Documentation - ft_linear_regression

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1 Definition of a Linear Regression

Linear regression is a statistical method used to model the relationship between one dependent variable (Y) and one or more independent variables (X). It assumes that this relationship is linear, meaning that changes in X are directly proportional to changes in Y.

Key Elements of Linear Regression:

- **Equation of a Line**:
  - For simple linear regression (one independent variable), the model is:

$$Y = mX + b + \epsilon$$

where:

    * Y: Dependent variable (outcome or target variable).
    * m: Independent variable (predictor or feature).
    * X: Slope of the line, representing the rate of change in Y for a one-unit change in X.
    * b: Intercept, the value of Y when $X = 0$.
    * ϵ: Error term, accounting for the deviation of actual values from the predicted line.

- **Multiple Linear Regression**:
  - If there are multiple independent variables (X_1, X_2, ..., X_n), the equation becomes:

    $$Y = b_0 + b_1X_1 + b_2X_2 + \cdots + b_nX_n + \epsilon$$

    where each b_i represents the coefficient (effect) of the corresponding feature.

- **Assumptions**:
  - The relationship between X and Y is linear.
  - The residuals (errors) are normally distributed.
  - Homoscedasticity: The variance of residuals is constant across all values of X.
  - Independence: Observations are independent of one another.

- **Goal**:
  - The primary goal of linear regression is to find the best-fit line that minimizes the sum of squared residuals (differences between observed and predicted values).

- **Applications**:
  - Predicting continuous outcomes, such as house prices, temperatures, or sales.

  - Understanding relationships between variables, such as how advertising spend affects sales.

In summary, linear regression is a foundational technique in statistics and machine learning that models and predicts outcomes based on a linear relationship between variables.

2 How it works

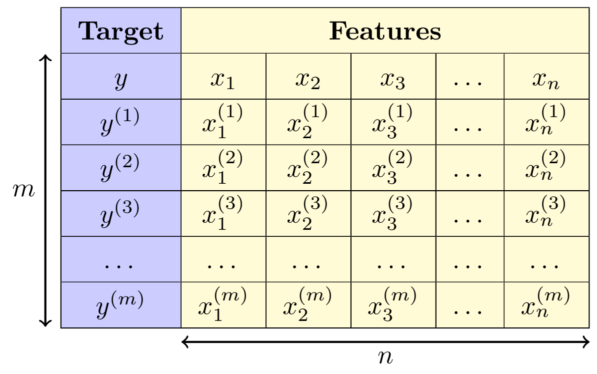
Linear regression is a supervised learning algorithm that predicts a continuous target variable by finding the best-fit linear relationship between input features and the target. Here's how it works using the **four fundamentals**:

- **Dataset**
  - **Definition**: The dataset consists of examples with input features (X) and their corresponding target value (Y).
    - Example: Predicting house prices (Y) based on features like size (X_1) and number of bedrooms (X_2).
  - The dataset must be:
    - Clean and properly formatted.
    - Split into **training** and **testing** sets to evaluate the model's performance.

  By convention:

  - m: Number of examples (rows).
  - n: Number of features (columns, excluding the target).

Dataset (x, y)



Target	**Features**				
y	x_1	x_2	x_3	$\dots$	x_n
$y^{(1)}$	$x_1^{(1)}$	$x_2^{(1)}$	$x_3^{(1)}$	$\dots$	$x_n^{(1)}$
$y^{(2)}$	$x_1^{(2)}$	$x_2^{(2)}$	$x_3^{(2)}$	$\dots$	$x_n^{(2)}$
$y^{(3)}$	$x_1^{(3)}$	$x_2^{(3)}$	$x_3^{(3)}$	$\dots$	$x_n^{(3)}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$y^{(m)}$	$x_1^{(m)}$	$x_2^{(m)}$	$x_3^{(m)}$	$\dots$	$x_n^{(m)}$

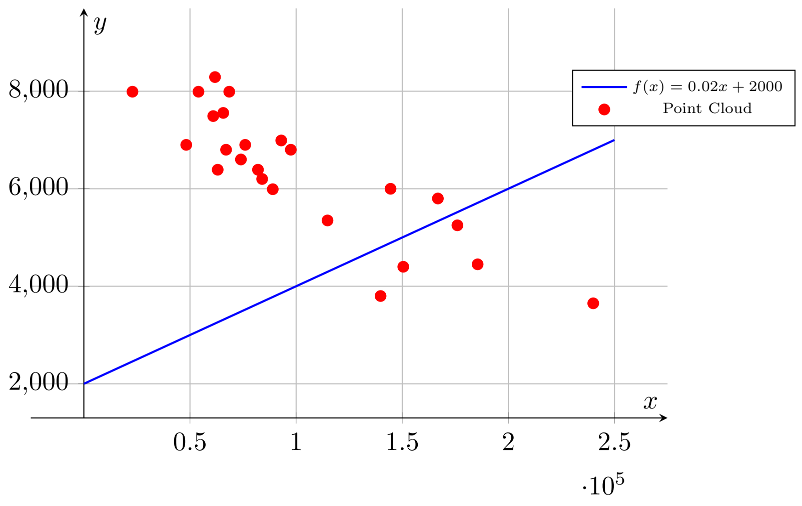
- **Model**
  - **Definition**: The model is a mathematical function that predicts Y (target) from X (features).
    - For simple linear regression:

$$\hat{Y} = mX + b$$

      - m: Slope (how much Y changes per unit of X).
      - b: Intercept (value of Y when $X = 0$).
    - For multiple linear regression:

$$\hat{Y} = b_0 + b_1X_1 + b_2X_2 + \cdots + b_nX_n$$

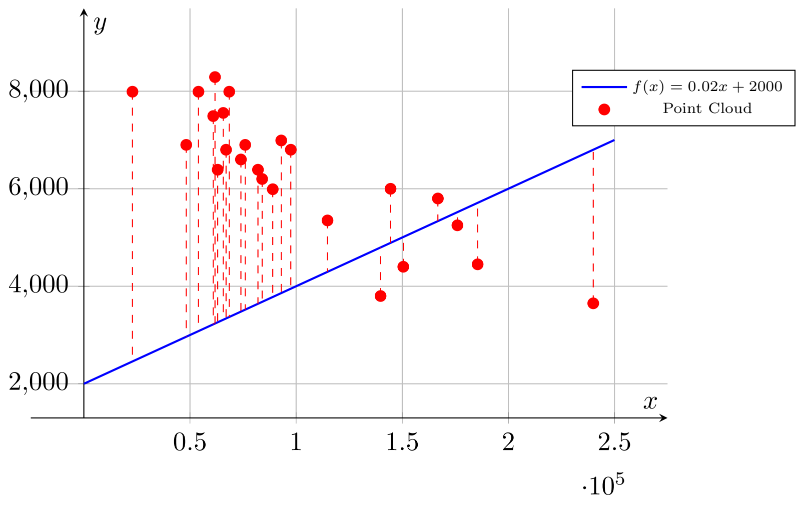
      - b_0: Intercept.
      - b_i: Coefficient of each feature (X_i).
  - **Purpose**: To represent the relationship between inputs (X) and output (Y) using a straight line (or hyperplane in multiple dimensions).



- **Cost Function**
  - **Definition**: The cost function measures the error between the predicted values ($\hat{Y}$) and the actual target values (Y).
    * For linear regression, the most common cost function is the **Mean Squared Error (MSE)**:

$$J(\theta) = \frac{1}{2m} \sum_{i=1}^{m} (\hat{Y}^{(i)} - Y^{(i)})^2$$

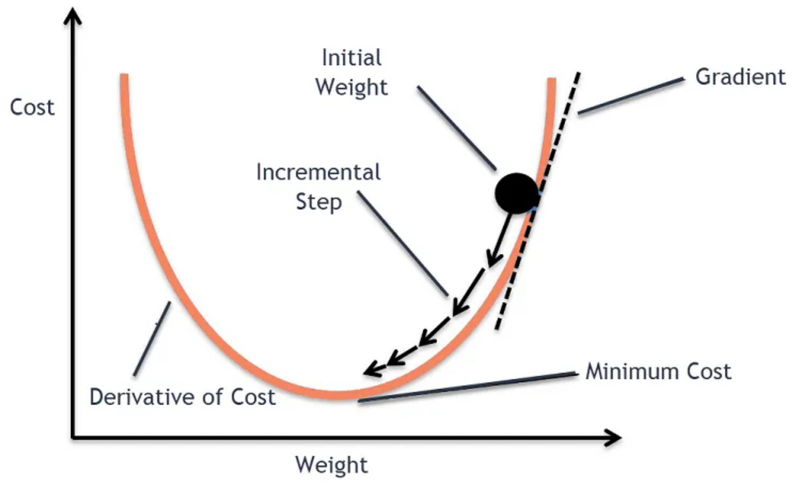
      · $J(\theta)$: Cost function value.
      · $\hat{Y}^{(i)}$: Predicted value for the i-th example.
      · $Y^{(i)}$: Actual target value for the i-th example.
      · m: Number of examples.
  - **Purpose**: To provide a metric that the algorithm minimizes to improve predictions. Lower cost indicates better model performance.



- **Minimization Algorithm**
  - **Definition**: The minimization algorithm adjusts the model's parameters (coefficients m, b or b_0, b_1, $\ldots$, b_n) to reduce the cost function $J(\theta)$.
  - The most common algorithm used is **Gradient Descent**:
    - Iteratively updates the parameters using the rule:

$$\theta = \theta - \alpha . \frac{\partial J(\theta)}{\partial \theta}$$

      - α: Learning rate (step size).
      - $\frac{\partial J(\theta)}{\partial \theta}$: Gradient of the cost function.
  - **Steps in Gradient Descent**:
    - Compute the cost $J(\theta)$ for the current parameters.
    - Calculate the gradient (direction of steepest ascent).
    - Update the parameters in the opposite direction of the gradient (descent).
    - Repeat until the cost converges to its minimum.



Summary

- **Dataset**: Provides input-output pairs to learn from.
- **Model**: Represents the linear relationship between inputs (X) and output (Y).
- **Cost Function**: Quantifies prediction errors to guide the optimization.
- **Minimization Algorithm**: Adjusts the model's parameters to minimize errors, finding the best-fit line.

By combining these four steps, linear regression learns to make accurate predictions based on the data.

3 Train the model

- **Feature Normalization**

```
def normalize_features(x)
```

  - **Purpose**: Normalize the input features (x) to have a mean of 0 and a standard deviation of 1. This is important to:

    * Improve the performance of gradient descent (features with large scales can slow convergence).
    * Make the model parameters more interpretable.
  - **How it works**:
$$x' = \frac{x - mean(x)}{std(x)}$$
  - Example: If x represents car mileage ranging from 0 to 200,000, normalization ensures that this large range doesn't dominate the training process.

```python
def denormalize_theta(theta, mean_x, std_x)
```

  - **Purpose**: Convert the normalized slope and intercept back to the original scale for interpretability.
  - **How it works**:
    * The slope is scaled by dividing by the standard deviation (std_x).
    * The intercept is adjusted by subtracting the influence of the slope on the mean of x.

- **Model**

```python
def model(X, theta)
```

  - **Purpose**: Predict the output ($\hat{y}$) using the linear regression equation:
$$\hat{y} = X.\theta$$
  - X includes the features and an intercept term (bias).
  - θ represents the model parameters: slope (θ_1) and intercept (θ_0).

- **Cost Function**

```python
def cost(X, y, theta)
```

- **Purpose**: Compute the **Mean Squared Error (MSE)** cost function to measure how well the model fits the data:

$$J(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(\hat{y}^{(i)} - y^{(i)})^2$$

  * m: Number of training examples.
  * Using $\frac{1}{2m}$ ensures the gradient is consistent in scale with other terms in the optimization process, making the training process smoother.
    The $\frac{1}{2}$ factor doesn't affect the optimization itself because it is a constant scaling factor.
    Minimizing $J(\theta)$ with $\frac{1}{2m}$ leads to the same parameter values θ as minimizing $J(\theta)$ with $\frac{1}{m}$.
    In summary, $\frac{1}{2m}$ is used instead of $\frac{1}{m}$ to make gradient calculations more elegant and computationally simpler without altering the results of the optimization.

- **Gradient Calculation**

```
def gradient(X, y, theta)
```

  - **Purpose**: Compute the gradients of the cost function with respect to θ_0 (intercept) and θ_1 (slope):
    * Gradient for θ_0:

$$\frac{\partial J}{\partial \theta_0} = \frac{1}{m}\sum_{i=1}^{m}(\hat{y}^{(i)} - y^{(i)})$$

    * Gradient for θ_1:

$$\frac{\partial J}{\partial \theta_1} = \frac{1}{m}\sum_{i=1}^{m}(\hat{y}^{(i)} - y^{(i)}).x^{(i)}$$

  - These gradients indicate the direction and magnitude of parameter updates.

- **Gradient Descent**

```python
def gradient_descent(X, y, theta, learning_rate, n_iterations)
```

  - **Purpose**: Optimize the parameters θ_0 and θ_1 by iteratively updating them in the direction of the negative gradient.
  - **How it works**:
    - Update rule:

$$\theta = \theta - \alpha.\nabla J(\theta)$$

      - α: Learning rate, controlling the step size.
      - $\nabla J(\theta)$: Gradient of the cost function.
    - Cost is computed and stored in each iteration for tracking convergence.
  - The formulas of the **subject** are:

$$tmp\theta_0 = \alpha.\frac{1}{m}\sum_{i=1}^{m}(\hat{y}^{(i)} - y^{(i)})$$

$$tmp\theta_1 = \alpha.\frac{1}{m}\sum_{i=1}^{m}(\hat{y}^{(i)} - y^{(i)}).x^{(i)}$$

  where:
    - α: Learning rate.
    - $\hat{y}^{(i)}$: estimatePrice(mileage[i]).
    - $y^{(i)}$: price[i].
    - $x^{(i)}$: mileage[i].

- **Evaluation Metrics**

```python
def coef_determination(y, pred)
```

  - **Formula**:

$$R^2 = 1 - \frac{Sum of Squared Residuals(SSR)}{Total Sum of Squares(SST)}$$

  - **Purpose**:

    * R^2 measures the proportion of variance in the target variable explained by the model.
    * $R^2 = 1$: Perfect fit.
    * $R^2 = 0$: No better than predicting the mean.

```python
def mean_squared_error(y_actual, y_predicted)
```

  - **Formula**:
$$MSE = \frac{1}{m}\sum_{i=1}^{m}(y^{(i)} - \hat{y}^{(i)})^2$$
  - **Purpose**:
    * MSE measures the average of the squared differences between actual and predicted values.
    * It penalizes larger errors more heavily than smaller ones due to the squaring.
    * It's useful for comparing model performance but not as interpretable because its units are the square of the target variable's units.

```python
def mean_absolute_error(y_actual, y_predicted)
```

  - **Formula**:
$$MAE = \frac{1}{m}\sum_{i=1}^{m}|y^{(i)} - \hat{y}^{(i)}|$$
  - **Purpose**:
    * MAE calculates the average of the absolute differences between actual and predicted values.
    * It provides an intuitive measure of error in the same unit as the target variable.
    * Less sensitive to outliers compared to MSE since it doesn't square the errors.

```python
def root_mean_squared_error(y_actual, y_predicted)
```

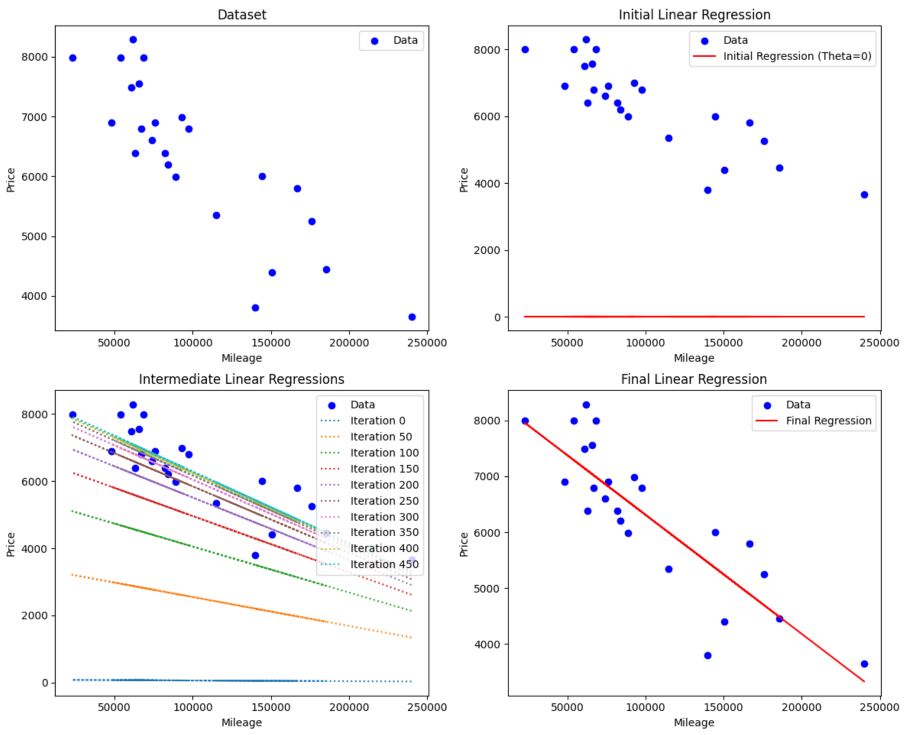
- **Formula**:

$$RMSE = \sqrt{MSE}$$

- **Purpose**:
  - RMSE is the square root of MSE, which brings the error back to the same unit as the target variable.
  - It combines the advantages of MSE (emphasizing larger errors) with interpretability in the target variable's units.
  - Often used in practice to compare model performance.

Metric	*PenalizesLargeErrors?*	*Unit*	*Interpretability*	*SensitivitytoOutliers*
MSE	*Yes*	*Squared*	*Low*	*High*
MAE	*No*	*Sameastarget*	*High*	*Low*
RMSE	*Yes*	*Sameastarget*	*Medium*	*Medium*
R^2	*No*	*Dimensionless*	*High(VarianceExplained)*	*N/A*

- **How It All Works Together**
  - **Normalize the features** to ensure faster and more stable gradient descent.
  - **Initialize** θ_0 **and** θ_1, and compute the initial cost.
  - Use **gradient descent** to iteratively update θ and minimize the cost.
  - Once training is complete:
    - Denormalize θ to interpret the slope and intercept in the original scale.
    - Evaluate the model's performance using R^2, MSE, MAE, and $RMSE$.
  - **Visualize** the results:
    - Plot the data points and the best-fit line.



4 Predict the model

```
def estimate_price(mileage, theta0, theta1)
```

- **Formula**:

$$Price = \theta_0 + (\theta_1.Mileage)$$

- **Purpose**:
  - The estimated price function is called with:
    - The user-provided mileage.
    - The loaded model parameters (θ_0 and θ_1).
    - θ_0: The baseline price.
    - θ_1: The price decrease per km.
- **Example**:

    - Assume the trained model get the values θ_0=8443.75 and θ_1=-0.0213
    - User enters a mileage of 50,000 km.
    - The program calculates:

$$Price = 8443.75 + (-0.0213 * 50000) = 8443.75 - 1065 = \$7378.75$$

Suppose that we get these values:

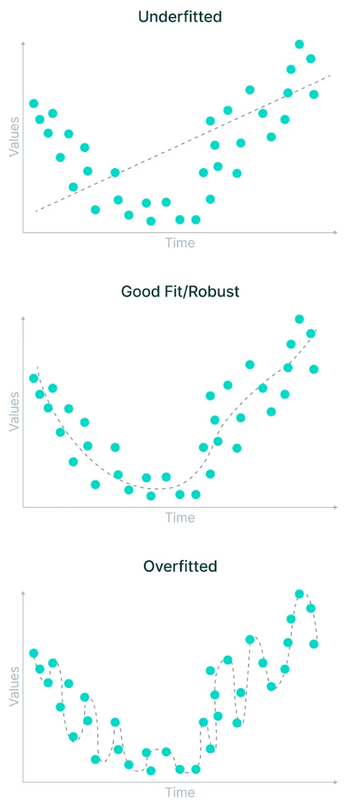
- **Mean Squared Error (*MSE*)**: 447428.88
  The squared error might look large, but it's in squared units and mainly useful for model comparisons.
- **Mean Absolute Error (*MAE*)**: 559.61
  On average, the predicted price is off by about $559.61.
- **Root Mean Squared Error (*RMSE*)**: 668.90
  Typical prediction error is around $668.90, but this value is more affected by outliers.
- **Coefficient of Determination (R^2)**: 73.19%
  Measures how well your model explains the variance in the target variable. Here, 73.19% of the variance in car prices is explained by the model (mileage as the feature).

5 What is Overfitting?

Overfitting happens when a machine learning model learns the training data **too well**, capturing not only the underlying patterns but also the **noise and random fluctuations** in the data. This results in a model that performs very well on the training data but poorly on new, unseen data.

- **How Overfitting Happens**:
    - **Overly Complex Models**:
        * A model with too many parameters or too much flexibility can "memorize" the training data instead of generalizing to unseen data.

    - Example: Using a high-degree polynomial in regression might perfectly fit all data points but create unrealistic predictions for new data.
  - **Limited or Noisy Data**:
    - When the dataset is too small, the model may mistake random noise or outliers for meaningful patterns.
  - **Poor Regularization**:
    - Without constraints, a model may prioritize fitting every detail of the data, including irrelevant patterns.
- **Signs of Overfitting**
  - **Low Training Error but High Test Error**:
    - The model performs exceptionally well on the training set (low error) but struggles on validation or test sets (high error).
  - **Excessively Complex Model**:
    - The model has too many features, layers, or degrees of freedom relative to the dataset size.
  - **Overly Wavy or Detailed Decision Boundary (in classification)**:
    - Instead of finding a general rule, the model may draw a boundary that precisely separates every point in the training set, even if it's unnecessary.
- **Visual Example**:



- **Consequences of Overfitting**:
  - Poor **generalization** to new data, leading to inaccurate predictions.
  - High **variance**, where the model is highly sensitive to small changes in the input data.
- **How to Prevent Overfitting**:
  - **Use More Data**:
    * Larger datasets help the model distinguish between real patterns and noise.
  - **Simplify the Model**:
    * Reduce the number of features, layers, or parameters.

    * Choose a simpler model (e.g., linear regression instead of polynomial regression).
- **Regularization**:
    * Add penalties to the model's complexity:
        - **L1 Regularization (Lasso)**: Shrinks less important feature coefficients to 0.
        - **L2 Regularization (Ridge)**: Penalizes large weights to prevent over-reliance on any feature.
    * Regularized cost function example:

$$J(\theta) = MSE + \lambda \sum_{i=1}^{m} \theta_i^2$$

        - λ: Regularization strength.
- **Cross-Validation**:
    * Split the data into training, validation, and test sets to evaluate performance on unseen data.
- **Early Stopping**:
    * In iterative algorithms (e.g., gradient descent), stop training once validation error starts increasing.
- **Data Augmentation**:
    * For image or text data, artificially increase the dataset size by generating new samples.

Summary:
Overfitting is when a model learns the noise or specific details of the training data rather than generalizing. It results in poor performance on unseen data. Preventing overfitting involves simplifying the model, using more data, applying regularization, and validating with unseen data to ensure the model generalizes well.