Documentation - matrix

Thibault Nguyen - thibnguy

January 2025

Contents

1 Exercise 0 - Add, Substract and Scale

1.1 Goal

The goal here is to make three functions:

1.1.1 Vectors Calculations

- compute the addition of 2 vectors.

```python
def add_vectors(v1, v2):
```

- compute the subtraction of a vector by another vector.

```python
def subtract_vectors(v1, v2):
```

- compute the scaling of a vector by a scalar.

```python
def scale_vector(v, scalar):
```

1.1.2 Matrices Calculations

- compute the addition of 2 matrices.

```python
def add_matrices(m1, m2):
```

- compute the subtraction of a matrix by another matrix.

```python
def subtract_matrices(m1, m2):
```

- compute the scaling of a matrix by a scalar.

```python
def scale_matrix(m, scalar):
```

1.2 Potential Usage Addition Vectors

- **Physics (Forces, Velocities)**:
  - Combining multiple forces acting on an object.
  - Adding velocities of moving objects to calculate resultant motion.
  - Example: If an object is moving north at 3 m/s and east at 4 m/s, the resultant velocity is:

$$v_{resultant} = v_1 + v_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

- **Graphics/Computer Vision**:
  - Moving an object in 3D space by adding a translation vector to its position vector.

- Combining multiple transformations in animations.

- **Navigation/Robotics**:
  - Adding displacement vectors to find the final position of a robot or object.

1.3 Formula Addition Vectors

To add two vectors, you add their corresponding components.

$$A = \left\langle A_1 \quad A_2 \; \ldots \; A_n \right\rangle \textbf{ and } B = \left\langle B_1 \quad B_2 \; \ldots \; B_n \right\rangle$$

The resultant vector $R = A + B$ is calculated as:

$$R = \sum_{i=1}^{n} \left\langle A_i + B_i \right\rangle$$

$$R = \left\langle A_1 + B_1 \quad A_2 + B_2 \; \ldots \; A_n + B_n \right\rangle$$

1.3.1 Example Addition Vectors

$$\begin{aligned} v_1 &= \left\langle 1 \quad 2 \quad 3 \right\rangle \text{ and } v_2 = \left\langle 4 \quad 5 \quad 6 \right\rangle \\ R &= \left\langle v_{1_1} + v_{2_1} \quad v_{1_2} + v_{2_2} \quad v_{1_3} + v_{2_3} \right\rangle \\ \Leftrightarrow R &= \left\langle 1+4 \quad 2+5 \quad 3+6 \right\rangle \\ \Leftrightarrow R &= \left\langle 5 \quad 7 \quad 9 \right\rangle . \end{aligned}$$

1.4 Potential Usage Substraction Vectors

- **Physics (Relative Motion):**:
  - Calculating relative velocity or displacement.
  - Example: If two objects are moving, their relative velocity is:

$$v_{relative} = v_1 - v_2$$

- **Engineering/Control Systems:**:
  - Computing error vectors in control systems, such as the difference between desired and actual positions.
- **Collision Detection (Games/Simulations)**:
  - Finding the difference between two positions to calculate the distance and detect potential collisions.
- **Graphics**:
  - Determining the direction vector between two points (e.g., camera and target in 3D rendering):

$$v_{direction} = p_{target} - p_{camera}$$

1.5 Formula Substraction Vectors

To substract two vectors, you substract their corresponding components.

$$A = \left\langle A_1 \quad A_2 \ \ldots \ A_n \right\rangle \textbf{ and } B = \left\langle B_1 \quad B_2 \ \ldots \ B_n \right\rangle$$

The resultant vector $R = A - B$ is calculated as:

$$R = \sum_{i=1}^{n} \left\langle A_i - B_i \right\rangle$$

$$R = \left\langle A_1 - B_1 \quad A_2 - B_2 \ \ldots \ A_n - B_n \right\rangle$$

1.5.1 Example Substraction Vectors

$$\begin{aligned} v_1 &= \left\langle 1 \quad 2 \quad 3 \right\rangle \text{ and } v_2 = \left\langle 4 \quad 5 \quad 6 \right\rangle \\ R &= \left\langle v_{1_1} - v_{2_1} \quad v_{1_2} - v_{2_2} \quad v_{1_3} - v_{2_3} \right\rangle \\ \Leftrightarrow R &= \left\langle 1 - 4 \quad 2 - 5 \quad 3 - 6 \right\rangle \\ \Leftrightarrow R &= \left\langle -3 \quad -3 \quad -3 \right\rangle. \end{aligned}$$

1.6 Potential Usage Scalar Vectors

- **Physics**:
  - Scaling vectors to adjust their magnitude while maintaining direction (e.g., scaling force or velocity vectors).
- **Graphics**:
  - Scaling an object's size by scaling its position or transformation vectors.
  - Adjusting lighting intensity in ray tracing.
- **Economics/Finance**:
  - Scaling economic data vectors (e.g., multiplying price vectors by a tax rate or exchange rate).
- **Robotics**:
  - Adjusting speed or distance traveled by scaling the movement vectors.

1.7 Formula Scalar Vectors

To scale a vector, you multiply the scalar to each component.

$$A = \left\langle A_1 \quad A_2 \ \dots \ A_n \right\rangle \textbf{ and } scalar = \lambda$$

The resultant vector $R = \lambda \cdot A$ is calculated as:

$$R = \left\langle \lambda \cdot A_1 \quad \lambda \cdot A_2 \ \dots \ \lambda \cdot A_n \right\rangle$$

1.7.1 Example Scalar Vectors

$$
\begin{aligned}
v_1 &= \left\langle 1 \quad 2 \quad 3 \right\rangle \text{ and } scalar = 2 \\
R &= \left\langle \lambda \cdot v_{1_1} \quad \lambda \cdot v_{1_2} \quad \lambda \cdot v_{1_3} \right\rangle \\
\Leftrightarrow R &= \left\langle 2 \cdot 1 \quad 2 \cdot 2 \quad 2 \cdot 3 \right\rangle \\
\Leftrightarrow R &= \left\langle 2 \quad 4 \quad 6 \right\rangle .
\end{aligned}
$$

1.8 Potential Usage Addition Matrices

- **Graphics (Transformations)**:
  - Adding transformation matrices to combine multiple transformations, such as translations or scaling operations, in animation or rendering.
- **Physics (Systems of Equations)**:
  - Adding stiffness or mass matrices in finite element analysis (e.g., structural mechanics).
- **Image Processing**:
  - Adding two images represented as matrices of pixel values (e.g., overlaying images or combining brightness adjustments).
- **Data Analysis**:
  - Summing multiple datasets represented as matrices (e.g., adding matrices of statistical data or experimental results).

1.9 Formula Addition Matrices

To add two matrices, you add their corresponding components. If the matrices A and B are of the same dimensions (say $m * n$)

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix} \textbf{ and } B = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1n} \\ B_{21} & B_{22} & \dots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \dots & B_{mn} \end{bmatrix}$$

The resultant matrix $C[i][j] = A[i][j] + B[i][j]$ is calculated as:

$$M = \sum_{i=1}^{m} \sum_{j=1}^{n} \Big\langle A[i][j] + B[i][j] \Big\rangle$$

$$M = A + B = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} & \dots & A_{1n} + B_{1n} \\ A_{21} + B_{21} & A_{22} + B_{22} & \dots & A_{2n} + B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} + B_{m1} & A_{m2} + B_{m2} & \dots & A_{mn} + B_{mn} \end{bmatrix}$$

1.9.1 Example Addition Matrices

$$\begin{aligned} m_1 &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } m_2 = \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \\ M &= \begin{bmatrix} m_{1_{11}} + m_{2_{11}} & m_{1_{12}} + m_{2_{12}} & m_{1_{13}} + m_{2_{13}} \\ m_{1_{21}} + m_{2_{21}} & m_{1_{22}} + m_{2_{22}} & m_{1_{23}} + m_{2_{23}} \end{bmatrix} \\ \Leftrightarrow M &= \begin{bmatrix} 1+7 & 2+8 & 3+9 \\ 4+10 & 5+11 & 6+12 \end{bmatrix} \\ \Leftrightarrow M &= \begin{bmatrix} 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}. \end{aligned}$$

1.10 Potential Usage Substraction Matrices

- **Graphics**:
  - Calculating the difference between transformation matrices to find relative transformations.
- **Engineering (Error Analysis)**:
  - Subtracting matrices to compute error matrices, such as the difference between measured and theoretical results.
- **Image Processing**:
  - Subtracting one image from another to detect differences or changes (e.g., motion detection, image masking).
- **Robotics**:
  - Determining the relative pose or state of a robot by subtracting one state matrix from another.

1.11 Formula Subctraction Matrices

To substract two matrices, you substract their corresponding components. If the matrices A and B are of the same dimensions (say $m * n$)

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix} \textbf{ and } B = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1n} \\ B_{21} & B_{22} & \dots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \dots & B_{mn} \end{bmatrix}$$

The resultant matrix $M[i][j] = A[i][j] - B[i][j]$ is calculated as:

$$M = \sum_{i=1}^{m} \sum_{j=1}^{n} \left\langle A[i][j] - B[i][j] \right\rangle$$

$$M = A - B = \begin{bmatrix} A_{11} - B_{11} & A_{12} - B_{12} & \dots & A_{1n} - B_{1n} \\ A_{21} - B_{21} & A_{22} - B_{22} & \dots & A_{2n} - B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} - B_{m1} & A_{m2} - B_{m2} & \dots & A_{mn} - B_{mn} \end{bmatrix}$$

1.11.1 Example Substraction Matrices

$$m_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } m_2 = \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

$$M = \begin{bmatrix} m_{1_{11}} - m_{2_{11}} & m_{1_{12}} - m_{2_{12}} & m_{1_{13}} - m_{2_{13}} \\ m_{1_{21}} - m_{2_{21}} & m_{1_{22}} - m_{2_{22}} & m_{1_{23}} - m_{2_{23}} \end{bmatrix}$$

$$\Leftrightarrow M = \begin{bmatrix} 1 - 7 & 2 - 8 & 3 - 9 \\ 4 - 10 & 5 - 11 & 6 - 12 \end{bmatrix}$$

$$\Leftrightarrow M = \begin{bmatrix} -6 & -6 & -6 \\ -6 & -6 & -6 \end{bmatrix}.$$

1.12 Potential Usage Scalar Matrices

- **Graphics**:
  - Scaling transformation matrices to modify object size or perspective effects.
- **Physics (Scaling Systems)**:
  - Scaling stiffness or mass matrices to simulate physical systems with different material properties.
- **Economics/Finance**:
  - Scaling cost or profit matrices to reflect inflation, tax rates, or currency conversion.
- **Image Processing**:

– Brightness adjustment by scaling pixel intensity matrices.

1.13 Formula Scalar Matrices

To scale a matrix, you multiply the scalar to each component.

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix} \textbf{ and } scalar = \lambda$$

The resultant vector $C = \lambda \cdot A$ is calculated as:

$$M = \begin{bmatrix} \lambda \cdot A_{11} & \lambda \cdot A_{12} & \dots & \lambda \cdot A_{1n} \\ \lambda \cdot A_{21} & \lambda \cdot A_{22} & \dots & \lambda \cdot A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda \cdot A_{m1} & \lambda \cdot A_{m2} & \dots & \lambda \cdot A_{mn} \end{bmatrix}$$

1.13.1 Example Scalar Matrices

$$m_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } scalar = 3$$

$$M = \begin{bmatrix} \lambda \cdot m_{1_{11}} & \lambda \cdot m_{1_{12}} & \lambda \cdot m_{1_{13}} \\ \lambda \cdot m_{1_{21}} & \lambda \cdot m_{1_{22}} & \lambda \cdot m_{1_{23}} \end{bmatrix}$$

$$\Leftrightarrow M = \begin{bmatrix} 3 \cdot 1 & 3 \cdot 2 & 3 \cdot 3 \\ 3 \cdot 4 & 3 \cdot 5 & 3 \cdot 6 \end{bmatrix}$$

$$\Leftrightarrow M = \begin{bmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{bmatrix}.$$

2 Exercise 1 - Linear Combination

2.1 Goal

The linear combination of two vectors involves multiplying each vector by a scalar and then adding the resulting vectors.

```
def linear_combination(vectors, scalars):
```

2.2 Potential Usage

- **Graphics (Blending and Transformations)**:
  - **Blending Transformations**:
    - Linear combination is used to interpolate between transformation matrices for smooth transitions in animations.
    - Example: Transitioning an object from one transformation A_1 to another A_2 can use:

$$C = (1 - t) \cdot A_1 + t \cdot A_2,\ 0 \leq t \leq 1$$

  - **Perspective Interpolation**:
    - Blending perspectives in 3D graphics by linearly combining projection matrices.
- **Physics and Engineering**:
  - **Combining Systems**:
    - In finite element analysis, linear combinations of stiffness or mass matrices are used to represent composite materials or coupled systems.
  - **Superposition Principle**:
    - The principle of superposition uses linear combinations to compute the combined effect of multiple systems (e.g., forces or displacements).

- **Data Science and Machine Learning**:
  - **Weighted Summation**:
    * Linear combinations of data matrices (e.g., weighted averages of datasets) to emphasize or de-emphasize certain features.
  - **Principal Component Analysis (PCA)**:
    * PCA combines feature matrices as linear combinations of eigenvectors to reduce dimensionality.
- **Economics and Finance**:
  - **Portfolio Analysis**:
    * Combining profit or cost matrices using scalar weights to represent contributions of different investments.
  - **Forecasting**:
    * Predicting outcomes by taking weighted linear combinations of historical matrices.

2.3 Formula

To form a linear combination, you multiply each component of the first input by its scalar, each component of the second input by its scalar, and then add the results component-wise.

$A = \left\langle A_1 \quad A_2 \ \ldots \ A_n \right\rangle$ **and** $B = \left\langle B_1 \quad B_2 \ \ldots \ B_n \right\rangle$, **and** c_1 **and** c_2 are scalars.

The resultant vector $R = c_1 \cdot A + c_2 + B$ is calculated as:

$$R = \sum_{i=1}^{n} \left\langle c_1 \cdot A_i + c_2 \cdot B_i \right\rangle$$

$$R = \left\langle c_1 \cdot A_1 + c_2 \cdot B_1 \quad c_1 \cdot A_2 + c_2 \cdot B_2 \ \ldots \ c_1 \cdot A_n + c_2 \cdot B_n \right\rangle$$

2.3.1 Example

$$v_1 = \left\langle 1 \quad 2 \quad 3 \right\rangle \textbf{ and } v_2 = \left\langle 4 \quad 5 \quad 6 \right\rangle \textbf{ and } c_1 = 3 \textbf{ and } c_2 = 2$$
$$R = \left\langle c1 \cdot v_{1_1} + c2 \cdot v_{2_1} \quad c1 \cdot v_{1_2} + c2 \cdot v_{2_2} \quad c1 \cdot v_{1_3} + c2 \cdot v_{2_3} \right\rangle$$
$$\Leftrightarrow R = \left\langle 3 \cdot 1 + 2 \cdot 4 \quad 3 \cdot 2 + 2 \cdot 5 \quad 3 \cdot 3 + 2 \cdot 6 \right\rangle$$
$$\Leftrightarrow R = \left\langle 3 + 8 \quad 6 + 10 \quad 9 + 12 \right\rangle$$
$$\Leftrightarrow R = \left\langle 11 \quad 16 \quad 21 \right\rangle$$

3 Exercise 2 - Linear Interpolation

3.1 Goal

Linear interpolation (or lerp) between two vectors calculates a point along the line between them, based on a parameter t in the range [0, 1].

```
def linear_interpolation(v1, v2, t):
```

3.2 Potential Usage

- **Graphics (Transformations and Animations)**:
  - **Camera Transitions**:
    - Interpolating between two camera transformation matrices for smooth camera motion in 3D scenes.
    - Example: Transitioning from A (initial view) to B (target view).
  - **Object Transformations**:
    - Interpolating transformation matrices to animate objects smoothly between keyframes.
  - **Blending Projection Matrices**:
    - Combining two projection matrices for effects like zoom or perspective shifts.

- **Physics and Simulations**:
  - **State Interpolation**:
    * Blending two physical states (e.g., position or orientation matrices) for smooth transitions in simulations.
  - **Collision Detection**:
    * Interpolating matrices to estimate object positions over time for more accurate collision checks.
- **Data Visualization**:
  - **Morphing Data Representations**:
    * Interpolating between two matrices of data to create animations showing the progression or transition of datasets.
  - **Heatmap Blending**:
    * Blending two heatmaps (represented as matrices) to show intermediate states between two data distributions.
- **Machine Learning**:
  - **Gradient Visualization**:
    * Interpolating matrices of weights or outputs in neural networks to visualize how weights evolve during training.
  - **Model Ensembles**:
    * Interpolating between two model outputs (e.g., decision matrices) to combine predictions.

3.3 Formula

To perform linear interpolation, you blend each component of the first input with the corresponding component of the second input using the interpolation factor as the weight.

$$A = \left\langle A_1 \quad A_2 \ \dots \ A_n \right\rangle \textbf{ and } B = \left\langle B_1 \quad B_2 \ \dots \ B_n \right\rangle \textbf{ and } t \in [0, 1]$$

The resultant vector R = A + t * (B - A) is calculated as:

$$R = \sum_{i=1}^{n} \left\langle A_i + t \cdot (B_i - A_i) \right\rangle$$

$$R = \left\langle A_1 + t \cdot (B_1 - A_1) \quad A_2 + t \cdot (B_2 - A_2) \ \ldots \ A_n + t \cdot (B_n - A_n) \right\rangle$$

3.3.1 Example

Type: **Scalars**

$$\begin{aligned} & A = 0 \textbf{ and } B = 1 \textbf{ and } t = 1 \\ & R = 0 + 1 \cdot (1 - 0) \\ \Leftrightarrow\ & R = 0 + (1 - 0) \\ \Leftrightarrow\ & R = 1 \end{aligned}$$

Type: **Vectors**

$$\begin{aligned} & A = \left\langle 2 \quad 1 \right\rangle \textbf{ and } B = \left\langle 4 \quad 2 \right\rangle \textbf{ and } t = 0.3 \\ & R = \left\langle 2 + 0.3 \cdot (4 - 2) \quad 1 + 0.3 \cdot (2 - 1) \right\rangle \\ \Leftrightarrow\ & R = \left\langle 2 + (1.2 - 0.6) \quad 1 + (0.6 - 0.3) \right\rangle \\ \Leftrightarrow\ & R = \left\langle 2 + 0.6 \quad 1 + 0.3 \right\rangle \\ \Leftrightarrow\ & R = \left\langle 2.6 \quad 1.3 \right\rangle \end{aligned}$$

Type: **Matrix**

$$
\begin{aligned}
A &= \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \textbf{ and } B = \begin{bmatrix} 20 & 10 \\ 30 & 40 \end{bmatrix} \textbf{ and } t = 0.5 \\
M &= \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} + 0.5 \cdot \begin{bmatrix} 20-2 & 10-1 \\ 30-3 & 40-4 \end{bmatrix} \\
\Leftrightarrow M &= \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} + 0.5 \cdot \begin{bmatrix} 18 & 9 \\ 27 & 36 \end{bmatrix} \\
\Leftrightarrow M &= \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 9 & 4.5 \\ 13.5 & 18 \end{bmatrix} \\
\Leftrightarrow M &= \begin{bmatrix} 11 & 5.5 \\ 16.5 & 22 \end{bmatrix}
\end{aligned}
$$

4 Exercise 3 - Dot Product

4.1 Goal

The dot product (also known as the scalar product) is an operation that takes two vectors and returns a scalar.

```python
def dot(u, v):
```

4.2 Potential Usage

- **Physics and Mechanics**:
  - **Work Done by a Force**:
    - The dot product is used to calculate the work done by a force F moving an object along a displacement d:

$$W = F \cdot d$$

  - **Projection of Forces**:
    * To find the component of a force in a given direction:

$$F_{parallel} = \frac{F \cdot d}{\|d\|}$$

  - **Moment of Inertia and Stress Analysis**:
    * In mechanics, dot products between stress and strain matrices represent the internal energy or deformation work.

- **Data Science and Machine Learning**:
  - **Similarity Between Data Points**:
    * The dot product is used to compute cosine similarity, which measures the angle between two data vectors. For matrices, the dot product can be extended to compare rows or columns.
  - **Matrix Factorization**:
    * The dot product is a core operation in matrix factorization techniques like Singular Value Decomposition (SVD) or Principal Component Analysis (PCA).

- **Computer Graphics**:
  - **Lighting Calculations**:
    * The dot product is used to compute the intensity of light on a surface by measuring the angle between the light direction and the surface normal.
  - **Projections**:
    * Projecting a 3D point onto a 2D plane often involves dot products to determine the shadow or projection length.

- **Economics and Finance**:
  - **Portfolio Analysis**:
    * Dot products are used to compute the total value of a portfolio by multiplying weights (percentages) with returns.

– **Utility and Cost Analysis**:

  * The dot product represents weighted summations, like computing costs for quantities of goods.

4.3 Formula

To calculate the dot product, you multiply each pair of corresponding components from two vectors and then add the results together

$$A = \left\langle A_1 \quad A_2 \ \ldots \ A_n \right\rangle \textbf{ and } B = \left\langle B_1 \quad B_2 \ \ldots \ B_n \right\rangle$$

The resultant vector R = A.B is calculated as:

$$R = \sum_{i=1}^{n} A_i \cdot B_i$$

$$R = A_1 \cdot B_1 + A_2 \cdot B_2 + \cdots + A_n \cdot B_n$$

4.3.1 Example

$$\begin{aligned} & A = \left\langle -1 \quad 6 \right\rangle \textbf{ and } B = \left\langle 3 \quad 2 \right\rangle \\ & R = -1 \cdot 3 + 6 \cdot 2 \\ \Leftrightarrow\ & R = -3 + 12 \\ \Leftrightarrow\ & R = 9 \end{aligned}$$

5 Exercise 4 - Norm

5.1 Goal

The goal here is to make three functions:

- compute the Norm-1 of a vector.

```python
def norm_1(vector):
```

- compute the Norm-2 of a vector.

```python
def norm_2(vector):
```

- compute the Norm-∞ of a vector.

```python
def norm_inf(vector):
```

5.2 Potential Usage Norm-1

- **Data Science and Machine Learning**:
  - **Feature Importance**:
    * Norm-1 is used to evaluate the magnitude of feature weights in machine learning models, particularly in LASSO regression, where it encourages sparsity by penalizing the norm-1 of the coefficients.
  - **Regularization**:
    * Norm-1 is commonly used as a penalty term in optimization to enforce sparsity (e.g., fewer active features or parameters).z
- **Computer Graphics**:
  - **Bounding Box Approximations**:
    * Norm-1 is used to compute the distance between objects in terms of their axis-aligned bounding boxes (AABBs), which simplifies collision detection.
  - **Pixel Value Summation**:
    * Norm-1 of an image matrix measures the intensity across all pixels in a specific axis (e.g., summing pixel values column by column).
- **Physics and Engineering**:
  - **Energy Computation**:
    * In physical systems, the norm-1 is used to compute energy dissipation or accumulation when values are constrained to specific axes.

- **Error Analysis**:
  - Norm-1 is used to measure errors in numerical solutions, particularly for systems that exhibit axis-aligned behavior.

- **Robotics and Pathfinding**:
  - **Manhattan Distance**:
    - Norm-1 is a core part of calculating the Manhattan distance between two points, which is useful for robots navigating in grid-based environments.
  - **Trajectory Optimization**:
    - Minimizing the norm-1 of control matrices reduces sharp turns or excessive movement along one axis.

5.3 Formula Norm-1

Also called the Manhattan norm or taxicab norm, it is the sum of the absolute values of the vector's components.

$$A = \left\langle A_1 \quad A_2 \ \ldots \ A_n \right\rangle$$

The resultant vector R = $||A||_1$ is calculated as:

$$||A||_1 = \sum_{i=1}^{n} |A_i|$$

$$||A||_1 = |A_1| + |A_2| + \cdots + |A_n|$$

5.3.1 Example Norm-1

$$A = \left\langle -1 \quad -2 \right\rangle$$

$$R = |-1| + |-2|$$

$$\Leftrightarrow R = 1 + 2$$

$$\Leftrightarrow R = 3$$

5.4 Potential Usage Norm-2

- **Physics and Mechanics**:
  - **Energy Computation**:
    * Norm-2 is used to calculate the total energy of a system when elements represent forces, velocities, or other physical quantities.
  - **Distance Measurement**:
    * Norm-2 provides the shortest path (straight-line distance) between two points in Euclidean space.
  - **Stress Analysis**:
    * Norm-2 is used to quantify the stress in materials using the matrix representation of stresses.
- **Computer Graphics**:
  - **Vector Magnitude**:
    * Norm-2 is used to normalize vectors (e.g., surface normals) in 3D rendering for lighting and shading calculations.
  - **Pixel Intensity**:
    * In image processing, Norm-2 of pixel matrices is used to calculate brightness or contrast.
  - **Camera and Object Transformations**:
    * Used to compute transformations' magnitudes or normalize directional vectors.
- **Data Science and Machine Learning**:
  - **Error Analysis**:
    * Norm-2 is used to measure the magnitude of residuals (e.g., in regression models) to evaluate the model's accuracy.
  - **Regularization**:
    * In Ridge regression, Norm-2 is penalized to prevent overfitting.

- **Similarity and Distance**:
  - The Euclidean distance is calculated using Norm-2 to measure closeness between data points or vectors.

- **Robotics and Pathfinding**:
  - **Trajectory Optimization:**:
    - Norm-2 minimizes the total energy or distance in robotic movement.
  - **Path Planning**:
    - Used to find the shortest path between two points in Euclidean space.

5.5 Formula Norm-2

Also called the Euclidean norm, it is the square root of the sum of the squares of the components. It represents the standard length of the vector.

$$A = \left\langle A_1 \quad A_2 \ \ldots \ A_n \right\rangle$$

The resultant vector R = $||A||_2$ is calculated as:

$$||A||_2 = \sqrt{\sum_{i=1}^{n} A_i^2}$$

$$||A||_2 = \sqrt{A_1^2 + A_2^2 + \cdots + A_n^2}$$

5.5.1 Example Norm-2

$$\begin{aligned}
& A = \left\langle -1 \quad -2 \right\rangle \\
& R = \sqrt{(-1)^2 + (-2^2)} \\
\Leftrightarrow\ & R = \sqrt{1+4} \\
\Leftrightarrow\ & R = \sqrt{5} \\
\Leftrightarrow\ & R = 2.2360679775
\end{aligned}$$

5.6 Potential Usage Norm-∞

- **Physics and Mechanics**:
  - **Load Balancing**:
    - Norm-∞ is used to identify the maximum load or stress on a system by analyzing row-wise quantities.
  - **Energy Distribution**:
    - It helps identify the maximum energy contribution in a specific direction (row-wise analysis).
- **Computer Graphics**:
  - **Bounding Box Approximation**:
    - Norm-∞ can be used to determine the maximum deviation along an axis, helping to construct bounding volumes for objects.
  - **Pixel Value Range**:
    - In image processing, Norm-∞ measures the row with the highest intensity values to determine brightness extremes.
- **Data Science and Machine Learning**:
  - **Error Analysis**:
    - Norm-∞ evaluates the worst-case error in residuals by taking the maximum row-wise deviation.
  - **Feature Importance**:
    - In models, Norm-∞ can measure the row with the largest cumulative feature weight.
- **Optimization and Control Systems**:
  - **Constraint Enforcement**:
    - Norm-∞ is used in optimization to enforce maximum constraints in control systems or linear programming problems.
  - **Stability Analysis**:
    - It measures the worst-case deviation in control systems, ensuring robustness.

5.7 Formula Norm-∞

Also called the maximum norm, it is the largest absolute value of the components of the vector.

$$A = \left\langle A_1 \quad A_2 \ \ldots \ A_n \right\rangle$$

The resultant vector R = $||A||_\infty$ is calculated as:

$$||A||_\infty = max(|A_i|)$$

$$||A||_\infty = max(|A_1|\,,\ |A_2|\,,\ \ldots,\ |A_n|)$$

5.7.1 Example Norm-∞

$$\begin{aligned} &A = \left\langle -1 \quad -2 \right\rangle \\ &R = max(|-1|\,,\ |-2|) \\ \Leftrightarrow\ &R = max(1,\ 2) \\ \Leftrightarrow\ &R = 2 \end{aligned}$$

6 Exercise 5 - Cosine

6.1 Goal

The cosine formula is used to compute the cosine of the angle θ between two vectors A and B. It is based on the dot product and the magnitudes (norms) of the vectors.

```
def angle_cos(u, v):
```

6.2 Potential Usage

- **Data Science and Machine Learning**:
  - **Similarity Measures**:

- * Cosine similarity is widely used to compare feature vectors (e.g., documents, images, user profiles) in machine learning and information retrieval.
  * Example: Comparing user preferences in a recommendation system.
- **Clustering**:
  * In clustering algorithms, cosine similarity is used to group data points with similar directions (e.g., hierarchical clustering).
- **Dimensionality Reduction**:
  * Measures the alignment between feature vectors for PCA or SVD.

- **Computer Graphics**:
  - **Lighting Calculations**:
    * Cosine is used to compute the intensity of light hitting a surface by measuring the angle between the light source direction and the surface normal.
  - **Projections**:
    * Cosine similarity helps determine the alignment of vectors in 3D space for transformations.
- **Robotics and Pathfinding**:
  - **Direction Alignment**:
    * Cosine measures the alignment of movement directions, optimizing robotic motion paths.
  - **Trajectory Planning**:
    * Used to compare the intended direction with the actual direction of movement.
- **Text and Natural Language Processing (NLP)**:
  - **Document Similarity**:

    * Cosine similarity is used to measure the similarity between term-frequency vectors of documents (e.g., bag-of-words or TF-IDF models).

  - **Semantic Analysis**:
    * Compares word embeddings (vector representations of words) to analyze semantic similarity.

6.3 Formula

To calculate the cosine of the angle between two vectors, you divide their dot product by the product of their magnitudes.

$$A = \left\langle A_1 \quad A_2 \ \ldots \ A_n \right\rangle \textbf{ and } B = \left\langle B_1 \quad B_2 \ \ldots \ B_n \right\rangle$$

The resultant cosine R = cos(θ) is calculated as:

$$cos(\theta) = \frac{A \cdot B}{||A||_2 \, ||B||_2}$$

Where:

- $A \cdot B$ is the dot product of A and B:

$$A \cdot B = \sum_{i=1}^{n} A_i \cdot B_i$$

$$A \cdot B = A_1 \cdot B_1 + A_2 \cdot B_2 + \cdots + A_n \cdot B_n$$

- $\|A\|_2$ and $\|B\|_2$ are the Euclidian norms (Norm-2) of the vectors:

$$\|A\|_2 = \sqrt{\sum_{i=1}^{n} A_i^2}, \quad \|B\|_2 = \sqrt{\sum_{i=1}^{n} B_i^2}$$

$$\|A\|_2 = \sqrt{A_1^2 + A_2^2 + \cdots + A_n^2}, \quad \|B\|_2 = \sqrt{B_1^2 + B_2^2 + \cdots + B_n^2}$$

6.3.1 Example

$$
\begin{aligned}
A &= \left\langle 1 \quad 2 \quad 3 \right\rangle \textbf{ and } B = \left\langle 4 \quad 5 \quad 6 \right\rangle \\
R &= \frac{1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6}{\sqrt{1^2 + 2^2 + 3^2} \cdot \sqrt{4^2 + 5^2 + 6^2}} \\
\Leftrightarrow R &= \frac{4 + 10 + 18}{\sqrt{1 + 4 + 9} \cdot \sqrt{16 + 25 + 36}} \\
\Leftrightarrow R &= \frac{32}{\sqrt{14} \cdot \sqrt{77}} \\
\Leftrightarrow R &= \frac{32}{\sqrt{1078}} \\
\Leftrightarrow R &= 0.9746318462
\end{aligned}
$$

7 Exercise 6 - Cross Product

7.1 Goal

The cross product (or vector product) is an operation on two vectors in 3D space that produces a third vector that is perpendicular to both input vectors.

```python
def cross_product(u, v):
```

7.2 Potential Usage

- **Physics and Mechanics**:
  - **Torque**:
    - The cross product computes the torque (τ) generated by a force (F) applied at a position (r):

$$\tau = r \cdot F$$

  - **Angular Momentum**:
    - The angular momentum of a particle is given by the cross product of its position vector and momentum:

$$L = r \cdot p$$

- **Magnetic Forces**:
  - The force on a charged particle in a magnetic field is computed using the cross product:

$$F = q(v \cdot B)$$

    where q is the charge, v is the velocity, and B is the magnetic field.

- **Computer Graphics**:
  - **Surface Normals**:
    - The cross product is used to compute the normal vector of a surface by taking the cross product of two edges of a triangle:

$$n = e_1 \cdot e_2$$

  - **Lighting and Shading**:
    - Cross products are used in lighting calculations to determine how light interacts with surfaces by using surface normals.
  - **Rotation Axes**:
    - The axis of rotation for two direction vectors is calculated using the cross product.

- **Robotics and Pathfinding**:
  - **Trajectory Planning**:
    - Cross products are used to calculate perpendicular vectors for trajectory adjustments in 3D space.
  - **Orientation and Alignment**:
    - Determines the orientation of a plane or robot arm by computing cross products of vectors.

- **Mathematics and Geometry**:

- **Area of a Parallelogram**:
  * The magnitude of the cross product gives the area of a parallelogram formed by two vectors:

$$Area = \|u \cdot v\|$$

- **Volume of a Parallelepiped**:
  * The volume of a parallelepiped is the scalar triple product:

$$Volume = w(u \cdot v)$$

    where w is the third vector

7.3 Formula

To compute the cross product of two vectors, you calculate the determinant of a matrix formed by the unit vectors and the components of the two vectors.

$$A = \left\langle A_x \quad A_y \quad A_z \right\rangle \textbf{ and } B = \left\langle B_x \quad B_y \quad B_z \right\rangle$$

The resultant R = A x B is calculated as:

$$R = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Where:

- i, j, k are the unit vectors in the $x-$, $y-$, $z-$ directions.

$A \text{ x } B = i(A_y \cdot B_z - A_z \cdot B_y) - j(A_x \cdot B_z - A_z \cdot B_x) + k(A_x \cdot B_y - A_y \cdot B_x)$
In vector form:
$A \text{ x } B = A_y \cdot B_z - A_z \cdot B_y,\ A_z \cdot B_x - A_x * B_z,\ A_x \cdot B_y - A_y \cdot B_x$

7.3.1 Example

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \textbf{ and } B = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} \\
R &= \left\langle 2 \cdot 6 - 3 \cdot 5 \quad 3 \cdot 4 - 1 \cdot 6 \quad 1 \cdot 5 - 2 \cdot 41 \right\rangle \\
\Leftrightarrow R &= \left\langle 12 - 15 \quad 12 - 6 \quad 5 - 8 \right\rangle \\
\Leftrightarrow R &= \left\langle -3 \quad 6 \quad - 3 \right\rangle
\end{aligned}
$$

8 Exercise 7 - Linear Map, Matrix Multiplication

8.1 Goal

The goal here is to make two functions:

- compute the multiplication of a matrix by a vector.

```
def mul_vec(matrix, vector):
```

- compute the multiplication of a matrix by another matrix.

```
def mul_mat(m1, m2):
```

8.2 Potential Usage Dot Vector

- **Physics and Mechanics**:
  - **Work Done by a Force**:
    - The dot product is used to calculate the work (W) done by a force (F) acting along a displacement (d):

$$W = F \cdot d$$

- **Projection of Forces**:
  * The dot product finds the component of a force in a specific direction:
$$F_{parallel} = \frac{F \cdot d}{\|d\|}$$
- **Electric and Magnetic Fields**:
  * In electromagnetism, the dot product computes the flux of a field across a surface.

- **Computer Graphics**:
  - **Lighting Calculations**:
    * The dot product is used to determine the intensity of light on a surface by computing the angle between the surface normal and the light direction.
  - **Projections**:
    * To project one vector onto another:

$$\text{Projection of } u \text{ onto } v = \frac{u \cdot v}{\|v\|^2} \cdot v$$

- **Data Science and Machine Learning**:
  - **Similarity Measures**:
    * The dot product forms the basis for cosine similarity, a common metric for comparing vectors in high-dimensional space (e.g., text or user profiles).
  - **Linear Regression**:
    * Predictions in linear regression models are calculated using the dot product between feature vectors and weight vectors.
- **Robotics and Pathfinding**:
  - **Alignment and Direction**:
    * The dot product determines if two vectors are aligned (positive dot product) or opposed (negative dot product).

- **Trajectory Planning**:
  * Used to calculate the deviation or alignment between desired and actual movement directions.

8.3 Formula Matrix Dot Vector

To multiply a matrix by a vector, you compute each component of the resulting vector by taking the dot product of the matrix's rows with the vector.

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix} \textbf{ and } v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

The resultant R = A.v is calculated as:

$$R_i = \sum_{j=1}^{n} A_{ij} \cdot v_j$$

for $i = 1, 2, \dots, m$.

$$R = \begin{bmatrix} A_{11} * v_1 + A_{12} * v_2 + \cdots + A_{1n} * v_n \\ A_{21} * v_1 + A_{22} * v_2 + \cdots + A_{2n} * v_n \\ \vdots \\ A_{m1} * v_1 + A_{m2} * v_2 + \cdots + A_{mn} * v_n \end{bmatrix}$$

8.3.1 Example Matrix Dot Vector

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \textbf{ and } v = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$R = \begin{bmatrix} 2 \cdot 4 + (-2) \cdot 2 \\ (-2) \cdot 4 + 2 \cdot 2 \end{bmatrix}$$

$$\Leftrightarrow R = \begin{bmatrix} 8 - 4 \\ -8 + 4 \end{bmatrix}$$

$$\Leftrightarrow R = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

8.4 Potential Usage Matrix Multiplication

- **Physics and Mechanics**:
  - **Transformations**:
    * Matrix multiplication is used to apply transformations like rotations, scaling, and translations to physical systems or coordinates.
  - **Coupled Systems**:
    * Models representing coupled systems (e.g., electrical networks or mechanical linkages) use matrix multiplication to combine component equations.
- **Computer Graphics**:
  - **3D Transformations**:
    * Matrix multiplication is essential for combining transformations like translation, rotation, and scaling in 3D rendering.
  - **Projection Matrices**:
    * Multiplying points by projection matrices maps 3D objects to 2D space for visualization.
  - **Animation**:

    * Animations use multiplication to apply transformations to objects over time.

- **Data Science and Machine Learning**:
  - **Feature Transformation**:
    * Multiplying feature matrices with weight matrices transforms data in machine learning models (e.g., in neural networks).
  - **Dimensionality Reduction**:
    * Principal Component Analysis (PCA) and Singular Value Decomposition (SVD) rely on matrix multiplication to reduce data dimensions.
  - **Covariance Computation**:
    * The covariance matrix, which measures relationships between variables, is computed using matrix multiplication.

- **Robotics and Pathfinding**:
  - **Kinematics**:
    * Robot arm movement is computed using matrix multiplication to apply transformations and calculate positions in 3D space.
  - **Path Planning**:
    * Multiplying state transition matrices with control matrices models robot motion in dynamic systems.

- **Economics and Finance**:
  - **Portfolio Analysis**:
    * Multiplying weight matrices with price or return matrices computes portfolio values.
  - **Input-Output Models**:
    * In economic modeling, input-output matrices represent interactions between industries, with results computed via matrix multiplication.

8.5 Formula Matrix Multiplication

To multiply two matrices, you compute each component of the resulting matrix by taking the dot product of each row of the first matrix with each column of the second matrix.

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix} \textbf{ and } B = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1n} \\ B_{21} & B_{22} & \dots & B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \dots & B_{mn} \end{bmatrix}$$

The resultant $M = A \cdot B$ is calculated as:

$$M[i][j] = \sum_{k=1}^{n} A[i][k] \cdot B[k][j]$$

Where:

- A is mxn
- B is nxp
- M is mxp

$$M = \begin{bmatrix} A_{11} \cdot B_{11} + A_{12} \cdot B_{21} + \cdots + A_{1n} \cdot B_{n1} & A_{11} \cdot B_{12} + A_{12} \cdot B_{22} + \cdots + A_{1n} \cdot B_{n2} & \cdots & A_{11} \cdot B_{1p} + A_{12} \cdot B_{2p} + \cdots + A_{1n} \cdot B_{np} \\ A_{21} \cdot B_{11} + A_{22} \cdot B_{21} + \cdots + A_{2n} \cdot B_{n1} & A_{21} \cdot B_{12} + A_{22} \cdot B_{22} + \cdots + A_{2n} \cdot B_{n2} & \cdots & A_{21} \cdot B_{1p} + A_{22} \cdot B_{2p} + \cdots + A_{2n} \cdot B_{np} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} \cdot B_{11} + A_{m2} \cdot B_{21} + \cdots + A_{mn} \cdot B_{n1} & A_{m1} \cdot B_{12} + A_{m2} \cdot B_{22} + \cdots + A_{mn} \cdot B_{n2} & \cdots & A_{m1} \cdot B_{1p} + A_{m2} \cdot B_{2p} + \cdots + A_{mn} \cdot B_{np} \end{bmatrix}$$

8.5.1 Example Matrix Multiplication

$$A = \begin{bmatrix} 3 & -5 \\ 6 & 8 \end{bmatrix} \textbf{ and } B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

$$M = \begin{bmatrix} 3 \cdot 2 + (-5) \cdot 4 & 3 \cdot 1 + (-5) \cdot 2 \\ 6 \cdot 2 + 8 \cdot 4 & 6 \cdot 1 + 8 \cdot 2 \end{bmatrix}$$

$$\Leftrightarrow M = \begin{bmatrix} 6 - 20 & 3 - 10 \\ 12 + 32 & 6 + 16 \end{bmatrix}$$

$$\Leftrightarrow M = \begin{bmatrix} -14 & -7 \\ 44 & 22 \end{bmatrix}$$

9 Exercise 8 - Trace

9.1 Goal

The trace of a matrix A, denoted as $tr(A)$, is the sum of the elements on its main diagonal.

```
def trace(u):
```

9.2 Potential Usage

- **Physics and Mechanics**:
  - **Tensor Analysis**:
    - The trace of a stress or strain tensor represents the volumetric strain in materials, indicating changes in volume.
  - **Inertia and Momentum**:

* In mechanics, the trace of the moment of inertia tensor can simplify computations for rotational dynamics.

- **Heat and Energy**:
  * In thermodynamics, traces of matrices are used to calculate internal energy or heat in certain systems.

- **Computer Graphics**:
  - **Transformations**:
    * The trace of a transformation matrix can indicate the scaling factor or the total effect of a 3D transformation.
  - **Eigenvalue Approximation**:
    * The trace is the sum of eigenvalues of a matrix, which can be useful for approximate calculations in transformations.

- **Data Science and Machine Learning**:
  - **Covariance Matrix Analysis**:
    * The trace of a covariance matrix represents the total variance in a dataset.
  - **Regularization**:
    * The trace norm is used in optimization problems to encourage low-rank solutions.
  - **Dimensionality Reduction**:
    * In PCA, the trace of the feature matrix indicates the total energy or information in the data.

- **Mathematics and Geometry**:
  - **Matrix Properties**:
    * The trace is invariant under cyclic permutations ($trace(AB) = trace(BA)$), making it useful in theoretical proofs.
  - **Eigenvalue Analysis**:

        * Since the trace equals the sum of the eigenvalues, it provides insights into the spectrum of the matrix.

- **Economics and Finance**:
  - **Portfolio Analysis**:
    * The trace of a covariance matrix in portfolio management represents the total risk of all assets.
  - **Input-Output Models**:
    * In economic systems, the trace can represent total internal consumption within industries.

9.3 Formula

To calculate the trace of a matrix, you sum all the elements on its main diagonal.

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix}$$

The resultant $R = tr(A)$ is calculated as:

$$R = \sum_{i=1}^{n} A_{ii}$$

$$R = A_{11} + A_{22} + \cdots + A_{mn}$$

9.3.1 Example

$$A = \begin{bmatrix} -2 & -8 & 4 \\ 1 & -23 & 4 \\ 0 & 6 & 4 \end{bmatrix}$$

$$R = -2 + (-23) + 4$$

$$\Leftrightarrow R = -21$$

10 Exercise 9 - Transpose

10.1 Goal

The transpose of a matrix A, denoted as $A^\top$ (or sometimes A^T), is obtained by flipping the matrix over its diagonal. In other words, the rows of A become the columns of $A^\top$, and the columns of A become the rows of $A^\top$.

```
def transpose(matrix):
```

10.2 Potential Usage

- **Physics and Mechanics**:
  - **Symmetry in Tensors**:
    * The transpose is used to verify symmetry in stress or strain tensors, important for mechanical equilibrium and material analysis.
  - **Change of Basis**:
    * Transpose matrices are used in coordinate transformations, particularly when switching between bases.
  - **Moment of Inertia**:
    * The transpose of inertia tensors simplifies rotational calculations in mechanics.
- **Computer Graphics**:
  - **Matrix Transformations**:
    * The transpose of a transformation matrix is used in rendering pipelines, particularly for inverse transformations.
  - **Normals Transformation**:
    * In lighting and shading, the transpose of the inverse transformation matrix is used to transform surface normals correctly.
  - **Vertex Data Layout**:
    * Switching between column-major and row-major order in matrix storage or transformations.

- **Data Science and Machine Learning**:
  - **Covariance Matrix**:
    - The transpose is used to compute the covariance matrix:

      $$\sum = \frac{1}{n} X^\top X$$

      Where X is the data matrix.
  - **Linear Regression**:
    - In regression models, the transpose of the feature matrix is used to solve the normal equations:

      $$\hat{\beta} = (X^\top X)^{-1} X^\top y$$

  - **Feature Engineering**:
    - Transposing data matrices reorganizes data for algorithms that require specific input formats.
- **Robotics and Pathfinding**:
  - **Transformation Matrices**:
    - The transpose of rotation matrices is used to calculate inverse rotations efficiently.
  - **Kinematics**:
    - Transpose matrices are used in forward and inverse kinematic calculations for robotic arms.
- **Economics and Finance**:
  - **Input-Output Analysis**:
    - The transpose reorganizes data matrices to analyze relationships between sectors (e.g., outputs as rows, inputs as columns).
  - **Portfolio Management**:
    - In matrix-based financial models, transposing matrices aligns data for computations like returns or risk assessments.

10.3 Formula

To transpose a matrix, you swap its rows with its columns.

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix}$$

The resultant $R = A^\top$ is calculated as:

$$R = (A^\top)_{ij} = A_{ji}$$

$$R = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{m1} \\ A_{12} & A_{22} & \dots & A_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{mn} \end{bmatrix}$$

10.3.1 Example

$$A = \begin{bmatrix} -2 & -8 & 4 \\ 1 & -23 & 4 \\ 0 & 6 & 4 \end{bmatrix}$$

$$R = \begin{bmatrix} -2 & 1 & 0 \\ -8 & -23 & 6 \\ 4 & 4 & 4 \end{bmatrix}$$

11 Exercise 10 - Row-Echelon Form

11.1 Goal

The row echelon form (REF) of a matrix is a simplified form that results from applying a series of row operations to a matrix. These operations aim to achieve a specific structure for the matrix. Here we will do the reduced row echelon form (RREF), where all elements above and under a pivot become 0.

```python
def row_echelon_form(u):
```

11.2 Potential Usage

- **Solving Systems of Linear Equations**:
  - **Gaussian Elimination**:
    * RREF is used to solve systems of linear equations by converting the augmented matrix into a form where solutions can be read directly.
  - **Identifying Solutions**:
    * The RREF form indicates whether the system has a unique solution, infinite solutions, or no solution (e.g., inconsistent equations).
- **Linear Algebra**:
  - **Matrix Rank**:
    * The number of non-zero rows in the RREF form gives the rank of the matrix, which indicates the number of linearly independent rows or columns.
  - **Basis Determination**:
    * RREF is used to determine a basis for the row space or column space of a matrix.
- **Data Science and Machine Learning**:
  - **Feature Selection**:
    * RREF can identify linearly dependent columns in a dataset, enabling dimensionality reduction.
  - **Regression Analysis**:

    * Used to simplify the design matrix in linear regression to determine whether predictors are linearly dependent.

- **Robotics and Control Systems**:
  - **State Equations**:
    * RREF is used to simplify state-space representations for linear systems.
  - **Inverse Kinematics**:
    * Helps in determining whether a system of equations has solutions for robotic arm positions and movements.
- **Cryptography**:
  - **Matrix Inversions**:
    * RREF simplifies the inversion of matrices used in cryptographic algorithms, such as Hill cipher encryption and decryption.
  - **Error Detection**:
    * Used in coding theory to analyze generator and parity-check matrices.

11.3 Formula

To transform a matrix A into its row echelon form U, follow these steps:

- **Choose a Pivot**: Locate the first non-zero entry in the first column as the pivot. If the entire column is zero, move to the next column.
- **Row Swapping (if needed)**: Swap rows to ensure the pivot is non-zero.
- **Scale the Pivot Row**: Scale the row so that the pivot becomes 1 (optional for REF but required for RREF).
- **Eliminate Below the Pivot**: Subtract multiples of the pivot row from the rows below to make all entries below the pivot zero.
- **Repeat**: Move to the next row and column and repeat the process until the entire matrix is in row echelon form.

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \dots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \dots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \dots & A_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & A_{m3} & \dots & A_{mn} \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & * & * & \dots & * \\ 0 & 1 & * & \dots & * \\ 0 & 0 & 1 & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

11.3.1 Example

$$A = \begin{bmatrix} 8 & 5 & -2 & 4 & 28 \\ 4 & 2.5 & 20 & 4 & -4 \\ 8 & 5 & 1 & 4 & 17 \end{bmatrix}$$

- **Pivot in the first Row**: Scale the first row by $\frac{1}{8}$

$$R_1 \to \frac{1}{8}R_1 \quad U = \begin{bmatrix} \frac{8}{8} & \frac{5}{8} & \frac{-2}{8} & \frac{4}{8} & \frac{28}{8} \\ 4 & 2.5 & 20 & 4 & -4 \\ 8 & 5 & 1 & 4 & 17 \end{bmatrix}$$

$$\Leftrightarrow U = \begin{bmatrix} 1 & \frac{5}{8} & -\frac{1}{4} & \frac{1}{2} & \frac{7}{2} \\ 4 & 2.5 & 20 & 4 & -4 \\ 8 & 5 & 1 & 4 & 17 \end{bmatrix}$$

- **Eliminate Below the Pivot**: Substract $4 \cdot R_1$ from R_2 and $8 \cdot R_1$ from R_3

$$R_2 \to R_2 - 4 \cdot R_1,\ R_3 \to R_3 - 8 \cdot R_1 \quad U = \begin{bmatrix} 1 & \frac{5}{8} & -\frac{1}{4} & \frac{1}{2} & \frac{7}{2} \\ 4 - 4 \cdot 1 & 2.5 - 4 \cdot \frac{5}{8} & 20 - 4 \cdot (-\frac{1}{4}) & 4 - 4 \cdot \frac{1}{2} & -4 - 4 \cdot \frac{7}{2} \\ 8 - 8 \cdot 1 & 5 - 8 \cdot \frac{5}{8} & 1 - 8 \cdot (-\frac{1}{4}) & 4 - 8 \cdot \frac{1}{2} & 17 - 8 \cdot \frac{7}{2} \end{bmatrix}$$

$$\Leftrightarrow U = \begin{bmatrix} 1 & \frac{5}{8} & -\frac{1}{4} & \frac{1}{2} & \frac{7}{2} \\ 0 & \frac{5}{2} - \frac{5}{2} & 20 + 1 & 4 - 2 & -4 - 14 \\ 0 & 5 - 5 & 1 + 2 & 4 - 4 & 17 - 28 \end{bmatrix}$$

$$\Leftrightarrow U = \begin{bmatrix} 1 & \frac{5}{8} & -\frac{1}{4} & \frac{1}{2} & \frac{7}{2} \\ 0 & 0 & 21 & 2 & -18 \\ 0 & 0 & 3 & 0 & -11 \end{bmatrix}$$

- **Make the second Pivot**: Divide R_2 by 21

$$R_2 \to \frac{1}{21} \cdot R_2 \quad U = \begin{bmatrix} 1 & \frac{5}{8} & -\frac{1}{4} & \frac{1}{2} & \frac{7}{2} \\ 0 & 0 & \frac{21}{21} & \frac{2}{21} & -\frac{18}{21} \\ 0 & 0 & 3 & 0 & -11 \end{bmatrix}$$

$$\Leftrightarrow U = \begin{bmatrix} 1 & \frac{5}{8} & -\frac{1}{4} & \frac{1}{2} & \frac{7}{2} \\ 0 & 0 & 1 & \frac{2}{21} & -\frac{6}{7} \\ 0 & 0 & 3 & 0 & -11 \end{bmatrix}$$

- **Eliminate Above the Pivot**: Substract $-\frac{1}{4} \cdot R_2$ from R_1

$$R_1 \to R_1 - (-\frac{1}{4}) \cdot R_2 \quad U = \begin{bmatrix} 1 & \frac{5}{8} & -\frac{1}{4} - (-\frac{1}{4} \cdot 1) & \frac{1}{2} - (-\frac{1}{4} \cdot \frac{2}{21}) & \frac{7}{2} - (-\frac{1}{4} \cdot (-\frac{6}{7})) \\ 0 & 0 & 1 & \frac{2}{21} & -\frac{6}{7} \\ 0 & 0 & 3 & 0 & -11 \end{bmatrix}$$

$$\Leftrightarrow U = \begin{bmatrix} 1 & \frac{5}{8} & 0 & \frac{1}{2} + \frac{2}{84} & \frac{7}{2} - \frac{6}{28} \\ 0 & 0 & 1 & \frac{2}{21} & -\frac{6}{7} \\ 0 & 0 & 3 & 0 & -11 \end{bmatrix}$$

$$\Leftrightarrow U = \begin{bmatrix} 1 & \frac{5}{8} & 0 & \frac{21}{42} + \frac{1}{42} & \frac{49}{14} - \frac{3}{14} \\ 0 & 0 & 1 & \frac{2}{21} & -\frac{6}{7} \\ 0 & 0 & 3 & 0 & -11 \end{bmatrix}$$

$$\Leftrightarrow U = \begin{bmatrix} 1 & \frac{5}{8} & 0 & \frac{22}{42} & \frac{46}{14} \\ 0 & 0 & 1 & \frac{2}{21} & -\frac{6}{7} \\ 0 & 0 & 3 & 0 & -11 \end{bmatrix}$$

$$\Leftrightarrow U = \begin{bmatrix} 1 & \frac{5}{8} & 0 & \frac{11}{21} & \frac{23}{7} \\ 0 & 0 & 1 & \frac{2}{21} & -\frac{6}{7} \\ 0 & 0 & 3 & 0 & -11 \end{bmatrix}$$

- **Eliminate Below the Pivot**: Substract $3 \cdot R_2$ from R_3

$$R_3 \to R_3 - 3 \cdot R_2 \quad U = \begin{bmatrix} 1 & \frac{5}{8} & 0 & \frac{11}{21} & \frac{23}{7} \\ 0 & 0 & 1 & \frac{2}{21} & -\frac{6}{7} \\ 0 & 0 & 3 - 3 \cdot 1 & 0 - 3 \cdot \frac{2}{21} & -11 - 3 \cdot (-\frac{6}{7}) \end{bmatrix}$$

$$\Leftrightarrow U = \begin{bmatrix} 1 & \frac{5}{8} & 0 & \frac{11}{21} & \frac{23}{7} \\ 0 & 0 & 1 & \frac{2}{21} & -\frac{6}{7} \\ 0 & 0 & 0 & -\frac{6}{21} & -\frac{77}{7} + \frac{18}{7} \end{bmatrix}$$

$$\Leftrightarrow U = \begin{bmatrix} 1 & \frac{5}{8} & 0 & \frac{11}{21} & \frac{23}{7} \\ 0 & 0 & 1 & \frac{2}{21} & -\frac{6}{7} \\ 0 & 0 & 0 & -\frac{2}{7} & -\frac{59}{7} \end{bmatrix}$$

- **Make the third Pivot**: Divide R_3 by $-\frac{2}{7}$

$$R_3 \to \frac{1}{-\frac{2}{7}} \cdot R_3 = -\frac{7}{2} \cdot R_3 \quad U = \begin{bmatrix} 1 & \frac{5}{8} & 0 & \frac{11}{21} & \frac{23}{7} \\ 0 & 0 & 1 & \frac{2}{21} & -\frac{6}{7} \\ 0 & 0 & 0 & -\frac{2}{7} \cdot (-\frac{7}{2}) & -\frac{59}{7} \cdot (-\frac{7}{2}) \end{bmatrix}$$

$$\Leftrightarrow U = \begin{bmatrix} 1 & \frac{5}{8} & 0 & \frac{11}{21} & \frac{23}{7} \\ 0 & 0 & 1 & \frac{2}{21} & -\frac{6}{7} \\ 0 & 0 & 0 & 1 & \frac{59}{2} \end{bmatrix}$$

- **Eliminate Above the Pivot**: Substract $\frac{11}{21} \cdot R_3$ from R_1:

$$R_1 \to R_1 - \frac{11}{21} \cdot R_3 \quad U = \begin{bmatrix} 1 & \frac{5}{8} & 0 & \frac{11}{21} - \frac{11}{21} \cdot 1 & \frac{23}{7} - \frac{11}{21} \cdot \frac{59}{2} \\ 0 & 0 & 1 & \frac{2}{21} & -\frac{6}{7} \\ 0 & 0 & 0 & 1 & \frac{59}{2} \end{bmatrix}$$

$$\Leftrightarrow U = \begin{bmatrix} 1 & \frac{5}{8} & 0 & 0 & \frac{23}{7} - \frac{649}{42} \\ 0 & 0 & 1 & \frac{2}{21} & -\frac{6}{7} \\ 0 & 0 & 0 & 1 & \frac{59}{2} \end{bmatrix}$$

$$\Leftrightarrow U = \begin{bmatrix} 1 & \frac{5}{8} & 0 & 0 & \frac{138}{42} - \frac{649}{42} \\ 0 & 0 & 1 & \frac{2}{21} & -\frac{6}{7} \\ 0 & 0 & 0 & 1 & \frac{59}{2} \end{bmatrix}$$

$$\Leftrightarrow U = \begin{bmatrix} 1 & \frac{5}{8} & 0 & 0 & -\frac{511}{42} \\ 0 & 0 & 1 & \frac{2}{21} & -\frac{6}{7} \\ 0 & 0 & 0 & 1 & \frac{59}{2} \end{bmatrix}$$

$$\Leftrightarrow U = \begin{bmatrix} 1 & \frac{5}{8} & 0 & 0 & -\frac{73}{6} \\ 0 & 0 & 1 & \frac{2}{21} & -\frac{6}{7} \\ 0 & 0 & 0 & 1 & \frac{59}{2} \end{bmatrix}$$

- **Eliminate Above the Pivot**: Substract $\frac{2}{21} \cdot R_3$ from R_2:

$$R_2 \to R_2 - \frac{2}{21} \cdot R_3 \quad U = \begin{bmatrix} 1 & \frac{5}{8} & 0 & 0 & -\frac{73}{6} \\ 0 & 0 & 1 & \frac{2}{21} - \frac{2}{21} \cdot 1 & -\frac{6}{7} - \frac{2}{21} \cdot \frac{59}{2} \\ 0 & 0 & 0 & 1 & \frac{59}{2} \end{bmatrix}$$

$$\Leftrightarrow U = \begin{bmatrix} 1 & \frac{5}{8} & 0 & 0 & -\frac{73}{6} \\ 0 & 0 & 1 & 0 & -\frac{18}{21} - \frac{59}{21} \\ 0 & 0 & 0 & 1 & \frac{59}{2} \end{bmatrix}$$

$$\Leftrightarrow U = \begin{bmatrix} 1 & \frac{5}{8} & 0 & 0 & -\frac{73}{6} \\ 0 & 0 & 1 & 0 & -\frac{77}{21} \\ 0 & 0 & 0 & 1 & \frac{59}{2} \end{bmatrix}$$

$$\Leftrightarrow U = \begin{bmatrix} 1 & \frac{5}{8} & 0 & 0 & -\frac{73}{6} \\ 0 & 0 & 1 & 0 & -\frac{11}{3} \\ 0 & 0 & 0 & 1 & \frac{59}{2} \end{bmatrix}$$

$$\Leftrightarrow U \approx \begin{bmatrix} 1 & 0.625 & 0 & 0 & -12.166666666666668 \\ 0 & 0 & 1 & 0 & -3.666666666666667 \\ 0 & 0 & 0 & 1 & 29.500000000000004 \end{bmatrix}$$

12 Exercise 11 - Determinant

12.1 Goal

The determinant of a square matrix A provides a scalar value that summarizes certain properties of the matrix, such as invertibility.

```
def determinant(u):
```

12.2 Potential Usage

- **Linear Algebra**:
  - **Invertibility**:
    - A matrix A is invertible if and only if $det(A) \neq 0$.
  - **Matrix Properties**:

    * The determinant provides insight into the rank of the matrix (a zero determinant indicates linear dependence among rows or columns).

- **Physics and Mechanics**:
  - **Change of Volume**:
    * The determinant of a transformation matrix represents the scaling factor for volume when the transformation is applied.
  - **Tensor Analysis**:
    * In mechanics, determinants are used to verify if a tensor transformation preserves or reverses orientation.
  - **Jacobian Determinant**:
    * The determinant of the Jacobian matrix in coordinate transformations quantifies how much a transformation stretches or shrinks space.
- **Computer Graphics**:
  - **Transformation**:
    * Determinants of transformation matrices determine whether the transformation preserves or flips the orientation of objects.
  - **3D Rendering**:
    * Used to compute whether a scaling or projection transformation affects the handedness of a 3D coordinate system.
- **Data Science and Machine Learning**:
  - **Covariance Matrices**:
    * The determinant of a covariance matrix measures data dispersion. A very small determinant indicates near-linear dependence among variables.
  - **Optimization**:

      * Determinants are used to compute volumes of feasible regions in optimization problems.

- **Robotics and Pathfinding**:
  - **Kinematics and Dynamics**:
    * The determinant is used to check the feasibility of robot arm movements or configurations by analyzing Jacobian matrices.
  - **Singularity Analysis**:
    * A zero determinant of the Jacobian indicates singularity, where the robot loses degrees of freedom.

12.3 Formula

To find the determinant of a matrix, you combine its elements using a specific pattern of multiplication and addition based on its size.

$$A = \begin{bmatrix} a \end{bmatrix}$$

Where: A is 1x1.
The resultant $R = det(A)$ is calculated as:

$$R = a$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Where: A is 2x2.
The resultant $R = det(A)$ is calculated as:

$$R = a \cdot d - b \cdot c$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Where: A is 3x3.

The resultant $R = det(A)$ is calculated as:

$$R = a \cdot (e \cdot i - f \cdot h) - b \cdot (d \cdot i - f \cdot g) + c \cdot (d \cdot h - e \cdot g)$$

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}$$

Where: A is nxn.

The resultant $R = det(A)$ is calculated as:

$$R = \sum_{j=1}^{n} (-1)^{1+j} \cdot A_{ij} \cdot det(M_{1j})$$

Where:

- A_{ij} is the element in the first row and $j-th$ column.
- M_{ij} is the minor matrix, obtained by removing the first row and $j-th$ column from A.

12.3.1 Example

To find the determinant of a matrix, you combine its elements using a specific pattern of multiplication and addition based on its size.

$$A = \begin{bmatrix} 8 & 5 & -2 & 4 \\ 4 & 2.5 & 20 & 4 \\ 8 & 5 & 1 & 4 \\ 28 & -4 & 17 & 1 \end{bmatrix}$$

$$R = A_{11} \cdot det(M_{11}) - A_{12} \cdot det(M_{12}) + A_{13} \cdot det(M_{13}) - A_{14} \cdot det(M_{14})$$

$$\Leftrightarrow R = 8 \cdot \begin{bmatrix} 2.5 & 20 & 4 \\ 5 & 1 & 4 \\ -4 & 17 & 1 \end{bmatrix} - 5 \cdot \begin{bmatrix} 4 & 20 & 4 \\ 8 & 1 & 4 \\ 28 & 17 & 1 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 4 & 2.5 & 4 \\ 8 & 5 & 4 \\ 28 & -4 & 1 \end{bmatrix} - 4 \cdot \begin{bmatrix} 4 & 2.5 & 20 \\ 8 & 5 & 1 \\ 28 & -4 & 17 \end{bmatrix}$$

- Compute $det(M_{11})$

$$M_{11} = \begin{bmatrix} 2.5 & 20 & 4 \\ 5 & 1 & 4 \\ -4 & 17 & 1 \end{bmatrix}$$

$$det(M_{11}) = M_{11_{11}} \cdot det(M_{11,11}) - M_{11_{12}} \cdot det(M_{11,12}) + M_{11_{13}} \cdot det(M_{11,13})$$

$$\Leftrightarrow det(M_{11}) = 2.5 \cdot \begin{bmatrix} 1 & 4 \\ 17 & 1 \end{bmatrix} - 20 \cdot \begin{bmatrix} 5 & 4 \\ -4 & 1 \end{bmatrix} + 4 \cdot \begin{bmatrix} 5 & 1 \\ -4 & 17 \end{bmatrix}$$

- Compute $det(M_{11,11})$, $det(M_{11,12})$ and $det(M_{11,13})$

$$
\begin{aligned}
& det(M_{11,11}) = M_{11,11_{11}} \cdot M_{11,11_{22}} - M_{11,11_{12}} \cdot M_{11,11_{21}} \\
\Leftrightarrow\ & det(M_{11,11}) = 1 \cdot 1 - 4 \cdot 17 \\
\Leftrightarrow\ & det(M_{11,11}) = 1 - 68 \\
\Leftrightarrow\ & det(M_{11,11}) = -67 \\
& det(M_{11,12}) = M_{11,12_{11}} \cdot M_{11,12_{22}} - M_{11,12_{12}} \cdot M_{11,12_{21}} \\
\Leftrightarrow\ & det(M_{11,12}) = 5 \cdot 1 - 4 \cdot (-4) \\
\Leftrightarrow\ & det(M_{11,12}) = 5 + 16 \\
\Leftrightarrow\ & det(M_{11,12}) = 21 \\
& det(M_{11,13}) = M_{11,13_{11}} \cdot M_{11,13_{22}} - M_{11,13_{12}} \cdot M_{11,13_{21}} \\
\Leftrightarrow\ & det(M_{11,12}) = 5 \cdot 17 - 1 \cdot (-4) \\
\Leftrightarrow\ & det(M_{11,12}) = 85 + 4 \\
\Leftrightarrow\ & det(M_{11,12}) = 89
\end{aligned}
$$

- Compute $det(M_{11})$

$$
\begin{aligned}
& det(M_{11}) = 2.5 \cdot (-67) - 20 \cdot 21 + 4 \cdot 89 \\
\Leftrightarrow\ & det(M_{11}) = -167.5 - 420 + 356 \\
\Leftrightarrow\ & det(M_{11}) = -231.5
\end{aligned}
$$

- Compute $det(M_{12})$

$$M_{12} = \begin{bmatrix} 4 & 20 & 4 \\ 8 & 1 & 4 \\ 28 & 17 & 1 \end{bmatrix}$$

$$det(M_{12}) = 4 \cdot \begin{bmatrix} 1 & 4 \\ 17 & 1 \end{bmatrix} - 20 \cdot \begin{bmatrix} 8 & 4 \\ 28 & 1 \end{bmatrix} + 4 \cdot \begin{bmatrix} 8 & 1 \\ 28 & 17 \end{bmatrix}$$

$$\Leftrightarrow det(M_{12}) = 4 \cdot (1 \cdot 1 - 4 \cdot 17) - 20 \cdot (8 \cdot 1 - 4 \cdot 28) + 4 \cdot (8 \cdot 17 - 1 \cdot 28)$$

$$\Leftrightarrow det(M_{12}) = 4 \cdot (1 - 68) - 20 \cdot (8 - 112) + 4 \cdot (136 - 28)$$

$$\Leftrightarrow det(M_{12}) = 4 \cdot (-67) - 20 \cdot (-104) + 4 \cdot 108$$

$$\Leftrightarrow det(M_{12}) = -268 + 2080 + 432$$

$$\Leftrightarrow det(M_{12}) = 2244$$

- Compute $det(M_{13})$

$$M_{13} = \begin{bmatrix} 4 & 2.5 & 4 \\ 8 & 5 & 4 \\ 28 & -4 & 1 \end{bmatrix}$$

$$det(M_{13}) = 4 \cdot \begin{bmatrix} 5 & 4 \\ -4 & 1 \end{bmatrix} - 2.5 \cdot \begin{bmatrix} 8 & 4 \\ 28 & 1 \end{bmatrix} + 4 \cdot \begin{bmatrix} 8 & 5 \\ 28 & -4 \end{bmatrix}$$

$$\Leftrightarrow det(M_{13}) = 4 \cdot (5 \cdot 1 - 4 \cdot (-4)) - 2.5 \cdot (8 \cdot 1 - 4 \cdot 28) + 4 \cdot (8 \cdot (-4) - 5 \cdot 28)$$

$$\Leftrightarrow det(M_{13}) = 4 \cdot (5 + 16) - 2.5 \cdot (8 - 112) + 4 \cdot (-32 - 140)$$

$$\Leftrightarrow det(M_{13}) = 4 \cdot 21 - 2.5 \cdot (-104) + 4 \cdot (-172)$$

$$\Leftrightarrow det(M_{13}) = 84 + 260 - 688$$

$$\Leftrightarrow det(M_{13}) = -344$$

- Compute $det(M_{14})$

$$M_{14} = \begin{bmatrix} 4 & 2.5 & 20 \\ 8 & 5 & 1 \\ 28 & -4 & 17 \end{bmatrix}$$

$$det(M_{14}) = 4 \cdot \begin{bmatrix} 5 & 1 \\ -4 & 17 \end{bmatrix} - 2.5 \cdot \begin{bmatrix} 8 & 1 \\ 28 & 17 \end{bmatrix} + 20 \cdot \begin{bmatrix} 8 & 5 \\ 28 & -4 \end{bmatrix}$$

$$\Leftrightarrow det(M_{14}) = 4 \cdot (5 \cdot 17 - 1 \cdot (-4)) - 2.5 \cdot (8 \cdot 17 - 1 \cdot 28) + 20 \cdot (8 \cdot (-4) - 5 \cdot 28)$$

$$\Leftrightarrow det(M_{14}) = 4 \cdot (85 + 4) - 2.5 \cdot (136 - 28) + 20 \cdot (-32 - 140)$$

$$\Leftrightarrow det(M_{14}) = 4 \cdot 89 - 2.5 \cdot 108 + 20 \cdot (-172)$$

$$\Leftrightarrow det(M_{14}) = 356 - 270 - 3440$$

$$\Leftrightarrow det(M_{14}) = -3354$$

- Now you can compute $R = det(A) = A_{11} \cdot det(M_{11}) - A_{12} \cdot det(M_{12}) + A_{13} \cdot det(M_{13}) - A_{14} \cdot det(M_{14})$

$$A = \begin{bmatrix} 8 & 5 & -2 & 4 \\ 4 & 2.5 & 20 & 4 \\ 8 & 5 & 1 & 4 \\ 28 & -4 & 17 & 1 \end{bmatrix}$$

$$R = 8 \cdot (-231.5) - 5 \cdot 2244 + (-2) \cdot (-344) - 4 \cdot (-3354)$$

$$\Leftrightarrow R = -1852 - 11220 + 688 + 13416$$

$$\Leftrightarrow R = 1032$$

13 Exercise 12 - Inverse

13.1 Goal

The inverse of a square matrix A (denoted as A^{-1} exists if and only if A is invertible, which happens when the determinant of A is non-zero ($det(A) \neq 0$)

```
def inverse(u):
```

13.2 Potential Usage

- **Linear Algebra**:
  - **Solving Linear Systems**:
    - The inverse is used to solve systems of equations:

$$x = A^{-1} \cdot b$$

  - **Matrix Equations**:
    - Inverse matrices simplify solving equations involving matrix variables, such as:

$$AX = B => X = A^{-1}B$$

- **Physics and Mechanics**:
  - **Coordinate Transformations**:
    - The inverse is used to reverse transformations, such as converting coordinates back to the original space.
  - **Tensor Analysis**:
    - The inverse of a tensor transformation matrix reverses its effects in mechanics and material science.
  - **Dynamics**:
    - In control systems, inverse matrices are used to compute the inverse dynamics of a mechanical system.
- **Computer Graphics**:
  - **Transformations**:
    - The inverse matrix is used to reverse 3D transformations (e.g., rotating back, scaling down, or undoing a perspective projection).
  - **Lighting Calculations**:

* To transform surface normals, the inverse transpose of the transformation matrix is often required:

$$N_{transformed} = (A^{-1})^\top N$$

- **Data Science and Machine Learning**:
  - **Regression Analysis**:
    * In linear regression, the coefficients are computed using the inverse of the design matrix:

$$\hat{\beta} = (X^\top X)^{-1} X^\top y$$

  - **Covariance Matrix**:
    * The inverse of a covariance matrix is used in multivariate analysis to compute the precision matrix, representing conditional dependencies.
  - **Optimization**:
    * Inverse matrices are used in solving optimization problems involving quadratic forms.
- **Robotics and Pathfinding**:
  - **Inverse Kinematics**:
    * The inverse Jacobian matrix is used to compute joint angles for a robotic arm to reach a desired position.
  - **Path Planning**:
    * Inverse transformations are used to calculate the required movements for robots to return to their initial states.
- **Cryptography**:
  - **Hill Cipher**:
    * The inverse of the key matrix is used for decrypting messages in the Hill cipher.

13.3 Formula

To find the inverse of a matrix, you divide the adjugate of the matrix by its determinant.

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix}$$

The cofactor $C_{ij} = (-1)^{i+j} \cdot det(M_{ij})$ is calculated as:

$$A^{-1} = \frac{1}{det(A)} \cdot adj(A)$$

Where:

- $det(A)$: Determinant of the matrix A
- $adj(A)$: Adjugate (or adjoint) of A, which is the transpose of the cofactor matrix.

$$Cofactor\ Matrix = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{mn} \end{bmatrix}$$

$$Cofactor\ Matrix = \begin{bmatrix} det(M_{11}) & -det(M_{12}) & \cdots & (+/-)det(M_{1n}) \\ -det(M_{21}) & det(M_{22}) & \cdots & (+/-)det(M_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (+/-)det(M_{m1}) & (+/-)det(M_{m2}) & \cdots & (+/-)det(M_{mn}) \end{bmatrix}$$

$$adj(A) = Cofactor\ Matrix^{\top}$$

13.3.1 Example

$$A = \begin{bmatrix} 8 & 5 & -2 \\ 4 & 7 & 20 \\ 7 & 6 & 1 \end{bmatrix}$$

$$det(A) = 8 \cdot \begin{bmatrix} 7 & 20 \\ 6 & 1 \end{bmatrix} - 5 \cdot \begin{bmatrix} 4 & 20 \\ 7 & 1 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 4 & 7 \\ 7 & 6 \end{bmatrix}$$

$$\Leftrightarrow det(A) = 8 \cdot (7 \cdot 1 - 20 \cdot 6) - 5 \cdot (4 \cdot 1 - 20 \cdot 7) + (-2) \cdot (4 \cdot 6 - 7 \cdot 7)$$

$$\Leftrightarrow det(A) = 8 \cdot (7 - 120) - 5 \cdot (4 - 140) + (-2) \cdot (24 - 49)$$

$$\Leftrightarrow det(A) = 8 \cdot (-113) - 5 \cdot (-136) + (-2) \cdot (-25)$$

$$\Leftrightarrow det(A) = -904 + 680 + 50$$

$$\Leftrightarrow det(A) = -174$$

$$\Leftrightarrow det(A) \neq 0 \rightarrow \text{inversible.}$$

$$Cofactor\ Matrix = \begin{bmatrix} det(M_{11}) & -det(M_{12}) & det(M_{13}) \\ -det(M_{21}) & det(M_{22}) & -det(M_{23}) \\ det(M_{31}) & -det(M_{32}) & det(M_{33}) \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} 7 & 20 \\ 6 & 1 \end{bmatrix} \quad M_{12} = \begin{bmatrix} 4 & 20 \\ 7 & 1 \end{bmatrix} \quad M_{13} = \begin{bmatrix} 4 & 7 \\ 7 & 6 \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} 5 & -2 \\ 6 & 1 \end{bmatrix} \quad M_{22} = \begin{bmatrix} 8 & -2 \\ 7 & 1 \end{bmatrix} \quad M_{23} = \begin{bmatrix} 8 & 5 \\ 7 & 6 \end{bmatrix}$$

$$M_{31} = \begin{bmatrix} 5 & -2 \\ 7 & 20 \end{bmatrix} \quad M_{32} = \begin{bmatrix} 8 & -2 \\ 4 & 20 \end{bmatrix} \quad M_{33} = \begin{bmatrix} 8 & 5 \\ 4 & 7 \end{bmatrix}$$

$$Cofactor\ Matrix = \begin{bmatrix} 7 \cdot 1 - 20 \cdot 6 & -(4 \cdot 1 - 20 \cdot 7) & 4 \cdot 6 - 7 \cdot 7 \\ -(5 \cdot 1 - (-2) \cdot 6) & 8 \cdot 1 - (-2) \cdot 7 & -(8 \cdot 6 - 5 \cdot 7) \\ 5 \cdot 20 - (-2) \cdot 7 & -(8 \cdot 20 - (-2) \cdot 4) & 8 \cdot 7 - 5 \cdot 4 \end{bmatrix}$$

$$\Leftrightarrow Cofactor\ Matrix = \begin{bmatrix} 7 - 120 & -(4 - 140) & 24 - 49 \\ -(5 + 12) & 8 + 14 & -(48 - 35) \\ 100 + 14 & -(160 + 8) & 56 - 20 \end{bmatrix}$$

$$\Leftrightarrow Cofactor\ Matrix = \begin{bmatrix} -113 & 136 & -25 \\ -17 & 22 & -13 \\ 114 & -168 & 36 \end{bmatrix}$$

$$adj(A) = Cofactor\ Matrix^{\top}$$

$$\Leftrightarrow adj(A) = \begin{bmatrix} -113 & 136 & -25 \\ -17 & 22 & -13 \\ 114 & -168 & 36 \end{bmatrix}^{\top}$$

$$\Leftrightarrow adj(A) = \begin{bmatrix} -113 & -17 & 114 \\ 136 & 22 & -168 \\ -25 & -13 & 36 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-174} \cdot \begin{bmatrix} -113 & -17 & 114 \\ 136 & 22 & -168 \\ -25 & -13 & 36 \end{bmatrix}$$

$$\Leftrightarrow A^{-1} = \begin{bmatrix} \frac{1}{-174} \cdot (-113) & \frac{1}{-174} \cdot (-17) & \frac{1}{-174} \cdot 114 \\ \frac{1}{-174} \cdot 136 & \frac{1}{-174} \cdot 22 & \frac{1}{-174} \cdot (-168) \\ \frac{1}{-174} \cdot (-25) & \frac{1}{-174} \cdot (-13) & \frac{1}{-174} \cdot 36 \end{bmatrix}$$

$$\Leftrightarrow A^{-1} = \begin{bmatrix} \frac{-113}{-174} & \frac{-17}{-174} & \frac{114}{-174} \\ \frac{136}{-174} & \frac{22}{-174} & \frac{-168}{-174} \\ \frac{-25}{-174} & \frac{-13}{-174} & \frac{36}{-174} \end{bmatrix}$$

$$\Leftrightarrow A^{-1} = \begin{bmatrix} \frac{113}{174} & \frac{17}{174} & \frac{19}{29} \\ -\frac{68}{87} & -\frac{11}{87} & \frac{28}{29} \\ \frac{25}{174} & \frac{13}{174} & -\frac{6}{29} \end{bmatrix}$$

$$\Leftrightarrow A^{-1} \approx \begin{bmatrix} 0.6494252873563219 & 0.09770114942528736 & -0.6551724137931034 \\ -0.7816091954022989 & -0.12643678160919541 & 0.9655172413793104 \\ 0.14367816091954022 & 0.07471264367816093 & -0.20689655172413793 \end{bmatrix}$$

14 Exercise 13 - Rank

14.1 Goal

The rank of a matrix A is the maximum number of linearly independent rows or columns in A. It provides a measure of the matrix's dimensionality and is often denoted as $rank(A)$.

```
def rank(u):
```

14.2 Potential Usage

- **Linear Algebra**:
  - **Solving Systems of Linear Equations**:
    - The rank helps determine the number of solutions:

        - if $rank(A) = rank([A|b])$, the system is consistent.
        - if $rank(A) < rank([A|b])$, the system is inconsistent.
  - **Column and Row Spaces**:
    * The rank indicates the dimension of the row space and column space of the matrix.
  - **Invertibility**:
    * A square matrix is invertible if and only if $rank(A) = n$ (full rank).

- **Physics and Mechanics**:
  - **Degrees of Freedom**:
    * The rank of a Jacobian matrix in mechanics determines the degrees of freedom of a system.
  - **Constraints in Systems**:
    * The rank reveals whether a system of forces or equations has redundant constraints or dependencies.

- **Computer Graphics**:
  - **Transformation Matrices**:
    * The rank of transformation matrices determines if they can fully describe rotations, scaling, and translations in 3D space.
  - **Projection Matrices**:
    * The rank determines the dimensionality of the projection, such as reducing 3D to 2D.

- **Data Science and Machine Learning**:
  - **Dimensionality Reduction**:
    * The rank is used to identify linearly dependent features, enabling dimensionality reduction (e.g., in PCA).
  - **Covariance Matrix Analysis**:

    * The rank of a covariance matrix indicates the number of principal components with non-zero variance.
  - **Matrix Factorization**:
    * In techniques like Singular Value Decomposition (SVD), the rank determines the number of significant components.

- **Robotics and Control Systems**:
  - **Feasibility of Motion**:
    * The rank of the Jacobian matrix determines whether a robot arm can move in the desired direction or reach a target position.
  - **Singularity Analysis**:
    * A Jacobian with reduced rank indicates singular configurations where the robot loses degrees of freedom.

- **Cryptography**:
  - **Linear Dependence**:
    * The rank is used to analyze the properties of key matrices in encryption and decryption algorithms.

14.3 Formula

To find the rank of a matrix, you count the maximum number of linearly independent rows or columns.

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix}$$

The resultant $R = rank(A)$ is calculated as:
$R =$ Number of pivots in the Reduced Row Echelon Form of A

14.3.1 Example

$$
\begin{aligned}
A &= \begin{bmatrix} 8 & 5 & -2 & 4 & 28 \\ 4 & 2.5 & 20 & 4 & -4 \\ 8 & 5 & 1 & 4 & 17 \end{bmatrix} \\
\Leftrightarrow row\_echelon(A) &= \begin{bmatrix} 1 & \frac{5}{8} & 0 & 0 & -\frac{73}{6} \\ 0 & 0 & 1 & 0 & -\frac{11}{3} \\ 0 & 0 & 0 & 1 & \frac{59}{2} \end{bmatrix} \\
row\_echelon(A) &\rightarrow 3pivots \\
rank(A) &= 3
\end{aligned}
$$

15 Exercise 14 - Bonus: Projection Matrix

15.1 Goal

A projection matrix is used to project vectors onto a subspace, such as a line or a plane. The formula depends on whether the projection is onto a line or a subspace defined by a set of vectors.

```python
def projection(fov, ratio, near, far):
```

15.2 Potential Usage

- **Linear Algebra**:
  - **Vector Projections**:
    - Projects a vector onto a subspace to simplify linear system solutions or find the closest approximation in least squares problems.
  - **Orthogonality**:
    - Projection matrices preserve orthogonality properties, making them useful in theoretical analyses.
- **Physics and Mechanics**:

- **Force Decomposition**:
  - Projects force vectors onto specific directions, helping to analyze components of forces in different axes.
- **Stress and Strain Analysis**:
  - Projection matrices are used to isolate components of stress or strain tensors along particular directions.

- **Computer Graphics**:
  - **2D and 3D Projections**:
    - Maps 3D points to 2D space for rendering and visualization, such as in perspective or orthographic projection.
  - **Lighting Calculations**:
    - Projects vectors onto a plane or surface to calculate shadows or reflections.

- **Data Science and Machine Learning**:
  - **Dimensionality Reduction**:
    - Projects high-dimensional data onto a lower-dimensional subspace, such as in Principal Component Analysis (PCA).
  - **Feature Selection**:
    - Projection matrices are used to identify or create new features in transformed spaces.

- **Robotics and Control Systems**:
  - **Path Planning**:
    - Projects robot configurations into feasible subspaces to simplify motion planning.
  - **Kinematics**:
    - Used in control systems to isolate movement along desired directions or planes.

- **Economics and Finance**:
  - **Portfolio Optimization**:
    - Projects asset returns onto a space of feasible portfolios based on constraints like risk or diversification.
  - **Input-Output Analysis**:
    - Projects economic activity onto subspaces for modeling inter-sector relationships.

15.3 Formula

The perspective projection matrix is calculated as:

$$P = \begin{bmatrix} \frac{f}{ratio} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & -\frac{far+near}{far-near} & -\frac{2 \cdot far \cdot near}{far-near} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Where:

- $f = \frac{1}{tan fov/2}$: The cotangent of half the field of view (FOV).
- $ratio$: The aspect ratio of the viewport (width/height).
- $near$: The distance to the near clipping plane.
- far: The distance to the far clipping plane.

Step-by-Step Derivation of the Matrix:

- **Field of View (FOV)**: The FOV determines the vertical viewing angle, and its cotangent defines how "narrow" or "wide" the frustum is:

$$f = \frac{1}{tan(fov/2)}$$

- **Aspect Ratio (Ratio)**: To account for the aspect ratio *ratio*, the horizontal scaling factor is divided by *ratio*:

$$Horizontal\ Scale = \frac{f}{ratio}$$

- **Depth Range**: The projection maps the near and far planes into normalized device coordinates (NDC), which range from $[-1, 1]$. The depth components are:

$$P_{z,z} = \frac{far + near}{far - near},\ P_{z,w} = \frac{2 \cdot far \cdot near}{far - near}$$

- **Perspective Divide**: The matrix includes a -1 in the third row to handle the perspective division in homogeneous coordinates, ensuring proper depth scaling.

15.3.1 Subject Parenthesis

In the subject, you will have to download the 'display_linux.tar.gz' folder. Unzip it and you will have to save the projection matrix into a file named 'proj'. 'proj' need to be in the same folder as 'display' (binary file gave in the 'display_linux.tar.gz' folder). Then you will just need to run 'display' to make it works.

15.4 Example

- FOV: 60 degrees radians
- ratio: 1.0
- near: 1.0
- far: 10.0

$$P = \begin{bmatrix} \frac{f}{ratio} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & -\frac{far+near}{far-near} & -\frac{2 \cdot far \cdot near}{far-near} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\Leftrightarrow P = \begin{bmatrix} \frac{\frac{1}{tan(rad(60)/2)}}{1.0} & 0 & 0 & 0 \\ 0 & \frac{1}{tan(rad(60)/2)} & 0 & 0 \\ 0 & 0 & -\frac{10.0+1.0}{10.0-1.0} & -\frac{2 \cdot 10.0 \cdot 1.0}{10.0-1.0} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\Leftrightarrow P = \begin{bmatrix} \frac{1}{tan(rad(60)/2)} & 0 & 0 & 0 \\ 0 & \frac{1}{tan(rad(60)/2)} & 0 & 0 \\ 0 & 0 & -\frac{10.0+1.0}{10.0-1.0} & -\frac{2 \cdot 10.0 \cdot 1.0}{10.0-1.0} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\Leftrightarrow P = \begin{bmatrix} \frac{1}{tan(\pi/6)} & 0 & 0 & 0 \\ 0 & \frac{1}{tan(\pi/6)} & 0 & 0 \\ 0 & 0 & -\frac{11.0}{9.0} & -\frac{20.0}{9.0} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\Leftrightarrow P \approx \begin{bmatrix} 1.73205 & 0 & 0 & 0 \\ 0 & 1.73205 & 0 & 0 \\ 0 & 0 & -1.22222 & -2.22222 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

16 Exercise 15 - Bonus: Complex Vector Spaces

16.1 Goal

Do all the previous exercises again, this time interpreting $\mathbb{K}$ in the function signatures, not as the field $\mathbb{R}$ of real numbers, but as the field $\mathbb{C}$ of complex numbers.

16.2 What are Complex Numbers?

Complex numbers extend the idea of real numbers by introducing a new dimension. They are numbers of the form:

$$z = a + bi$$

Where:

- a is the **real part** (a real number)
- b is the **imaginary part** (also a real number)
- i is the **imaginary unit**, defined by $i^2 = -1$

16.2.1 Why Complex Numbers?

Complex numbers were introduced to solve equations that do not have solutions in the set of real numbers. For example, the equation $x^2 + 1 = 0$ has no real solution because the square of a real number cannot be negative. By defining $i = \sqrt{-1}$, we can write the solutions as $x = \pm i$.

16.2.2 Polar Form of Complex Numbers

A complex number can also be represented in **polar form**:

$$z = r(\cos\theta + i\sin\theta)$$

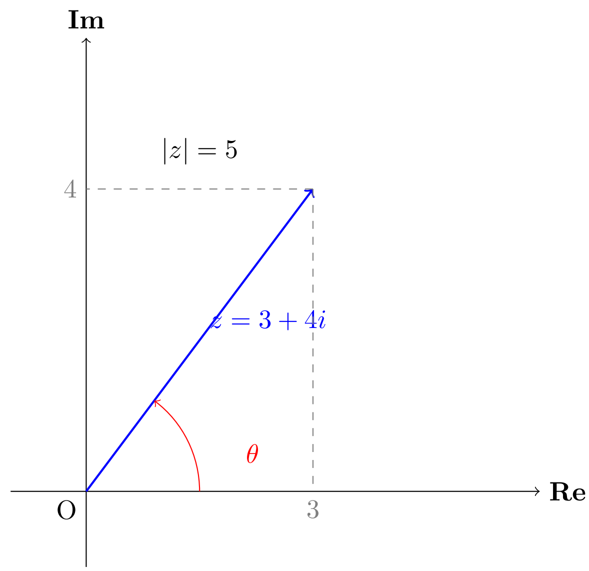
Where:

- $r = |z| = \sqrt{a^2 + b^2}$ is the magnitude or modulus of z.
- $\theta = tan^{-1}(\frac{b}{a})$ is the argument (angle).

In this form, multiplication and division become simpler:

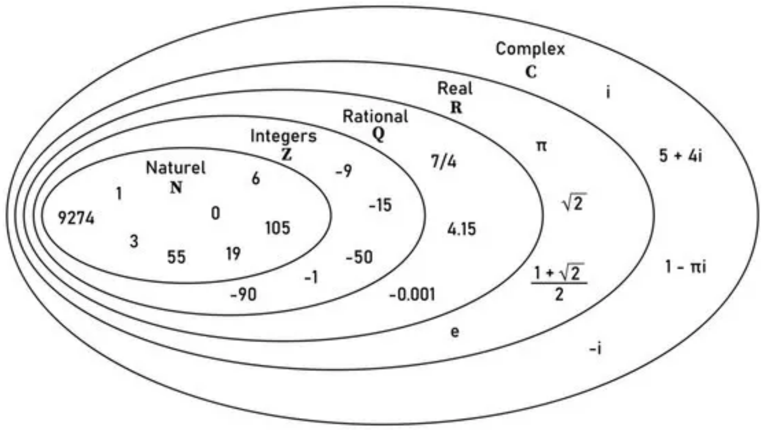
- $z_1 z_2 = r_1 r_2[cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2)]$
- $\frac{z_1}{z_2} = \frac{r_1}{r_2}[cos(\theta_1 - \theta_2) + isin(\theta_1 - \theta_2)]$

16.2.3 Visualizing Complex Numbers



16.2.4 Applications of Complex Numbers

- **Engineering and Physics**:
  - Used in signal processing, electrical engineering (e.g., AC circuit analysis), and quantum mechanics.
- **Mathematics**:
  - Fundamental in solving polynomials and analyzing functions.
- **Computer Graphics**:
  - Complex numbers simplify transformations and rotations in 2D.



16.3 Again and again

In this subsection, we will do all the previous exercises again.

16.4 Exercise 0

16.4.1 Addition Vectors Complex Numbers

To add complex numbers, the process is the same as with real numbers: simply add the real parts together and the imaginary parts together.

$$v_1 = \left\langle 1+2i \quad 3+4i \quad 5+6i \right\rangle \textbf{ and } v_2 = \left\langle 6-7i \quad 8-9i \quad 10-11i \right\rangle$$
$$R = \left\langle (1+2i)+(6-7i) \quad (3+4i)+(8-9i) \quad (5+6i)+(10-11i) \right\rangle$$
$$\Leftrightarrow R = \left\langle (1+6)+(2i-7i) \quad (3+8)+(4i-9i) \quad (5+10)+(6i-11i) \right\rangle$$
$$\Leftrightarrow R = \left\langle 7-5i \quad 11-5i \quad 15-5i \right\rangle$$

16.4.2 Substraction Vectors Complex Numbers

To subtract complex numbers, the process is similar to real numbers: subtract the real parts and subtract the imaginary parts.

$$
\begin{aligned}
v_1 = \left\langle 1+2i \quad 3+4i \quad 5+6i \right\rangle \textbf{ and } v_2 &= \left\langle 6-7i \quad 8-9i \quad 10-11i \right\rangle \\
R &= \left\langle (1+2i)-(6-7i) \quad (3+4i)-(8-9i) \quad (5+6i)-(10-11i) \right\rangle \\
\Leftrightarrow R &= \left\langle (1-6)+(2i+7i) \quad (3-8)+(4i+9i) \quad (5-10)+(6i+11i) \right\rangle \\
\Leftrightarrow R &= \left\langle -5+9i \quad -5+13i \quad -5+17i \right\rangle
\end{aligned}
$$

16.4.3 Scalar Vectors Complex Numbers

To multiply a complex number by a scalar, the process is straightforward: multiply the scalar with both the real part and the imaginary part of the complex number.

$$
\begin{aligned}
v_1 &= \left\langle 1+2i \quad 3+4i \quad 5+6i \right\rangle \textbf{ and } scalar = 2 \\
R &= \left\langle 2(1+2i) \quad 2(3+4i) \quad 2(5+6i) \right\rangle \\
\Leftrightarrow R &= \left\langle 2\cdot 1+2\cdot 2i \quad 2\cdot 3+2\cdot 4i \quad 2\cdot 5+2\cdot 6i \right\rangle \\
\Leftrightarrow R &= \left\langle 2+4i \quad 6+8i \quad 10+12i \right\rangle
\end{aligned}
$$

16.4.4 Addition Matrices Complex Numbers

To add matrices of complex numbers, the process is similar to adding regular matrices: add the corresponding real parts together and the corresponding imaginary parts together element by element.

$$A = \begin{bmatrix} 1+2i & 2+3i & 3+4i \\ 4+5i & 5+6i & 6+7i \end{bmatrix} \textbf{ and } B = \begin{bmatrix} 7-8i & 8-9i & 9-10i \\ 10-11i & 11-12i & 12-13i \end{bmatrix}$$

$$R = \begin{bmatrix} (1+2i)+(7-8i) & (2+3i)+(8-9i) & (3+4i)+(9-10i) \\ (4+5i)+(10-11i) & (5+6i)+(11-12i) & (6+7i)+(12-13i) \end{bmatrix}$$

$$\Leftrightarrow R = \begin{bmatrix} (1+7)+(2i-8i) & (2+8)+(3i-9i) & (3+9)+(4i-10i) \\ (4+10)+(5i-11i) & (5+11)+(6i-12i) & (6+12)+(7i-13i) \end{bmatrix}$$

$$\Leftrightarrow R = \begin{bmatrix} 8-6i & 10-6i & 12-6i \\ 14-6i & 16-6i & 18-6i \end{bmatrix}$$

16.4.5 Substraction Matrices Complex Numbers

To subtract matrices of complex numbers, the process is similar to subtracting regular matrices: subtract the corresponding real parts and the corresponding imaginary parts element by element.

$$A = \begin{bmatrix} 1+2i & 2+3i & 3+4i \\ 4+5i & 5+6i & 6+7i \end{bmatrix} \textbf{ and } B = \begin{bmatrix} 7-8i & 8-9i & 9-10i \\ 10-11i & 11-12i & 12-13i \end{bmatrix}$$

$$R = \begin{bmatrix} (1+2i)-(7-8i) & (2+3i)-(8-9i) & (3+4i)-(9-10i) \\ (4+5i)-(10-11i) & (5+6i)-(11-12i) & (6+7i)-(12-13i) \end{bmatrix}$$

$$\Leftrightarrow R = \begin{bmatrix} (1-7)+(2i+8i) & (2-8)+(3i+9i) & (3-9)+(4i+10i) \\ (4-10)+(5i+11i) & (5-11)+(6i+12i) & (6-12)+(7i+13i) \end{bmatrix}$$

$$\Leftrightarrow R = \begin{bmatrix} -6+10i & -6+12i & -6+14i \\ -6+16i & -6+18i & -6+20i \end{bmatrix}$$

16.4.6 Scalar Matrices Complex Numbers

To multiply a matrix of complex numbers by a scalar, the process is straightforward: multiply the scalar with both the real and imaginary parts of each element in the matrix.

$$
\begin{aligned}
A &= \begin{bmatrix} 1+2i & 2+3i & 3+4i \\ 4+5i & 5+6i & 6+7i \end{bmatrix} \textbf{ and } scalar = 3 \\
R &= \begin{bmatrix} 3(1+2i) & 3(2+3i) & 3(3+4i) \\ 3(4+5i) & 3(5+6i) & 3(6+7i) \end{bmatrix} \\
\Leftrightarrow R &= \begin{bmatrix} 3\cdot 1+3\cdot 2i & 3\cdot 2+3\cdot 3i & 3\cdot 3+3\cdot 4i \\ 3\cdot 4+3\cdot 5i & 3\cdot 5+3\cdot 6i & 3\cdot 6+3\cdot 7i \end{bmatrix} \\
\Leftrightarrow R &= \begin{bmatrix} 3+6i & 6+9i & 9+12i \\ 12+15i & 15+18i & 18+21i \end{bmatrix}
\end{aligned}
$$

16.5 Exercise 1

To perform a linear combination of complex numbers, the process is simple: multiply each complex number by its respective scalar, then add the resulting complex numbers by combining their real and imaginary parts separately.

$$
\begin{aligned}
v_1 &= \left\langle 1+2i \quad 3+4i \quad 5+6i \right\rangle \textbf{ and } v_2 = \left\langle 6-7i \quad 8-9i \quad 10-11i \right\rangle \textbf{ and } c_1 = 3 \textbf{ and } c_2 = 2 \\
R &= \left\langle 3(1+2i)+2(6-7i) \quad 3(3+4i)+2(8-9i) \quad 3(5+6i)+2(10-11i) \right\rangle \\
\Leftrightarrow R &= \left\langle (3\cdot 1+2\cdot 6)+(3\cdot 2i-2\cdot 7i) \quad (3\cdot 3+2\cdot 8)+(3\cdot 4i-2\cdot 9i) \quad (3\cdot 5+2\cdot 10)+(3\cdot 6i-2\cdot 11i) \right\rangle \\
\Leftrightarrow R &= \left\langle (3+12)+(6i-14i) \quad (9+16)+(12i-18i) \quad (15+20)+(18i-22i) \right\rangle \\
\Leftrightarrow R &= \left\langle 15-8i \quad 25-6i \quad 35-4i \right\rangle
\end{aligned}
$$

16.6 Exercise 2

To perform a linear interpolation of complex numbers, the process is straightforward: interpolate the real parts separately and the imaginary parts separately using the same interpolation factor.

Type: **Scalar**

$$
\begin{aligned}
& A = 0 + 1i \textbf{ and } B = 1 + 2i \textbf{ and } t = 1 \\
& R = (0 + 1i) + 1((1 + 2i) - (0 + 1i)) \\
\Leftrightarrow\ & R = 1i + (1 + 2i) - 1i \\
\Leftrightarrow\ & R = 1 + 2i
\end{aligned}
$$

Type: **Vector**

$$
\begin{aligned}
& A = \left\langle 1 + 1i \quad 2 + 2i \right\rangle \textbf{ and } B = \left\langle 3 + 3i \quad 4 + 4i \right\rangle \textbf{ and } t = 0.5 \\
& R = \left\langle (1 + 1i) + 0.5((3 + 3i) - (1 + 1i)) \quad (2 + 2i) + 0.5((4 + 4i) - (2 + 2i)) \right\rangle \\
\Leftrightarrow\ & R = \left\langle (1 + 1i) + (0.5((3 - 1) + (3i - 1i))) \quad (2 + 2i) + (0.5((4 - 2) + (4i - 2i))) \right\rangle \\
\Leftrightarrow\ & R = \left\langle (1 + 1i) + (0.5 \cdot (2 + 2i)) \quad (2 + 2i) + (0.5(2 + 2i)) \right\rangle \\
\Leftrightarrow\ & R = \left\langle (1 + 1i) + (1 + 1i) \quad (2 + 2i) + (1 + 1i) \right\rangle \\
\Leftrightarrow\ & R = \left\langle (1 + 1) + (1i + 1i) \quad (2 + 1) + (2i + 1i) \right\rangle \\
\Leftrightarrow\ & R = \left\langle 2 + 2i \quad 3 + 3i \right\rangle
\end{aligned}
$$

Type: **Matrix**:

$$A = \begin{bmatrix} 1+1i & 2+2i \\ 3+3i & 4+4i \end{bmatrix} \textbf{ and } B = \begin{bmatrix} 5+5i & 6+6i \\ 7+7i & 8+8i \end{bmatrix} \textbf{ and } t = 0.5$$

$$R = \begin{bmatrix} (1+1i)+0.5((5+5i)-(1+1i)) & (2+2i)+0.5((6+6i)-(2+2i)) \\ (3+3i)+0.5((7+7i)-(3+3i)) & (4+4i)+0.5((8+8i)-(4+4i)) \end{bmatrix}$$

$$\Leftrightarrow R = \begin{bmatrix} (1+1i)+0.5((5-1)+(5i-1i)) & (2+2i)+0.5((6-2)+(6i-2i)) \\ (3+3i)+0.5((7-3)+(7i-3i)) & (4+4i)+0.5((8-4)+(8i-4i)) \end{bmatrix}$$

$$\Leftrightarrow R = \begin{bmatrix} (1+1i)+0.5(4+4i) & (2+2i)+0.5(4+4i) \\ (3+3i)+0.5(4+4i) & (4+4i)+0.5(4+4i) \end{bmatrix}$$

$$\Leftrightarrow R = \begin{bmatrix} (1+1i)+(2+2i) & (2+2i)+(2+2i) \\ (3+3i)+(2+2i) & (4+4i)+(2+2i) \end{bmatrix}$$

$$\Leftrightarrow R = \begin{bmatrix} (1+2)+(1i+2i) & (2+2)+(2i+2i) \\ (3+2)+(3i+2i) & (4+2)+(4i+2i) \end{bmatrix}$$

$$\Leftrightarrow R = \begin{bmatrix} 3+3i & 4+4i \\ 5+5i & 6+6i \end{bmatrix}$$

16.7 Exercise 3

To compute the dot product of two vectors of complex numbers, multiply each pair of corresponding elements, taking the complex conjugate of the elements in one of the vectors, and then sum the results.
The conjugate of a complex number $z = a + bi$, where a is the real part and b is the imaginary part, is given by:

$$\overline{z} = a - bi$$

In the conjugate, the sign of the imaginary part is reversed, while the real part remains unchanged. For example:

- The conjugate of $3 + 4i$ is $3 - 4i$
- The conjugate of $-2 - 5i$ is $-2 + 5i$

Conjugates are useful in various operations with complex numbers, such as division and finding the magnitude.

$$A = \left\langle A_1 \quad A_2 \right\rangle \textbf{ and } B = \left\langle B_1 \quad B_2 \right\rangle$$

The resultant $R = A \cdot B$ is calculated as:

$$R = A_1 \cdot \overline{B_1} + A_2 \cdot \overline{B_2}$$

$$\begin{aligned}
& A = \left\langle -1 - 2i \quad 6 + 1i \right\rangle \textbf{ and } B = \left\langle 3 + 3i \quad 2 - 2i \right\rangle \\
& R = (-1 - 2i) \cdot (3 - 3i) + (6 + 1i) \cdot (2 + 2i) \\
\Leftrightarrow\ & R = ((-1)(3 - 3i) + (-2i)(3 - 3i)) + (6(2 - 2i) + 1i(2 + 2i)) \\
\Leftrightarrow\ & R = ((-3 + 3i) + (-6i + 6i^2)) + ((12 - 12i) + (2i + 2i^2)) \\
\Leftrightarrow\ & R = ((-3 + 3i) + (-6i - 6)) + ((12 - 12i) + (2i - 2)) \\
\Leftrightarrow\ & R = ((-3 - 6) + (-6i + 3i)) + ((12 - 2) + (2i + 12i)) \\
\Leftrightarrow\ & R = (-9 - 3i) + (10 + 14i) \\
\Leftrightarrow\ & R = (-9 + 10) + (14i - 3i) \\
\Leftrightarrow\ & R = 1 + 11i
\end{aligned}$$

16.8 Exercise 4

16.8.1 Norm-1 Complex Numbers

To compute the norm-1 of a vector of complex numbers, sum the absolute values of each complex number in the vector. The absolute value of a complex number $z = a + bi$ is given by:

$$|z| = \sqrt{a^2 + b^2}$$

So, for a vector of complex numbers $v = [z_1, z_2, \ldots, z_n]$, the norm-1 is:

$$\|v\|_1 = |z_1| + |z_2| + \cdots + |z_n|$$

$$\begin{aligned}
A &= \left\langle -1 - 1i \quad -2 - 2i \right\rangle \\
R &= \sqrt{(-1)^2 + (-1)^2} + \sqrt{(-2)^2 + (-2)^2} \\
\Leftrightarrow R &= \sqrt{1+1} + \sqrt{4+4} \\
\Leftrightarrow R &= \sqrt{2} + \sqrt{8} \\
\Leftrightarrow R &= \sqrt{2} + 2\sqrt{2} \\
\Leftrightarrow R &= 3\sqrt{2} \\
\Leftrightarrow R &\approx 4.242640687119286
\end{aligned}$$

16.8.2 Norm-2 Complex Numbers

To compute the norm-2 (also called the Euclidean norm) of a vector of complex numbers, take the square root of the sum of the squared absolute values of the complex numbers in the vector. The absolute value of a complex number $z = a + bi$ is given by:

$$|z| = \sqrt{a^2 + b^2}$$

For a vector of complex numbers $v = [z_1, z_2, \ldots, z_n]$, the norm-2 is:

$$\|v\|_2 = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}$$

$$
\begin{aligned}
& A = \left\langle -1-1i \quad -2-2i \right\rangle \\
& R = \sqrt{((-1)^2 + (-1)^2) + ((-2)^2 + (-2)^2)} \\
\Leftrightarrow\ & R = \sqrt{(1+1) + (4+4)} \\
\Leftrightarrow\ & R = \sqrt{2+8} \\
\Leftrightarrow\ & R = \sqrt{10} \\
\Leftrightarrow\ & R \approx 3.1622776601683795
\end{aligned}
$$

16.8.3 Norm-∞ Complex Numbers

To compute the norm-infinity (also called the max norm) of a vector of complex numbers, find the largest absolute value among all the complex numbers in the vector. The absolute value of a complex number $z = a + bi$ is given by:

$$|z| = \sqrt{a^2 + b^2}$$

For a vector of complex numbers $v = [z_1, z_2, \ldots, z_n]$, the norm-$\infty$ is:

$$\|v\|_\infty = max(|z_1|, |z_2|, \ldots, |z_n|)$$

$$
\begin{aligned}
& A = \left\langle -1-1i \quad -2-2i \right\rangle \\
& R = max(\sqrt{((-1)^2 + (-1)^2)}, \sqrt{(-2)^2 + (-2)^2}) \\
\Leftrightarrow\ & R = max(\sqrt{(1+1)}, \sqrt{4+4}) \\
\Leftrightarrow\ & R = max(\sqrt{2}, \sqrt{8}) \\
\Leftrightarrow\ & R = max(\sqrt{2}, 2\sqrt{2}) \\
\Leftrightarrow\ & R = 2\sqrt{2} \\
\Leftrightarrow\ & R \approx 2.8284271247461903
\end{aligned}
$$

16.9 Exercise 5

To compute the cosine of the angle between two vectors of complex numbers, use the dot product and the norm-2 (Euclidean norm) of the vectors. The formula is:

$$cos(\theta) = \frac{Re(v \cdot w^*)}{\|v\|_2 \cdot \|w\|_2}$$

Steps:

- **Dot Product with Conjugate**: Compute the dot product of the vector v with the conjugate of w:

  $$v \cdot w^* = \sum_{i=1}^{n} v_i \cdot \overline{w_i}$$

  where $\overline{w_i}$ is the conjugate of the $i-th$ element of w.

- **Real Part**: Take the real part of the dot product: $Re(v \cdot w^*)$.

- **Norm-2**: Compute the Euclidean norms of v and w:

  $$\|v\|_2 = \sqrt{\sum_{i=1}^{n} |v_i|^2},\ \|w\|_2 = \sqrt{\sum_{i=1}^{n} |w_i|^2}$$

- **Combine**: Substitute into the formula:

  $$cos(\theta) = \frac{Re(v \cdot w^*)}{\|v\|_2 \cdot \|w\|_2}$$

$$
\begin{aligned}
A &= \left\langle 1+1i \quad 2+2i \quad 3+3i \right\rangle \textbf{ and } B = \left\langle 4+4i \quad 5+5i \quad 6+6i \right\rangle \\
cos(\theta) &= \frac{Re(A \cdot \overline{B})}{\|A\|_2 \cdot \|B\|_2} \\
Re(A \cdot \overline{B}) &= (1+1i)(4-4i) + (2+2i)(5-5i) + (3+3i)(6-6i) \\
(1+1i)(4-4i) &= (1 \cdot 4) + (1 \cdot (-4i)) + (1i \cdot 4) + (1i \cdot (-4i)) \\
\Leftrightarrow (1+1i)(4-4i) &= 4 - 4i + 4i - 4(-1) \\
\Leftrightarrow (1+1i)(4-4i) &= 8 \\
(2+2i)(5-5i) &= (2 \cdot 5) + (2 \cdot (-5i)) + (2i \cdot 5) + (2i \cdot (-5i)) \\
\Leftrightarrow (2+2i)(5-5i) &= 10 - 10i + 10i + 10 \\
\Leftrightarrow (2+2i)(5-5i) &= 20 \\
(3+3i)(6-6i) &= (3 \cdot 6) + (3 \cdot (-6i)) + (3i \cdot 6) + (3i \cdot (-6i)) \\
\Leftrightarrow (3+3i)(6-6i) &= 18 - 18i + 18i + 18 \\
\Leftrightarrow (3+3i)(6-6i) &= 36 \\
Re(A \cdot \overline{B}) &= 8 + 20 + 36 \\
\Leftrightarrow Re(A \cdot \overline{B}) &= 64 \\
\|A\|_2 &= \sqrt{(1^2+1^2) + (2^2+2^2) + (3^2+3^2)} \\
\Leftrightarrow \|A\|_2 &= \sqrt{(1+1) + (4+4) + (9+9)} \\
\Leftrightarrow \|A\|_2 &= \sqrt{2+8+18} \\
\Leftrightarrow \|A\|_2 &= \sqrt{28} \\
\|B\|_2 &= \sqrt{(4^2+4^2) + (5^2+5^2) + (6^2+6^2)} \\
\Leftrightarrow \|B\|_2 &= \sqrt{(16+16) + (25+25) + (36+36)} \\
\Leftrightarrow \|B\|_2 &= \sqrt{32+50+72} \\
\Leftrightarrow \|B\|_2 &= \sqrt{154} \\
cos(\theta) &= \frac{64}{\sqrt{28} \cdot \sqrt{154}} \\
\Leftrightarrow cos(\theta) &= \frac{64}{\sqrt{4312}} \\
\Leftrightarrow cos(\theta) &\approx 0.9746318461970762
\end{aligned}
$$

16.10 Exercise 6

To compute the cross product of vectors of complex numbers, the process is similar to real vectors: compute the determinant for each component, treating the real and imaginary parts of the complex numbers in the same way.

$$\begin{aligned}
A &= \left\langle 1+2i \quad -3+4i \quad 5-6i \right\rangle \textbf{ and } B = \left\langle -7+8i \quad 9-10i \quad -11+12i \right\rangle \\
R &= \left\langle A_1 \cdot B_2 - A_2 \cdot B_1 \quad A_2 \cdot B_0 - A_0 \cdot B_2 \quad B_0 \cdot B_1 - A_1 \cdot B_0 \right\rangle \\
\Leftrightarrow R &= \left\langle \begin{matrix} (-3+4i) \cdot (-11+12i) - (5-6i) \cdot (9-10i) \\ (5-6i) \cdot (-7+8i) - (1+2i) \cdot (-11+12i) \\ (1+2i) \cdot (9-10i) - (-3+4i) \cdot (-7+8i) \end{matrix} \right\rangle
\end{aligned}$$

$$\begin{aligned}
(-3+4i) \cdot (-11+12i) &= (-3 \cdot (-11)) + (-3 \cdot 12i) + (4i \cdot (-11)) + (4i \cdot 12i) \\
\Leftrightarrow (-3+4i) \cdot (-11+12i) &= 33 - 36i - 44i - 48 \\
\Leftrightarrow (-3+4i) \cdot (-11+12i) &= -15 - 80i
\end{aligned}$$

$$\begin{aligned}
(5-6i) \cdot (9-10i) &= (5 \cdot 9) + (5 \cdot (-10i)) + ((-6i) \cdot 9) + ((-6i) \cdot (-10i)) \\
\Leftrightarrow (5-6i) \cdot (9-10i) &= 45 - 50i - 54i - 60 \\
\Leftrightarrow (5-6i) \cdot (9-10i) &= -15 - 104i
\end{aligned}$$

$$\begin{aligned}
(-3+4i) \cdot (-11+12i) - (5-6i) \cdot (9-10i) &= (-15-80i) - (-15-104i) \\
\Leftrightarrow (-3+4i) \cdot (-11+12i) - (5-6i) \cdot (9-10i) &= (-15-80i) + (15+104i) \\
\Leftrightarrow (-3+4i) \cdot (-11+12i) - (5-6i) \cdot (9-10i) &= (-15+15) + (-80i+104i) \\
\Leftrightarrow (-3+4i) \cdot (-11+12i) - (5-6i) \cdot (9-10i) &= 24i
\end{aligned}$$

$$
\begin{aligned}
&(5-6i)\cdot(-7+8i) = (5\cdot(-7)) + (5\cdot 8i) + ((-6i)\cdot(-7)) + ((-6i)\cdot 8i)\\
\Leftrightarrow\ &(5-6i)\cdot(-7+8i) = -35 + 40i + 42i + 48\\
\Leftrightarrow\ &(5-6i)\cdot(-7+8i) = 13 + 82i
\end{aligned}
$$

$$
\begin{aligned}
&(1+2i)\cdot(-11+12i) = (1\cdot(-11)) + (1\cdot 12i) + (2i\cdot(-11)) + (2i\cdot 12i)\\
\Leftrightarrow\ &(1+2i)\cdot(-11+12i) = -11 + 12i - 22i - 24\\
\Leftrightarrow\ &(1+2i)\cdot(-11+12i) = -35 - 10i
\end{aligned}
$$

$$
\begin{aligned}
&(5-6i)\cdot(-7+8i) - (1+2i)\cdot(-11+12i) = (13+82i) - (-35-10i)\\
\Leftrightarrow\ &(5-6i)\cdot(-7+8i) - (1+2i)\cdot(-11+12i) = (13+82i) + (35+10i)\\
\Leftrightarrow\ &(5-6i)\cdot(-7+8i) - (1+2i)\cdot(-11+12i) = (13+35) + (82i+10i)\\
\Leftrightarrow\ &(5-6i)\cdot(-7+8i) - (1+2i)\cdot(-11+12i) = 48 + 92i
\end{aligned}
$$

$$
\begin{aligned}
&(1+2i)\cdot(9-10i) = (1\cdot 9) + (1\cdot(-10i)) + (2i\cdot 9) + (2i\cdot(-10i))\\
\Leftrightarrow\ &(1+2i)\cdot(9-10i) = 9 - 10i + 18i + 20\\
\Leftrightarrow\ &(1+2i)\cdot(9-10i) = 29 + 8i
\end{aligned}
$$

$$
\begin{aligned}
&(-3+4i)\cdot(-7+8i) = ((-3)\cdot(-7)) + ((-3)\cdot 8i) + (4i\cdot(-7)) + (4i\cdot 8i)\\
\Leftrightarrow\ &(-3+4i)\cdot(-7+8i) = 21 - 24i - 28i - 32\\
\Leftrightarrow\ &(-3+4i)\cdot(-7+8i) = -11 - 52i
\end{aligned}
$$

$$
\begin{aligned}
&(1+2i)\cdot(9-10i) - (-3+4i)\cdot(-7+8i) = (29+8i) - (-11-52i)\\
\Leftrightarrow\ &(1+2i)\cdot(9-10i) - (-3+4i)\cdot(-7+8i) = (29+8i) + (11+52i)\\
\Leftrightarrow\ &(1+2i)\cdot(9-10i) - (-3+4i)\cdot(-7+8i) = (29+11) + (8i+52i)\\
\Leftrightarrow\ &(1+2i)\cdot(9-10i) - (-3+4i)\cdot(-7+8i) = 40 + 60i
\end{aligned}
$$

$$R = \left\langle \begin{matrix} (-3+4i)\cdot(-11+12i)-(5-6i)\cdot(9-10i) \\ (5-6i)\cdot(-7+8i)-(1+2i)\cdot(-11+12i) \\ (1+2i)\cdot(9-10i)-(-3+4i)\cdot(-7+8i) \end{matrix} \right\rangle$$

$$\Leftrightarrow R = \left\langle 24i \quad 48+92i \quad 40+60i \right\rangle$$

16.11 Exercise 7

16.11.1 Multiplication Matrices by Vectors Complex Numbers

To multiply a matrix by a vector of complex numbers, the process is the same as with real numbers: multiply each row of the matrix by the corresponding elements of the vector, summing the products, while treating the real and imaginary parts as usual.

$$A = \begin{bmatrix} 2 & -0-2i \\ -0-2i & 2 \end{bmatrix} \textbf{ and } B = \begin{bmatrix} 4+1i \\ 2-3i \end{bmatrix}$$

$$R = \begin{bmatrix} 2\cdot(4+1i)+(-2i)\cdot(2-3i) \\ (-2i)\cdot(4+1i)+2\cdot(2-3i) \end{bmatrix}$$

$$\Leftrightarrow R = \begin{bmatrix} (8+2i)+(-4i-6) \\ (-8i+2)+(4-6i) \end{bmatrix}$$

$$\Leftrightarrow R = \begin{bmatrix} (8-6)+(2i-4i) \\ (4+2)+(-8i-6i) \end{bmatrix}$$

$$\Leftrightarrow R = \begin{bmatrix} 2-2i \\ 6-14i \end{bmatrix}$$

16.11.2 Multiplication Matrices Complex Numbers

To multiply a matrix by another matrix of complex numbers, the process is the same as with real numbers: multiply the rows of the first matrix by the columns of the second matrix, summing the products, while treating the real and imaginary parts as usual.

$$
\begin{aligned}
A &= \begin{bmatrix} 3 & -0-5i \\ 6+2i & 8 \end{bmatrix} \textbf{ and } B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \\
R &= \begin{bmatrix} 3\cdot 2 + (-5i)\cdot 4 & 3\cdot 1 + (-5i)\cdot 2 \\ (6+2i)\cdot 2 + 8\cdot 4 & (6+2i)\cdot 1 + 8\cdot 2 \end{bmatrix} \\
\Leftrightarrow R &= \begin{bmatrix} 6-20i & 3-10i \\ 12+4i+32 & 6+2i+16 \end{bmatrix} \\
\Leftrightarrow R &= \begin{bmatrix} 6-20i & 3-10i \\ 44+4i & 22+2i \end{bmatrix}
\end{aligned}
$$

16.12 Exercise 8

To compute the trace of a matrix of complex numbers, the process is the same as with real numbers: sum the elements on the main diagonal, treating the real and imaginary parts as usual.

$$
\begin{aligned}
A &= \begin{bmatrix} -0-2i & -8+1i & 4-4i \\ 1+2i & -0-23i & 4 \\ 0 & 6-1i & 4+3i \end{bmatrix} \\
R &= (-2i) + (-23i) + (4+3i) \\
\Leftrightarrow R &= 4-22i
\end{aligned}
$$

16.13 Exercise 9

To compute the transpose of a matrix of complex numbers, the process is the same as with real numbers: interchange the rows and columns of the matrix, keeping the real and imaginary parts unchanged.

$$A = \begin{bmatrix} -0-2i & -8+1i & 4-4i \\ 1+2i & -0-23i & 4 \\ 0 & 6-1i & 4+3i \end{bmatrix}$$

$$R = \begin{bmatrix} -0-2i & 1+2i & 0 \\ -8+1i & -0-23i & 6-1i \\ 4-4i & 4 & 4+3i \end{bmatrix}$$

16.14 Exercise 10

To compute the reduced row echelon form of a matrix of complex numbers, the process is the same as with real numbers: perform row operations (swapping, scaling, and adding multiples of rows) to transform the matrix into its canonical form, treating the real and imaginary parts as usual.

$$A = \begin{bmatrix} 1+1i & 2 \\ 2+2i & 4+4i \end{bmatrix}$$

$$\textbf{First Pivot}: R_1 = \frac{1}{1+1i} \cdot R_1$$

$$\Leftrightarrow A = \begin{bmatrix} \frac{1+1i}{1+1i} & \frac{2}{1+1i} \\ 2+2i & 4+4i \end{bmatrix}$$

Here we have to divide 2 complex numbers.

$$\frac{z_1}{z_2} = \frac{z_1 \cdot \overline{z_2}}{z_2 \cdot \overline{z_2}}$$

$$z_2 = a + bi \textbf{ and } \overline{z_2} = a - bi$$

So

$$z_2 \cdot \overline{z_2} = (a+bi) \cdot (a-bi) = a^2 - bi^2 \text{ (}\textit{Take a look at remarkable identity})$$

$$\Leftrightarrow A = \begin{bmatrix} 1 & \frac{2\cdot(1-1i)}{(1+1i)\cdot(1-1i)} \\ 2+2i & 4+4i \end{bmatrix}$$

$$\Leftrightarrow A = \begin{bmatrix} 1 & \frac{2-2i}{1^2-(1i)^2} \\ 2+2i & 4+4i \end{bmatrix}$$

$$\Leftrightarrow A = \begin{bmatrix} 1 & \frac{2-2i}{1+1} \\ 2+2i & 4+4i \end{bmatrix}$$

$$\Leftrightarrow A = \begin{bmatrix} 1 & \frac{2-2i}{2} \\ 2+2i & 4+4i \end{bmatrix}$$

$$\Leftrightarrow A = \begin{bmatrix} 1 & 1-1i \\ 2+2i & 4+4i \end{bmatrix}$$

$$\textbf{Eliminate Under Pivot} : R_2 = R_2 - (2+2i)\cdot R_1$$

$$\Leftrightarrow A = \begin{bmatrix} 1 & 1-1i \\ (2+2i) - 1\cdot(2+2i) & (4+4i)-(1-1i)(4+4i) \end{bmatrix}$$

$$\Leftrightarrow A = \begin{bmatrix} 1 & 1-1i \\ 0 & (4+4i)-(4+4i-4i+4) \end{bmatrix}$$

$$\Leftrightarrow A = \begin{bmatrix} 1 & 1-1i \\ 0 & 4+4i-8 \end{bmatrix}$$

$$\Leftrightarrow A = \begin{bmatrix} 1 & 1-1i \\ 0 & -4+4i \end{bmatrix}$$

$$\textbf{Second Pivot} : R_2 = \frac{1}{-4+4i}\cdot R_2$$

$$\Leftrightarrow A = \begin{bmatrix} 1 & 1-1i \\ 0 & \frac{-4+4i}{-4+4i} \end{bmatrix}$$

$$\Leftrightarrow A = \begin{bmatrix} 1 & 1-1i \\ 0 & 1 \end{bmatrix}$$

$$\textbf{Eliminate Above the Pivot} : R_1 = R_1 - (1-1i)\cdot R_2$$

$$\Leftrightarrow A = \begin{bmatrix} 1 & (1-1i)-(1-1i) \\ 0 & 1 \end{bmatrix}$$

$$\Leftrightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

16.15 Exercise 11

To compute the determinant of a matrix of complex numbers, the process is the same as with real numbers: expand along a row or column using minors and cofactors, treating the real and imaginary parts as usual.

$$A = \begin{bmatrix} 8+1i & 5-2i & -2+3i \\ 4-1i & 7+4i & 20-5i \\ 7+2i & 6-3i & 1+4i \end{bmatrix}$$

$$R = A_{11} \cdot det(M_{11}) - A_{12} \cdot det(M_{12}) + A_{13} \cdot det(M_{13})$$

- Compute $det(M_{11})$

$$M_{11} = \begin{bmatrix} 7+4i & 20-5i \\ 6-3i & 1+4i \end{bmatrix}$$

$$det(M_{11}) = (7+4i)(1+4i) - (20-5i)(6-3i)$$

$$\Leftrightarrow det(M_{11}) = (7+28i+4i-16) - (120-60i-30i-15)$$

$$\Leftrightarrow det(M_{11}) = (-9+32i) - (105-90i)$$

$$\Leftrightarrow det(M_{11}) = -9+32i-105+90i$$

$$\Leftrightarrow det(M_{11}) = -114+122i$$

- Compute $det(M_{12})$

$$M_{12} = \begin{bmatrix} 4-1i & 20-5i \\ 7+2i & 1+4i \end{bmatrix}$$

$$det(M_{12}) = (4-1i)(1+4i) - (20-5i)(7+2i)$$

$$\Leftrightarrow det(M_{12}) = (4+16i-1i+4) - (140+40i-35i+10)$$

$$\Leftrightarrow det(M_{12}) = (8+15i) - (150+5i)$$

$$\Leftrightarrow det(M_{12}) = 8+15i-150-5i$$

$$\Leftrightarrow det(M_{12}) = -142+10i$$

- Compute $det(M_{13})$

$$M_{13} = \begin{bmatrix} 4-1i & 7+4i \\ 7+2i & 6-3i \end{bmatrix}$$

$$det(M_{13}) = (4-1i)(6-3i) - (7+4i)(7+2i)$$

$$\Leftrightarrow det(M_{13}) = (24-12i-6i-3) - (49+14i+28i-8)$$

$$\Leftrightarrow det(M_{13}) = (21-18i) - (41+42i)$$

$$\Leftrightarrow det(M_{13}) = 21-18i-41-42i$$

$$\Leftrightarrow det(M_{13}) = -20-60i$$

$$\Leftrightarrow R = (8+1i)(-114+122i) - (5-2i)(-142+10i) + (-2+3i)(-20-60i)$$

$$\Leftrightarrow R = (-912+976i-114i-122) - (-710+50i+284i+20) + (40+120i-60i+180)$$

$$\Leftrightarrow R = (-1034+862i) - (-690+334i) + (220+60i)$$

$$\Leftrightarrow R = -1034+862i+690-334i+220+60i$$

$$\Leftrightarrow R = -124+558i$$

16.16 Exercise 12

To compute the inverse of a matrix of complex numbers, the process is the same as with real numbers: calculate the determinant and the adjugate matrix, then divide each element of the adjugate by the determinant, treating the real and imaginary parts as usual.

$$A = \begin{bmatrix} 8+1i & 5-2i & -2+3i \\ 4-1i & 7+4i & 20-5i \\ 7+2i & 6-3i & 1+4i \end{bmatrix}$$

$$det(A) = -124 + 558i$$

$$\Leftrightarrow det(A) \neq 0 \rightarrow \text{inversible.}$$

$$Cofactor\ Matrix = \begin{bmatrix} det(M_{11}) & -det(M_{12}) & det(M_{13}) \\ -det(M_{21}) & det(M_{22}) & -det(M_{23}) \\ det(M_{31}) & -det(M_{32}) & det(M_{33}) \end{bmatrix}$$

$$det(M_{11}) = -114 + 122i$$

$$det(M_{12}) = -142 + 10i$$

$$det(M_{13}) = -20 - 60i$$

$$det(M_{21}) = (5-2i)(1+4i) - (-2+3i)(6-3i)$$

$$\Leftrightarrow det(M_{21}) = (5+20i-2i+8) - (-12+6i+18i+9)$$

$$\Leftrightarrow det(M_{21}) = (13+18i) - (-3+24i)$$

$$\Leftrightarrow det(M_{21}) = 13+18i+3-24i$$

$$\Leftrightarrow det(M_{21}) = 16 - 6i$$

$$
\begin{aligned}
&det(M_{22}) = (8+1i)(1+4i) - (-2+3i)(7+2i) \\
\Leftrightarrow\ &det(M_{22}) = (8+32i+1i-4) - (-14-4i+21i-6) \\
\Leftrightarrow\ &det(M_{22}) = (4+33i) - (-20+17i) \\
\Leftrightarrow\ &det(M_{22}) = 4+33i+20-17i \\
\Leftrightarrow\ &det(M_{22}) = 24+16i
\end{aligned}
$$

$$
\begin{aligned}
&det(M_{23}) = (8+1i)(6-3i) - (5-2i)(7+2i) \\
\Leftrightarrow\ &det(M_{23}) = (48-24i+6i+3) - (35+10i-14i+4) \\
\Leftrightarrow\ &det(M_{23}) = (51-18i) - (39-4i) \\
\Leftrightarrow\ &det(M_{23}) = 51-18i-39+4i \\
\Leftrightarrow\ &det(M_{23}) = 12-14i
\end{aligned}
$$

$$
\begin{aligned}
&det(M_{31}) = (5-2i)(20-5i) - (-2+3i)(7+4i) \\
\Leftrightarrow\ &det(M_{31}) = (100-25i-40i-10) - (-14-8i+21i-12) \\
\Leftrightarrow\ &det(M_{31}) = (90-65i) - (-26+13i) \\
\Leftrightarrow\ &det(M_{31}) = 90-65i+26-13i \\
\Leftrightarrow\ &det(M_{31}) = 116-78i
\end{aligned}
$$

$$
\begin{aligned}
&det(M_{32}) = (8+1i)(20-5i) - (-2+3i)(4-1i) \\
\Leftrightarrow\ &det(M_{32}) = (160-40i+20i+5) - (-8+2i+12i+3) \\
\Leftrightarrow\ &det(M_{32}) = (165-20i) - (-5+14i) \\
\Leftrightarrow\ &det(M_{32}) = 165-20i+5-14i \\
\Leftrightarrow\ &det(M_{32}) = 170-34i
\end{aligned}
$$

$$det(M_{33}) = (8+1i)(7+4i) - (5-2i)(4-1i)$$
$$\Leftrightarrow det(M_{33}) = (56+32i+7i-4) - (20-5i-8i-2)$$
$$\Leftrightarrow det(M_{33}) = (52+39i) - (18-13i)$$
$$\Leftrightarrow det(M_{33}) = 52+39i-18+13i$$
$$\Leftrightarrow det(M_{33}) = 34+52i$$

$$\Leftrightarrow Cofactor\ Matrix = \begin{bmatrix} -114+122i & 142-10i & -20-60i \\ -16+6i & 24+16i & -12+14i \\ 116-78i & -170+34i & 34+52i \end{bmatrix}$$
$$adj(A) = Cofactor\ Matrix^{\top}$$
$$\Leftrightarrow adj(a) = \begin{bmatrix} -114+122i & -16+6i & 116-78i \\ 142-10i & 24+16i & -170+34i \\ -20-60i & -12+14i & 34+52i \end{bmatrix}$$

$$A^{-1} = \frac{1}{det(A)} \cdot adj(A)$$
$$\Leftrightarrow A^{-1} = \begin{bmatrix} \frac{-114+122i}{-124+558i} & \frac{-16+6i}{-124+558i} & \frac{116-78i}{-124+558i} \\ \frac{142-10i}{-124+558i} & \frac{24+16i}{-124+558i} & \frac{-170+34i}{-124+558i} \\ \frac{-20-60i}{-124+558i} & \frac{-12+14i}{-124+558i} & \frac{34+52i}{-124+558i} \end{bmatrix}$$
$$\Leftrightarrow A^{-1} = \begin{bmatrix} \frac{(-114+122i)\cdot(-124-558i)}{(-124+558i)\cdot(-124-558i)} & \frac{(-16+6i)\cdot(-124-558i)}{(-124+558i)\cdot(-124-558i)} & \frac{(116-78i)\cdot(-124-558i)}{(-124+558i)\cdot(-124-558i)} \\ \frac{(142-10i)\cdot(-124-558i)}{(-124+558i)\cdot(-124-558i)} & \frac{(24+16i)\cdot(-124-558i)}{(-124+558i)\cdot(-124-558i)} & \frac{(-170+34i)\cdot(-124-558i)}{(-124+558i)\cdot(-124-558i)} \\ \frac{(-20-60i)\cdot(-124-558i)}{(-124+558i)\cdot(-124-558i)} & \frac{(-12+14i)\cdot(-124-558i)}{(-124+558i)\cdot(-124-558i)} & \frac{(34+52i)\cdot(-124-558i)}{(-124+558i)\cdot(-124-558i)} \end{bmatrix}$$
$$\Leftrightarrow A^{-1} = \begin{bmatrix} \frac{12768+63612i-15128i+68076}{(-124)^2-(558i)^2} & \frac{1984+8928i-744i+3348}{(-124)^2-(558i)^2} & \frac{-14384-64728i+9672i-43524}{(-124)^2-(558i)^2} \\ \frac{-17608-79236i+1240i-5580}{(-124)^2-(558i)^2} & \frac{-2976-13392i-1984i+8928}{(-124)^2-(558i)^2} & \frac{21080+94860i-4216i+18972}{(-124)^2-(558i)^2} \\ \frac{2480+11160i+7440i-33480}{(-124)^2-(558i)^2} & \frac{1488+6696i-1984i+7812}{(-124)^2-(558i)^2} & \frac{-4216-18972i-6448i+29016}{(-124)^2-(558i)^2} \end{bmatrix}$$

$$\Leftrightarrow A^{-1} = \begin{bmatrix} \frac{80844+48484i}{15376+311364} & \frac{5332+8184i}{15376+311364} & \frac{-57908-55056i}{15376+311364} \\ \frac{-23188-77996i}{15376+311364} & \frac{5952-15376i}{15376+311364} & \frac{40052+90644i}{15376+311364} \\ \frac{-31000+18600i}{15376+311364} & \frac{9300+4712i}{15376+311364} & \frac{24800-25420i}{15376+311364} \end{bmatrix}$$

$$\Leftrightarrow A^{-1} = \begin{bmatrix} \frac{80844+48484i}{326740} & \frac{5332+8184i}{326740} & \frac{-57908-55056i}{326740} \\ \frac{-23188-77996i}{326740} & \frac{5952-15376i}{326740} & \frac{40052+90644i}{326740} \\ \frac{-31000+18600i}{326740} & \frac{9300+4712i}{326740} & \frac{24800-25420i}{326740} \end{bmatrix}$$

$$\Leftrightarrow A^{-1} = \begin{bmatrix} \frac{80844}{326740} \cdot \frac{48484i}{326740} & \frac{5332+8184i}{326740} \cdot \frac{8184i}{326740} & \frac{-57908-55056i}{326740} \cdot \frac{-55056i}{326740} \\ \frac{-23188-77996i}{326740} \cdot \frac{-77996i}{326740} & \frac{5952-15376i}{326740} \cdot \frac{-15376i}{326740} & \frac{40052+90644i}{326740} \cdot \frac{90644i}{326740} \\ \frac{-31000+18600i}{326740} \cdot \frac{18600i}{326740} & \frac{9300+4712i}{326740} \cdot \frac{4712i}{326740} & \frac{24800-25420i}{326740} \cdot \frac{-25420i}{326740} \end{bmatrix}$$

16.17 Exercise 13

To compute the rank of a matrix of complex numbers, the process is the same as with real numbers: reduce the matrix to its row echelon form, then count the number of non-zero rows, treating the real and imaginary parts as usual.

$$A = \begin{bmatrix} 1+1i & 2 \\ 2+2i & 4+4i \end{bmatrix}$$

$$RREF = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$rank(A) = 2$$