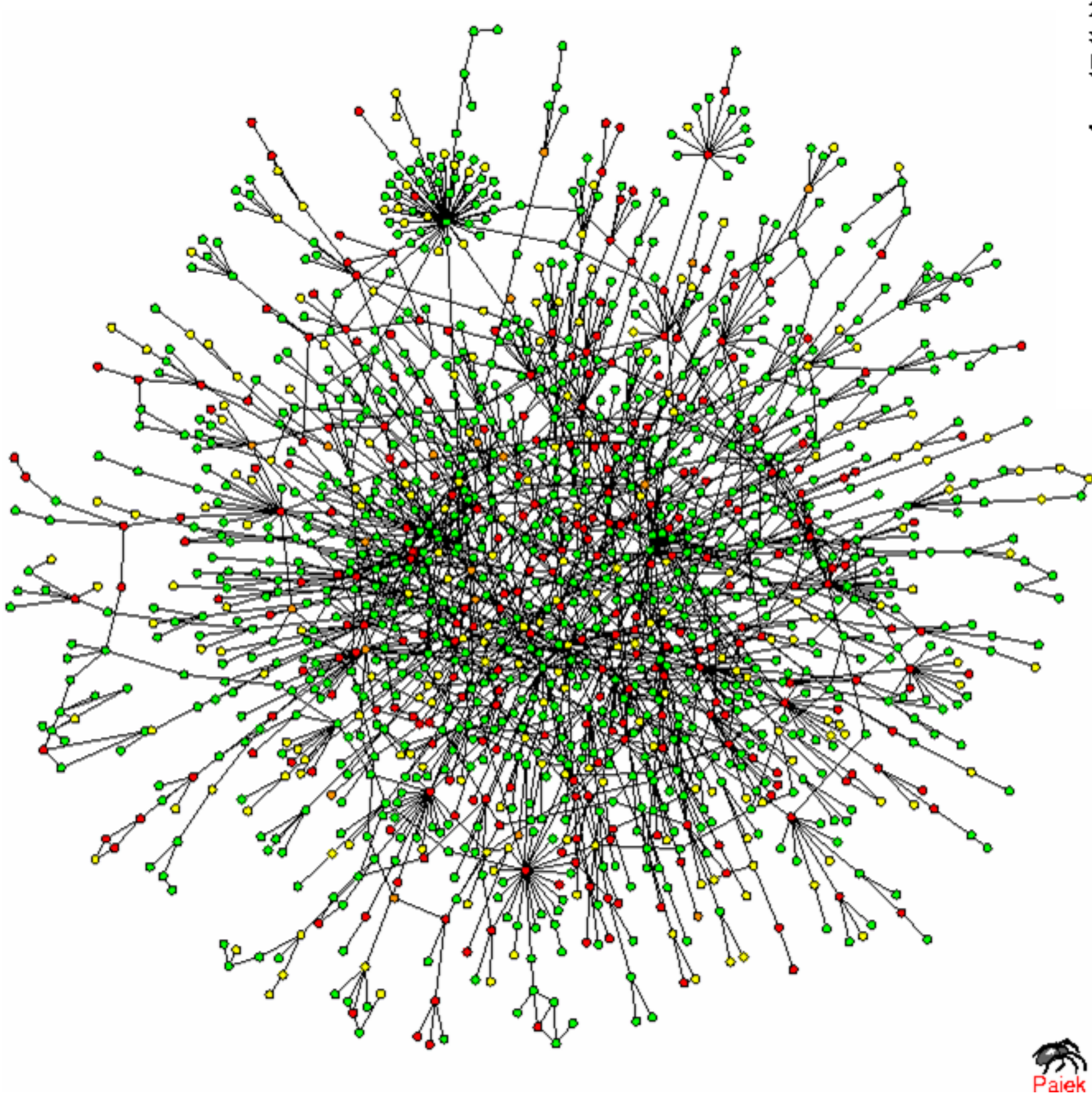


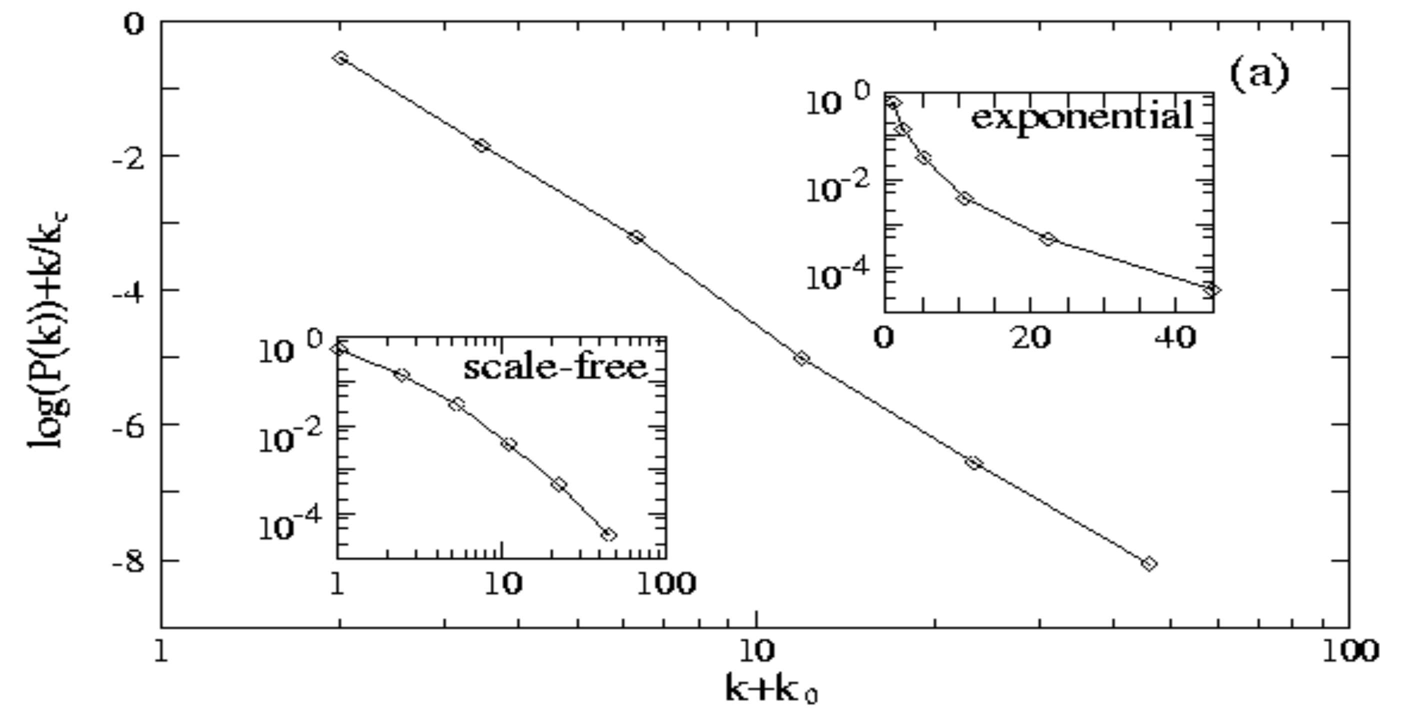
Scale-free networks - other examples

Protein networks

Jeong et.al. (2001)

- Nodes: proteins
- Links: physical interactions-binding





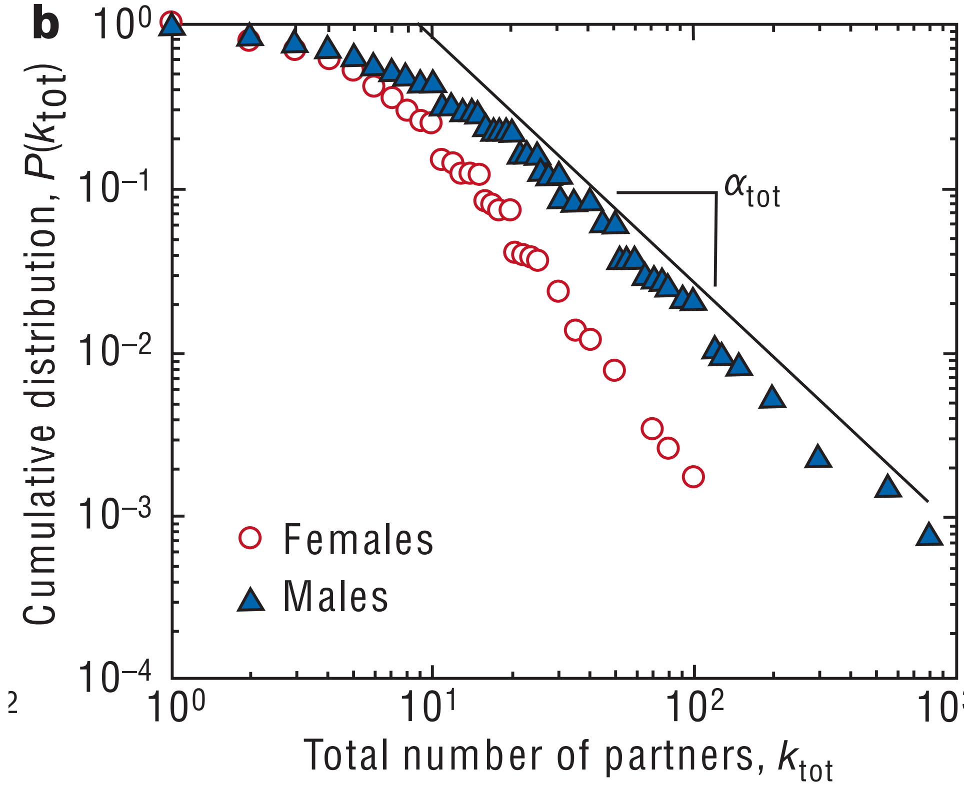
$$P(k) \sim (k+k_0)^{-\gamma} \exp\left(-\frac{k+k_0}{k_\tau}\right)$$

Scale-free networks - other examples

Sexual-interaction networks

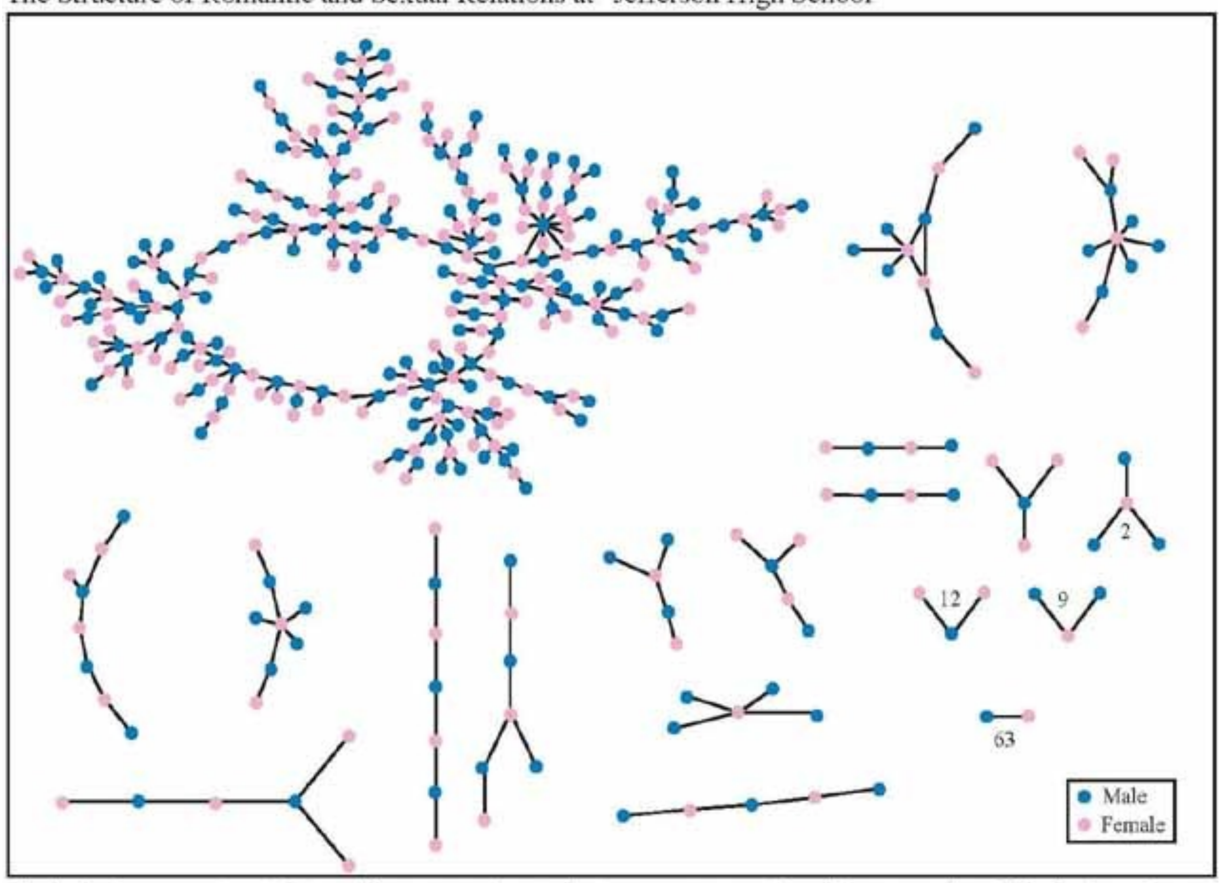
- Nodes: individuals
- Links: sexual incursion

Liljeros et.al. (2001)



Bearman et.al. (2004)

The Structure of Romantic and Sexual Relations at "Jefferson High School"

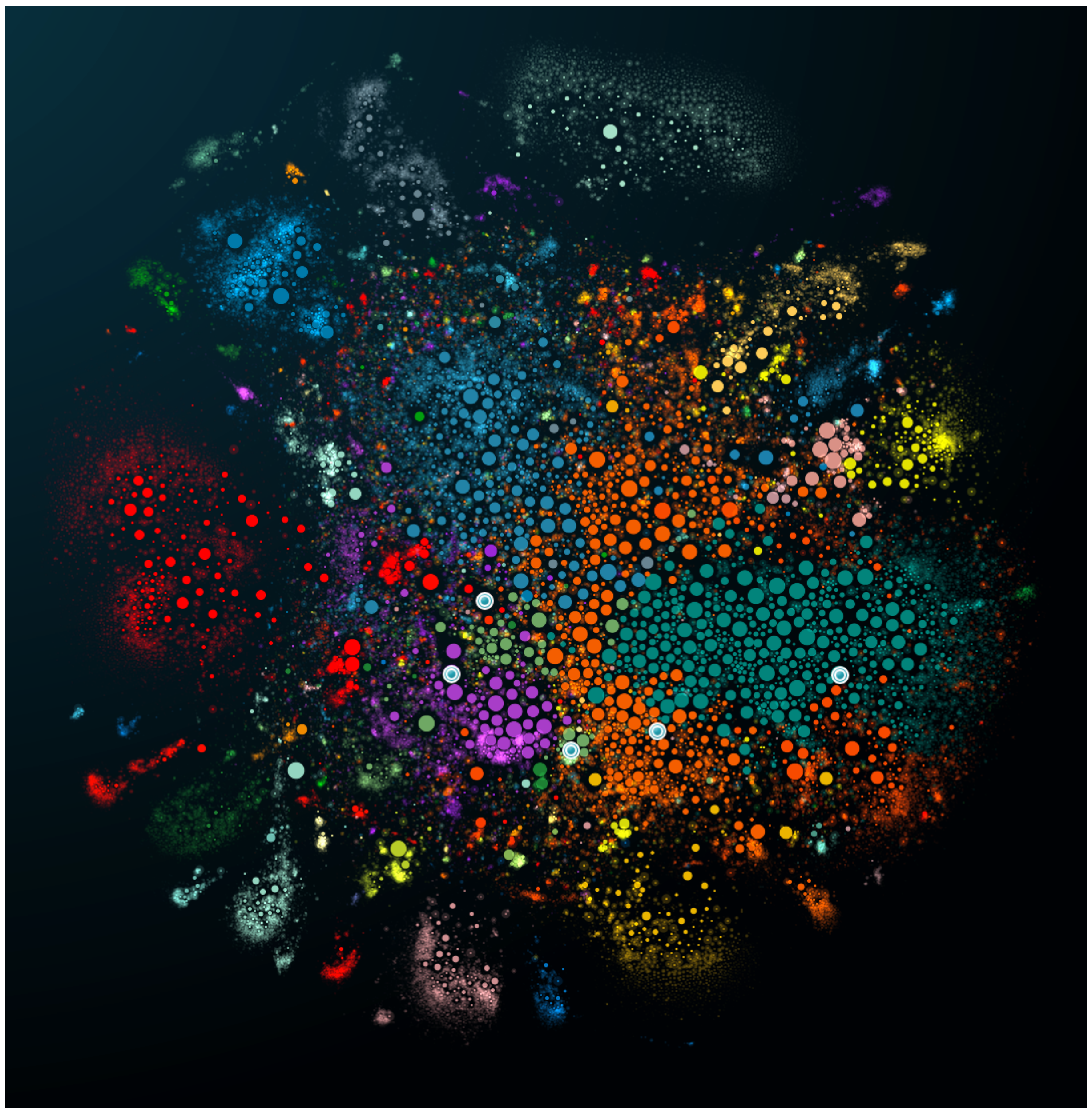


Each circle represents a student and lines connecting students represent romantic relations occuring within the 6 months preceding the interview. Numbers under the figure count the number of times that pattern was observed (i.e. we found 63 pairs unconnected to anyone else).

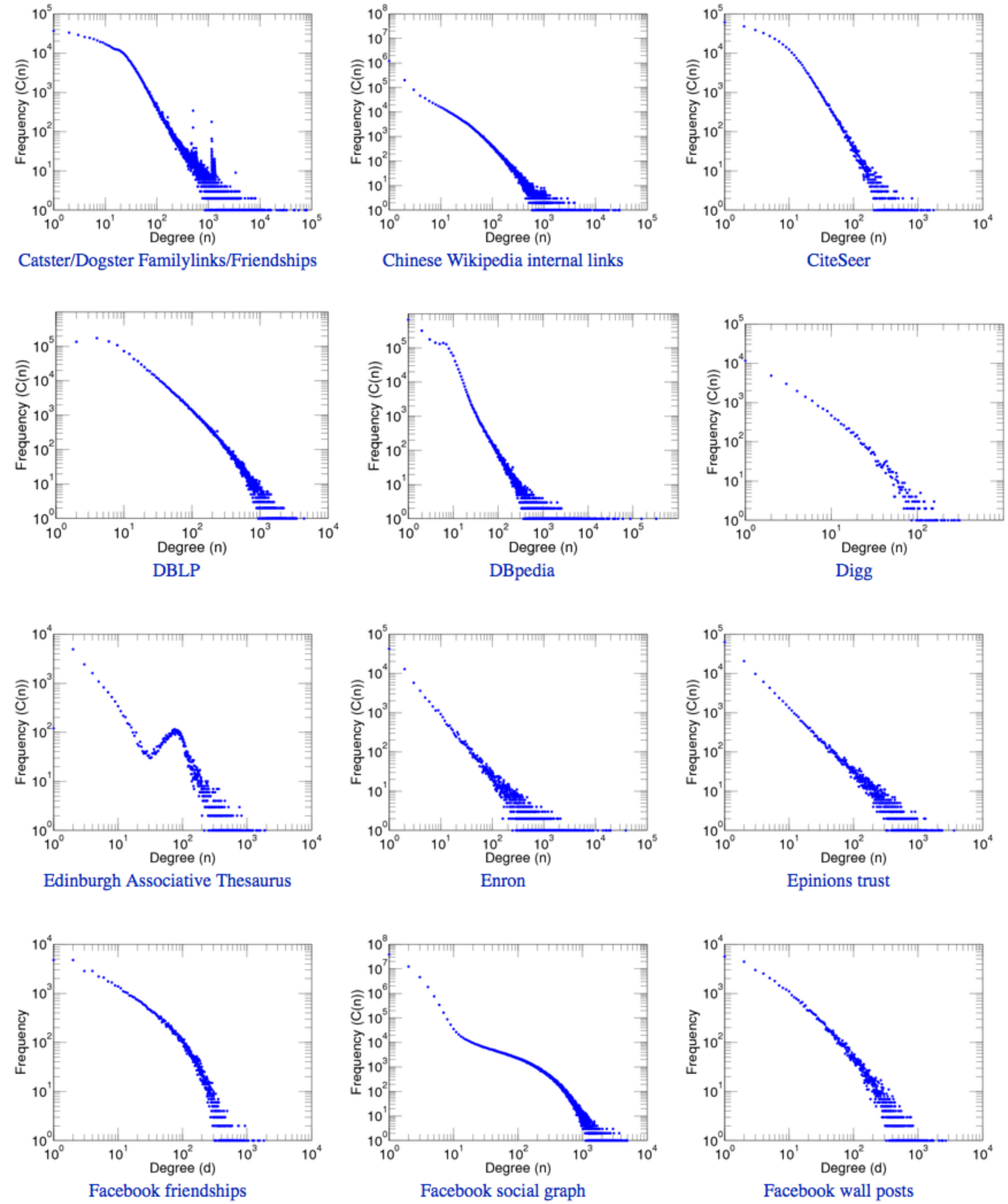
Scale-free networks - other examples

Online social networks

- Nodes: individuals
- Links: online interactions



Social network of Steam
http://85.25.226.110/mapper



Scale-free networks

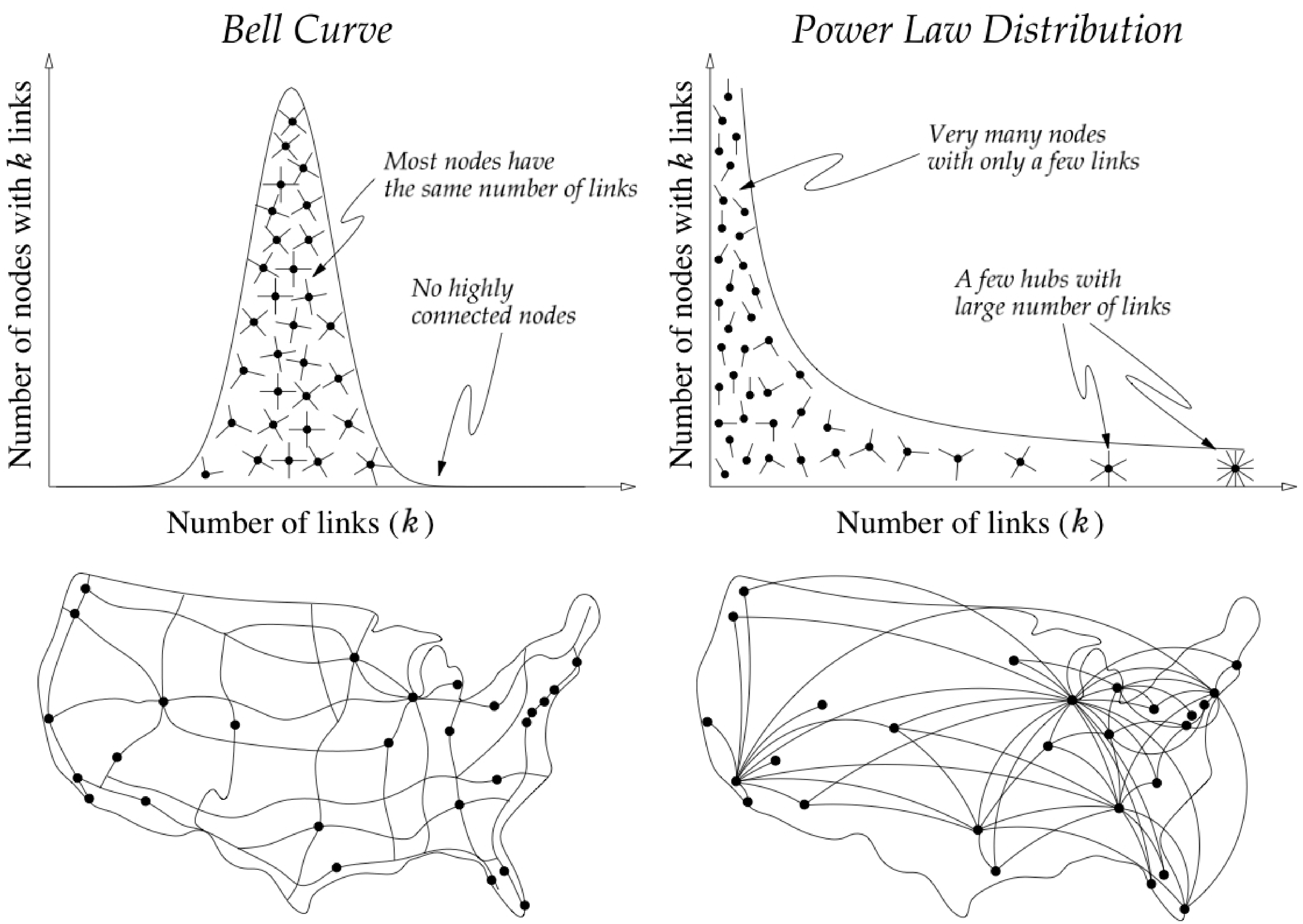
Network	Size	$\langle k \rangle$	κ	γ_{out}	γ_{in}	ℓ_{real}	ℓ_{rand}	ℓ_{pow}	Reference
WWW	325 729	4.51	900	2.45	2.1	11.2	8.32	4.77	Albert, Jeong, and Barabási 1999
WWW	4×10^7	7		2.38	2.1				Kumar *et al.*, 1999
WWW	2×10^8	7.5	4000	2.72	2.1	16	8.85	7.61	Broder *et al.*, 2000
WWW, site	260 000				1.94				Huberman and Adamic, 2000
Internet, domain*	3015–4389	3.42–3.76	30–40	2.1–2.2	2.1–2.2	4	6.3	5.2	Faloutsos, 1999
Internet, router*	3888	2.57	30	2.48	2.48	12.15	8.75	7.67	Faloutsos, 1999
Internet, router*	150 000	2.66	60	2.4	2.4	11	12.8	7.47	Govindan, 2000
Movie actors*	212 250	28.78	900	2.3	2.3	4.54	3.65	4.01	Barabási and Albert, 1999
Co-authors, SPIRES*	56 627	173	1100	1.2	1.2	4	2.12	1.95	Newman, 2001b
Co-authors, neuro.*	209 293	11.54	400	2.1	2.1	6	5.01	3.86	Barabási *et al.*, 2001
Co-authors, math.*	70 975	3.9	120	2.5	2.5	9.5	8.2	6.53	Barabási *et al.*, 2001
Sexual contacts*	2810			3.4	3.4				Liljeros *et al.*, 2001
Metabolic, *E. coli*	778	7.4	110	2.2	2.2	3.2	3.32	2.89	Jeong *et al.*, 2000
Protein, *S. cerev.**	1870	2.39		2.4	2.4				Jeong, Mason, *et al.*, 2001
Ythan estuary*	134	8.7	35	1.05	1.05	2.43	2.26	1.71	Montoya and Solé, 2000
Silwood Park*	154	4.75	27	1.13	1.13	3.4	3.23	2	Montoya and Solé, 2000
Citation	783 339	8.57			3				Redner, 1998
Phone call	53×10^6	3.16		2.1	2.1				Aiello *et al.*, 2000
Words, co-occurrence*	460 902	70.13		2.7	2.7				Ferrer i Cancho and Solé, 2001
Words, synonyms*	22 311	13.48		2.8	2.8				Yook *et al.*, 2001b

Albert, R. et.al. Rev. Mod. Phy. (2002)

Exponents of real-world networks are usually between 2 and 3

Scale-free distribution

What does it mean?



AL. Barabási, Linked (2002)

Degree fluctuations have no characteristic scale (scale invariant)

Scale-free distribution - discrete formalism

Degree distribution: $p_k = Ck^{-\gamma}$

$$\sum_{k=1}^{\infty} p_k = 1 \qquad C\sum_{k=1}^{\infty} k^{-\gamma} = 1 \qquad C = \frac{1}{\sum_{k=1}^{\infty} k^{-\gamma}} = \frac{1}{\zeta(\gamma)}$$

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \qquad \sigma = \Re(s) > 1.$$

Riemann Zeta function

$$p_k = \frac{k^{-\gamma}}{\zeta(\gamma)}$$

for k>0 (i.e. we assume that there are no disconnected nodes in the network)

Scale-free distribution - continuous formalism

$$P(k) = Ck^{-\gamma} \qquad k = [K_{\min}, \infty)$$

$$\int_{K_{\min}}^{\infty} P(k)dk = 1 \qquad C = \frac{1}{\int_{K_{\min}}^{\infty} k^{-\gamma} dk} = (\gamma - 1)K_{\min}^{\gamma - 1}$$

$$P(k) = (\gamma - 1)K_{\min}^{\gamma - 1} k^{-\gamma}$$

Scale-free distribution - divergencies

$$P(k) = Ck^{-\gamma} \qquad k = [k_{\min}, \infty)$$

$$\int_{k_{\min}}^{\infty} P(k) dk = 1$$

$$C = \frac{1}{\int_{k_{\min}}^{\infty} k^{-\gamma}\, dk} = (\gamma - 1) k_{\min}^{\gamma-1}$$

$$P(k) = (\gamma - 1) k_{\min}^{\gamma-1} k^{-\gamma}$$

The m*th* moment

$$< k^m >= \int_{k_{\min}}^{\infty} k^m P(k) dk \qquad < k^m >= (\gamma - 1) k_{\min}^{\gamma-1} \int_{k_{\min}}^{\infty} k^{m-\gamma}\, dk = \frac{(\gamma - 1)}{(m - \gamma + 1)} k_{\min}^{\gamma-1} \left[k^{m-\gamma+1} \right]_{k_{\min}}^{\infty}$$

Reminder

- average: $\langle k \rangle$ is the 1st moment
- STD: $\sigma(k) = \sqrt{\langle k^2 \rangle - \langle k \rangle^2}$ is calculable from the 1st and 2nd moment

Scale-free distribution - divergencies

The m*th* moment

$$< k^m >= \int_{k_{\min}}^{\infty} k^m P(k) dk \qquad < k^m >= (\gamma - 1) k_{\min}^{\gamma-1} \int_{k_{\min}}^{\infty} k^{m-\gamma}\, dk = \frac{(\gamma - 1)}{(m - \gamma + 1)} k_{\min}^{\gamma-1} \left[k^{m-\gamma+1} \right]_{k_{\min}}^{\infty}$$

Consequence

If m-γ+1<0: the integral converges

If m-γ+1>0, the integral diverges.

For a fixed γ this means that all moments with m>γ-1 diverge.

In other words…

γ	⟨k⟩	σ	
γ<2	divergent	divergent	anomalous regime
2≦γ<3	finite	divergent	scale-free regime
γ>3	finite	finite	random regime

Scale-free distribution - continuous formalism

Network	Size	$\langle k \rangle$	κ	γ_{out}	γ_{in}
WWW	325 729	4.51	900	2.45	2.1
WWW	4×10^7	7		2.38	2.1
WWW	2×10^8	7.5	4000	2.72	2.1
WWW, site	260 000				1.94
Internet, domain*	3015–4389	3.42–3.76	30–40	2.1–2.2	2.1–2.2
Internet, router*	3888	2.57	30	2.48	2.48
Internet, router*	150 000	2.66	60	2.4	2.4
Movie actors*	212 250	28.78	900	2.3	2.3
Co-authors, SPIRES*	56 627	173	1100	1.2	1.2
Co-authors, neuro.*	209 293	11.54	400	2.1	2.1
Co-authors, math.*	70 975	3.9	120	2.5	2.5
Sexual contacts*	2810			3.4	3.4
Metabolic, *E. coli*	778	7.4	110	2.2	2.2
Protein, *S. cerev.**	1870	2.39		2.4	2.4
Ythan estuary*	134	8.7	35	1.05	1.05
Silwood Park*	154	4.75	27	1.13	1.13
Citation	783 339	8.57			3
Phone call	53×10^6	3.16		2.1	2.1
Words, co-occurrence*	460 902	70.13		2.7	2.7
Words, synonyms*	22 311	13.48		2.8	2.8

Albert, R. et.al. Rev. Mod. Phy. (2002)

- Exponents of real-world networks are usually between 2 and 3
- $\Rightarrow \langle k^2 \rangle$ diverges if $N \to \infty$
- Consequently:

$$\sigma_k = (\langle k^2 \rangle - \langle k \rangle^2)^{1/2} \to \infty$$

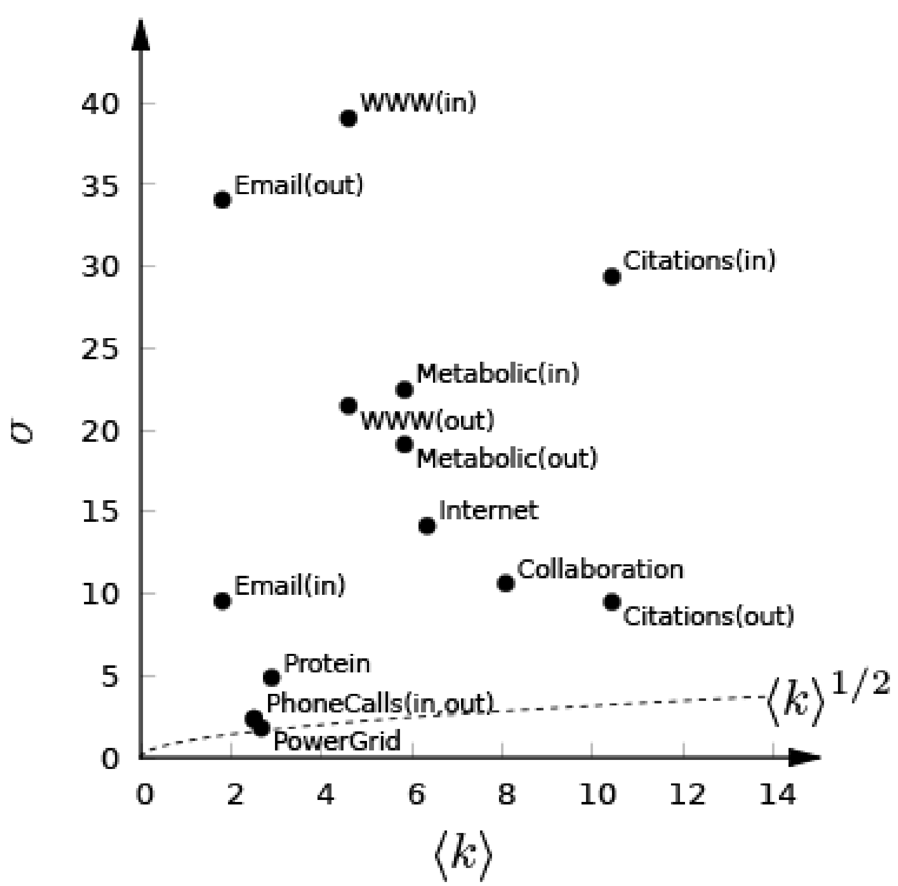
$$k = \langle k \rangle \pm \sigma_k = \langle k \rangle \pm \infty$$

- Average values are meaningless since the fluctuations are are infinitely large

Scale-free networks

NETWORK	NL		$\langle k \rangle$ $\langle k_{in} \rangle = \langle k_{out} \rangle$	σ_{in}	σ_{out}	σ	γ_{in}	γ_{out}	γ
Internet	192,244	609,066	6.34	-	-	14.14	-	-	3.42*
WWW	325,729	1,497,134	4.60	39.05	21.48	-	2.31	2.00	-
Power Grid	4,941	6,594	2.67	-	-	1.79	-	-	Exp.
Mobile Phone Calls	36,595	91,826	2.51	2.39	2.32	-	4.69*	5.01*	-
Email	57,194	103,731	1.81	9.56	34.07	-	3.43*	2.03	-
Science Collaboration	23,133	93,439	8.08	-	-	10.63	-	-	3.35
Actor Network	702,388	29,397,908	83.71	-	-	200.86	-	-	2.12
Citation Network	449,673	4,689,479	10.43	29.37	9.49	-	3.03**	4.00	-
E. Coli Metabolism	1,039	5,802	5.58	22.46	19.12	-	2.43	2.90	-
Yeast Protein Interactions	2,018	2,930	2.90	-	-	4.88	-	-	2.89*

AL Barabási, Network Science Book (2013)



- Each network has larger degree fluctuation than equivalent random networks
- Exceptions:
  - Power-grid network - not scale-free
  - Mobile-call network - very high degree exponent

Scale-free distribution

- What if the network is FINITE? …this is the case in each real-world network
  - There is an expected maximum degree K_{max}

Estimating K_{max}

$$\int_{K_{\max}}^{\infty} P(k)dk \approx \frac{1}{N}$$

Why: the probability to have a node larger than $\boldsymbol{K_{max}}$ should not exceed the prob. to have one node, i.e. ***1/N*** fraction of all nodes

$$\int_{K_{\max}}^{\infty} P(k)dk = (\gamma-1)K_{\min}^{\gamma-1}\int_{K_{\max}}^{\infty} k^{-\gamma}dk = \frac{(\gamma-1)}{(-\gamma+1)}K_{\min}^{\gamma-1}\left[k^{-\gamma+1}\right]_{K_{\max}}^{\infty} = \frac{K_{\min}^{\gamma-1}}{K_{\max}^{\gamma-1}} \approx \frac{1}{N}$$

$$K_{\max} = K_{\min}N^{\frac{1}{\gamma-1}}$$

- Because the $N<\infty$:
  - There is a cutoff of $P(k)$ at the maximum degree K_{max}
  - No node exists with degree larger than the system size

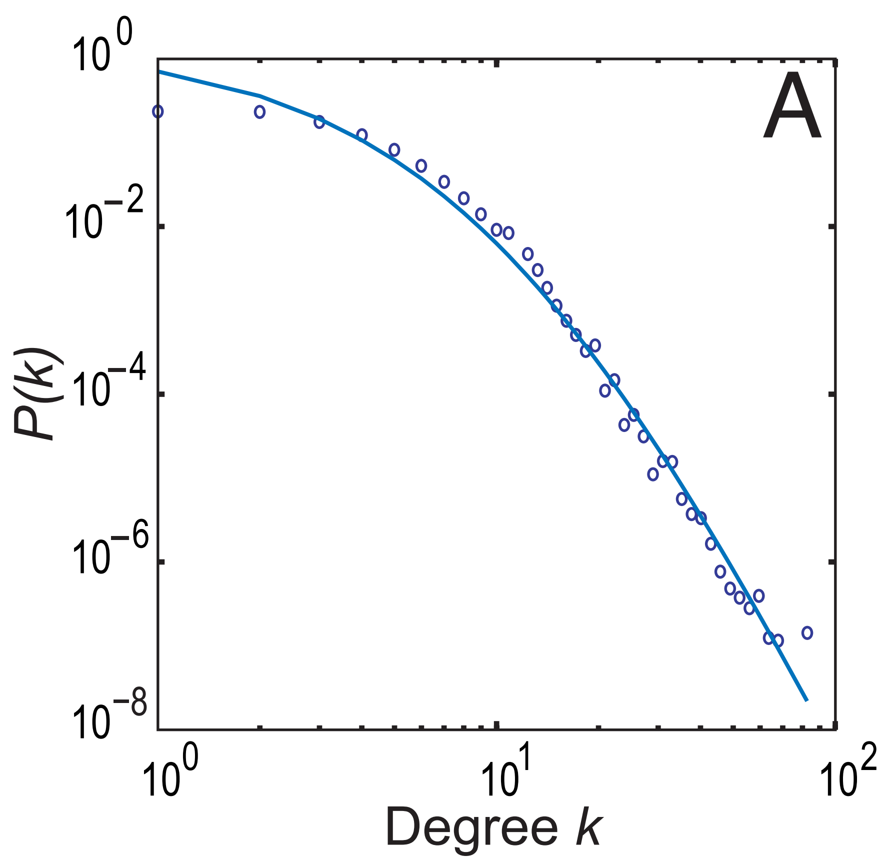
Scale-free networks

Why do most of the real networks have degree exponent between 2 and 3?

- To detect a scale-free network its degree distribution needs to span through several (at least 2-3) orders of magnitude ⇒ $K_{max}\sim 10^3$
- This constraints the system size very large if the degree exponent is large
- Example: let's choose $\gamma=5$, $K_{min}=1$ and $K_{max}\sim 10^3$

$$K_{\max} = K_{\min} N^{\frac{1}{\gamma-1}}$$

$$N = \left(\frac{K_{\max}}{K_{\min}}\right)^{\gamma-1} \approx 10^{12}$$



Onnela, et.al., PNAS (2007)

- Example: Mobile-call networks
  - $N=4.6\text{x}10^8$
  - $\gamma=6.2$

Scale-free networks - distances

$$
<l> \sim \begin{cases} const. & \gamma = 2 \\ \dfrac{\ln \ln N}{\ln(\gamma - 1)} & 2 < \gamma < 3 \\ \dfrac{\ln N}{\ln \ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}
$$

Ultra Small World

Small World

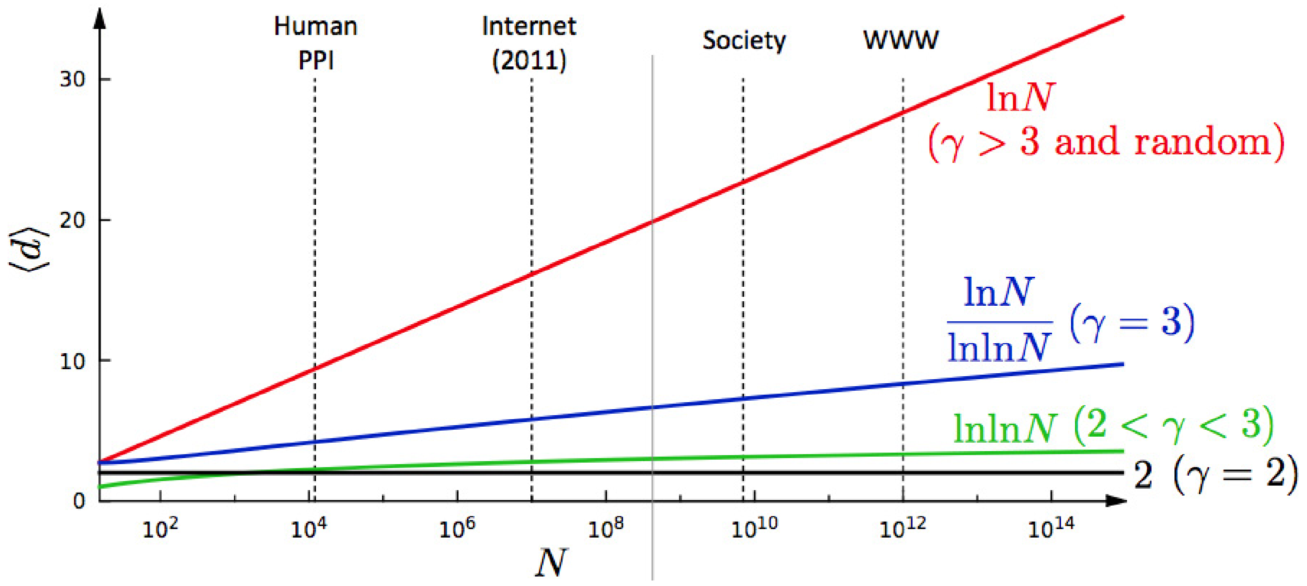
$\gamma = 2$: Size of the biggest hub is of order O(N). Most nodes can be connected within two layers of it, thus the average path length will be independent of the system size.

$2 < \gamma < 3$: The average path length increases slower than logarithmically. In a random network all nodes have comparable degree, thus most paths will have comparable length. In a scale-free network the vast majority of the path go through the few high degree hubs, reducing the distances between nodes.

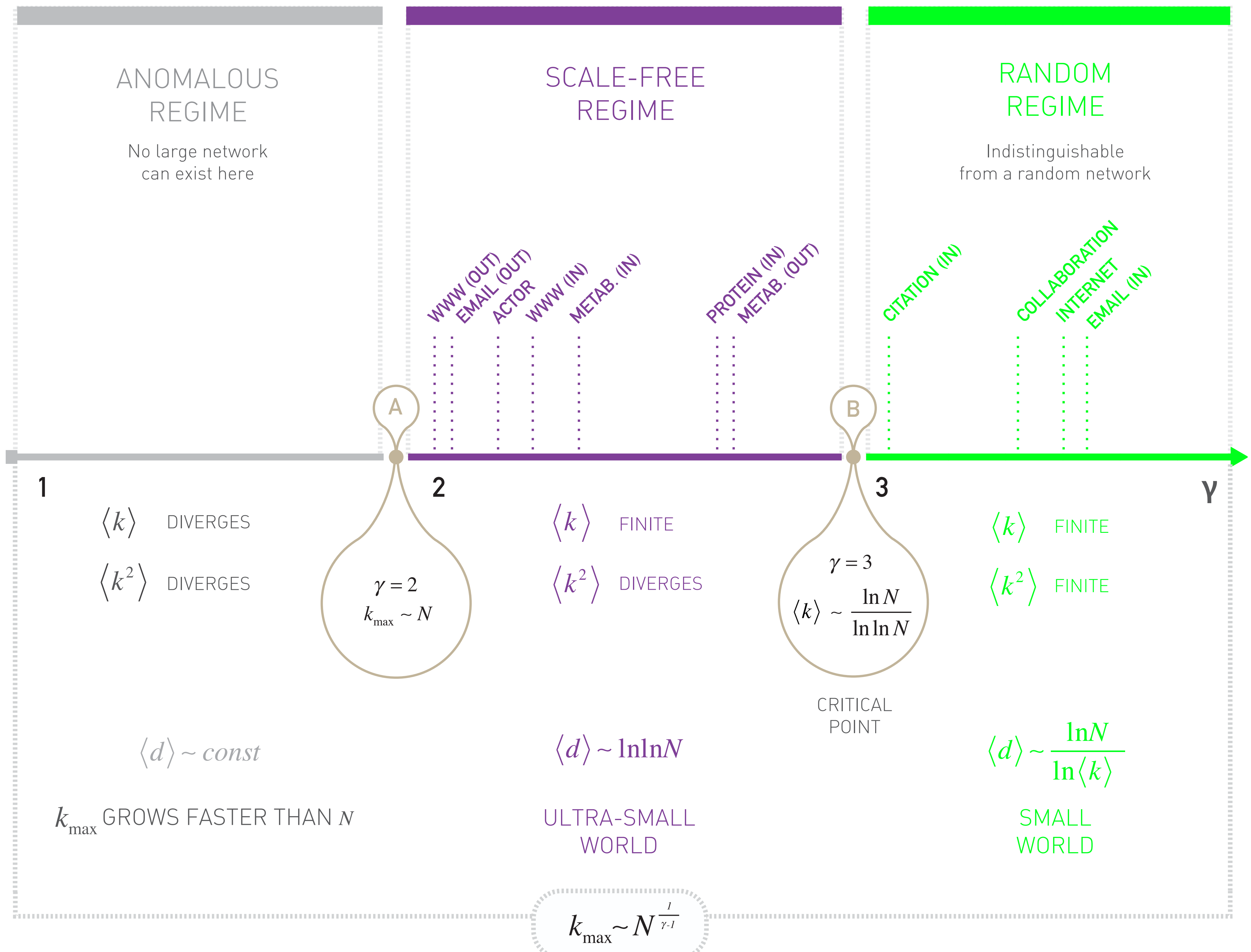
$\gamma = 3$: Some key models produce γ=3, so the result is of particular importance for them. This was first derived by Bollobas and collaborators for the network diameter in the context of a dynamical model, but it holds for the average path length as well.

$\gamma > 3$: The second moment of the distribution is finite, thus in many ways the network behaves as a random network. Hence the average path length follows the result that we derived for the random network model earlier.

Cohen, Havlin Phys. Rev. Lett. 90, 58701(2003); Cohen, Havlin and ben-Avraham, in Handbook of Graphs and Networks, Eds. Bornholdt and Shuster (Willy-VCH, NY, 2002) Chap. 4; Confirmed also by: Dorogovtsev et al (2002), Chung and Lu (2002); (Bollobas, Riordan, 2002; Bollobas, 1985; Newman, 2001



Scale-free networks - summary

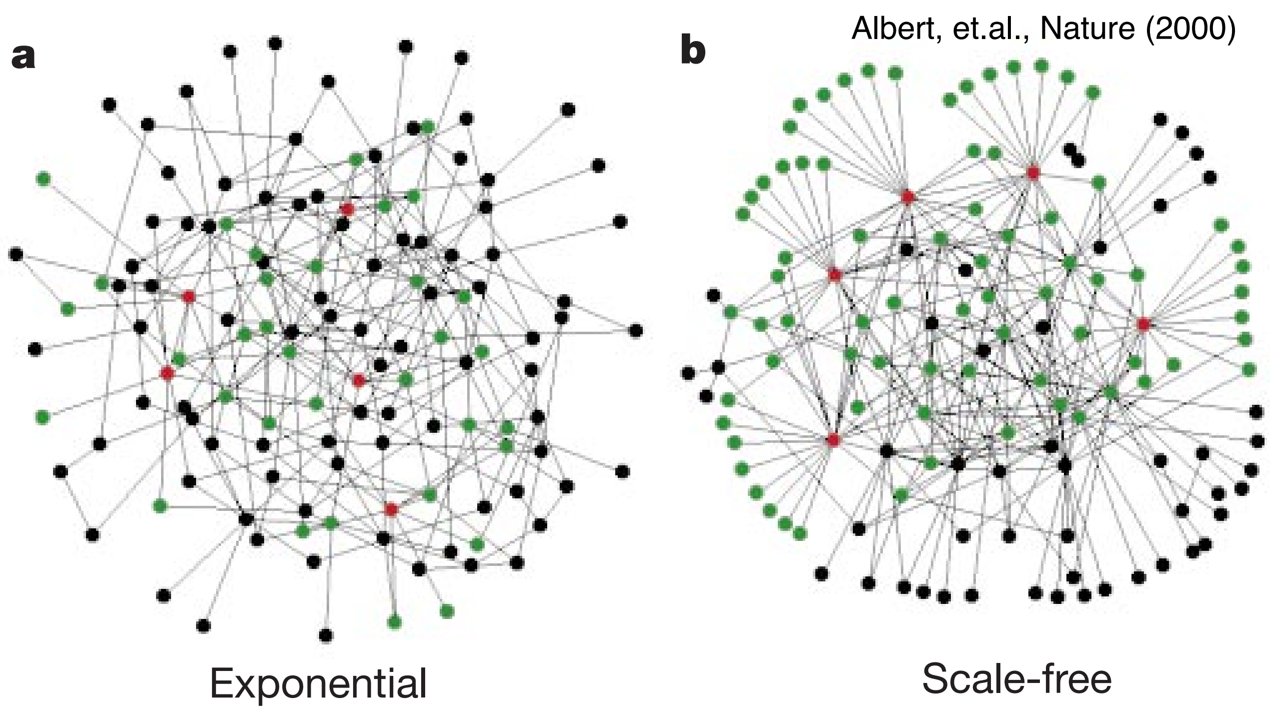


Slide from CCNR Complex Networks Course
A. L. Barabási 2014

Scale-free networks - role of hubs

Network robustness and attack tolerance

- How network topology is resistant agains failure and targeted attacks



(a) Poisson random graph

(b) Scale-free network

Both networks have the same parameters:

- N=130
- <k>=3.3

Numerical experiment:

1. Take a connected network
2. Remove nodes one at a time
3. Observe the size of the LCC

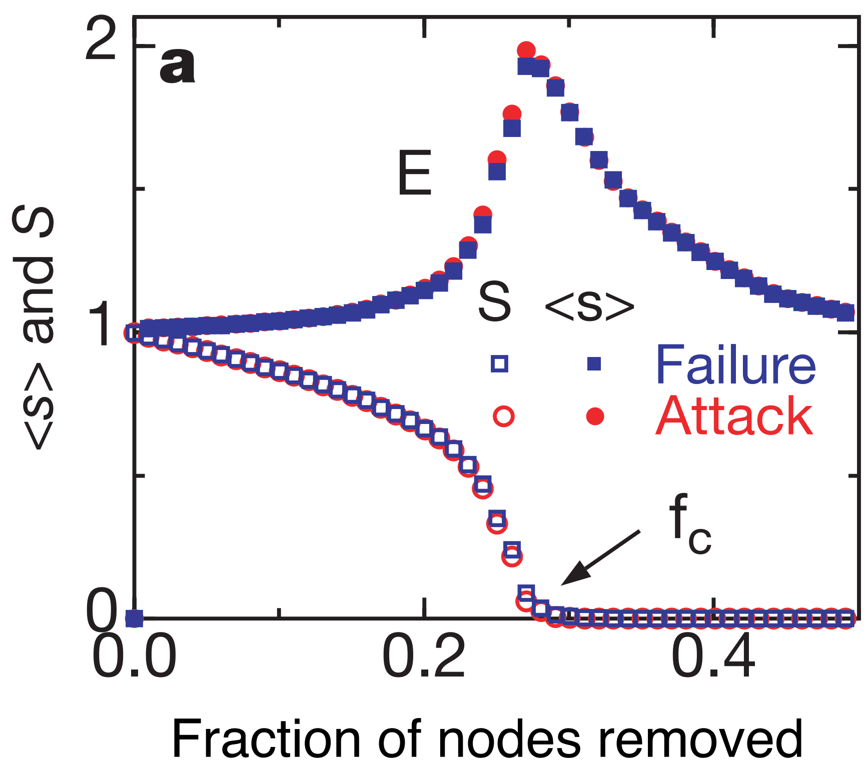
Node removal strategies:

1. Remove nodes randomly ("failures")
2. Remove nodes in descending order of their degrees, i.e. hubs first ("attacks")

Scale-free networks - role of hubs

Network robustness and attack tolerance

Albert, et.al., Nature (2000)



- S: relative size of the LCC
- $<s>$: average size of components other than LCC

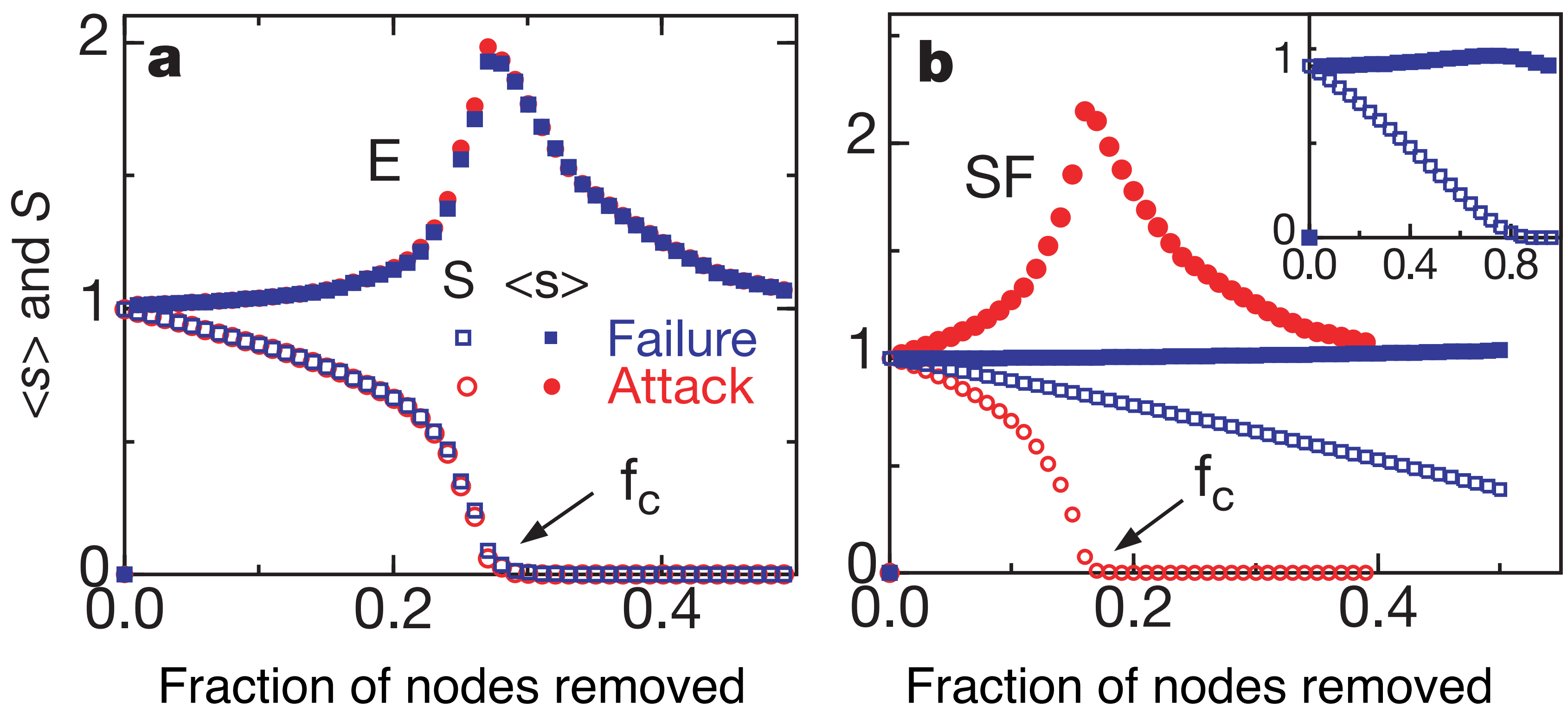
Poisson random graph

- Both removal methods give the same result
- The network falls apart after a finite fraction of nodes are removed ($S \to 0$)

Scale-free networks - role of hubs

Network robustness and attack tolerance

Albert, et.al., Nature (2000)



- S: relative size of the LCC
- $<s>$: average size of components other than LCC

Poisson random graph

- Both removal methods give the same result
- The network falls apart after a finite fraction of nodes are removed ($S \to 0$)

Scale-free network

- Robust against random removal (blue)
- Vulnerable against targeted attacks

Consequences:

- Internet is still working even several servers are out of service
- Random vaccination is not effective in case of epidemic spreading

The Barabási-Albert model

of scale-free networks

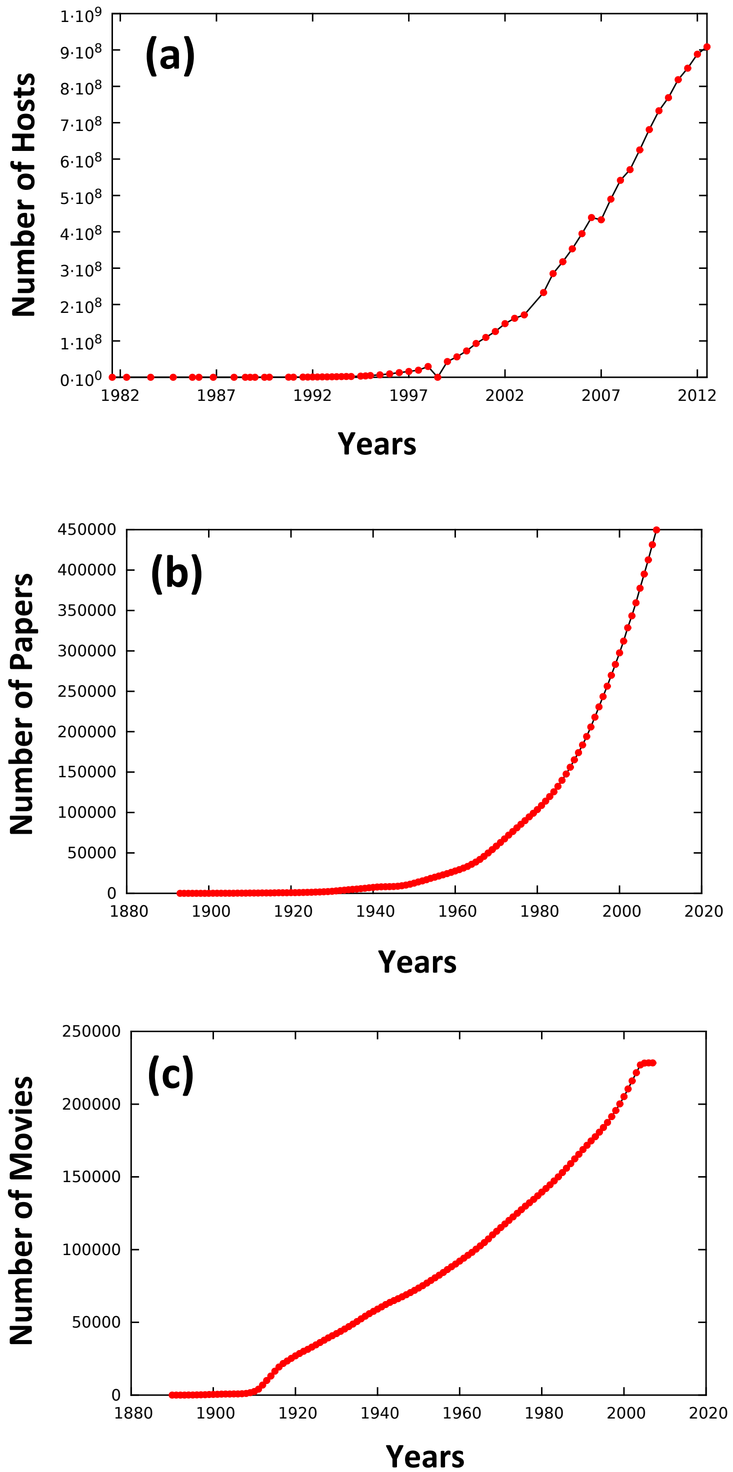
Emergence of hubs

What did we miss with the earlier network models?

1. Networks are evolving
   - Networks are not static but growing in time as new nodes are entering the system

2. Preferential attachement
   - Nodes are not connected randomly but tends to link to more attractive nodes

- Pólya urn model (1923)
- Yule process (1925)
- Zipf's law (1941)
- Cumulative advantage (1968)
- Preferential attachement (1999)
- Pareto's law - 80/20 rule
- The rich get richer phenomena
- etc.



AL Barabási, Network Science Book (2013)

The Barabási-Albert model

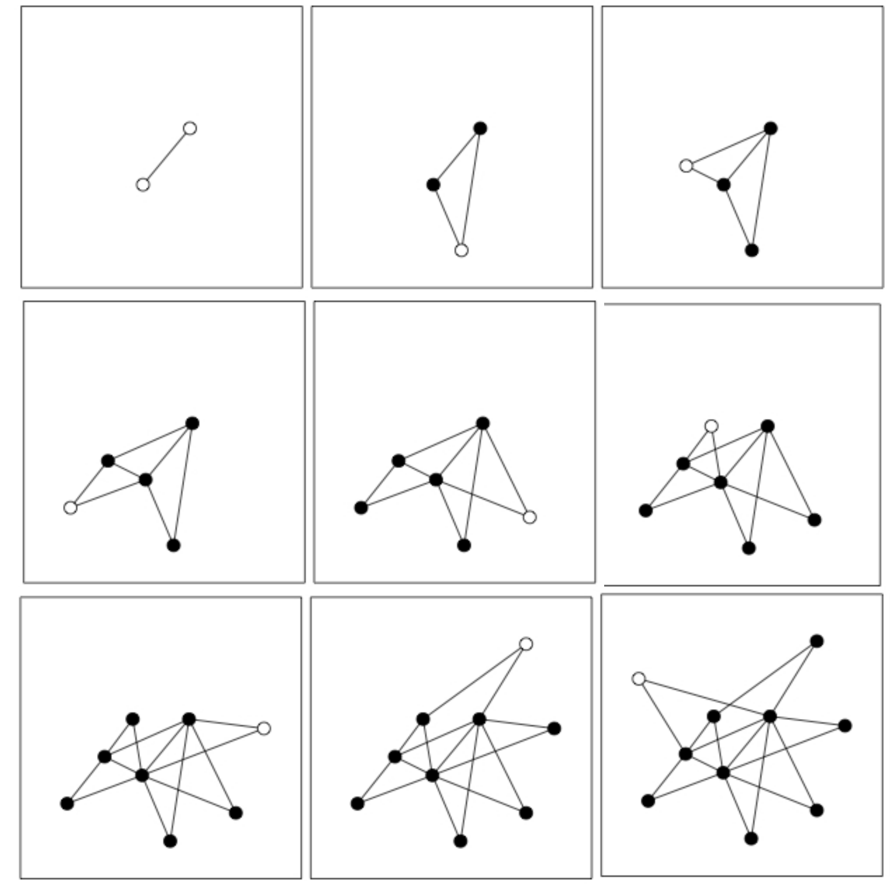
1. Start with m_0 connected nodes

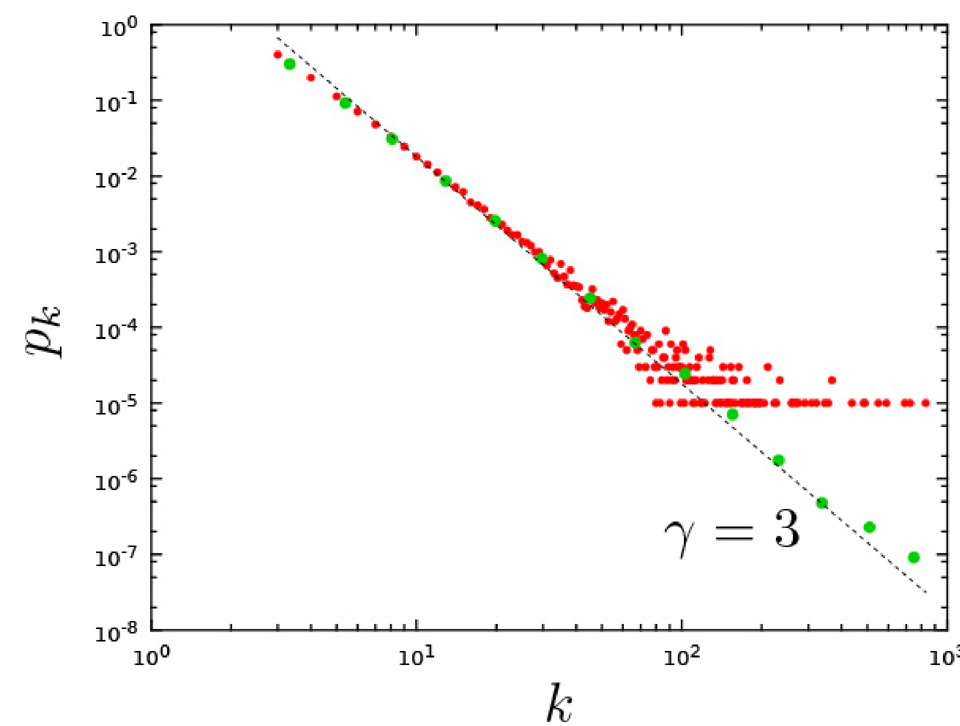
2. At each timestep we add a new node with m ($\leq m_0$) links that connect the new node to m nodes already in the network.

3. The probability $\pi(k)$ that one of the links of the new node connects to node i depends on the degree k_i of node i as

$$\Pi(k_i) = \frac{k_i}{\sum_j k_j}$$

- The emerging network will be scale-free with degree exponent $\gamma=3$ independently from the choice of m_0 and m





AL Barabási, Network Science Book (2013)

The BA model - emergence of hubs

- The time evolution of the degree of node i which joins the network at t_i

$$\frac{\partial k_i}{\partial t} = m\Pi(k_i) = m\frac{k_i}{\sum_{j=1}^{N-1} k_j}$$

- sum runs through each node other than the newly added node i

$$\sum_{j=1}^{N-1} k_j = 2mt - m$$

- thus we can write $\partial k_i/\partial t$ as

$$\frac{\partial k_i}{\partial t} = \frac{k_i}{2t-1}$$

- for large t the term (-1) can be neglected and we can rearrange as

$$\frac{\partial k_i}{k_i} = \frac{1}{2}\frac{\partial t}{t}$$

- We can write it as an integrate

$$\int_{t_i}^{t} \frac{1}{k_i(t)} \partial k_i(t) = \int_{t_i}^{t} \frac{1}{2}\frac{1}{t}\partial t$$

- and use $k_i(t_i)=m$ (the degree of node i is m when it joins the network)

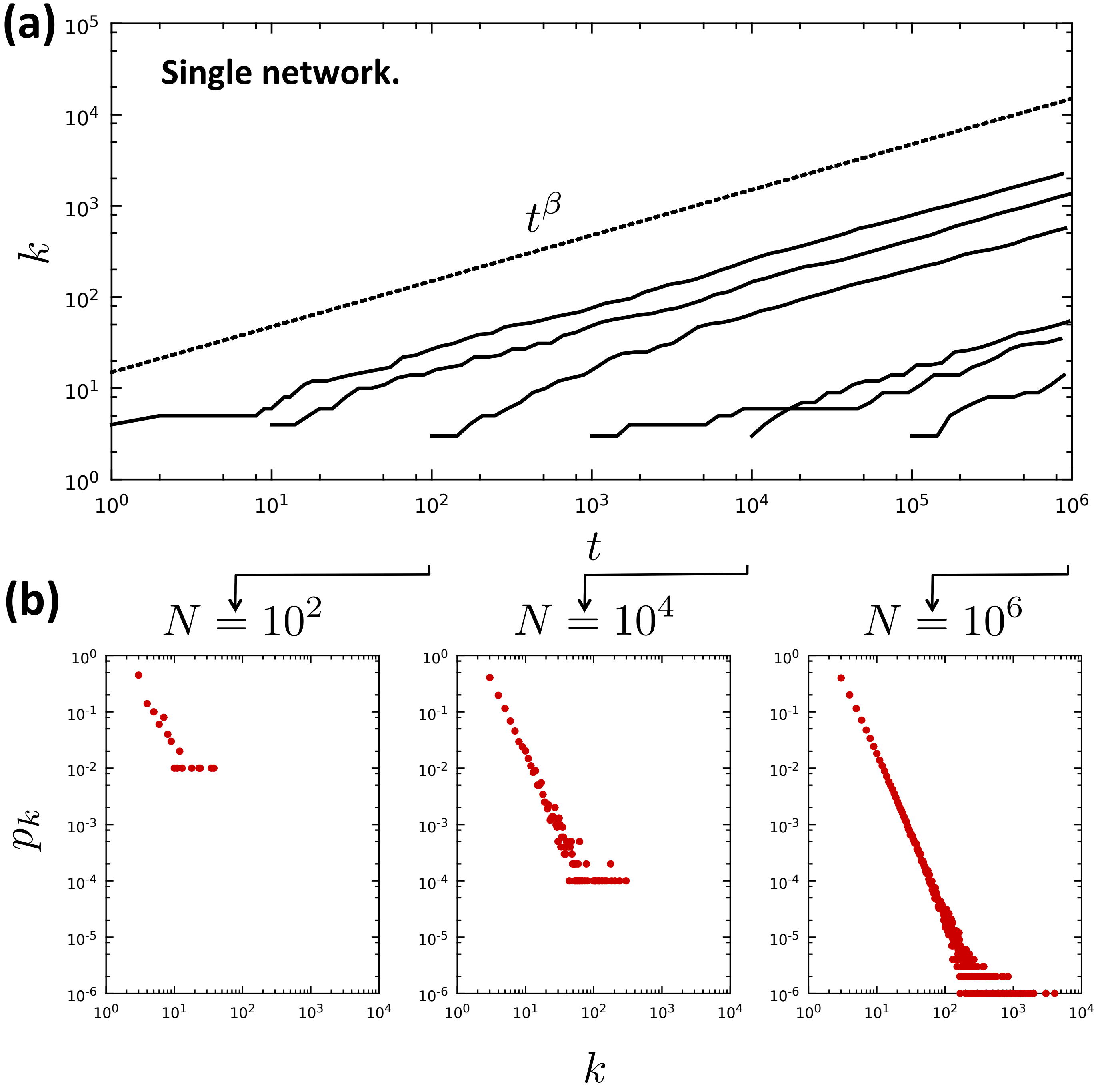
$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\beta}$$

- with exponent $\beta=1/2$

Consequences:

- The degree of each node increases as a power-law with exponent $\beta=1/2$
- The growth of the network is sub-linear
- Earlier a node was added larger its degree due to its earlier arrival and not because it grows feaster

The BA model - emergence of hubs



Consequences:

- The degree of each node increases as a power-law with exponent $\beta=1/2$
- The growth of the network is sub-linear
- Earlier a node was added larger its degree due to its earlier arrival and not because it grows feaster

Rich-get-richer mechanism

AL Barabási, Network Science Book (2013)

The BA model - degree distribution

- We have seen that

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\beta} \quad \text{where } \beta=1/2$$

- The (cumulative) probability the degree k_i of node i is smaller than k at time t:

$$P[k_i(t)<k] = P\left(t_i > \frac{m^{1/\beta}t}{k^{1/\beta}}\right)$$

- Assume that we add a node with equal time intervals, thus the probability of the arrival of each node is constant

$$P(t_i) = \frac{1}{m_0+t}$$

- Thus substituting this to above

$$P\left(t_i > \frac{m^{1/\beta}t}{k^{1/\beta}}\right) = 1 - P\left(t_i \leq \frac{m^{1/\beta}t}{k^{1/\beta}}\right) =$$

$$= 1 - \frac{m^{1/\beta}t}{k^{1/\beta}(t+m_0)}$$

- Hence the degree distribution is the derivative of the cumulative distribution

$$P(k) = \frac{\partial P[k_i(t)<k]}{\partial k} = \frac{2m^{1/\beta}t}{m_0+t}\frac{1}{k^{1/\beta+1}}$$

- Which asymptotically scales ($t\rightarrow\infty$) as

$$P(k) \sim 2m^{1/\beta}k^{-\gamma} \quad \text{with} \quad \gamma = \frac{1}{\beta}+1=3$$

solution by A.-L.Barabási, R. Albert and H. Jeong, Physica A 272, 173 (1999)

- Another solution by

Krapivsky, Redner, Leyvraz, PRL 2000
Dorogovtsev, Mendes, Samukhin, PRL 2000
Bollobas et al, Random Struc. Alg. 2001

$$P(k) = \frac{2m(m+1)}{k(k+1)(k+2)}$$

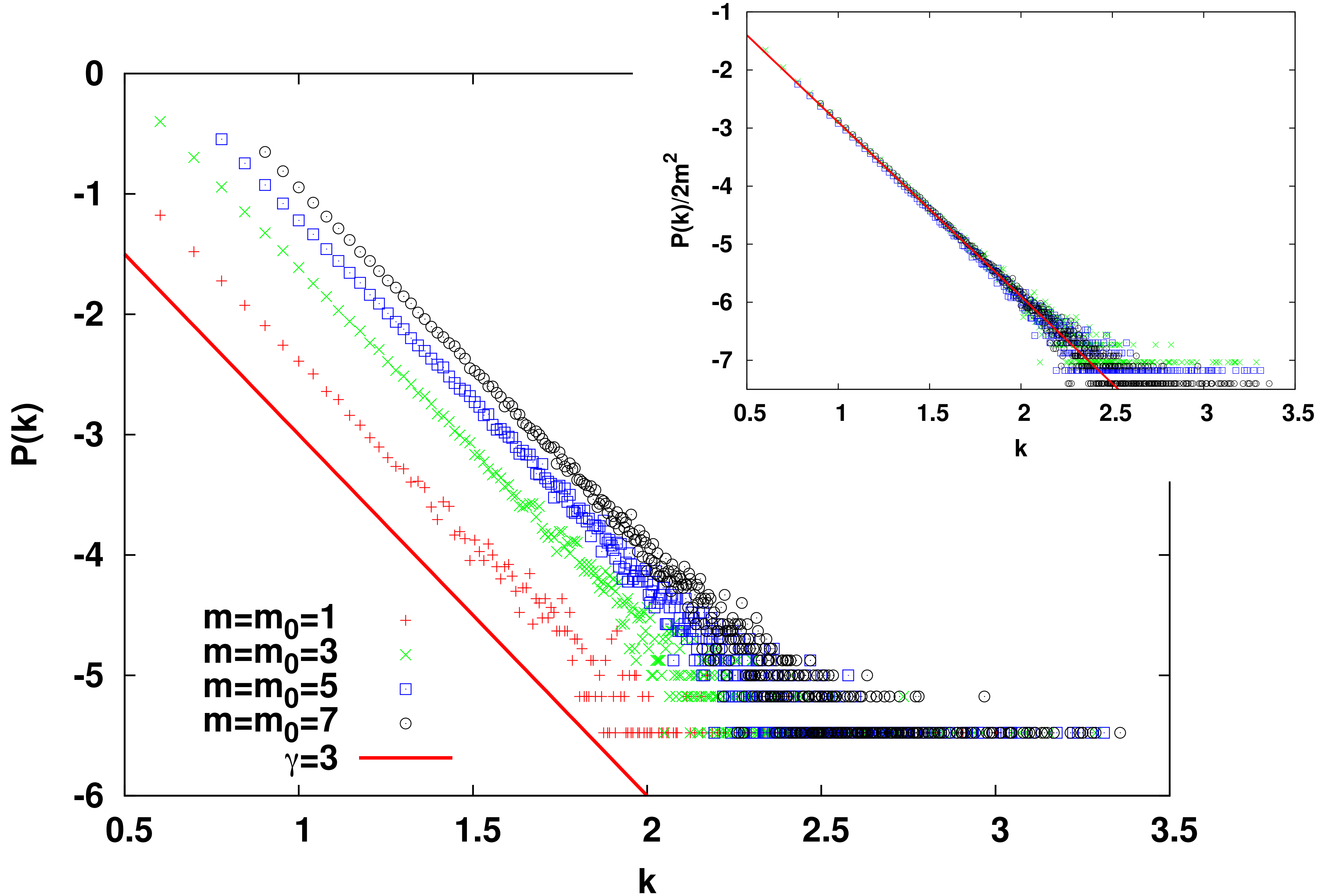
- Which asymptotically scales ($k>>1$) as

$$P(k) \sim k^{-3}$$

The BA model - degree distribution

Consequences:

- The degree exponent is independent of m
- The degree exponent is stationer in time and the degree distribution is time independent
- The exponent is very close to the exponents of real networks



The BA model - path length

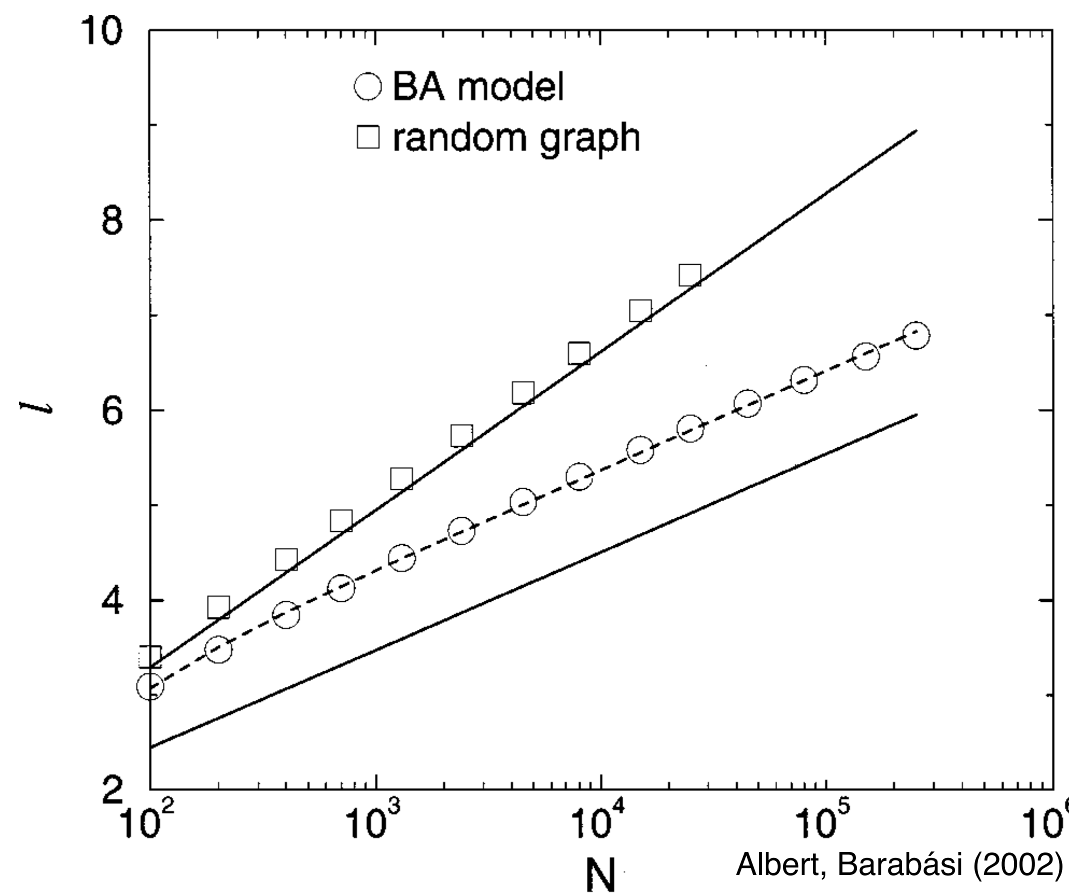
$$
\langle l \rangle \sim \begin{cases} const. & \gamma = 2 \\ \dfrac{\ln\ln N}{\ln(\gamma-1)} & 2<\gamma<3 \\ \dfrac{\ln N}{\ln\ln N} & \gamma = 3 \\ \ln N & \gamma > 3 \end{cases}
$$

	$const.$	$\gamma = 2$	Size of the biggest hub is of order O(N). Most nodes can be connected within two layers of it, thus the average path length will be independent of the system size.
Ultra Small World	$\frac{\ln\ln N}{\ln(\gamma-1)}$	$2<\gamma<3$	The average path length increases slower than logarithmically. In a random network all nodes have comparable degree, thus most paths will have comparable length. In a scale-free network the vast majority of the path go through the few high degree hubs, reducing the distances between nodes.
	$\frac{\ln N}{\ln\ln N}$	$\gamma = 3$	Some key models produce γ=3, so the result is of particular importance for them. This was first derived by Bollobas and collaborators for the network diameter in the context of a dynamical model, but it holds for the average path length as well.
Small World	$\ln N$	$\gamma > 3$	The second moment of the distribution is finite, thus in many ways the network behaves as a random network. Hence the average path length follows the result that we derived for the random network model earlier.

$$
\langle l \rangle = \frac{\ln N}{\ln\ln N}
$$

Ultra Small World network

Bollobás, Riordan (2001)



Albert, Barabási (2002)

The BA model - clustering coefficient

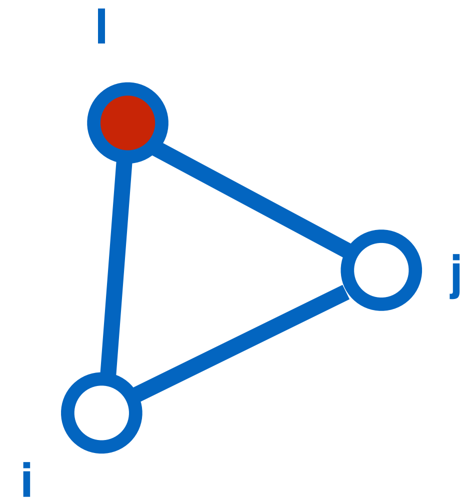
- Denote the probability to have a link between nodes i and j with $P(i,j)$
- The probability to have a triangle between i,j and l:

$$P(i,j)P(i,l)P(j,l)$$

$$C = \frac{\#\triangle}{\frac{k(k-1)}{2}}$$

- The expected number of tringles:

$$Nr_l(\triangleleft) = \int_{i=1}^{N} di \int_{j=1}^{N} dj P(i,j) P(i,l) P(j,l)$$



- Calculation of $P(i,j)$:
  - Let's assume that node j arrives at time $t_j=j$
  - The probability that it links to a node i with degree k_i is given by the preferential attachment rule:

$$P(i,j) = m\Pi(k_i(j)) = m\frac{k_i(j)}{\sum_{l=1}^{j} k_l} = m\frac{k_i(j)}{2mj}$$

The BA model - clustering coefficient

$$P(i,j) = m\Pi(k_i(j)) = m\frac{k_i(j)}{\sum_{l=1}^{j} k_l} = m\frac{k_i(j)}{2mj}$$

- We also know that:

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{1/2} = m\left(\frac{j}{i}\right)^{1/2}$$

Using again the notation $t_j=j$ and $t_i=i$ for nodes j and i

$$P(i,j) = \frac{m}{2}(ij)^{-\frac{1}{2}}$$

$$Nr_l(\triangleleft) = \int_{i=1}^{N} di \int_{j=1}^{N} dj P(i,j)P(i,l)P(j,l) = \frac{m^3}{8}\int_{i=1}^{N} di \int_{j=1}^{N} dj (ij)^{-\frac{1}{2}}(il)^{-\frac{1}{2}}(jl)^{-\frac{1}{2}} = \frac{m^3}{8l}\int_{i=1}^{N}\frac{di}{i}\int_{j=1}^{N}\frac{dj}{j} = \frac{m^3}{8l}(\ln N)^2$$

- Consequently $C = \dfrac{\frac{m^3}{8l}(\ln N)^2}{k_l(k_l-1)/2}$ where $k_l(t) = m\left(\frac{N}{l}\right)^{1/2}$ Which is the degree of node l at current time $t=N$

- Let's approximate $k_l(k_l-1) \approx k_l^2 = m^2\frac{N}{l}$ thus

$$\boxed{C = \frac{m}{4}\frac{(\ln N)^2}{N}}$$

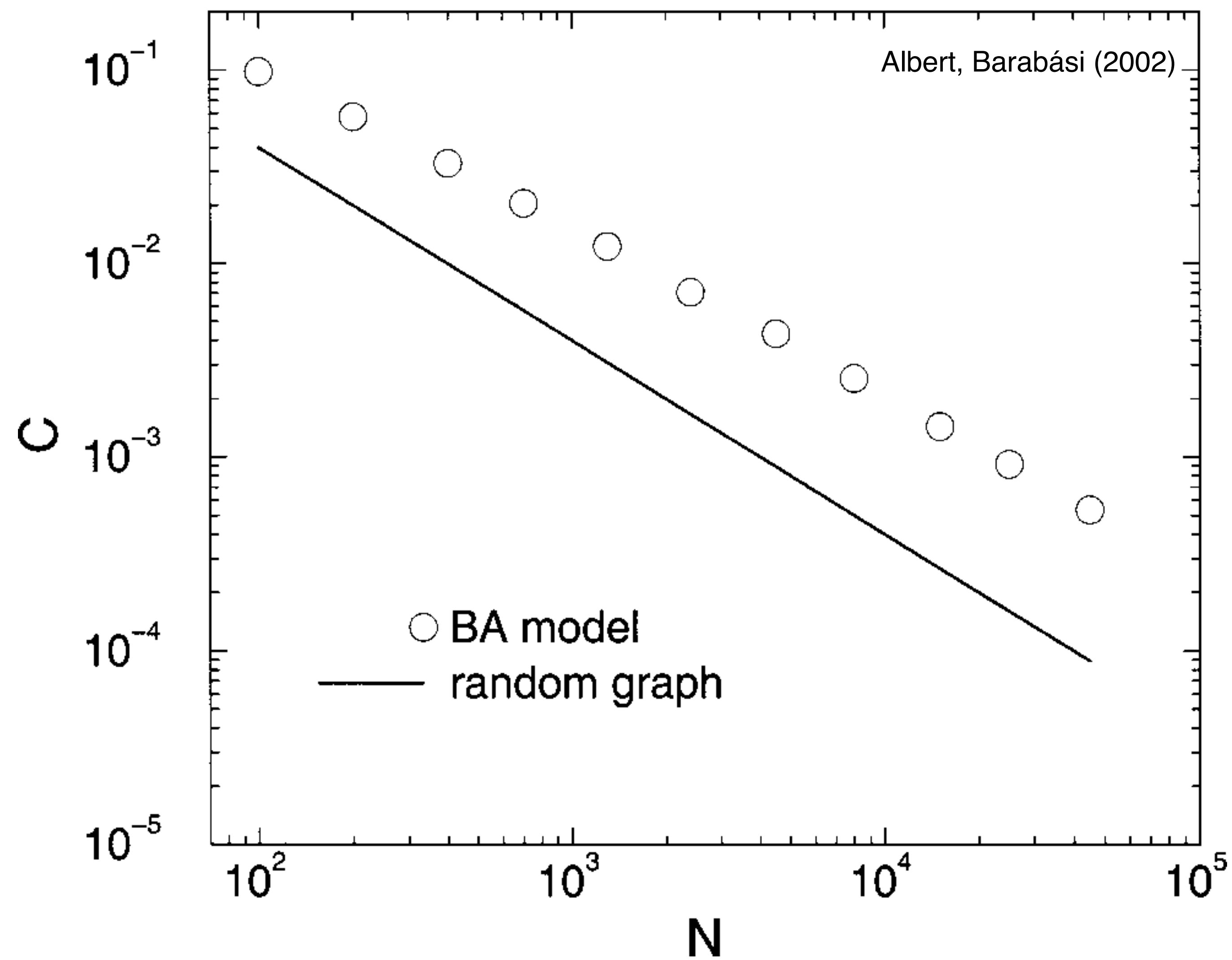
Konstantin Klemm, Victor M. Eguiluz, Phys. Rev. E 65, 057102 (2002)

The BA model - clustering coefficient

- The clustering coefficient decreases with the system size as

$$C = \frac{m}{4}\frac{(\ln N)^2}{N}$$

- It is still 5 times more than for random graphs



Degree correlations:

- The BA model is inducing non-trivial degree correlations due to its definition

$$n_{kl} \simeq k^{-2} l^{-2}$$

ER Random Network - catch up

Network	Degree distribution	Path length	Clustering coefficient
Real world networks	broad	short	large
Regular lattices	constant	long	large
ER random networks	Poissonian	short	small
WS small-world networks	exponential	short	large
BA scale-free networks	power-law	short	N dependent

Do we need both growth and preferential attachment?

model A

Growth

~~Preferential attachement~~

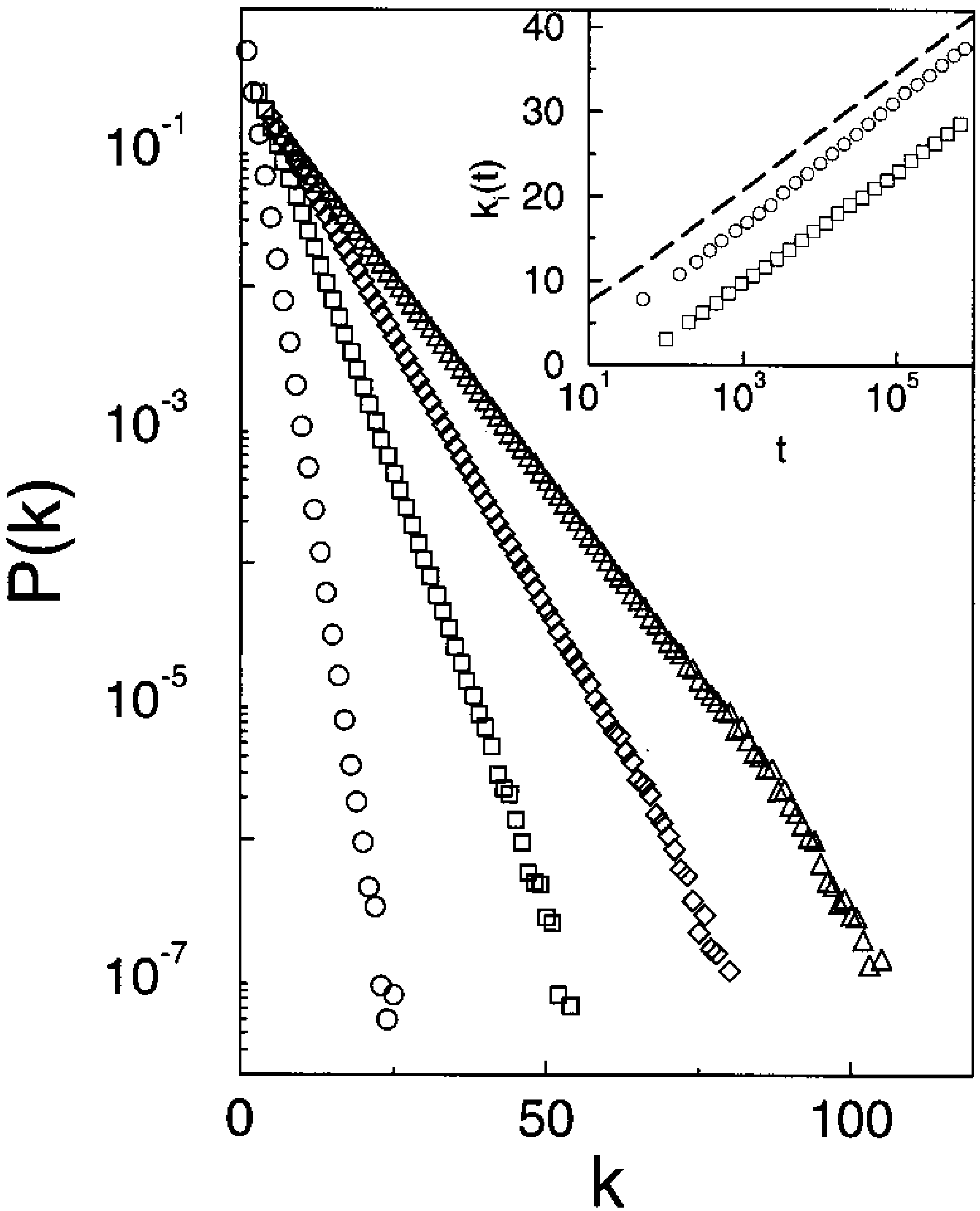
$\Pi(k_i)$: uniform

$$\frac{\partial k_i}{\partial t} = A\Pi(k_i) = \frac{m}{m_0 + t - 1}$$

$$k_i(t) = m\ln\left(\frac{m_0 + t - 1}{m + t_i - 1}\right) + m$$

$$P(k) = \frac{e}{m}\exp\left(-\frac{k}{m}\right) \sim e^{-k}$$

It is not power-law!!!

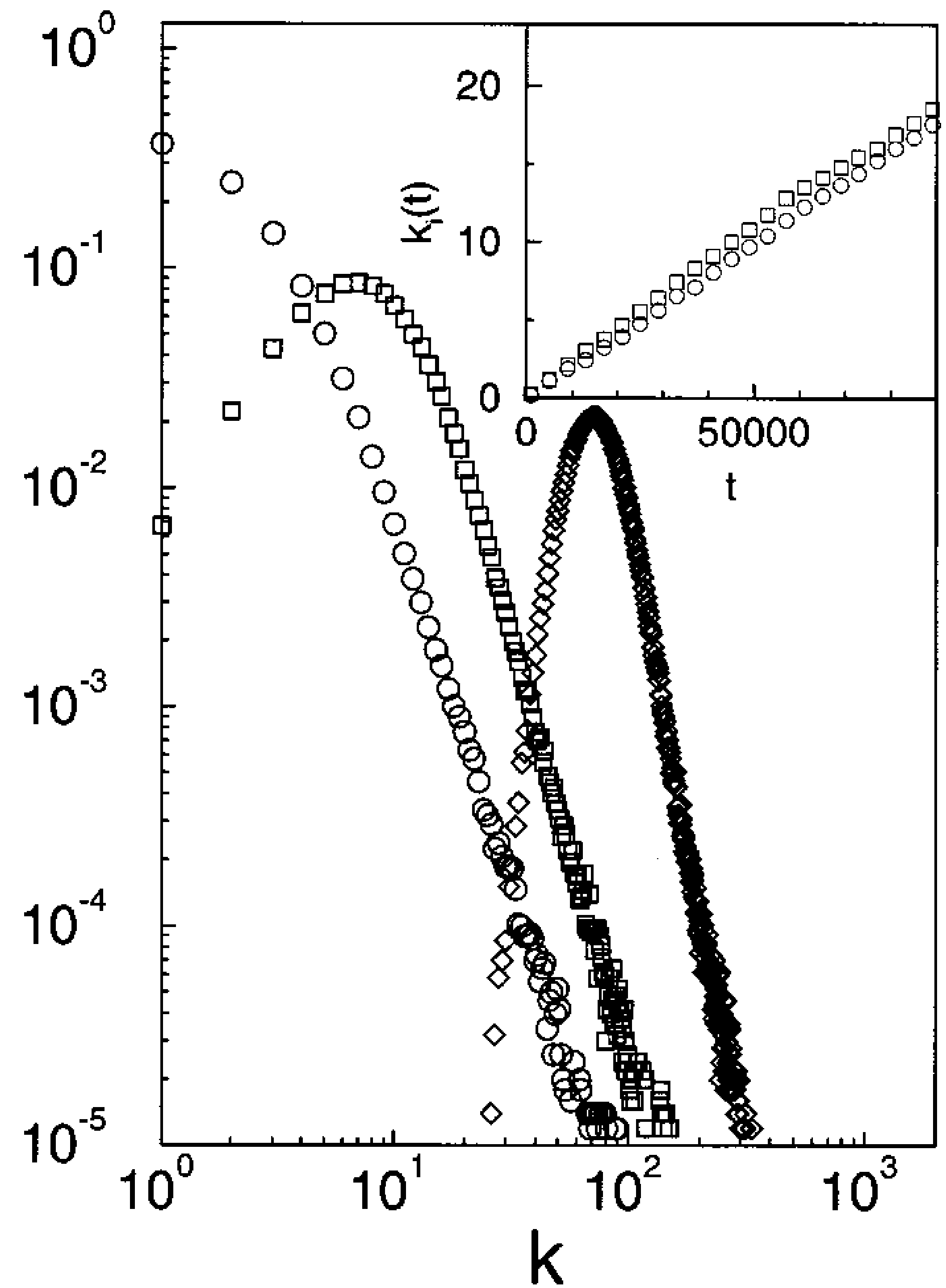


model B

~~Growth~~ **Preferential attachement**

$$\frac{\partial k_i}{\partial t} = A\Pi(k_i) + \frac{1}{N} = \frac{N}{N-1}\frac{k_i}{2t} + \frac{1}{N}$$

$$k_i(t) = \frac{2(N-1)}{N(N-2)}t + Ct^{\frac{N}{2(N-1)}} \sim \frac{2}{N}t$$



The P(k) evolves as:

1. power-law (initially)
2. Gaussian
3. Fully connected

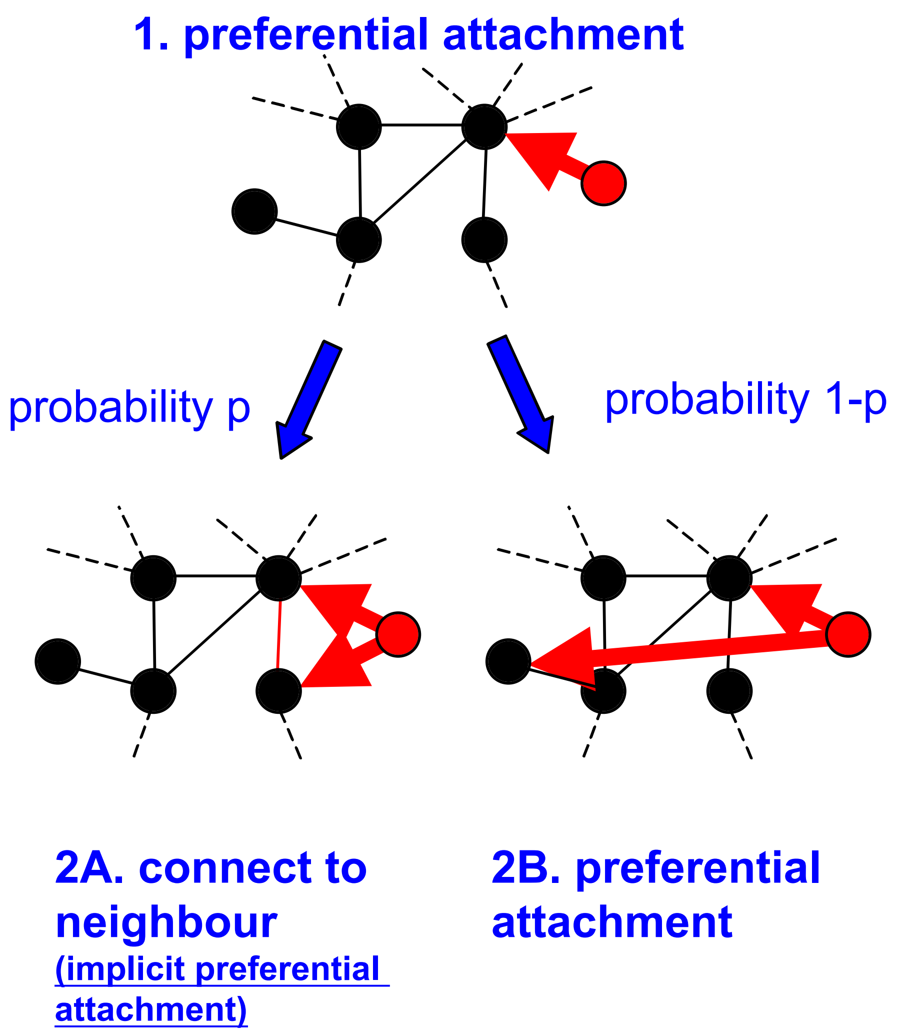
It is not stationer!!!

Other scale-free models 1

The Holme-Kim model

- Motivation: more realistic clustering coefficient

1. Take a small seed network
2. Create a new vertex with m edges
3. Connect the first of the m edges to existing vertices with a probability proportional to their degree k (just like BA)
4. With probability p, connect the next edge to a random neighbour of the vertex of step 3., otherwise do 3. again
5. Repeat 2.-4. until the network has grown to desired size of N vertices



$$C(k) \propto \frac{1}{k}$$

for large N, ie clustering more realistic! This type of clustering is found in many real-world networks.

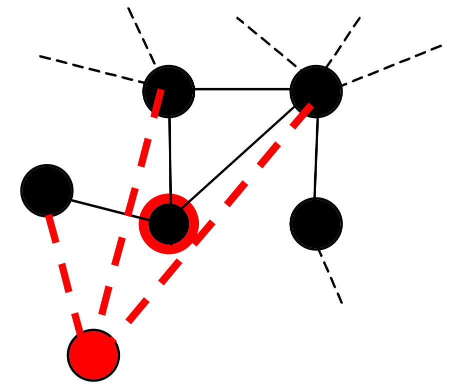
Other scale-free models 1

The vertex-copying model

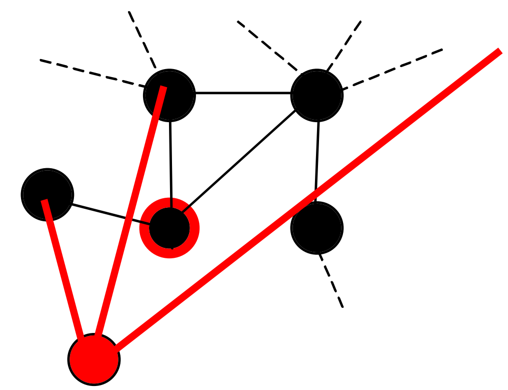
- Motivation:
  - Citations network or WWW where links are often copied
  - Local explanation to preferential attachement

1. Take a small seed network
2. Pick a random vertex
3. Make a copy of it
4. With probability p, move each edge of the copy to point to a random vertex
5. Repeat 2.-4. until the network has grown to desired size of N vertices

1. copy a vertex



2. rewire edges with *p*



- Asymptotically scale-free with exponent $\gamma \geq 3$