1 Difference in fecundity and mortality

We assume two populations of bacteria, having both logistic growth with different growth rates and carrying capacities. Individuals from any population can switch to the other population.
• $P(t)$ is the population of P (Pathogenic) bacteria at time t
• $M(t)$ is the population of M (Mutualistic) bacteria at time t
• f_{P} is the fecundity rate of P bacteria
• f_{M} is the fecundity rate of M bacteria. We have $f_{\mathrm{M}} < f_{\mathrm{P}}$
• μ_{P} is the mortality rate of P bacteria
• μ_{M} is the mortality of M bacteria. We have $\mu_{\mathrm{M}} < \mu_{\mathrm{P}}$
• K is the carrying capacity
• k_1 is the switching rate from P to M
• k_{-1} is the switching rate from M to P

$$P \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} M$$

The system of differential equation describing the system is:

$$\begin{cases} \dfrac{\mathrm{d}P}{\mathrm{d}t} = (f_{\mathrm{P}}(1-k_1)P + f_{\mathrm{M}}k_{-1}M)\left(1 - \dfrac{\alpha P + M}{K}\right) - \mu_{\mathrm{P}}P & \text{(1a)} \\ \dfrac{\mathrm{d}M}{\mathrm{d}t} = (f_{\mathrm{M}}(1-k_{-1})M + f_{\mathrm{P}}k_1P)\left(1 - \dfrac{\alpha P + M}{K}\right) - \mu_{\mathrm{M}}M & \text{(1b)} \\ P(0) = P_0 & \text{(1c)} \\ M(0) = 0. & \text{(1d)} \end{cases}$$

The stable state (equilibrium) can be analytically derived and we a simple formula for the ratio P over M at equilibrium (independent of K and α):

$$\frac{M_{\mathrm{eq}}}{P_{\mathrm{eq}}} = \Theta_2 - \Theta_1 + \sqrt{(\Theta_2 - \Theta_1)^2 + \Delta} \tag{2}$$

Where

$$\Theta_1 = \frac{f_{\mathrm{P}}(1-k_1)}{2f_{\mathrm{M}}k_{-1}} \tag{3}$$

$$\Theta_2 = \frac{\mu_{\mathrm{P}}(1-k_{-1})}{2\mu_{\mathrm{M}}k_{-1}} \tag{4}$$

$$\Delta = 4\frac{k_1 f_{\mathrm{P}}\mu_{\mathrm{P}}}{k_{-1}f_{\mathrm{M}}\mu_{\mathrm{M}}} \tag{5}$$

also we have:

$$\frac{P_{\mathrm{eq}}}{M_{\mathrm{eq}}} = \frac{\Theta_1 - \Theta_2 + \sqrt{(\Theta_2 - \Theta_1)^2 + \Delta}}{\Delta} \tag{6}$$