Homework 1

Thibault Schowing
Introduction to Signal and Image Processing

March 17, 2020

1 Fourier transform

For the following continuous functions $f(x)$, sketch the magnitude spectrum $|F(u)|$, i.e. the absolute value of their Fourier Transform (sketches by hand are sufficient):

- The impulse function $f_1(x) = 1/2 \cdot \delta(x)$:

  The FT of this is $\frac{1}{2}\int_{-\inf}^{\inf} \delta(x) \cdot e^{-i2\pi ux} dx$

  Since the integral of the Dirac function is 1, the magnitude spectrum of $f_1(x)$ is $\frac{1}{2}$:

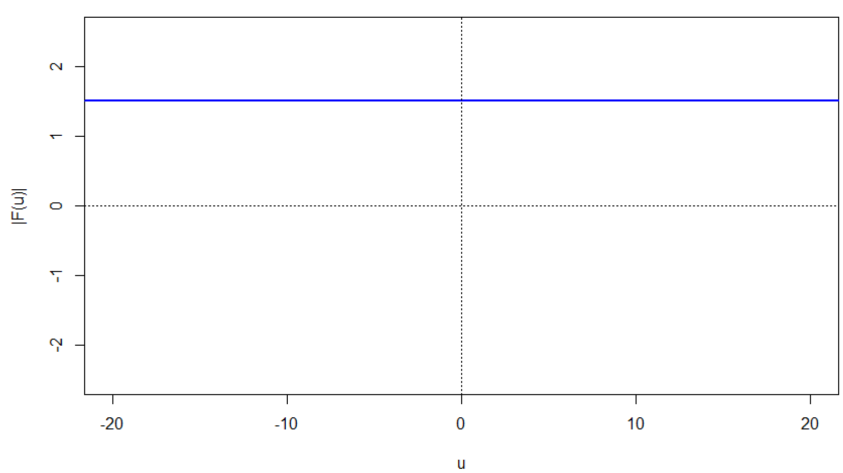

Figure 1:

- the function $f_2(x) = 2 \cdot cos(2\pi u_a x) + cos(2\pi u_b x)$, with the frequencies $u_b > u_a > 0$.

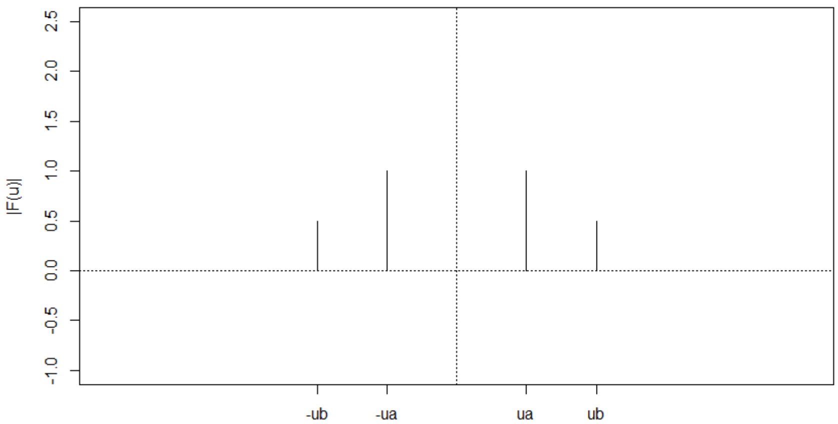

Figure 2:

2 Lloyd-Max quantization

$$\epsilon = \sum_{k=1}^{K} \int_{z_k}^{z_{k-1}} (z - q_k)^2 p(z) dz$$

2.1 [1 point]

Show that

$$z_k = \frac{q_{k-1} + q_k}{2}$$

We first take the derivative and set it to 0:

$$\frac{\delta \epsilon}{\delta z_k} = (z_k - q_{k-1})^2 \cdot p(z_k) - (z_k - q_k)^2 \cdot p(z_k) = 0$$

We separate both parts and get rid of the $p(z_k)$:

$$(z_k - q_k)^2 = (z_k - q_{k-1})^2$$

We develop and get:

$$z_k^2 - 2z_k q_k + q^2 = z_k^2 - 2z_k q_{k-1} + q_{k-1}^2$$

We can get rid of the z_k^2:

$$-2z_k q_k + q_k^2 = -2z_k q_{k-1} + q_{k-1}^2$$

and group the terms:

$$2(-z_k q_k + z_k q_{k-1}) = q_{k-1}^2 - q_k^2$$

$$2z_k(q_{k-1} - q_k) = q_{k-1}^2 - q_k^2$$

We can use the identity to develop the right part and keep only z_k on the left:

$$z_k = \frac{q_{k-1}^2 - q_k^2}{2(q_{k-1} - q_k)} = \frac{(q_{k-1} + q_k)(q_{k-1} - q_k)}{2(q_{k-1} - q_k)}$$

The $(q_{k-1} - q_k)$ cancel each other and so we get:

$$z_k = \frac{q_{k-1} + q_k}{2}$$

2.2 [1 point]

Show that

$$q_k = \frac{\int_{z_k}^{z_{k+1}} z \cdot p(z)dz}{\int_{z_k}^{z_{k+1}} p(z)dz}$$

We start with the partial derivative with respect to q_k as seen in the lecture and minimize it:

$$\frac{\delta \epsilon}{\delta q_k} = -2 \int_{z_k}^{z_{k+1}} (z - q_k) \cdot p(z)dz = 0$$

Then separate the sum in two parts:

$$-2(\int_{z_k}^{z_{k+1}} z \cdot p(z)dz - q_k \int_{z_k}^{z_{k+1}} p(z)dz) = 0$$

Divide by 2:

$$\int_{z_k}^{z_{k+1}} z \cdot p(z)dz = q_k \int_{z_k}^{z_{k+1}} p(z)dz$$

And we finally get:

$$q_k = \frac{\int_{z_k}^{z_{k+1}} z \cdot p(z)dz}{\int_{z_k}^{z_{k+1}} p(z)dz}$$

2.3 [1 point]

For the first iteration we have $q_1 = 0.3$, $q_2 = 0.8$, $z_1 = 0$, $z_3 = 1$. So we calculate according to the formulas above:

$$z_2 = \frac{0.3 + 0.8}{2} = 0.55$$

And we also need q_2, knowing $p(z) = 1$:

$$q_k = \frac{\int_{z_k}^{z_{k+1}} z \cdot p(z)dz}{\int_{z_k}^{z_{k+1}} p(z)dz} = \frac{\Big|_{z_2}^{z_3} \frac{z^2}{2}}{\Big|_{z_2}^{z_3} z}$$

After plugging the numbers and solve the integral we have:

$$q_1 = \frac{\frac{(0.55)^2 - 0^2}{2}}{0.55 - 0} = \frac{0.55}{2} = 0.275$$

$$q_2 = \frac{\frac{(1)^2 - (0.55)^2}{2}}{1 - 0.55} = \frac{(1 - 0.55)(1 + 0.55)}{2(1 - 0.55)} = 0.775$$

Iteration 2, we first recalculate z:

$$z_k = \frac{0.275 + 0.775}{2} = 0.525$$

And then the q's:

$$q_1 = \frac{0.525}{2} = 0.2625$$

$$q_2 = \frac{1 + 0.525}{2} = 0.7625$$

It will most probably converge this way: $z_2 = 0.5$, $q_1 = 0.25$ and $q_2 = 0.75$

3 Chamfer distance maps

The implementation has been made with the algorithm provided in the HW1. Before treating the picture, a one pixel margin is added on every line/row. After the chamferization, the margin is removed to have a picture with the original size.

Result:

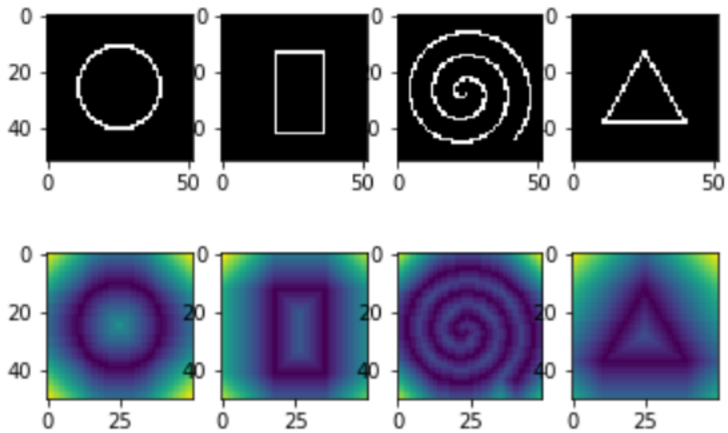

Figure 3: Result of the Chamfering algorithm

4 Bilinear interpolation

Here are presented the results with the grayscale and the RGB image.

Figure 4: Factor 2 interpolation. **Note:** to overcome the complicated LaTeX image display system, here I used the native rescaling of *includegraphics* command to illustrate my results.

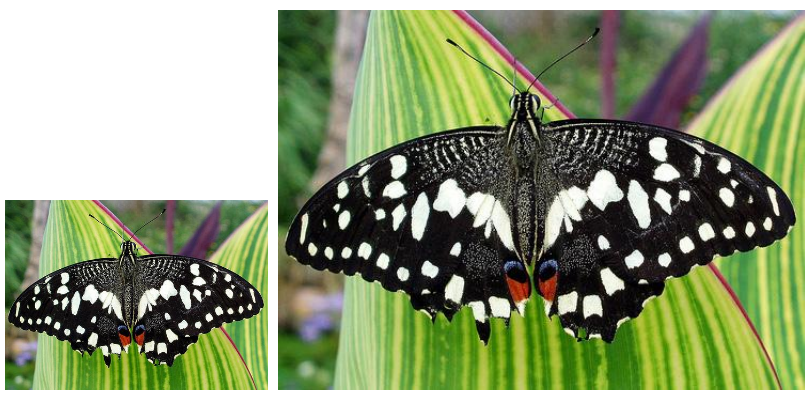

Figure 5: Factor 2 interpolation of an RGB image. **Note:** same as in figure 4