On the computation of Gröbner bases for pluriweighted-homogeneous systems

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ABSTRACT

In this paper, we examine the structure of systems which are weighted homogeneous for several systems of weights, and how it impacts Gröbner basis computations. We present different ways to compute Gröbner bases for systems with this structure, either directly or by reducing to existing structures. We also present optimization techniques which are suitable for this structure.

The most natural orderings to compute a Gröbner basis for systems with this structure are weighted orderings following the systems of weights, and we discuss the possibility to use the algorithms in order to directly compute a basis for such an order, regardless of the structure of the system.

We discuss applicable notions of regularity which could be used to evaluate the complexity of the algorithm, and prove that they are generic if non-empty. Finally, we present experimental data from a prototype implementation of the algorithms in SageMath.

CCS CONCEPTS

• **Computing methodologies → Algebraic algorithms**.

KEYWORDS

Gröbner bases, weighted-homogeneity, multihomogeneity, multiple systems of weights

1 INTRODUCTION

Gröbner bases are a very powerful tool for manipulating polynomial ideals. They give ways to describe the set of solutions of a system of polynomial equations, as well as obtaining various information of algebraic nature on the variety or the ideal. Since their introduction by Buchberger [7] in the 1960's, many algorithms have been developed for computing Gröbner bases, either as adaptations of Buchberger's algorithm [19, 21, 22] or by encoding polynomial

operations in linear algebra [12–15, 25]. A particularity of Gröbner basis computations that it is proved to be an extremely difficult problem in the worst case, yet this does not materialize in practice, for systems arising in applications. To better understand this phenomenon, research efforts have focused on identifying generic regularity properties of the polynomial systems ensuring better complexity bounds [5, 23, 24]. Those efforts were then extended to so-called structured polynomial systems, leading to dedicated algorithms with good complexity bounds.

Among such structures are multihomogeneous systems [8, 16, 29] and weighted homogeneous systems [17, 18]. Both those structures have their particularities. A Gröbner basis for multihomogeneous systems can be computed using a dedicated algorithm, splitting the matrices which appear into smaller matrices. Characterizing the regularity of those systems, and analysing the complexity of the algorithm, is however very complicated, and only done in some cases such as bilinear systems. For weighted-homogeneous systems, one can perform a change of variables and use state of the art algorithms, under standard regularity assumptions. The additional structure still leads to better complexity bounds. More recently, algorithms have been developed for computing sparse Gröbner bases [6, 19], including the weighted and multihomogeneous cases.

In this work, we consider a structure which generalizes both the multihomogeneous and weighted homogeneous structures, by considering systems which are weighted-homogeneous for several systems of weights. This structure can appear for instance in physics, where homogeneity is multidimensional, or in algorithms using a weighted order with systems which may already be homogeneous [9]. We present three different algorithms adapted from Matrix-F5 [4] following this structure and taking advantage of the sparse nature of the systems. The first one is dedicated to the pluriweighted-homogeneous structure, and considers the polynomials pluridegree by pluridegree. We also show that several optimizations are available for this algorithm, notably parallelism, and pruning some steps using signatures.

The second algorithm uses a change of variables to reduce the problem to the known multihomogeneous case. The resulting algorithm performs exactly the same steps as the dedicated algorithm, but such a transformation allows to take advantage of state of the art algorithms. Finally, we present the specialization of the sparse-Matrix-F5 [19] algorithm to our structure. Again, this allows to take advantage of existing algorithms, but with the caveat that the computation is restricted to the zero-dimensional case.

A general fact about Gröbner basis algorithms dedicated to a structure is that they compute a Gröbner basis for an order carefully chosen to ensure that the structure is preserved throughout the algorithm. In addition to those theoretical properties which make it

possible to obtain complexity bounds, this order usually gives the fastest Gröbner basis computation in practice. For some applications, having any Gröbner basis is enough, for instance for computing the Hilbert series of the ideal [2, 30]. Furthermore, multi-step strategies have been designed around this feature, combining those "easy" Gröbner basis computations with change of order algorithms in order to obtain a Gröbner basis for any wanted order. In the multihomogeneous case, this "easy" ordering is a graded block ordering, and in the weighted-homogeneous, it is a weight-graded ordering.

In the pluriweighted-homogeneous case, similarly to those two cases, the most natural orders, and the one the aforementioned algorithms are tailored to, are orders graded with respect to the systems of weights. It is now a classical fact that any ordering can be represented by such a system of weights. This opens the possibility of looking at the situation from the opposite angle, by choosing the weights which represent the desired monomial ordering. Of course, it is unlikely that the systems of interest are plurihomogeneous for those weights, but it is possible to adapt the algorithm to run on non-plurihomogeneous systems. This opens different questions from the usual cases, because it is unlikely that we can or want to avoid degree falls in this case.

The complexity of Gröbner basis algorithms is usually examined under some regularity assumptions which are proved to be satisfied for generic systems. We examine several existing regularity properties, and show that they are generic if not empty. This restriction is necessary and not unexpected, as this phenomenon was already present for the more specific multihomogeneous structure. As an opening towards complexity analyses, we show how to compute the Hilbert series of ideals under some of those hypotheses.

2 NOTATIONS

The following notations are used throughout the paper: K is a field, $n \in \mathbb{N}_{>0}$, and $A = K[X_1, \ldots, X_n] = K[\mathbf{X}]$. When the indexing is clear by the context, we shall frequently use bold shorthands for indexed tuples and products, such as $\mathbf{d} = (d_1, \ldots, d_k)$ or $\mathbf{X}^{\boldsymbol{\alpha}} = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$.

3 DEFINITIONS

3.1 Pluriweighted-homogeneity

We first recall the usual definitions of weighted homogeneity and multihomogeneity.

Definition 3.1. A *system of weights* on A is a tuple $W = (w_1, \ldots, w_n)$ in $\mathbb{Z}^n$. It defines a $\mathbb{Z}$-grading on A, by setting the *weighted degree* of a monomial to be

$$\deg_W(X_1^{\alpha_1} \cdots X_n^{\alpha_n}) = w_1 \alpha_1 + \cdots + w_n \alpha_n,$$

and setting the weighted degree of a polynomial as the maximum of the weighted degrees of monomials in its support. A polynomial is called *weighted-homogeneous* or *W-homogeneous* if all its monomials have the same W-degree. This defines a $\mathbb{N}$-grading on A.

Let $k \in \{1, \ldots, n\}$, and $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$ such that $n = n_1 + n_2 + \cdots + n_k$. Group the variables $X_1, \ldots, X_n$ accordingly, as $X_{1,1}, \ldots, X_{1,n_1}, X_{2,1}, \ldots, X_{2,n_2}, \ldots, X_{k,n_k}$. The *multidegree* of a monomial with respect to the partition $\mathbf{n}$ is

$$\text{mdeg}_{\mathbf{n}}(\mathbf{X}_{1,\bullet}^{\boldsymbol{\alpha}_1} \cdots \mathbf{X}_{k,\bullet}^{\boldsymbol{\alpha}_k}) = (\deg(\mathbf{X}_{1,\bullet}^{\boldsymbol{\alpha}_1}), \ldots, \deg(\mathbf{X}_{k,\bullet}^{\boldsymbol{\alpha}_k})).$$

A polynomial is called multihomogeneous (with respect to the partition $\mathbf{n}$) with multidegree $\mathbf{d}$ if all its monomials have multidegree $\mathbf{d}$. This defines a $\mathbb{N}^k$-grading on A.

In this paper, we consider the following generalization of those two notions.

Definition 3.2. Let $k \in \{1, \ldots, n\}$. A *k-plurisystem of weights* is a family $\mathbf{W} = (W_1, \ldots, W_k) \in (\mathbb{Z}^n)^k$ which is linearly independent over $\mathbb{Q}$. The $\mathbf{W}$-pluridegree of the monomial $\mathbf{X}^{\boldsymbol{\alpha}}$ is

$$\text{pdeg}_{\mathbf{W}}(m) = (\deg_{W_1}(m), \ldots, \deg_{W_k}(m)).$$

A polynomial f is called $\mathbf{W}$-plurihomogeneous (or pluriweighted-homogeneous) with pluridegre $\mathbf{d}$ if all the monomials in its support have pluridegree $\mathbf{d}$.

Example 3.3. Weighted homogeneity is a particular case of pluri-weighted homogeneity, by setting $k = 1$ and $W_1 = W$.

Multihomogeneity is also a particular case, with the plurisystem of weights $\mathbf{W}_{\mathbf{n}}^b = (W_{\mathbf{n},1}^b, \ldots, W_{\mathbf{n},k}^b)$ defined as

$$W_{\mathbf{n},i}^b = (\underbrace{0, \ldots \ldots, 0}_{n_1 + \cdots + n_{i-1}}, \underbrace{1, \ldots, 1}_{n_i}, \underbrace{0, \ldots \ldots, 0}_{n_{i+1} + \cdots + n_k}). \tag{1}$$

Just like in the multihomogeneous case, there is no canonical way to define the pluridegree of an arbitrary polynomial. It is a straightforward verification that this notion defines a $\mathbb{Z}^k$-grading on A. Plurihomogeneous components of a polynomial, and plurihomogeneous ideals are defined as usual for this graduation. For a plurihomogeneous ideal I (possibly 0), we denote by $(A/I)_{\mathbf{d}}$ the K-vector span of all plurihomogeneous polynomials of pluridegree $\mathbf{d}$ in A/I.

Notation 3.4. To avoid nested indices, we use a matrix notation for k-plurisystems of weights: we denote $\mathbf{W} = (w_{i,j}) \in \mathbb{Z}^{k \times n}$, with $W_i = \mathbf{w}_{i,\bullet} = (w_{i,1}, \ldots, w_{i,n})$. With this notation, the $\mathbf{W}$-pluridegree can be expressed as a matrix-vector product:

$$\deg_{\mathbf{W}}(X_1^{\alpha_1} \cdots X_n^{\alpha_n}) = \mathbf{W} \cdot \boldsymbol{\alpha}.$$

Definition 3.5. Two k-plurisystems of weights $\mathbf{W}, \mathbf{W}'$ are called *equivalent* if there exists an invertible matrix $P \in \mathbb{Q}^{k \times k}$ such that

$$\mathbf{W} = P \cdot \mathbf{W}'.$$

PROPOSITION 3.6. *Let $\mathbf{W}_1, \mathbf{W}_2$ be two k-plurisystems of weights. Then $\mathbf{W}_1$ and $\mathbf{W}_2$ are equivalent iff for all m_1, m_2 monomials in $K[\mathbf{X}]$,*

$$\deg_{\mathbf{W}_1}(m_1) = \deg_{\mathbf{W}_1}(m_2) \iff \deg_{\mathbf{W}_2}(m_1) = \deg_{\mathbf{W}_2}(m_2). \quad (2)$$

PROOF. Let $\alpha^{(1)}$ and $\alpha^{(2)}$ be the exponent vectors of m_1 and m_2 respectively. If $\mathbf{W}_1$ and $\mathbf{W}_2$ are equivalent, then there exists P invertible such that $\mathbf{W}_1 = P\mathbf{W}_2$. So for $i \in \{1, 2\}$, $\deg_{W_1}(m_i) = \alpha^{(i)} \cdot \mathbf{W}_1 = \alpha^{(i)} \cdot \mathbf{W}_2 \cdot P = \deg_{W_1}(m_i) \cdot P$. Equality (2) follows.

Conversely, assume that Property (2) holds for all monomials m_1, m_2. Let $v \in \mathbb{Q}^n$ be a vector in $\ker(\mathbf{W}_1)$. Clearing out the denominators, we can assume that $v \in \mathbb{Z}^n$. Separating the non-negative and non-positive coordinates, we can write $v = \alpha - \beta$ with $\alpha, \beta \in \mathbb{N}^n$. Then $\mathbf{W}_1 \alpha = \mathbf{W}_1 \beta$. Taking the corresponding monomials, using Property (2) and tracing back the previous computation, we get that $v \in \ker(\mathbf{W}_2)$. By symmetry, $\ker(\mathbf{W}_1) = \ker(\mathbf{W}_2)$. This implies that $\mathbf{W}_1$ and $\mathbf{W}_2$ have the same row space, and thus the same reduced row echelon form. So they are equal up to left-multiplication by an invertible matrix. □

3.2 Size-bounded plurisystems of weights

So far there is no restriction on the weights, and in particular they may be zero or negative, and lead to pluridegrees which are zero or negative. In this paper, we will restrict to multi-systems of weights such that the number of monomials at any $\mathbf{W}$-pluridegree is finite.

Definition 3.7. A k-system of weights $\mathbf{W} = (w_{i,j}) \in \mathbb{Z}^{n \times k}$ is called *positive* (resp. *non-negative*) if all the entries $w_{i,j}$ are positive (resp. *non-negative*). $\mathbf{W}$ is called *size-bounded* if for all $\mathbf{d} \in \mathbb{Z}^k$, there exists only finitely many monomials with $\mathbf{W}$-pluridegree $\mathbf{d}$.

We can characterize size-bounded systems of weights as follows.

Proposition 3.8. *Let $\mathbf{W}$ be a multi system of weights. $\mathbf{W}$ is not size-bounded if and only if there exists a non-trivial monomial with $\mathbf{W}$-pluridegree $(0, \ldots, 0)$.*

Proof. Assume that there exists a non-trivial monomial m_0 with $\mathbf{W}$-pluridegree $(0, \ldots, 0)$. Then m_0^k has $\mathbf{W}$-pluridegree $(0, \ldots, 0)$ for all $k \in \mathbb{N}$, and $\mathbf{W}$ is not size-bounded.

Conversely, assume that $\mathbf{W}$ is not size-bounded, and let $\mathbf{d}$ be a $\mathbf{W}$-pluridegree such that the set $M_{\mathbf{d}}$ of monomials of $\mathbf{W}$-pluridegree $\mathbf{d}$ is infinite. By Dixon's lemma, there exists a finite subset $B_{\mathbf{d}} \subseteq M_{\mathbf{d}}$ such that all elements of $M_{\mathbf{d}}$ are divisible by an element of $B_{\mathbf{d}}$. Since $M_{\mathbf{d}}$ is infinite, there exists a monomial $m \in M_{\mathbf{d}} \setminus B_{\mathbf{d}}$, and by construction it is divisible by $b \in B_{\mathbf{d}}$. The quotient m/b has $\mathbf{W}$-pluridegree $\mathbf{d} - \mathbf{d} = (0, \ldots, 0)$. □

In the rest of the paper, we will frequently restrict to systems of weights such that W_1 is positive, which are clearly size-bounded.

3.3 The plurigraded-reverse-lexicographical ordering

By analogy with the weighted and multihomogeneous cases, we define the following order, which will be a natural order for computing Gröbner bases. Recall that the *reverse-lexicographic ordering* is defined by

$$\mathbf{X}^{\alpha} <_{\text{revlex}} \mathbf{X}^{\beta} \iff \begin{cases} \alpha_n = \beta_n, \ldots, \alpha_{j+1} = \beta_{j+1} \\ \alpha_j > \beta_j. \end{cases}$$

Definition 3.9. Let $\mathbf{W} \in \mathbb{Z}^{k \times n}$ be a plurisystem of weights. The $\mathbf{W}$-*plurigraded reverse lexicographical ordering* on A is the order $<_{\mathbf{W}-\text{pgrevlex}}$, defined by

$$m <_{\mathbf{W}-\text{pgrevlex}} m' \iff \begin{cases} \text{pdeg}_{\mathbf{W}}(m) <_{\text{lex}} \text{pdeg}_{\mathbf{W}}(m'), \text{ or} \\ \text{same pluridegree and } m <_{\text{revlex}} m'. \end{cases}$$

Remark 3.10. The algorithm from Section 4 would also work with any other tie-breaker order instead of revlex.

Proposition 3.11. *Let $\mathbf{W} = (W_1, \ldots, W_k)$ be a plurisystem of weights. Then the $\mathbf{W}$-pgrevlex ordering is a total ordering, and it is compatible with the $\mathbb{Z}^k$-graduation induced by $\mathbf{W}$.*

Further assume that either of the following conditions hold:

(1) $\mathbf{W}$ is non-negative and size-bounded;

(2) W_1 is positive.

Then the $\mathbf{W}$-pgrevlex ordering is a monomial ordering (i.e., for any nontrivial monomial m, $1 <_{\mathbf{W}-\text{pgrevlex}} m$).

Proof. The $\mathbf{W}$-pgrevlex ordering is clearly a total ordering and compatible with the graduation.

Now let m be a nontrivial monomial with pluridegree $\mathbf{d}$, and assume that condition (1) holds. Since $\mathbf{W}$ is size-bounded, by Proposition 3.8, $\mathbf{d}$ is not zero. Furthermore, since $\mathbf{W}$ is non-negative, the non-zero coefficients of $\mathbf{d}$ are positive. In particular, $0 <_{\text{lex}} \mathbf{d}$, and thus $1 <_{\mathbf{W}-\text{pgrevlex}} m$.

If instead condition (2) holds, then the first coefficient of $\mathbf{d}$ is positive, with the same conclusion. □

Remark 3.12. Let $\mathbf{W}$ and $\mathbf{W}'$ be two equivalent plurisystems of weights. Let m and m' be two monomials of A. It is not true in general that $m <_{\mathbf{W}-\text{pgrevlex}} m'$ if and only if $m <_{\mathbf{W}'-\text{pgrevlex}} m'$.

With $n = 2$, take for example $\mathbf{W} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{W}' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \mathbf{W} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$. Then $1 <_{\mathbf{W}-\text{pgrevlex}} XY$ but $XY <_{\mathbf{W}'-\text{pgrevlex}} 1$. In particular, the order $<_{\mathbf{W}'-\text{pgrevlex}}$ is not even a monomial ordering.

Despite the remark, sinc equivalent plurisystems of weights define the same homogeneous components, Gröbner bases *for plurihomogeneous systems* do not depend on the choice of an equivalent plurisystem of weights.

Proposition 3.13. *Let $\mathbf{W}$ and $\mathbf{W}'$ be equivalent plurisystems of weights. Let I be a $\mathbf{W}$-plurihomogeneous ideal. Let L (resp. L') be the set of leading monomials of I w.r.t. $<_{\mathbf{W}-\text{pgrevlex}}$ (resp. $<_{\mathbf{W}'-\text{pgrevlex}}$). Then $L = L'$.*

In particular, if $G \subseteq I$ is a Gröbner basis of I w.r.t. $<_{\mathbf{W}-\text{pgrevlex}}$ then G is a Gröbner basis of I w.r.t $<_{\mathbf{W}'-\text{pgrevlex}}$.

Proof. Let $m \in L$, and let $f \in I$ such that $m = \text{lm}(I)$ w.r.t. $<_{\mathbf{W}-\text{pgrevlex}}$. Since I is a $\mathbf{W}$-plurihomogeneous ideal, all plurihomogeneous components of f lie in I, so w.l.o.g. we can assume that f is plurihomogeneous. In particular, all the terms in its support have the same $\mathbf{W}$-pluridegree, so m is also the leading monomial of m w.r.t. the grevlex order. And since $\mathbf{W}$ and $\mathbf{W}'$ are equivalent, f is also $\mathbf{W}'$-plurihomogeneous, and by the same argument, m is the leading monomial w.r.t. $<_{\mathbf{W}'-\text{pgrevlex}}$. So $m \in L'$. We conclude that $L \subseteq L'$, and by symmetry $L = L'$.

The consequences on the Gröbner bases is immediate, since Gröbner basis are characterized by their leading terms. □

If $k = n$, there is only one monomial at each $\mathbf{W}$-pluridegree, and the $\mathbf{W}$-pgrevlex is defined only in terms of the pluridegrees. More generally, it is a classical fact [27] that for any monomial ordering $<$ on A, there exists a plurisystem of weights $\mathbf{W} \in \mathbb{Z}^{n \times n}$ such that

$$m < m' \iff \text{pdeg}_{\mathbf{W}}(m) <_{\text{lex}} \text{pdeg}_{\mathbf{W}}(m') \iff m <_{\mathbf{W}-\text{pgrevlex}} m'.$$

4 MATRIX-F5 ALGORITHM

4.1 General Matrix-F5 algorithm

We will present different variants of the Matrix F5 algorithm adapted to the structure. To this end, we first present a very general form of the algorithm.

The algorithm itself is very close to the original Matrix-F5 algorithm, with a degree of freedom in the order for processing the new elements. That difference is delegated to the subroutine Algo-Steps, which will output a list of steps to process. Each such step is a triple $(\mathbf{d}, M, S)$ where:

Algorithm 1: General Matrix-F5 algorithm

Input : $F = (f_1, \ldots, f_r)$ a system of structured polynomials; $d_{\max}$ a bound

Output : G a Gröbner basis of F, truncated at the bound $d_{\max}$

1   Steps $\leftarrow$ AlgoSteps$(F, d_{\max})$;

2   $G \leftarrow \{(f_i, (1, i)) : i \in \{1, \ldots, r\}\}$;

3   Mac $\leftarrow$ ();

4   **for** $(\mathbf{d}, M, S)$ in Steps :

5   Mac$_\mathbf{d}$ $\leftarrow$ matrix with columns indexed by M and 0 rows;

6   **for** $i \in \{1, \ldots, r\}$:

7   **for** (m, j) in S with $j = i$:

8   Find an element g with signature (m'', i) in G ;

9   Add the coefficient vector of $\frac{m}{m''}g$ to the bottom of Mac$_d$, with signature (m, i)

10   Reduce Mac$_\mathbf{d}$ to row-echelon form, only reducing below the pivots and without permutations ;

11   Add to G all polynomials corresponding to a new pivot in Mac$_\mathbf{d}$ and which are not reducible by G, with the signature of their row ;

12   **return** G

- $\mathbf{d}$ is some "degree" for the step;
- M is the set of all monomials of "degree" $\mathbf{d}$ in A;
- S is a set of pairs $(m, i) \in \mathrm{Mon}(A) \times \{1, \ldots, m\}$ such that mf_i has "degree" $\mathbf{d}$.

The notion of "degree" can for example be from a $\mathbb{N}$-graduation on A (standard degree, weighted degree), any $\mathbb{N}^k$-graduation (multidegree, pluridegree) or even a graduation on a subalgebra of A containing all the polynomials in F (sparse degree [19], G-degree [20]).

The pairs (m, i) correspond to *signatures* [13], and with the notations above, we say that (m, i) has "degree" $\mathbf{d}$. Going in depth in the theory of signature Gröbner bases is out of the scope of this paper, so we only recall the basic requirements to prove correctness of the algorithm and of the criteria in Section 4.3. The interested reader will find more details in [1, 10, 13, 21, 28]. Signatures are ordered with the "degree over position over term": let (m, i) and (m, i') be two signatures with respective "degree" $\mathbf{d}$ and $\mathbf{d}'$,

$$(m, i) \prec (m', i') \iff \begin{cases} \mathbf{d} < \mathbf{d}' \\ \text{or } \mathbf{d} = \mathbf{d}' \text{ and } i < i' \\ \text{or } \mathbf{d} = \mathbf{d}' \text{ and } i = i' \text{ and } m < m'. \end{cases}$$

Note that signatures need not be totally ordered, if the "degrees" are not.

For $f \in \langle f_1, \ldots, f_r \rangle$, we say that f has signature (m, i) if it can be written

$$f = \alpha m f_i + \sum_{j \in \{1, \ldots, i\}} \sum_{\mu \in \mathrm{Mon}(A)} \alpha_{\mu, j} \mu f_j$$

with $\alpha, \alpha_{\mu, j} \in K$, and for all signature (μ, j) with $\alpha_{\mu, j} \neq 0$, $(\mu, j) \prec (m, i)$. For all signatures (m, i), we denote by $V_{(m,i)}$ (resp. $V_{\prec (m,i)}$) the K-vector space spanned by all polynomials with signature less than, equal or incomparable to (m, i) (resp. strictly smaller than (m, i)).

The reason for restricting reductions in line 10 is to keep the correspondence between signatures and rows in the echelon form. If this is not required, one can modify the algorithm as follows: at lines 8-9, form the row with mf_i; at line 10, use any reduction to row echelon form; and at line 11, disregard the signatures. With that modified algorithm, we end up reducing mf_i multiple times, but on the other hand it allows to use fast linear algebra for the computation of the row echelon form.

4.2 Plurihomogeneous Matrix-F5 algorithm

We now specialize Algorithm 1 to the $\mathbf{W}$-plurigrading. Let $\mathbf{W}$ be a plurisystem of weights such that W_1 is positive. Let $d_{\max} \in \mathbb{N}$. The algorithm will compute a Gröbner basis of F w.r.t. the $\mathbf{W}$-pgrevlex ordering, truncated at W_1-degree $d_{\max}$.

This is done by letting the routine AlgoSteps output steps following the grading. Thus, the "degree" $\mathbf{d}$ is chosen to be the $\mathbf{W}$-pluridegree, and the sets of monomials and signatures are chosen as those with pluridegree $\mathbf{d}$ for all $\mathbf{d}$ with $d_1 \leq d_{\max}$. Between steps, pluridegrees can be ordered using the lexicographic order, so that signatures are totally ordered, but we will see (Proposition 4.4) that it does not matter.

Remark 4.1. If W_1 is not positive but $\mathbf{W}$ is size-bounded, it is possible to design a suitable AlgoSteps function, by using a linear combination of the weights instead of W_1. Functionally, the algorithm is the same, for an equivalent system of weights.

If $\mathbf{W}$ is not size-bounded, the algorithm as presented will not work. However, algorithms such as F4 or F5, which build matrices at each step dynamically, based on the support of the polynomials being considered, would be able to follow a similar structure.

We now prove the main loop invariants for the algorithm without criteria, which ensure that the algorithm is correct. The proof is a transposition of the result in the homogeneous case[5], and we only outline the arguments.

THEOREM 4.2 (LOOP INVARIANTS). *Let $\mathbf{d}$ be a pluridegree with $d_1 \leq d_{\max}$.*

(1) *Let (m, i) be a signature of pluridegree $\mathbf{d}$ added in Mac$_\mathbf{d}$, then the polynomials associated to the rows above and up to that signature span $V_{(m,i)}$;*

(2) *When starting step $\mathbf{d}$ (line 4), any polynomial with signature at most (m, i) with (m, i) having pluridegree $< \mathbf{d}$ is reducible modulo G.*

PROOF. For the first invariant, we have to prove that for every signature (m, j), the polynomial $\frac{m}{m'}g$ added at line 9 lies in $V_{m,i}$. This is proved by induction on $\mathbf{d}$: the base case is clear, and by the induction hypothesis, the property holds for g' with signature (m', j), so it holds for $\frac{m}{m'}g$.

The second property is a consequence of the characterization of Gröbner bases using Macaulay matrices: since we pick all the possible signatures in $S_\mathbf{d}$ for each $\mathbf{d}$, by the first invariant, at the end of the loop, the rows of Mac$_\mathbf{d}$ span the space of polynomials of pluridegree $\mathbf{d}$, and they have distinct leading monomials. This implies that all the leading monomials of $\langle F \rangle$ with pluridegree $\mathbf{d}$ are a pivot in Mac$_\mathbf{d}$. Since again this is done for every $\mathbf{d}$, all the leading monomials of $\langle F \rangle$ with pluridegree at most $\mathbf{d}$ are a pivot in one of

the matrices computed so far. By construction of G, it implies that every polynomial with degree at most $\mathbf{d}$ is reducible by G. □

Remark 4.3. In particular, the correctness of the algorithm does not depend on the choice of g at line 4.

4.3 Optimizations

We present here a few optimizations which are available for the algorithm. First, if W_1 is a positive system of weights, one can parallelize the computations like in the multihomogeneous case.

PROPOSITION 4.4. *Let $\mathbf{W}$ be a plurisystem of weights such that W_1 is positive. Let $\mathbf{d} = (d_1, \ldots, d_k)$ and $\mathbf{d}' = (d_1', \ldots, d_k')$ be two distinct pluridegrees, with $d_1 = d_1'$. Let $M_\mathbf{d}$ and $S_\mathbf{d}$ (resp. $M_{\mathbf{d}'}$ and $S_{\mathbf{d}'}$) the set of monomials and signatures for the step $\mathbf{d}$ (resp. $\mathbf{d}'$). Then no monomial of $M_\mathbf{d}$ divides any monomial of $M_{\mathbf{d}'}$, and no signature of $S_\mathbf{d}$ divides any signature of $S_{\mathbf{d}'}$.*

PROOF. For a contraction, let $\mu \in M_\mathbf{d}$ and $\mu' \in M_{\mathbf{d}'}$ such that there exists a monomial v with $\mu' = v\mu$. If $\mu = \mu'$, the monomials would have the same pluridegree, so $v \neq 1$. Since W_1 is a positive system of weights, $\text{wdeg}_{W_1}(v) > 0$. So $d_1' = \text{wdeg}_{W_1}(\mu') = \text{wdeg}_{W_1}(v) + \text{wdeg}_{W_1}(\mu) > \text{wdeg}_{W_1}(\mu) = d_1$, which contradicts the assumption. The proof is the same for the signatures. □

The proposition implies that for a given value of d_1, the steps are independent from one another, and can be performed in parallel.

The next few optimizations concern signatures. First, we state the F5 criterion for plurihomogeneous systems.

THEOREM 4.5 (F5 CRITERION [13]). *Let (m, i) be a signature appearing in the course of the algorithm. If m lies in the ideal spanned by $f_1, \ldots, f_{i-1}$, then the row inserted with signature (m, i) is a linear combination of the rows inserted above.*

PROOF. Assume that we chose mf_i at line 9. The hypothesis means that there exists polynomials $r, p_1, \ldots, p_{i-1}$ such that
$$m + r = p_1 f_1 + \cdots + p_{i-1} f_{i-1},$$
and all the terms in the support of r are smaller than m. So
$$mf_i = f_i p_1 f_1 + \cdots + f_i p_{i-1} f_{i-1} + r f_i$$
and the row with signature (m, i) is linear combination of rows appearing above in the matrix, either with signature (μ, j) with $j < i$, or with signature (μ, i) with $\mu < m$.

In particular, $V_{(m,i)} = V_{\prec(m,i)}$, and by the first loop invariant, it remains true regardless of the choice made at line 8. □

This criterion can be used even with the algorithm simplified as described at the end of Section 4.1. If we do not do this simplification, and keep track of the signatures in the echelon form, we can also eliminate rows with the following Syzygy criterion, which essentially states that if the row associated to a signature is reduced to 0, the same will happen to all rows associated to a multiple of that signature.

THEOREM 4.6 (SYZYGY CRITERION, E.G. [28]). *Let (m, i) be a signature appearing in the course of the algorithm at pluridegree $\mathbf{d}$, such that the row associated to the signature (m, i) is a linear combination of the rows appearing above in the matrix at pluridegree $\mathbf{d}$. Let μ be a monomial. Then $(m\mu, i)$ is a linear combination of the rows appearing above in the matrix at pluridegree $\mathbf{d}' = \mathbf{d} + \text{pdeg}_\mathbf{W}(\mu)$.*

PROOF. By the loop invariant of the algorithm, the rows above the row corresponding to $(m\mu, i)$ in $\text{Mac}_{\mathbf{d}'}$ span $V_{\prec(m\mu, i)}$. By the order on the signatures, this space contains the span of polynomials which can be written as μp, where $p \in V_{\prec(m,i)}$. Since the polynomial associated to the row with signature $(m\mu, i)$ lies in the latter space, it lies in the former. □

The last optimization criterion of this section relies on the observation that at a given step, if all the signatures have a non-trivial gcd, it means that the matrix is exactly the same as a matrix which has been considered at a previous step. This is something that can appear in general weighted-homogeneous and multihomogeneous cases too, but this should be rare. On the other hand, for pluriweighted-homogeneous systems, it is inherent to the subdivision that some steps will have very few signatures, and very few ways to shift those signatures up.

PROPOSITION 4.7. *Let $(\mathbf{d}, M_\mathbf{d}, S_\mathbf{d})$ be a step in the course of the algorithm. Let μ be the greatest common divisor of all the monomial part of signatures in $S_\mathbf{d}$. If $\mu \neq 1$, then the step will not yield any new polynomial for the Gröbner basis.*

PROOF. Let $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k)$ be the pluridegree of μ. By Remark 4.3, we can assume that the rows of $\text{Mac}_\mathbf{d}$ are formed by adding mf_i for each signature (m, i). Then all polynomials associated to those rows are of the form $\mu m' f_i$ where $m' f_i$ has pluridegree $\mathbf{d} - \boldsymbol{\delta}$. So after reducing $\text{Mac}_\mathbf{d}$ to row echelon form, for any polynomial g associated to a row in $\text{Mac}_\mathbf{d}$, there exists g' with pluridegree $\mathbf{d} - \boldsymbol{\delta}$ such that $g = \mu g'$. By the second loop invariant, g' is reducible modulo G, and therefore so is g. By construction, g will not be added to the basis. □

4.4 Reducing to multihomogeneous systems

In the previous section, we presented an algorithm for computing a Gröbner basis for pluriweighted-homogeneous systems. In this section, we will see that we can emulate the behavior of this algorithm by using state-of-the-art algorithms for multihomogeneous systems.

Those algorithms follow the structure of Algorithm 1, for the plurisystem of weights $\mathbf{W}_\mathbf{n}^{b*} = (W_0, W_{\mathbf{n},1}^b, \ldots, W_{\mathbf{n},k-1}^b)$ with for $i > 0$, $W_{\mathbf{n},i}^b$ as in Eq. (1), and $W_0 = (1, \ldots, 1) = \sum_{i=1}^k W_{\mathbf{n},i}^b$. This plurisystem of weights is equivalent to the plurisystem of weights $\mathbf{W}_\mathbf{n}^b$.

Definition 4.8. Let $\mathbf{W}$ be a non-negative k-system of weights in $\mathbb{Z}^{n \times k}$. Consider the algebra $B = K[Y_{1,1}, \ldots, Y_{n,1}, Y_{1,2}, \ldots, \ldots, Y_{n,k}]$ in nk variables. Let $\mathbf{p}$ be the partition $kn = n + n + \cdots + n$, and consider the corresponding multigrading defined by grouping the variables $Y_{\bullet, j}$ together for all $j \in \{1, \ldots, k\}$. We define the morphism $\mathcal{F}_\mathbf{W} : A \to B$ with
$$\mathcal{F}_\mathbf{W}(f)(\mathbf{Y}) = f\left(\mathbf{Y}_{1,\bullet}^{\mathbf{w}_{\bullet,1}}, \ldots, \mathbf{Y}_{n,\bullet}^{\mathbf{w}_{\bullet,n}}\right).$$

Example 4.9. With the plurisystem of weights $\mathbf{W} = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{smallmatrix}\right)$,
$$\mathcal{F}_\mathbf{W}(X_1 X_2^2 + X_1^2 X_3) = Y_{1,1} Y_{1,2} Y_{2,1} Y_{2,2}^2 + Y_{1,1}^2 Y_{1,2}^2 Y_{3,1} Y_{3,2}^3,$$
and it is multihomogeneous with degree 3 in $Y_{1,1}, Y_{2,1}, Y_{3,1}$, and degree 5 in $Y_{1,2}, Y_{2,2}, Y_{3,2}$.

PROPOSITION 4.10. *The map $\mathcal{F}_{\mathbf{W}}$ is a morphism, and sends pluri-homogeneous polynomials in A with pluridegree $\mathbf{d}$ onto multihomogeneous polynomials in B with multidegree $\mathbf{d}$. Furthermore, this morphism sends the order $<_{\mathbf{W}\text{-pgrevlex}}$ on the order $<_{\mathbf{W}_{\mathbf{p}}^{\mathbf{b}}\text{-pgrevlex}}$.*
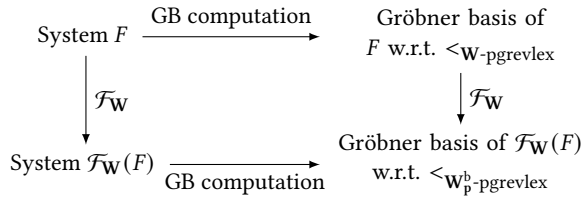
The main observation for the computation of Gröbner bases is that this morphism commutes with S-polynomials.

PROPOSITION 4.11. *Let f and g be two polynomials. Then*

$$\text{S-Pol}(\mathcal{F}_{\mathbf{W}}(f), \mathcal{F}_{\mathbf{W}}(g)) = \mathcal{F}_{\mathbf{W}}(\text{S-Pol}(f, g)).$$

This property allows to compute Gröbner bases through $\mathcal{F}_{\mathbf{W}}$.

THEOREM 4.12. *Let $F = (f_1, \ldots, f_m)$ be a tuple of polynomials in A. Endow the algebra A with the monomial order $<_{\mathbf{W}\text{-pgrevlex}}$. Let $\mathcal{F}_{\mathbf{W}}(F) = (\mathcal{F}_{\mathbf{W}}(f_1), \ldots, \mathcal{F}_{\mathbf{W}}(f_m))$ in B. Endow the algebra B with the plurisystem of weights $\mathbf{W}_{\mathbf{p}}^{\mathbf{b}}$ (cf. Eq. (1)). Then the following operations commute:*

System F $\xrightarrow{\text{GB computation}}$ Gröbner basis of F w.r.t. $<_{\mathbf{W}\text{-pgrevlex}}$

$\downarrow \mathcal{F}_{\mathbf{W}}$ $\qquad\qquad\qquad\qquad\qquad \downarrow \mathcal{F}_{\mathbf{W}}$

System $\mathcal{F}_{\mathbf{W}}(F)$ $\xrightarrow{\text{GB computation}}$ Gröbner basis of $\mathcal{F}_{\mathbf{W}}(F)$ w.r.t. $<_{\mathbf{W}_{\mathbf{p}}^{\mathbf{b}}\text{-pgrevlex}}$

PROOF. It is a consequence of the fact that a Gröbner basis can be computed by a sequence of S-polynomials and reductions (themselves a particular case of S-polynomial). By Proposition 4.11, the result of a sequence of such calculations, run on polynomials of the form $\mathcal{F}_{\mathbf{W}}(f)$, will be polynomials in the image of $\mathcal{F}_{\mathbf{W}}$. And by Proposition 4.10, the orders are compatible, so that applying $\mathcal{F}_{\mathbf{W}}$ preserves the propriety of being a Gröbner basis. □

So we can compute a Gröbner basis of $\mathcal{F}_{\mathbf{W}}(F)$ using state-of-the-art algorithms for multihomogeneous Gröbner bases, and invert $\mathcal{F}_{\mathbf{W}}$ to recover the wanted basis.

In practice, as mentioned, algorithms for multihomogeneous systems compute a basis for the $\mathbf{W}_{\mathbf{p}}^{\mathbf{b}*}$ ordering, which is equivalent to the $\mathbf{W}_{\mathbf{p}}^{\mathbf{b}}$ ordering. By Proposition 3.13, it does not change the result of a Gröbner basis computation if the input is multihomogeneous, and it amounts to a mere reordering of the matrices considered in the course of the algorithm.

The algorithm does not take into account the inherent sparsity of systems arising from the application of $\mathcal{F}_{\mathbf{W}}$. This leads to both larger matrices and to more matrices. This can be handled by restricting the computation to monomials appearing in the image of $\mathcal{F}_{\mathbf{W}}$. In doing so, there are only finitely monomials of given W_1-degree, which allows the algorithm to run directly with the weights $\mathbf{W}_{\mathbf{p}}^{\mathbf{b}}$. With those changes, the multihomogeneous algorithms encodes exactly the same operations, in the same order, as the plurihomogeneous algorithm.

This is similar to the weighted homogeneous case, where taking advantage of the structure in matrix algorithms requires to restrict to monomials in the image of the transformation.

4.5 Sparse Gröbner basis algorithms

One last approach worth mentioning is the use of sparse Gröbner basis algorithms, and in particular their specialization to multihomogeneous systems. We recall basic facts about the sparse variant of the Matrix-F5 algorithm in that case. Let $\mathbf{n}$ be a partition of n, and let F be a multihomogeneous system of polynomials with multidegree $\mathbf{d}$. For simplicity, we assume that all polynomials have the same multidegree $\mathbf{d}$. The sparse Gröbner basis algorithm can be emulated by running Algorithm 1 with AlgoSteps a function returning, for all $d \in \{1, \ldots, d_{\max}\}$, the monomials obtained by products of d monomials of multidegree $\mathbf{d}$, and for signatures the pairs (m, i) with m a product of $d - 1$ monomials of multidegree $\mathbf{d}$. We refer the reader to the seminal papers [6, 19] for more details.

This algorithm does not in general compute a Gröbner basis, but a *sparse Gröbner basis*. The notion of sparse Gröbner basis is relative to the choice of the monomials in the computation. It is in particular suitable for listing the solutions of a system which is zero-dimensional in the multihomogeneous sense. However, systems arising from the $\mathcal{F}_{\mathbf{W}}$ transformation will almost never be zero-dimensional: a system of n equations in n variables becomes a system of n equations in kn variables.

It is also possible to run the sparse Matrix-F5 algorithm directly on the plurihomogeneous system. In practice, it means that given a system F of $\mathbf{W}$-plurihomogeneous polynomials with the same $\mathbf{W}$-pluridegree $\mathbf{d}$, the function AlgoSteps function would return, at each degree d, the monomials obtained by the products of d monomials of pluridegree $\mathbf{d}$, and similarly for the signature. If the system is zero-dimensional in the sparse sense, this allows to compute a sparse Gröbner basis, and then to recover the solutions using the sparse FGLM algorithm.

4.6 Affine systems

Using this algorithm for systems which are not plurihomogeneous can be useful from at least two perspectives. First, similar to the usual cases, it may happen that that the polynomials in a system have a large plurihomogeneous component of maximal pluridegree (say, for a lexicographical comparison), so that using the structure to guide the computation is advantageous. Second, with the observation at the end of Section 3.3, it is possible to represent *any* monomial order as a $\mathbf{W}$-pgrevlex order for some $\mathbf{W}$. Assuming that W_1 is positive, Algorithm 1 allows to compute a Gröbner basis for those orders, by also adding lower pluridegree monomials to M. In that case, it may happen that there is only one monomial for each system of weights, and any non-trivial system is affine.

Remark 4.13. If W_1 is not positive, by Remark 3.12, it is not enough to find an equivalent system of weights such that W_1 is positive.

Adapting the algorithm to deal with those affine situations is more complicated. Even in the homogeneous case (standard grading), the Matrix-F5 algorithm does not behave nicely with affine polynomials. Indeed, it is proved [11] that whenever the degree of the polynomials falls, the resulting polynomials can no longer be used for the F5 criterion, and that conversely, the F5 criterion may not catch all reductions to zero even for regular sequences.

For this reason, affine systems are usually studied under the

stronger hypothesis of *regularity in the affine sense* [3], which encodes that the highest degree components of the polynomials form a regular sequence, and thus that the F5 criterion is able to avoid any degree fall.

This approach could work for polynomials whose highest pluridegree components satisfy the regularity conditions in Section 5. However, when looking at the systems for a fixed weight-matrix order, rather than letting the structure of the system guide the choice of the weights, it is expected that degree falls will happen. Besides the obvious case of n linearly independent systems of weights, for example, for the weights defining an elimination ordering, any actual elimination will be a degree fall.

In the usual case with the standard degree, when degree falls cannot be avoided, there are two approaches. First, one can homogenize the system. In that case, degree falls cannot happen, and the F5-criterion is able to avoid reductions to zero. However, this leads to working with polynomials of degree much higher than necessary. This approach would work in our case, using k homogenization variables.

The other approach, is to follow the degree falls and insert them in the matrices computed at the appropriate degree. This invalidates the criteria, but it avoids computing polynomials of degree higher than necessary and it is the most efficient approach in practice.

5 REGULARITY AND HILBERT MULTISERIES

5.1 Definitions

In this section, we give different possible definitions of regularity, by analogy with known cases.

Definition 5.1. Let $\mathbf{W} \in \mathbb{N}^{k \times n}$ be a non-negative multi-system of weights, and let $A = K[X_1, \ldots, X_n]$. Let $F = (f_1, \ldots, f_r) \in A^r$ be a sequence of $\mathbf{W}$-plurihomogeneous polynomials, with respective $\mathbf{W}$-pluridegree $\mathbf{d}_i$. For all $i \in \{1, \ldots, r\}$, let I_i be the ideal $\langle f_1, \ldots, f_i \rangle$.

For all $i \in \{2, \ldots, r\}$, and for all $\mathbf{W}$-pluridegree $\mathbf{d}$, we consider the multiplication map

$$\phi_i^{(\mathbf{d})} : \quad (A/I_{i-1})_{\mathbf{d}} \quad \rightarrow \quad (A/I_{i-1})_{\mathbf{d}+\mathbf{d}_i}$$
$$f \quad \mapsto \quad f_i f.$$

Let $f \in A_{\mathbf{d}}$ be such that $f_i f \in I_{i-1}$, we say that:

(1) f is a *trivial divisor of* 0 *in* A/I_{i-1} if $f \in I_{i-1}$;
(2) f is a *semi-trivial divisor of* 0 *in* A/I_{i-1} if $\deg_{\mathbf{W}}(f) = \mathbf{d}$ and $\phi_i^{(\mathbf{d})}$ is surjective;
(3) f is an *eliminable divisor of* 0 *in* A/I_{i-1} if the leading term of f is divisible by the leading term of a semi-trivial divisor of 0 in A/I_{i-1}.

We also say that:

(1) F is a *regular sequence* if for all i, all divisors of 0 in A/I_{i-1} are trivial;
(2) F is a *semi-regular sequence* if for all i, all divisors of 0 in A/I_{i-1} are trivial or semi-trivial (i.e., for all $\mathbf{d}$, $\phi_i^{\mathbf{d}}$ is either injective or surjective);
(3) F is a *weakly regular sequence* if for all i, all divisors of 0 in A/I_{i-1} are trivial, semi-trivial or eliminable.

We make a few notes on the relations between this definition and pre-existing notions:

(1) The definition of regular sequences does not depend on the grading, and is the same as the usual definition, including in the homogeneous [25] and weighted homogeneous [17] cases.
(2) The definition of semi-regular sequences does depend on the grading. For $\mathbb{N}$-gradings, it specializes to the usual definition in the homogeneous [4, 26] and some weighted homogeneous [18] cases. In those cases, if $r \leq n$, there cannot exist semi-trivial divisors of 0, and weak regularity, semi-regularity and regularity are equivalent.
(3) If $r \geq i > n$, in the homogeneous and some weighted homogeneous cases, for semi-regular sequences, it is known that there exists a (weighted) degree $\delta^{(i)}$ such that for all $d < \delta^{(i)}$, $\phi_i^{(d)}$ is injective, and for all $d \geq \delta_i$, $\phi_i^{(d)}$ is surjective. So in those cases, weak regularity implies semi-regularity.

In Algorithm 1 different types of divisors of zero correspond to different types of reductions to zero:

(1) if f is a trivial divisor of 0 in A/I_{i-1}, then $(\mathrm{lm}(f), i)$ is a signature eliminated by the F5 criterion;
(2) if f is a semi-trivial divisor of 0 in A/I_{i-1}, then the matrix Mac_d after adding the signatures in i has more rows than columns, and has full rank; if there are as many pivots as columns above the row associated to signature $(\mathrm{lm}(f), i)$, it is possible to avoid that reduction to zero;
(3) if f is an eliminable divisor of 0 in A/I_{i-1}, then $(\mathrm{lm}(f), i)$ is a signature eliminated by the syzygy criterion.

5.2 Hilbert multiseries

Definition 5.2. Let $k, n \in \mathbb{N}$, let $\mathbf{W} \in \mathbb{Z}^{k \times n}$ be a size-bounded k-plurisystem of weights. We consider the algebra $A = K[\mathbf{X}]$ with the $\mathbf{W}$-plurigrading. Let $I \subset A$ be a $\mathbf{W}$-plurihomogeneous ideal. The *Hilbert multiseries* of A/I is the formal series

$$\mathrm{HS}_{A/I}(T_1, \ldots, T_k) = \sum_{\mathbf{d} = (d_1, \ldots, d_k) \in \mathbb{N}^k} \dim_K((A/I)_{\mathbf{d}}) T_1^{d_1} \cdots T_k^{d_k}.$$

PROPOSITION 5.3. *The Hilbert multiseries of the algebra A is*

$$\mathrm{HS}_A(\mathbf{T}) = \frac{1}{\left(1 - T_1^{w_{1,1}} T_2^{w_{2,1}} \cdots T_k^{w_{k,1}}\right) \cdots \left(1 - T_1^{w_{1,n}} \cdots T_k^{w_{k,n}}\right)}.$$

PROOF. It is proved similarly to the case of a single weight. Note that $\dim(A_{\mathbf{d}})$ is equal to the number of monomials of multi-$\mathbf{W}$-degree $\mathbf{d}$. The sum of all monomials in A can be expressed as a fraction

$$\sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} \mathbf{X}^{\boldsymbol{\alpha}} = \frac{1}{(1 - X_1) \cdots (1 - X_n)},$$

and the result follows after substituting X_i by $\mathbf{T}^{\mathbf{w}_{\bullet,i}} X_i$, grouping by degree in $\mathbf{T}$ and evaluating at $X_i = 1$. □

The next theorem describes a way to compute the Hilbert multiseries of a quotient by an ideal generated by a regular or semi-regular sequence.

THEOREM 5.4. *With the same notations, let $(f_1, \ldots, f_r) \in A^r$ be a sequence of $\mathbf{W}$-plurihomogeneous polynomials, spanning an ideal I, with respective pluridegree $\mathbf{d}_i$. For $i \in \{0, \ldots, m\}$, let I_i be the ideal spanned by $(f_1, \ldots, f_i)$, with $I_0 = A$ and $I_r = I$.*

F is a regular sequence iff for all i,

$$\mathsf{HS}_{A/I_i}(\mathbf{T}) = (1-\mathbf{T^{d_1}})\mathsf{HS}_{A/I_{i-1}}(\mathbf{T}) = \frac{(1 - \mathbf{T^{d_1}}) \cdots (1 - \mathbf{T^{d_m}})}{(1 - \mathbf{T}^{W_{\bullet,1}}) \cdots (1 - \mathbf{T}^{W_{\bullet,n}})}.$$

F is a semi-regular sequence iff for all i,

$$\mathsf{HS}_{A/I_i}(\mathbf{T}) = \left\lfloor (1 - \mathbf{T^{d_1}})\mathsf{HS}_{A/I_{i-1}}(\mathbf{T}) \right\rfloor,$$

where $\lfloor S \rfloor$ is obtained by removing negative coefficients from S.

PROOF. The proof is similar to the usual case, so we only outline the argument. Let $c_{\mathbf{d},i} = \dim(A/I_i)_{\mathbf{d}}$. The multiplication map $\phi_i^{(\mathbf{d})}$ defines an exact sequence

$$0 \to K_{\mathbf{d}} \to (A/I_{i-1})_{\mathbf{d}} \to (A/I_{i-1})_{\mathbf{d}+\mathbf{d}_i} \to (A/I_i)_{\mathbf{d}+\mathbf{d}_i} \to 0$$

where $K_{\mathbf{d}} = \mathrm{Ker}(\phi_i^{(\mathbf{d})})$. This gives that

$$c_{\mathbf{d}+\mathbf{d}_i,i} = c_{\mathbf{d}+\mathbf{d}_i,i-1} - c_{\mathbf{d},i-1} - \dim(K_{\mathbf{d}}). \tag{3}$$

F is a regular sequence iff $K_{\mathbf{d}} = 0$. F is a semi-regular sequence if and only if either $K_{\mathbf{d}} = 0$ (if $\phi_i^{(\mathbf{d})}$ is injective) or $\dim_K((A/I_i)_{\mathbf{d}+\mathbf{d}_i}) = 0$ (if $\phi_i^{(\mathbf{d})}$ is surjective, i.e., $c_{\mathbf{d},i-1} - c_{\mathbf{d}+\mathbf{d}_i,i-1} \le 0$). In either case, multiplying Eq. (3) by $\mathbf{T}^{\mathbf{d}-\mathbf{d}_i}$ and summing over $\mathbf{d}$ gives the result. The closed form for the regular case follows by induction. $\square$

Remark 5.5. For semi-regular sequences, contrary to the usual case, it is not necessarily true that one can remove all coefficients after the *first* negative coefficient. As such, it is not clear whether it holds that

$$\mathsf{HS}_{A/I_i}(\mathbf{T}) = \left\lfloor \frac{(1 - \mathbf{T^{d_1}}) \cdots (1 - \mathbf{T^{d_m}})}{(1 - \mathbf{T}^{W_{\bullet,1}}) \cdots (1 - \mathbf{T}^{W_{\bullet,n}})} \right\rfloor.$$

Getting a closed form for the Hilbert series of semi-regular and weakly regular sequences, as well as extracting from those series complexity parameters such as the number of solutions or the degree of the polynomials in the Gröbner basis, will be the object of future research.

5.3 Genericity

THEOREM 5.6. Let $\mathbf{d}_1, \ldots, \mathbf{d}_r$ be pluridegrees such that there exists at least one regular (resp. semi-regular, resp. weakly regular) pluriweighted-homogeneous sequence of with respective pluridegree $\mathbf{d}_i$. Then, among pluriweighted-homogeneous sequences with respective pluridegree $\mathbf{d}_i$, regular (resp. semi-regular, resp. weakly regular) sequences are generic.

PROOF. In all three cases, the property is a statement on the Macaulay matrices built by Algorithm 1, at each pluridegree. Each of the coefficients, both before and after reducing to row-echelon form, can be written as a rational function in the coefficients of the input polynomial, in the Zariski-open set defined by the denominators not vanishing. So the Macaulay matrix has a generic shape.

All three properties state that the Macaulay matrices, after removing rows with the criteria, have full rank. Since the rank is upper semi-continuous, if there exists a system for which the Macaulay matrix at pluridegree $\mathbf{d}$ has maximal rank, so does the generic Macaulay matrix. This implies that the properties are generic. $\square$

In general, the properties are not always generic, as the third example in Table 1 shows. It is not unexpected, as it was already the case with weighted-homogeneous systems that regular sequences are not generic when there does not exist enough monomials at the wanted weighted degrees. Based on experiments, we expect however that it is possible to identify large classes of pluridegrees such that some of the regularity properties are generic, in particular for non-degenerate cases with $r \le n - k$.

5.4 Relation to the dimension

By analogy with the weighted- and multihomogeneous cases, it is possible to define a suitable geometric ambient space to talk about the dimension of a variety defined by a plurihomogeneous system. One defines the relation $\sim_{\mathbf{W}}$ on $K^n \setminus \{(0, \ldots, 0)\}$ defined by

$$\mathbf{x} \sim_{\mathbf{W}} \mathbf{y} \iff \exists \boldsymbol{\lambda} \in K^n \text{ s.t. } \forall i, y_i = \lambda_1^{w_{1,i}} \cdots \lambda_k^{w_{k,i}} x_i.$$

This relation does not depend on the choice of a plurisystem of weights equivalent to $\mathbf{W}$. The pluriweighted projective space $\mathbb{P}_{\mathbf{W}}(K)$ is then defined as

$$\mathbb{P}_{\mathbf{W}}(K) = K^n \setminus \{(0, \ldots, 0)\}/\sim_{\mathbf{W}},$$

and similarly, given a plurihomogeneous ideal, the pluriprojective variety $V_{\mathbf{W}}(I)$ is defined as $V(I) \setminus \{(0, \ldots, 0)\}/\sim_{\mathbf{W}}$. The natural notion of dimension is then

$$\dim(V_{\mathbf{W}}(I)) = \dim(V(I)) - k.$$

In particular, a pluriprojective variety with dimension 0 is a finite set of points modulo $\sim_{\mathbf{W}}$.

Since the notion of regularity does not depend on the grading, a regular sequence of length m defines a pluriprojective variety with dimension $n - m - k$ if $m \le n - k$, and an empty pluriprojective variety otherwise. It is not clear whether the same holds for semi-regular and weakly regular sequences.

6 EXPERIMENTAL RESULTS

We wrote a toy implementation[1] of Algorithm 1 specialized to the plurihomogeneous case. In the following table, we briefly report on the largest total degree reached, the number and size of the matrices, the number of parallelized steps, and the number of reductions to zero, for generic systems. We report it for three algorithms: Algorithm 1 with and without filtering out steps with a non-strivial GCD, and the state of the art (following the W_1-homogeneous structure). If the values in a row are indicated as ">", it means that the computation was run with the given value of $d_{\max}$, but that the result is not a Gröbner basis. The number of reductions to zero is the total number of reductions done by the algorithm with the two criteria, and the number in parentheses is the number of rank defects in the Macaulay matrices (syzygies which are neither trivial, semi-trivial or eliminable).

Beyond the usual signature criteria, it appears that eliminating steps with a non-trivial GCD eliminates quite a number of small matrices. The resulting speed-up is likely to be due to skipping the overhead of spawning parallel processes for trivial computations.

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[1] https://gitlab.com/thibaut.verron/plurihomo-gb

Table 1: Experimental data on generic plurihomogeneous systems

Weights	Pluridegree	Algo.	d_{max}	Max deg	#Steps	Max. #mtx/step	Max mtx size	Reds to zero	Time (s)
$\left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{smallmatrix}\right)$	2 pols, $(100, 50)$	Gcd filter	190	76	10	1	400	0	1.29
		No gcd filter	190	76	91	147	400	0	186.89
		Default	>150	>64	>51	1	>900k	>0	>754.36
$\left(\begin{smallmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 1 & 1 \end{smallmatrix}\right)$	2 pols, $(30, 30)$	Gcd filter	71	53	42	5	23k	0	11.14
		No gcd filter	71	53	42	65	23k	0	33.83
		Default	>45	>29	>16	1	>1100k	>0	>343.47
$\left(\begin{smallmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 1 & 1 \end{smallmatrix}\right)$	3 pols, $(30, 30)$	Gcd filter	85	44	56	6	80k	40 (9)	33.94
		No gcd filter	85	44	56	88	80k	40 (9)	88.91
		Default	>45	>34	>16	1	>1651k	>0	>517.19

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