Example run of Algorithm SigMöller

As an illustration, consider the ring $A = \mathbb{Z}[x, y]$ with the lexicographical ordering with $x > y$, and the ideal generated by $f_1 = 3xy + x + y^2$ and $f_2 = x^2$.

The algorithm maintains a signature Gröbner basis G and a queue of saturated pairs $\mathcal{P}$. Both are finite ordered sequences (lists), which we denote with square brackets, *e.g.* $G = [\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_t]$. The elements of G are pairs (polynomial, signature), for which we use the notations $\mathbf{g}_i = (g_i, \mathfrak{s}(\mathbf{g}_i))$. To simplify the notations, given a basis $G = [\mathbf{g}_1, \dots, \mathbf{g}_t]$, m a monomial of A and $k \in \mathbb{N}$, we use the notation

$$J_m^{(k)} = \mathrm{Sat}(m; \mathrm{LM}(g_1), \dots, \mathrm{LM}(g_k)) = \left\{ i \,:\, i \in \{1..k\} \,:\, \mathrm{LM}(g_i) \mid m \right\}$$

for the saturated sets.

In a saturated set, we will use the notation * to denote those indices which contribute the maximal signature. For example, in the saturated set $J = \{1, 2, 4^*, 5, 8^*\}$, we would have

$$S_J \simeq \frac{M(J)}{\mathrm{LM}(g_4)} \mathfrak{s}(\mathbf{g}_4) \simeq \frac{M(J)}{\mathrm{LM}(g_8)} \mathfrak{s}(\mathbf{g}_8) \gneq \frac{M(J)}{\mathrm{LM}(g_i)} \mathfrak{s}(\mathbf{g}_i) \text{ for } i \in \{1, 2, 5\}.$$

If only one index is marked with a star, the saturated set is regular and that index is the signature index.

The algorithm starts with an empty basis $G = []$ and an empty queue of regular saturated sets $\mathcal{P} = []$. We first add f_1 with signature $\mathbf{e}_1$ to G, that is, we add the element $\mathbf{g}_1 = (3xy + x + y^2, \mathbf{e}_1)$ to G.

We observe that no saturated sets can be formed, so $G_1 = [\mathbf{g}_1]$ is a weak signature Gröbner basis of $\langle f_1 \rangle$.

We then introduce f_2, with signature $\mathbf{e}_2$. It cannot be reduced modulo G, so we add to the basis the element $\mathbf{g}_2 = (x^2, \mathbf{e}_2)$.

To form regular saturated sets, we consider all possible least common multiples of leading monomials of g_i involving g_2. Here, the set of leading monomials is $\{xy, x^2\}$, and the only non-trivial LCM that we can form is x^2y. For each least common multiple m, the set

$$J = J_m^{(2)} = \mathrm{Sat}(m; \mathrm{LM}(g_1), \mathrm{LM}(g_2)) = \{i \in \{1, 2\} \,:\, \mathrm{LM}(g_i) \text{ divides } m\}$$

is a saturated set, with $M(J) = m$. Here, for $m = x^2y$, we get $J_1 = J_{x^2y}^{(2)} = \{1, 2^*\}$.

We have $M(J_1) = x^2y = x\mathrm{LM}(g_1) = y\mathrm{LM}(g_2)$. We multiply the corresponding signatures and we compare: here, $x\mathfrak{s}(\mathbf{g}_1) = x\mathbf{e}_1 \lneq y\mathfrak{s}(\mathbf{g}_2) = y\mathbf{e}_2$, so the presignature of J_1 is $S_{J_1} = y\mathbf{e}_2$, and it is regular with signature index 2.

We now compute a S-polynomial associated to J_1, namely $h_3 = 3yg_2 - xg_1 = -x^2 - xy^2$ with signature $\mathfrak{s}(\mathbf{h}_3) = 3y\mathfrak{s}(\mathbf{g}_2) = 3y\mathbf{e}_2$. Since $\mathrm{LM}(h_3) = -\mathrm{LM}(g_2)$ and $\mathfrak{s}(\mathbf{h}_3) \gneq \mathfrak{s}(\mathbf{g}_2)$, h_3 is regular $\mathfrak{s}$-reducible modulo G, and the result is $g_3 = -xy^2$. It still has signature $3y\mathbf{e}_2$, because we only performed a regular $\mathfrak{s}$-reduction. We add to G the element $\mathbf{g}_3 = (-xy^2, 3y\mathbf{e}_2)$.

The next few steps are identical, so we give a fast-forward version:

4. Regular saturated set $J_2 = J_{xy^2}^{(3)} = \{1, 3^*\}$ with $M(J_2) = xy^2$ and $S_{J_2} = y\mathbf{e}_2$
   $\to \mathbf{g}_4 = (xy + y^3, 9y\mathbf{e}_2)$;

5. Regular saturated set $J_3 = J_{xy}^{(4)} = \{1, 4^*\}$ with $M(J_3) = xy$ and $S_{J_3} = y\mathbf{e}_2$
   $\to \mathbf{g}_5 = (-x + 3y^3 - y^2, 27y\mathbf{e}_2)$;

6. Regular saturated set $J_4 = J_{xy}^{(5)} = \{1, 4, 5^*\}$ with $M(J_4) = xy$ and $S_{J_4} = y^2\mathbf{e}_2$
   $\to \mathbf{g}_6 = (3y^4, 27y^2\mathbf{e}_2)$;

7. Regular saturated set $J_5 = J_{xy^2}^{(4)} = \{1, 3, 4^*\}$ with $M(J_5) = xy^2$ and $S_{J_5} = y^2\mathbf{e}_2$
   $\to \mathbf{g}_7 = (y^4, 9y^2\mathbf{e}_2)$.

Both J_3 and J_5 were added to $\mathcal{P}$ after Step 4 (construction of $\mathbf{g}_4$). But at Step 5, since $S_{J_3} \precneqq S_{J_5}$, we have to consider J_3 first, and keep J_5 for later. After Step 5, $\mathcal{P}$ contains both J_4 and J_5, whose presignatures are incomparable: $S_{J_4} \simeq S_{J_5}$. So we could have considered J_5 before J_4, the result would still have been correct.

After introducing $\mathbf{g}_7$, the basis G has 7 elements:

$$G = [(3xy + \dots, \mathbf{e}_1), (x^2, \mathbf{e}_2), (-xy^2, 3y\mathbf{e}_2), (xy + \dots, 9y\mathbf{e}_2), \\ (-x + \dots, 27y\mathbf{e}_2), (3y^4, 27y^2\mathbf{e}_2), (y^4, 9y^2\mathbf{e}_2)].$$

The queue $\mathcal{P}$ is not empty at this point, but before continuing, we need to form regular saturated sets using the latest addition $\mathbf{g}_7$.

We go through the same process as before to form saturated sets:

1. List all possible least common multiples of leading monomials involving $\mathrm{LM}(g_7)$: those are y^4, xy^4, x^2y^4.

2. For each of them, compute the corresponding saturated set:
   - y^4 gives $J_{y^4}^{(7)} = \{6^*, 7^*\}$ with presignature $y^2\mathbf{e}_2$;
   - xy^4 gives $J_{xy^4}^{(7)} = \{1, 3, 4, 5, 6^*, 7^*\}$, with presignature $xy^2\mathbf{e}_2$;
   - x^2y^4 gives $J_{x^2y^4}^{(7)} = \{1, 2, 3, 4, 5, 6^*, 7^*\}$ with presignature $x^2y^2\mathbf{e}_2$.

None of those 3 saturated sets is regular: there is always a signature collision between $\mathbf{g}_6$ and $\mathbf{g}_7$. For example, with $J_{x^4}^{(7)}$, $\mathfrak{s}(\mathbf{g}_6) \simeq \mathfrak{s}(\mathbf{g}_7) \simeq y^2\mathbf{e}_2$.

So we need to make them regular, which is done by forming new sets with just one of the colliding signatures. From $J_{y^4}^{(7)}$, we could form $\{6^*\}$ and $\{7^*\}$, which are trivial.

From $J_{xy^4}^{(7)}$, we can form the regular saturated sets $\{1, 3, 4, 5, 6^*\}$ and $\{1, 3, 4, 5, 7^*\}$. Since the set $\{1, 3, 4, 5, 6^*\}$ does not contain 7, it is already in $\mathcal{P}$. And we add the new regular saturated set $\{1, 3, 4, 5, 7^*\}$ to $\mathcal{P}$.

Similarly, from $J_{x^2y^4}^{(7)}$, we find the new regular saturated set $\{1, 2, 3, 4, 5, 7^*\}$ to add to $\mathcal{P}$.

Then we continue with the regular saturated set $J_6 = \{1, 3, 4, 5^*\}$ with $M(J_2) = x^2y$ and $S(J_2) = 27y^3\mathbf{e}_2$. It gives rise to $h_8 = 3y^5 + y^4$ with signature $\mathfrak{s}(\mathbf{h}_8) = 27y^3\mathbf{e}_2$.

Since $\mathrm{LM}(h_8) = y\mathrm{LM}(g_6)$ and $\mathfrak{s}(\mathbf{h}_8) = y\mathfrak{s}(\mathbf{g}_6)$, we know that h_8 is 1-singular reducible modulo G and can be discarded. Note that we only needed to compare the leading *monomials* (without coefficients) of h_8 and g_6, and not verify whether there is an actual linear combination eliminating that term.

Remark 0.1. If we had considered J_5 before J_4 at Step 6, $\mathbf{g}_7$ would have been built before $\mathbf{g}_6$, and $\mathbf{g}_6$ would have been discarded for being 1-singular reducible modulo $\mathbf{g}_7$. In that case, the non-regular saturated sets $J_{y^4}^{(7)}$, $J_{xy^4}^{(7)}$ and $J_{x^2y^4}^{(7)}$ would never have been considered.

The remainder of the run proceeds differently, depending on whether the F5 criterion (Prop. 5.8) is implemented. If it is not, the remaining regular saturated sets all give rise to polynomials regular $\mathfrak{s}$-reducing to 0, and the algorithm terminates, returning the 7-elements basis written above.

On the other hand, if the F5 criterion is implemented, those reductions to zero are excluded. Let us illustrate it with the next saturated set in the queue: $J_7 = \{2, 5^*\}$, with signature $27xy\mathbf{e}_2$. We need to test whether $27xy$ lies in the ideal of leading terms of $\langle f_1 \rangle$, which is equivalent to testing whether $27xy$ is reducible modulo the already computed basis $G_1 = \{(3xy^2 + x + y^2, \mathbf{e}_1)\}$. Since it is indeed reducible, this regular saturated set indeed satisfies the F5 criterion, and can be discarded without further calculation. The criterion eliminates all subsequent regular saturated sets in the same way.