Problem Set Three, CR15 ENS de Lyon. Course authorities : Karsai / Crespelle / Unicomb

Due date : 01/12/2016, by 11:59 p.m.

Complementary material : N/A

1 The Watts-Strogatz model

Erdös-Rényi graphs do not have two important properties observed in many real-world networks. Firstly,

- they do not generate local clustering and triadic closures. This is because there is a constant, random, and independent probability of two nodes being connected. Secondly,
- they do not account for the formation of hubs. Formally, the degree distribution of ER graphs converges to a Poisson distribution, rather than the power law observed in many real-world, scale-free networks.

The Watts-Strogatz model (1998) was designed to be the simplest possible solution to address the first of the two limitations. It accounts for clustering while retaining the short average path lengths of the ER model. It does so by interpolating between an ER graph and a regular ring lattice. Consequently, the model is able to at least partially replicate the 'small-world' phenomena present in a variety of networks, such as the power grid, the neural network of *C. elegans*, or networks of movie stars. The algorithm works as follows.

Given the number of nodes N, the mean degree k, which is assumed to be an even integer, and a special parameter β satisfying $0 \leq \beta \leq 1$, all such that $N \gg k \gg \log N \gg 1$, the model constructs an undirected graph with $Nk/2$ edges in the following way.

- One constructs a regular ring lattice, a graph with N nodes each connected to k neighbours, $k/2$ on each side. That is, if the nodes are labeled $n_0, \ldots, n_{N-1}$, there is an edge (n_i, n_j) if and only if

$$0 <\mid i-j \mid \bmod (n-k/2) \leq k/2 \tag{1}$$

- for every node $n_i = n_0, \ldots, n_{N-1}$ take every edge (n_i, n_j) with $i < j$, and rewire it with probability β. Rewiring is done by replacing (n_i, n_j) with (n_i, n_l) where l is chosen with uniform probability from all possible values that avoid self-loops and link duplication.

Illustrate concisely that this model accounts for clustering while retaining the short average path lengths of the ER model.

2 Barabási-Albert Model

The Barabási-Albert, or BA model is an algorithm for generating random scale-free networks using a preferential attachment mechanism. The algorithm is as follows.

- The network begins with a random, connected network of nodes. Then,
- new nodes are added to the network one at a time. Each new node is connected to existing nodes with a probability that is proportional to the number of links that the existing nodes already have. Formally, the probability that a new node is connected to an existing node u is

$$p_u = \frac{k_u}{\sum k_v}, \tag{2}$$

  where k_u is the degree of node u, and the sum is made over all pre-existing nodes v, such that the denominator results in twice the current number of edges in the network.

Heavily linked nodes, or hubs, tend to quickly accumulate even more links, while nodes with only a few links are unlikely to be chosen as the destination for a new link. As such, new nodes have a higher likelihood of attaching themselves to heavily linked nodes.

Plot the degree distribution $P(k)$ for a fixed N of a network thus generated, and calculate the quantity that shows the BA model produces scale-free networks. Is it a small-world network? Why, or why not?