Series 1 - solutions: population growth models

Exercise 1. *Let us assume that a population $N(t)$ follows the exponential growth model (Matlhusian model) with intrinsic growth rate parameter $r > 0$ and initial population size $N(0) > 0$. We define the doubling time T_2 as the time needed for the population to double in size: $N(\tau + T_2) = 2N(\tau)$.*

1) Relate the doubling time T_2 to the intrinsic growth rate r. Is this relationship dependent on the time τ?

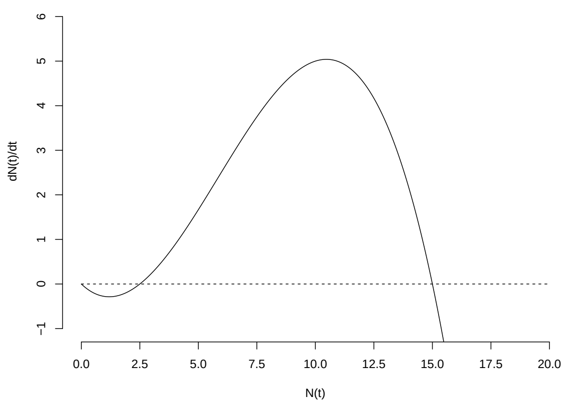
By definition $N(\tau + T_2) = 2N(\tau)$, with $N(\tau) = N(0)e^{r\tau}$. We obtain that $N(0)e^{r(\tau+T_2)} = 2N(0)e^{r\tau}$, which after simplification reads $e^{rT_2} = 2$. By taking the logarithm we obtain finally that $T_2 = ln(2)/r$. Consequently, the relationship does not depend on τ

2) Could the doubling time T_2 be defined for a population following the logistic growth (Verhulst model)?

We first remark that if it would be possible to defined the doubling time, it should be only for an initial condition $0 < N(0) < K$. Indeed for $N(0) > K$, $N(t)$ is a decreasing function.

In general it is impossible to define properly the doubling time T_2 for the logistic growth. This is so because at some time t we will have $N(t) > K/2$, and as the trajectories converge to K the population size cannot anymore double.

Exercise 2. *Let us assume that the differential equation describing the population dynamic $N(t)$ is represented by the following curve:*



1. Write a plausible differential equation $\frac{dN}{dt} = \dots$.

As the function crosses the horizontal axis in three points ($N = 0$, $N = 2.5$ and $N = 15$), a plausible equation could be a polynomial of degree three of the form $\frac{dN}{dt} = aN(2.5 - N)(N - 15)$ with $a > 0$ a positive parameter (here $a = 0.2/15$).

2. As for the logistic growth model seen during the lecture, describe qualitatively the trajectories of this dynamical system. In particular, describe the trajectories for the initial conditions given by $N(0) = 0, 0.2, 2.5, 5, 15,$ and 18.

For $N(0) = 0$, we have $N(t) = 0$ for all time t. For $N(0) = 0.2$, the trajectory will decrease and converge to 0. For $N(0) = 2.5$, we have $N(t) = 2.5$ for all time t. For $N(t) = 5$, the trajectory will increase and converge to 15. For $N(0) = 15$, we have $N(t) = 15$ for all time t. Finally for $N(0) = 18$, the trajectory will decrease and converge to 15.

3. Find all the equilibrium points and describe their nature (stable or unstable).

There are three equilibrium points given by the zeros of the differential equation. They are given by $N^* = 0$, 2.5, and 15. The two equilibrium points $N^* = 0$ and 15 are stable, while the equilibrium point $N^* = 2.5$ is unstable.

Exercise 3. *Let us assume that a population dynamic $N(t)$ is given by the following differential equation:*

$$\frac{dN(t)}{dt} = rN(t)\left(1 - \frac{N(t)}{K}\right) - H,$$

with $H > 0$ a parameter representing the harvesting rate.

1) Find the equilibrium points and determine their nature (stable or unstable). You will have to discus the equilibrium points under the two conditions $H > rK/4$ and $H \leq rK/4$.

2) Finally describe the trajectories of this dynamical system under the two conditions.

We first consider the case where $H > rK/4$. In that case the differential equation looks like:

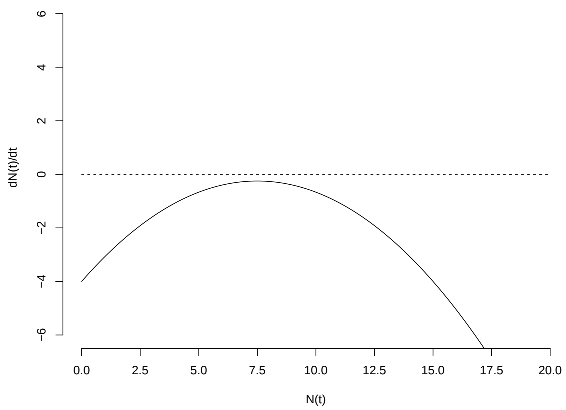


Figure 1: Example for $r = 1$, $K = 14$, and $H = 4$

From this figure we can see that there is no equilibrium point and that the derivative is always negative. Therefore all trajectories converge to minus infinity, i.e, $\lim_{t\to\infty} N(t) = -\infty$.

In the case $H \leq rK/4$, the differential equations looks like:

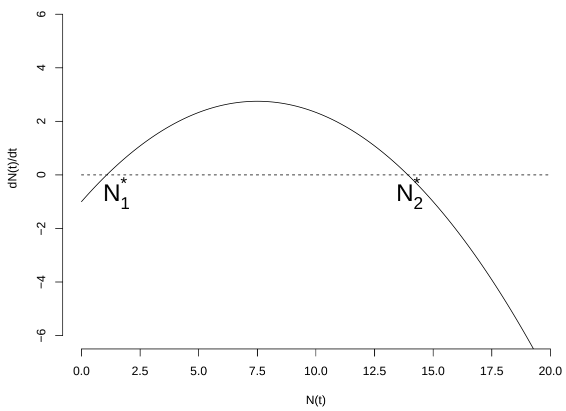


Figure 2: Example for $r = 1$, $K = 14$, and $H = 1$

We can see that there are two equilibrium points N_1^* and N_2^*. We can also see that N_1^* is unstable, while N_2^* is stable. These two equilibrium points can be computed as follow. We have to solve the equation $rN\left(1 - \frac{N}{K}\right) - H = 0$. This equation is a polynomial of degree two, $-(r/K)N^2 + rN - H = 0$. The solutions are given by $N_1^* = K/2(1 - \sqrt{1 - 4H/rK})$ and $N_2^* = K/2(1 + \sqrt{1 - 4H/rK})$.

Depending on the initial condition $N(0)$, we have five types of trajectories:

1. *If $N(0) < N_1^*$, the trajectory decreases and $\lim_{t\to\infty} N(t) = -\infty$*
2. *If $N(0) = N_1^*$, we have $N(t) = N_1^*$, as it is an equilibrium point.*
3. *If $N_1^* < N(0) < N_2^*$, the trajectory increases and $\lim_{t\to\infty} N(t) = N_2^*$*

4. *If* $N(0) = N_2^*$*, we have* $N(t) = N_2^*$*, as it is an equilibrium point.*

5. *If* $N(0) > N_2^*$*, the trajectory decreases and* $\lim_{t\to\infty} N(t) = N_2^*$

Finally in the limit case where $H = rK/4$*, we have only one equilibrium point at which* $N_1^* = N_2^*$*. If* $N(0) > N^*$*, the population decreases to* N^**; if* $N(0) < N^*$*, the population decreases to* $\lim_{t\to\infty} N(t) = -\infty$*. This type of equilibrium is called a "saddle point".*