# Working with valuations

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# Part 1: Signature Gröbner bases over Tate algebras

joint work with Xavier Caruso[1] and Tristan Vaccon[2]

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# Precision and Gröbner bases

- Question: in $\mathbb{R}[X]$, reduce $f = X^2$ modulo $g = 0.01X - 1$

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- It terminates, but...

- $g \simeq 1$, but $f \bmod g \not\simeq 0$

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\end{tikzcd}
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- Another way?

$$f = X^2 \xrightarrow{+X^2 g} 0.01X^3 \xrightarrow{+0.01X^3 g} 0.0001X^4 \xrightarrow{\cdots} \cdots$$

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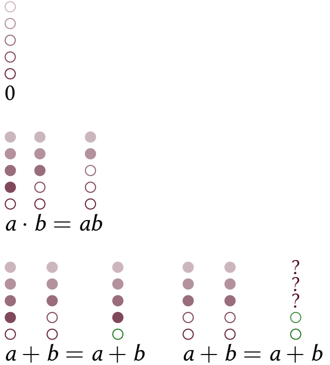
- This work: make sense of this process for convergent power series in $\mathbb{Z}_p[[X]]$

# Valued fields and rings: basic definitions

Valuation: function val : $k \to \mathbb{Z} \cup \{\infty\}$ with:

- $\mathrm{val}(a) = \infty \iff a = 0$
- $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$
- $\mathrm{val}(a + b) \geq \min(\mathrm{val}(a), \mathrm{val}(b))$




Examples: $1$ $\quad \pi \quad$ $a = a_3\pi^3 + a_4\pi^4 + \dots$ $\quad$ $\mathrm{val}(a) = 3$

$b = b_{-3}\pi^{-3} + b_{-2}\pi^{-2} + \dots$ $\quad$ $\mathrm{val}(b) = -3$

# Examples of valued fields and rings

| Ring $K^\circ$ $\underset{\text{val} \geq 0}{\overset{\text{Frac}}{\rightleftarrows}}$ Field $K$ | | Uniformizer $\pi$ | Residue field $K^\circ/\pi$ |
|---|---|---|---|
| $\mathbb{Z}_{(p)}$ | $\mathbb{Q}$ | $p$ prime | $\mathbb{F}_p$ |
| $\mathbb{Z}_p$ | $\mathbb{Q}_p$ | $p$ prime | $\mathbb{F}_p$ |
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- Metric and topology defined by "$a$ is small" $\iff$ "$\mathrm{val}(a)$ is large"
- In a complete valuation ring, a series is convergent iff its general term goes to 0:

$$\sum_{n=0}^{0} a_n = a_0$$

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## Tate Series

$\mathbf{X} = X_1, \ldots, X_n$

### Definition

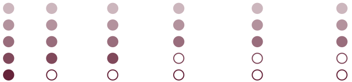
- $K\{\mathbf{X}\}^\circ$ = ring of series in $\mathbf{X}$ with coefficients in $K^\circ$ converging for all $\mathbf{x} \in K^\circ$
  = ring of power series whose general coefficients tend to 0

### Motivation

- Introduced by Tate in 1971 for rigid geometry
  ($p$-adic equivalent of the bridge between algebraic and analytic geometry over $\mathbb{C}$)

### Examples

- Polynomials (finite sums are convergent)



- $\displaystyle\sum_{i,j=0}^{\infty} \pi^{i+j} X^i Y^j = 1 + \pi X + \pi Y + \pi^2 X^2 + \pi^2 XY + \pi^2 Y^2 + \cdots$

- Not a Tate series: $\displaystyle\sum_{i=0}^{\infty} X^i = 1 + 1X + 1X^2 + 1X^3 + \cdots$

# Term ordering for Tate algebras

$\mathbf{X}^{\mathbf{i}} = X_1^{i_1} \cdots X_n^{i_n}$

- Starting from a usual monomial ordering $1 <_m \mathbf{X}^{\mathbf{i}_1} <_m \mathbf{X}^{\mathbf{i}_2} <_m \ldots$
- We define a term ordering putting more weight on large coefficients

Usual term ordering:

$$\pi \cdot 1 <_m 1\,\mathbf{X}^{\mathbf{i}_1} <_m \pi\,\mathbf{X}^{\mathbf{i}_2} <_m \pi^2\,\mathbf{X}^{\mathbf{i}_3} <_m \cdots$$

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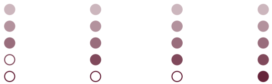

$$\cdots < \pi^2\,\mathbf{X}^{\mathbf{i}_3} < \pi \cdot \underbrace{1 < \pi\,\mathbf{X}^{\mathbf{i}_2}}_{<_m} < 1\,\mathbf{X}^{\mathbf{i}_1} < \cdots$$

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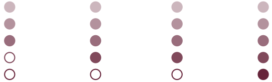

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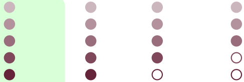
- It has infinite descending chains, but they converge to zero
- Tate series always have a leading term

$\mathrm{LT}(f)$



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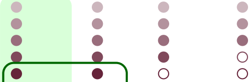
$\mathrm{LT}(f)$



$$f = \boxed{a_2 XY + a_1 X} + a_0 \cdot 1 + a_3 X^2 Y^2 + \dots$$

$$\overline{f} = \overline{a_2} XY + \overline{a_1} X$$

- Isomorphism
$$\begin{array}{rcl} K\{\mathbf{X}\}^\circ / \langle \pi \rangle & \simeq & \mathbb{F}[\mathbf{X}] \\ f & \mapsto & \overline{f} \end{array}$$

  compatible with the term order

# Gröbner bases

- Standard definition once the term order is defined:

$$G \text{ is a Gröbner basis of } I \iff \text{for all } f \in I, \text{ there is } g \in G \text{ s.t. } \mathrm{LT}(g) \text{ divides } \mathrm{LT}(f)$$

- Standard equivalent characterizations:
  1. $G$ is a Gröbner basis of $I$
  2. for all $f \in I$, $f$ is reducible modulo $G$
  3. for all $f \in I$, $f$ reduces to zero modulo $G$ $\exists$ sequence of reductions converging to $0$

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  If $I$ is saturated: — $\pi f \in I \implies f \in I$

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- Every Tate ideal has a finite Gröbner basis
- It can be computed using the usual algorithms (reduction, Buchberger, $F_4$)
- In practice, the algorithms run with finite precision and without loss of precision

No division by $\pi$

# Buchberger's algorithm

$G \leftarrow \{f_1, \ldots, f_m\}$<br>
$B \leftarrow \{$S-pol of $g_1$ and $g_2$ for $g_1, g_2 \in G\}$<br>
While $B \neq \varnothing$:<br>
    Pop $v$ from $B$<br>
    $w \leftarrow$ reduction of $v$ modulo $G$<br>
    If $w = 0$:<br>
        Pass<br>
    Else:<br>
        $B \leftarrow B \cup \{$S-pol of $w$ and $g$ for $g \in G\}$<br>
        $G \leftarrow G \cup \{w\}$<br>
Return $G$

# Why signatures?

Problem: useless and redundant computations, infinite reductions to 0

Example with a S-polynomial

$p = p_1 f_1 + p_2 f_2 + \cdots + p_k f_k + \cdots + p_m f_m$ $\qquad$ $q = q_1 f_1 + q_2 f_2 + \cdots + q_l f_l + \cdots + q_m f_m$

$\text{S-Pol}(p, q) = \mu p - \nu q$

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- $1^{\text{st}}$ idea: keep track of the representation of the ideal elements
  [Möller, Mora, Traverso 1992]

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$$\mathbf{p} = p_1 \mathbf{e}_1 + p_2 \mathbf{e}_2 + \cdots + p_k \mathbf{e}_k + \cdots + p_m \mathbf{e}_m \qquad \mathbf{q} = q_1 \mathbf{e}_1 + q_2 \mathbf{e}_2 + \cdots + q_l \mathbf{e}_l + \cdots + q_m \mathbf{e}_m$$

$$\text{S-Pol}(p, q) = \mu p - \nu q$$

$$\text{S-Pol}(\mathbf{p}, \mathbf{q}) = \mu\left(p_1 \mathbf{e}_1 + \cdots + p_k \mathbf{e}_k + \cdots + p_m \mathbf{e}_m\right) - \nu\left(q_1 \mathbf{e}_1 + \cdots + q_l \mathbf{e}_l + \cdots + q_m \mathbf{e}_m\right)$$

# Why signatures?

Problem: useless and redundant computations, infinite reductions to 0

- 1st idea: keep track of the representation of the ideal elements
  [Möller, Mora, Traverso 1992]
- 2nd idea: the largest term of the representation is enough
  [Faugère 2002 ; Gao, Volny, Wang 2010 ; Arri, Perry 2011... Eder, Faugère 2017]

## Example with a S-polynomial

$$p = p_1 f_1 + p_2 f_2 + \cdots + p_k f_k + \cdots + 0\, f_m \qquad q = q_1 f_1 + q_2 f_2 + \cdots + q_l f_l + \cdots + 0\, f_m$$

$$\begin{aligned} \mathbf{p} &= p_1 \mathbf{e}_1 + p_2 \mathbf{e}_2 + \cdots + p_k \mathbf{e}_k + \cdots + 0\, \mathbf{e}_m \\ &= \mathrm{LT}(p_k)\mathbf{e}_k + \text{smaller terms} \end{aligned} \qquad \begin{aligned} \mathbf{q} &= q_1 \mathbf{e}_1 + q_2 \mathbf{e}_2 + \cdots + q_l \mathbf{e}_l + \cdots + 0\, \mathbf{e}_m \\ &= \mathrm{LT}(q_l)\mathbf{e}_l + \text{smaller terms} \end{aligned}$$

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$$\begin{aligned} \text{S-Pol}(\mathbf{p}, \mathbf{q}) &= \mu\,(p_1 \mathbf{e}_1 + \cdots + p_k \mathbf{e}_k + \cdots + 0\, \mathbf{e}_m) - \nu\,(q_1 \mathbf{e}_1 + \cdots + q_l \mathbf{e}_l + \cdots + 0\, \mathbf{e}_m) \\ &= \mu \mathrm{LT}(p_k)\mathbf{e}_k - \nu \mathrm{LT}(q_l)\mathbf{e}_l + \text{smaller terms} \end{aligned}$$

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$$p = p_1 f_1 + p_2 f_2 + \cdots + p_k f_k + \cdots + 0\, f_m \qquad q = q_1 f_1 + q_2 f_2 + \cdots + q_l f_l + \cdots + 0\, f_m$$

$$\mathbf{p} = p_1 \mathbf{e}_1 + p_2 \mathbf{e}_2 + \cdots + p_k \mathbf{e}_k + \cdots + 0\, \mathbf{e}_m \qquad \mathbf{q} = q_1 \mathbf{e}_1 + q_2 \mathbf{e}_2 + \cdots + q_l \mathbf{e}_l + \cdots + 0\, \mathbf{e}_m$$

$$= \mathrm{LT}(p_k)\mathbf{e}_k + \text{smaller terms} \qquad = \mathrm{LT}(q_l)\mathbf{e}_l + \text{smaller terms}$$

$$\text{S-Pol}(p, q) = \mu p - \nu q$$

$$\begin{aligned} \text{S-Pol}(\mathbf{p}, \mathbf{q}) &= \mu \left(p_1 \mathbf{e}_1 + \cdots + p_k \mathbf{e}_k + \cdots + 0\, \mathbf{e}_m\right) - \nu \left(q_1 \mathbf{e}_1 + \cdots + q_l \mathbf{e}_l + \cdots + 0\, \mathbf{e}_m\right) \\ &= \mu \mathrm{LT}(p_k)\mathbf{e}_k - \nu \mathrm{LT}(q_l)\mathbf{e}_l + \text{smaller terms} \\ &= \mu \mathrm{LT}(p_k)\mathbf{e}_k + \text{smaller terms} \quad \text{if } \mu \mathrm{LT}(p_k)\mathbf{e}_k \gneqq \nu \mathrm{LT}(q_l)\mathbf{e}_l \end{aligned}$$

# Why signatures?

Problem: useless and redundant computations, infinite reductions to 0

- $1^{\text{st}}$ idea: keep track of the representation of the ideal elements
  [Möller, Mora, Traverso 1992]
- $2^{\text{nd}}$ idea: the largest term of the representation is enough
  [Faugère 2002 ; Gao, Volny, Wang 2010 ; Arri, Perry 2011... Eder, Faugère 2017]

## Example with a S-polynomial

$$p = p_1 f_1 + p_2 f_2 + \cdots + p_k f_k + \cdots + 0\, f_m$$
$$\mathbf{p} = p_1 \mathbf{e}_1 + p_2 \mathbf{e}_2 + \cdots + p_k \mathbf{e}_k + \cdots + 0\, \mathbf{e}_m$$
$$= \mathrm{LT}(p_k)\mathbf{e}_k + \text{smaller terms}$$

$\mathfrak{s}(p) =$ signature of $p$

$$q = q_1 f_1 + q_2 f_2 + \cdots + q_l f_l + \cdots + 0\, f_m$$
$$\mathbf{q} = q_1 \mathbf{e}_1 + q_2 \mathbf{e}_2 + \cdots + q_l \mathbf{e}_l + \cdots + 0\, \mathbf{e}_m$$
$$= \mathrm{LT}(q_l)\mathbf{e}_l + \text{smaller terms}$$

$$\text{S-Pol}(p, q) = \mu p - \nu q$$
$$\text{S-Pol}(\mathbf{p}, \mathbf{q}) = \mu\left(p_1 \mathbf{e}_1 + \cdots + p_k \mathbf{e}_k + \cdots + 0\, \mathbf{e}_m\right) - \nu\left(q_1 \mathbf{e}_1 + \cdots + q_l \mathbf{e}_l + \cdots + 0\, \mathbf{e}_m\right)$$
$$= \mu \mathrm{LT}(p_k)\mathbf{e}_k - \nu \mathrm{LT}(q_l)\mathbf{e}_l + \text{smaller terms}$$
$$= \mu \mathrm{LT}(p_k)\mathbf{e}_k + \text{smaller terms} \quad \text{if } \mu \mathrm{LT}(p_k)\mathbf{e}_k \gneq \nu \mathrm{LT}(q_l)\mathbf{e}_l$$

Regular S-polynomial

# Buchberger's algorithm, with signatures

$G \leftarrow \{(\mathbf{e}_1, f_1), \dots, (\mathbf{e}_m, f_m)\}$
$B \leftarrow \{$S-pol of $p_1$ and $p_2$ for $p_1, p_2 \in G\}$
While $B \neq \varnothing$:
    Pop $(\mathbf{u}, v)$ from $B$ with smallest $\mathbf{u}$
    $w \leftarrow$ regular reduction of $(\mathbf{u}, v)$ modulo $G$
    If $w = 0$:
        Pass
    Else:
        $B \leftarrow B \cup \{$regular S-pol of $(\mathbf{u}, w)$ and $p$ for $p \in G\}$
        $G \leftarrow G \cup \{(\mathbf{u}, w)\}$
Return $G$

# Buchberger's algorithm, with signatures

1. $G \leftarrow \{(\mathbf{e}_1, f_1), \dots, (\mathbf{e}_m, f_m)\}$
2. $B \leftarrow \{$S-pol of $p_1$ and $p_2$ for $p_1, p_2 \in G\}$
3. While $B \neq \varnothing$:
4. Pop $(\mathbf{u}, v)$ from $B$ with smallest $\mathbf{u}$
5. $w \leftarrow$ regular reduction of $(\mathbf{u}, v)$ modulo $G$
6. If $w = 0$:
7. Pass
8. Else:
9. $B \leftarrow B \cup \{$regular S-pol of $(\mathbf{u}, w)$ and $p$ for $p \in G\}$
10. $G \leftarrow G \cup \{(\mathbf{u}, w)\}$
11. Return $G$

Need to order the signatures!

# Signature orderings

Signature orderings:

- Necessary for correctness and termination of the algorithms
- Different choices lead to different performances

Examples (polynomial case):

- $\mu \mathbf{e}_i <_{\text{pot}} \nu \mathbf{e}_j \iff i < j$, or if equal, $\mu < \nu$

  Position over Term

- $\mu \mathbf{e}_i <_{\text{top}} \nu \mathbf{e}_j \iff \mu < \nu$, or if equal, $i < j$

  Term over Position

- $\mu \mathbf{e}_i <_{\text{dopot}} \nu \mathbf{e}_j \iff \deg(p) < \deg(q)$, or if equal, $i < j$, or if equal, $\mu < \nu$

  Degree over Position over Term

# Signature orderings

Signature orderings:

- Necessary for correctness and termination of the algorithms
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Examples (polynomial case):

- $\mu\mathbf{e}_i <_{\text{pot}} \nu\mathbf{e}_j \iff i < j$, or if equal, $\mu < \nu$

  Position over Term

  - Theoretically convenient
  - Incremental
  - Rarely the most efficient

- $\mu\mathbf{e}_i <_{\text{top}} \nu\mathbf{e}_j \iff \mu < \nu$, or if equal, $i < j$

  Term over Position

  - Better in practice
  - Theoretically complicated

- $\mu\mathbf{e}_i <_{\text{dopot}} \nu\mathbf{e}_j \iff \deg(p) < \deg(q)$, or if equal, $i < j$, or if equal, $\mu < \nu$

  Degree over Position over Term

  - "F5-ordering" for homogeneous systems and degree order
  - Avoid going too high in degree, still incremental
  - Best of both worlds

# Buchberger's algorithm, incremental variant

$Q \leftarrow (f_1, \ldots, f_m)$
$G \leftarrow \varnothing$
For $f \in Q$
    $G \leftarrow G \cup \{f\}$
    $B \leftarrow \{$S-pol of $f$ and $g$ for $g \in G\}$
    While $B \neq \varnothing$:
        Pop $v$ from $B$
        $w \leftarrow$ reduction of $v$ modulo $G$
        If $w = 0$:
            Pass
        Else:
            $B \leftarrow B \cup \{$S-pol of $w$ and $g$ for $g \in G\}$
            $G \leftarrow G \cup \{w\}$
Return $G$

# Signature orderings for Tate series

Signature orderings:

- Necessary for correctness and termination of the algorithms
- Different choices lead to different performances
- Difficulty with Tate series: multiplying by $\pi$ should decrease the signature

Orders for Tate series:

- $\mu \mathbf{e}_i <_{\text{pot}} \nu \mathbf{e}_j \iff i < j$, or if equal, $\mu < \nu$

  Position over Term

- Theoretically convenient
- Incremental
- Rarely the most efficient

- $\mu \mathbf{e}_i <_{\text{top}} \nu \mathbf{e}_j \iff \mu < \nu$, or if equal, $i < j$

  Term over Position

- Better in practice
- Theoretically complicated

# Signature-based algorithm, PoT ordering

1. $Q \leftarrow (f_1, \ldots, f_m)$
2. $G \leftarrow \varnothing$
3. For $f \in Q$
4. $G_S \leftarrow \{(0, g) : g \in G_S\} \cup \{(1, f)\}$
5. $B \leftarrow \{$S-pol of $(1, f)$ and $p$ for $p \in G_S\}$
6. While $B \neq \varnothing$:
7. Pop $(u, v)$ from $B$ with smallest $u$
8. $w \leftarrow$ regular reduction of $(u, v)$ modulo $G_S$
9. If $w = 0$:
10. Pass
11. Else:
12. $B \leftarrow B \cup \{$regular S-pol of $(u, w)$ and $p$ for $p \in G_S\}$
13. $G_S \leftarrow G_S \cup \{(u, w)\}$
14. $G \leftarrow \{v : (u, v) \in G_S\}$
15. Return $G$

# Signature orderings for Tate series

Signature orderings:

- Necessary for correctness and termination of the algorithms
- Different choices lead to different performances
- Difficulty with Tate series: multiplying by $\pi$ should decrease the signature

Orders for Tate series:

- $\mu \mathbf{e}_i <_{\text{pot}} \nu \mathbf{e}_j \iff i < j$, or if equal, $\mu < \nu$

  Position over Term

  - Theoretically convenient
  - Incremental
  - Rarely the most efficient

- $\mu \mathbf{e}_i <_{\text{top}} \nu \mathbf{e}_j \iff \mu < \nu$, or if equal, $i < j$

  Term over Position

  - Better in practice
  - Theoretically complicated

- $\mu \mathbf{e}_i <_{\text{vopot}} \nu \mathbf{e}_j \iff \mathrm{val}(p) < \mathrm{val}(q)$, or if equal, $i < j$, or if equal, $\mu < \nu$

  Valuation over Position over Term

  - Analogue of the F5 ordering for the valuation
  - Allows to delay (or avoid) high valuation computations

# Signature-based algorithm, VoPoT ordering

1. $Q \leftarrow (f_1, \ldots, f_m)$
2. $G \leftarrow \varnothing$
3. While $\exists f \in Q$ with smallest valuation:
4. $G_S \leftarrow \{(0, g) : g \in G_S\} \cup \{(1, f)\}$
5. $B \leftarrow \{$S-pol of $(1, f)$ and $p$ for $p \in G_S\}$
6. While $B \neq \varnothing$:
7. Pop $(u, v)$ from $B$ with smallest $u$
8. $w \leftarrow$ regular reduction of $(u, v)$ modulo $G_S$
9. If $\mathrm{val}(w) > \mathrm{val}(f)$:
10. $Q \leftarrow Q \cup w$
11. Else:
12. $B \leftarrow B \cup \{$regular S-pol of $(u, w)$ and $p$ for $p \in G_S\}$
13. $G_S \leftarrow G_S \cup \{(u, w)\}$
14. $G \leftarrow \{v : (u, v) \in G_S\}$
15. Return $G$

# Conclusion and perspectives

Results:

- Possible to design signature-based algorithms for Tate algebras
- Two algorithms with two orders
- Implemented in Sage, working towards including them in the distribution

Future work:

- Reduction of Tate series is very different from reduction of polynomials
- Design algorithms to perform those reductions more efficiently
- Goal: being able to take advantage of e.g. delaying reductions in VoPoT

References:


- Caruso, Vaccon and Verron, ‘Gröbner bases over Tate algebras’ (2019)
- Caruso, Vaccon and Verron, ‘Signature-based algorithms for Gröbner bases over Tate algebras’ (2020) [preprint]

# Part 2: Integral bases of P-recursive sequences

joint work with Shaoshi Chen[1], Lixin Du[1,2] and Manuel Kauers[2]

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Séminaire Calcul Formel, Limoges

27 février 2020

# What does it mean to be integral?

| Ring $K^\circ$ $\underset{\text{val} \geq 0}{\overset{\text{Frac}}{\rightleftarrows}}$ | Field $K$ | Valuation at | |
|---|---|---|---|
| $\mathbb{Z}_p$ | $\mathbb{Q}_p$ | $p$ prime | Local integrality |
| $\mathbb{C}[[x-\alpha]]$ | $\mathbb{C}((x-\alpha))$ | $x-\alpha$ | |

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| $\mathbb{C}[[x-\alpha]]$ | $\mathbb{C}((x-\alpha))$ | $x - \alpha$ | |
| $\mathbb{Z}$ Denom. $\in \{\pm 1\}$ | $\mathbb{Q}$ | All primes | Global integrality |
| $\mathbb{C}[x]$ Denom. $\in \mathbb{C}$ | $\mathbb{C}(x)$ | All $x - \alpha$ | |

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| $\mathbb{C}[x]$ | Denom. $\in \mathbb{C}$ | $\mathbb{C}(x)$ | All $x-\alpha$ | |
| $\mathcal{O}_{\mathbb{C}[X]}$ | Denom. of minpoly $\in \mathbb{C}$ | $\overline{\mathbb{C}(x)}$ | All $x-\alpha$ | |

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| | | $\left\{\begin{matrix}\text{Sol. of diffeq}\\ \text{with poly. coefs.}\end{matrix}\right\}$ | | |

# What does it mean to be integral?

$$\text{Ring } K^\circ \;\underset{\text{val} \geq 0 \iff \text{“no pole”}}{\overset{\text{Frac}}{\rightleftarrows}}\; \text{Field } K$$

| Ring $K^\circ$ | Field $K$ | Valuation at | |
|---|---|---|---|
| $\mathbb{Z}_p$ | $\mathbb{Q}_p$ | $p$ prime | Local integrality |
| $\mathbb{C}[[x-\alpha]]$ | $\mathbb{C}((x-\alpha))$ | $x-\alpha$ | |
| $\mathbb{Z}$ Denom. $\in \{\pm 1\}$ | $\mathbb{Q}$ | All primes | Global integrality |
| $\mathbb{C}[x]$ Denom. $\in \mathbb{C}$ | $\mathbb{C}(x)$ | All $x-\alpha$ | |
| $\mathcal{O}_{\mathbb{C}[X]}$ Denom. of minpoly $\in \mathbb{C}$ $\iff$ “no pole” | $\overline{\mathbb{C}(x)}$ | All $x-\alpha$ | |
| | $\left\{\begin{array}{c}\text{Sol. of diffeq} \\ \text{with poly. coefs.}\end{array}\right\}$ | | |

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| $\mathcal{O}_{\mathbb{C}[X]}$ | Denom. of minpoly $\in \mathbb{C}$ $\iff$ “no pole” | $\overline{\mathbb{C}(x)}$ | All $x-\alpha$ | |
| “No pole” [Kauers Koutschan 2015] | | $\left\{\begin{array}{c}\text{Sol. of diffeq}\\ \text{with poly. coefs.}\end{array}\right\}$ | All $x-\alpha$ | |

# What does it mean to be integral?

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| | | | | |
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| $\mathbb{Z}_p$ | | $\mathbb{Q}_p$ | $p$ prime | Local integrality |
| $\mathbb{C}[[x-\alpha]]$ | | $\mathbb{C}((x-\alpha))$ | $x-\alpha$ | |
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| $\mathcal{O}_{\mathbb{C}[X]}$ | Denom. of minpoly $\in \mathbb{C}$ $\iff$ “no pole” | $\overline{\mathbb{C}(x)}$ | All $x-\alpha$ | |
| “No pole” [Kauers Koutschan 2015] | | $\left\{\begin{matrix}\text{Sol. of diffeq} \\ \text{with poly. coefs.}\end{matrix}\right\}$ | All $x-\alpha$ | |
| This work : “val $\geq 0$” | | $\left\{\begin{matrix}\text{Sol. of rec.} \\ \text{with poly. coefs.}\end{matrix}\right\}$ | | |

# Better framework: integral operators

Polynomial algebras:

- "Algebraic equations": $C(x)[y]$, commutative: $xy = yx$

Algebraic case (finite extension):

- Given $\alpha(x) \in \overline{C(x)}$ or equivalently given $P \in C[x][y]$
- Question: What are integral elements of $C(x)(\alpha) = C(x)[y]/\langle P \rangle$?
- Answer: $Q$ is integral iff for all $\alpha(x)$ sol of $P$, $(Q(\alpha))(x)$ does not have any pole
- Integral elements form a $C[x]$-algebra in $C(x)[y]$

Can we compute a basis of that set as a $C[x]$-module?

- Yes: Trager's algorithm, van Hoeij's algorithm
- Application: computation of integrals [Trager 1984]

# Better framework: integral operators

Polynomial and Ore algebras:

- "Algebraic equations": $C(x)[y]$, commutative: $xy = yx$
- "Differential equations": $C(x)\langle D\rangle$, non-commutative: $Dx = xD + 1$

Differential case:

- Given $L \in C[x]\langle D\rangle$
- Question: What are integral elements of $C(x)\langle D\rangle/\langle L\rangle$?
- Answer: $B$ is integral iff for all $\alpha(x)$ sol of $L$, $(B \cdot \alpha))(x)$ does not have any pole
- Integral elements form a $C[x]$-module in $C(x)\langle D\rangle$

Can we compute a basis of that $C[x]$-module?

- Yes: adaptation of van Hoeij's algorithm [Kauers, Koutschan 2015]
- Application: computation of integrals [Chen, van Hoeij, Kauers, Koutschan 2018]

# Better framework: integral operators

Polynomial and Ore algebras:

- “Algebraic equations”: $C(x)[y]$, commutative: $xy = yx$
- “Differential equations”: $C(x)\langle D\rangle$, non-commutative: $Dx = xD + 1$
- “Recurrence equations”: $C(n)\langle S\rangle$, non-commutative: $Sn = (n+1)S$

Recurrence case:

- Given $L \in C[x]\langle S\rangle$
- Question: What are integral elements of $C(x)\langle S\rangle/\langle L\rangle$?
- Answer: $B$ is integral iff for all $\alpha(x)$ sol of $L$, $(B \cdot \alpha))(x)$ ... ???

# Better framework: integral operators

Polynomial and Ore algebras:

- "Algebraic equations": $C(x)[y]$, commutative: $xy = yx$
- "Differential equations": $C(x)\langle D\rangle$, non-commutative: $Dx = xD + 1$
- "Recurrence equations": $C(n)\langle S\rangle$, non-commutative: $Sn = (n+1)S$

Recurrence case:

- Given $L \in C[x]\langle S\rangle$
- Question: What are integral elements of $C(x)\langle S\rangle/\langle L\rangle$?
- Answer: $B$ is integral iff for all $\alpha(x)$ sol of $L$, $(B \cdot \alpha))(x)$ has "valuation" $\geq 0$ everywhere
- Integral elements form a $C[x]$-module in $C(x)\langle D\rangle$

Can we compute a basis of that $C[x]$-module?

- Yes: adaptation of van Hoeij's algorithm [Chen, Du, Kauers, V. 2020]
- Application: computation of sums?

# Van Hoeij's algorithm for finding integral bases (differential case)

Local algorithm:

Input. $L \in C[x]\langle D\rangle$ with order $r$, $\alpha \in C$

Output. $B_1, \dots, B_r$ basis of $C(x)\langle D\rangle/\langle L\rangle$ integral at $\alpha$

1. $B_1, \dots, B_r \leftarrow$ basis of $C(x)\langle D\rangle/\langle L\rangle$
2. For $d \in \{1, \dots, r\}$:
3.     While $B_i$ is not integral at $\alpha$
4.         $B_i \leftarrow (x-\alpha)B_i$
5.     While there exists $a_1, \dots, a_{d-1} \in C$
   such that $A := \frac{1}{x-\alpha}(a_1B_1 + \cdots + a_{d-1}B_{d-1} - B_d)$ is integral at $\alpha$
6.         $B_d \leftarrow A$
7. Return $B_1, \dots, B_r$

## Van Hoeij's algorithm for finding integral bases (differential case)

Local algorithm:

**Input.** $L \in C[x]\langle D\rangle$ with order $r$, $\alpha \in C$

**Output.** $B_1, \ldots, B_r$ basis of $C(x)\langle D\rangle/\langle L\rangle$ integral at $\alpha$

1. $B_1, \ldots, B_r \leftarrow$ basis of $C(x)\langle D\rangle/\langle L\rangle$ $\qquad$ $B_1, \ldots, B_r \leftarrow 1, \ldots, D^{r-1}$
2. For $d \in \{1, \ldots, r\}$:
3. $\quad$ While $B_i$ is not integral at $\alpha$
4. $\quad\quad$ $B_i \leftarrow (x - \alpha)B_i$
5. $\quad$ While there exists $a_1, \ldots, a_{d-1} \in C$
   $\quad$ such that $A := \frac{1}{x-\alpha}(a_1 B_1 + \cdots + a_{d-1}B_{d-1} - B_d)$ is integral at $\alpha$
6. $\quad\quad$ $B_d \leftarrow A$
7. Return $B_1, \ldots, B_r$

# Van Hoeij's algorithm for finding integral bases (differential case)

$f_1, \ldots, f_r \in C((x-\alpha))$ basis of solutions of $L$

Local algorithm:

Input. $L \in C[x]\langle D\rangle$ with order $r$, $\alpha \in C$

Output. $B_1, \ldots, B_r$ basis of $C(x)\langle D\rangle/\langle L\rangle$ integral at $\alpha$

1. $B_1, \ldots, B_r \leftarrow$ basis of $C(x)\langle D\rangle/\langle L\rangle$ $\quad$ $B_1, \ldots, B_r \leftarrow 1, \ldots, D^{r-1}$
2. For $d \in \{1, \ldots, r\}$:
3. While $B_i$ is not integral at $\alpha$ $\quad$ $B_i \cdot f_j$ has a pole at $\alpha$ for some $j$
4. $B_i \leftarrow (x-\alpha)B_i$
5. While there exists $a_1, \ldots, a_{d-1} \in C$ such that $A := \frac{1}{x-\alpha}(a_1B_1 + \cdots + a_{d-1}B_{d-1} - B_d)$ is integral at $\alpha$
6. $B_d \leftarrow A$
7. Return $B_1, \ldots, B_r$

# Van Hoeij's algorithm for finding integral bases (differential case)

Local algorithm:

$f_1, \ldots, f_r \in C((x-\alpha))$
basis of solutions of $L$

Input. $L \in C[x]\langle D\rangle$ with order $r$, $\alpha \in C$

Output. $B_1, \ldots, B_r$ basis of $C(x)\langle D\rangle/\langle L\rangle$ integral at $\alpha$

1. $B_1, \ldots, B_r \leftarrow$ basis of $C(x)\langle D\rangle/\langle L\rangle$ — $B_1, \ldots, B_r \leftarrow 1, \ldots, D^{r-1}$
2. For $d \in \{1, \ldots, r\}$:
3. While $B_i$ is not integral at $\alpha$ — $B_i \cdot f_j$ has a pole at $\alpha$ for some $j$
4. $B_i \leftarrow (x-\alpha)B_i$
5. While there exists $a_1, \ldots, a_{d-1} \in C$
   such that $A := \frac{1}{x-\alpha}(a_1B_1 + \cdots + a_{d-1}B_{d-1} - B_d)$ is integral at $\alpha$
6. $B_d \leftarrow A$
7. Return $B_1, \ldots, B_r$

$$\iff \forall j,\ (a_1B_1 + \cdots + a_{d-1}B_{d-1} - B_d)\cdot f_j(\alpha) = 0$$
$$\iff \forall j,\ a_1B_1 \cdot f_j(\alpha) + \ldots a_{d-1}B_{d-1}\cdot f_j(\alpha) = B_d \cdot f_j(\alpha)$$

Linear system of equations

# Van Hoeij's algorithm for finding integral bases (differential case)

Local algorithm:

$f_1, \ldots, f_r \in C((x-\alpha))$
basis of solutions of $L$

Input. $L \in C[x]\langle D \rangle$ with order $r$, $\alpha \in C$

Output. $B_1, \ldots, B_r$ basis of $C(x)\langle D \rangle / \langle L \rangle$ integral at $\alpha$

1. $B_1, \ldots, B_r \leftarrow$ basis of $C(x)\langle D \rangle / \langle L \rangle$ $\quad B_1, \ldots, B_r \leftarrow 1, \ldots, D^{r-1}$
2. For $d \in \{1, \ldots, r\}$:
3. $\quad$ While $B_i$ is not integral at $\alpha$ $\quad B_i \cdot f_j$ has a pole at $\alpha$ for some $j$
4. $\quad\quad B_i \leftarrow (x - \alpha) B_i$
5. $\quad$ While there exists $a_1, \ldots, a_{d-1} \in C$

   such that $A := \frac{1}{x-\alpha}(a_1 B_1 + \cdots + a_{d-1} B_{d-1} - B_d)$ is integral at $\alpha$
6. $\quad\quad B_d \leftarrow A$
7. Return $B_1, \ldots, B_r$

$$\iff \forall j,\ (a_1 B_1 + \cdots + a_{d-1} B_{d-1} - B_d) \cdot f_j(\alpha) = 0$$
$$\iff \forall j,\ a_1 B_1 \cdot f_j(\alpha) + \ldots a_{d-1} B_{d-1} \cdot f_j(\alpha) = B_d \cdot f_j(\alpha)$$

Linear system of equations

Global algorithm: loop over all $\alpha \in C$

# Van Hoeij's algorithm for finding integral bases (differential case)

Local algorithm:

$f_1, \ldots, f_r \in C((x-\alpha))$
basis of solutions of $L$

Input. $L \in C[x]\langle D \rangle$ with order $r$, $\alpha \in C$

Output. $B_1, \ldots, B_r$ basis of $C(x)\langle D \rangle / \langle L \rangle$ integral at $\alpha$

1. $B_1, \ldots, B_r \leftarrow$ basis of $C(x)\langle D \rangle / \langle L \rangle$ — $B_1, \ldots, B_r \leftarrow 1, \ldots, D^{r-1}$
2. For $d \in \{1, \ldots, r\}$:
3. While $B_i$ is not integral at $\alpha$ — $B_i \cdot f_j$ has a pole at $\alpha$ for some $j$
4. $B_i \leftarrow (x - \alpha) B_i$
5. While there exists $a_1, \ldots, a_{d-1} \in C$ such that $A := \frac{1}{x-\alpha}(a_1 B_1 + \cdots + a_{d-1} B_{d-1} - B_d)$ is integral at $\alpha$
6. $B_d \leftarrow A$
7. Return $B_1, \ldots, B_r$

$$\iff \forall j,\ (a_1 B_1 + \cdots + a_{d-1} B_{d-1} - B_d) \cdot f_j(\alpha) = 0$$
$$\iff \forall j,\ a_1 B_1 \cdot f_j(\alpha) + \ldots a_{d-1} B_{d-1} \cdot f_j(\alpha) = B_d \cdot f_j(\alpha)$$

Linear system of equations

Global algorithm: loop over all $\alpha \in C$

Nothing happens at all but finitely many of them
(roots of the leading coefficient of $L$)

# Finding solutions of P-recursive sequences

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3 \in \mathbb{C}[n]\langle S\rangle$ with $Sn = (n+1)S$

- Natural action: $L$ acts on $\mathbb{C}^{\mathbb{Z}}$ via $(n \cdot u)_k = ku_k$, $(S \cdot u)_k = u_{k+1}$

If $(u_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is a solution, then for all $n \in \mathbb{Z}$:

- $(n-1)u_{n+3} = -(n-3)u_{n+1} - (n-1)(n+1)u_n$

| $\cdots$ | $-1$ | 0 | 1 | 2 | 3 | 4 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| | | 1 | 0 | 0 | | | |
| | | 0 | 1 | 0 | | | |
| | | 0 | 0 | 1 | | | |

# Finding solutions of P-recursive sequences

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3 \in \mathbb{C}[n]\langle S\rangle$ with $Sn = (n+1)S$

- Natural action: $L$ acts on $\mathbb{C}^{\mathbb{Z}}$ via $(n \cdot u)_k = ku_k$, $(S \cdot u)_k = u_{k+1}$

If $(u_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is a solution, then for all $n \in \mathbb{Z}$:

  - $(n-1)u_{n+3} = -(n-3)u_{n+1} - (n-1)(n+1)u_n$

| $\cdots$ | $-1$ | 0 | 1 | 2 | 3 | 4 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| | | 1 | 0 | 0 | $-1$ | | |
| | | 0 | 1 | 0 | 0 | | |
| | | 0 | 0 | 1 | $-3$ | | |

# Finding solutions of P-recursive sequences

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3 \in \mathbb{C}[n]\langle S\rangle$ with $Sn = (n+1)S$

- Natural action: $L$ acts on $\mathbb{C}^{\mathbb{Z}}$ via $(n \cdot u)_k = ku_k$, $(S \cdot u)_k = u_{k+1}$

If $(u_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is a solution, then for all $n \in \mathbb{Z}$:

- $(n-1)u_{n+3} = -(n-3)u_{n+1} - (n-1)(n+1)u_n$

| $\cdots$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $\cdots$ |
|---|---|---|---|---|---|---|---|
| | | $1$ | $0$ | $0$ | $-1$ | $\times$ | $0 = -2$ |
| | | $0$ | $1$ | $0$ | $0$ | $0$ | $0 = 0$ |
| | | $0$ | $0$ | $1$ | $-3$ | $\times$ | $0 = -6$ |

# Finding solutions of P-recursive sequences

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3 \in \mathbb{C}[n]\langle S\rangle$ with $Sn = (n+1)S$

- Natural action: $L$ acts on $\mathbb{C}^{\mathbb{Z}}$ via $(n \cdot u)_k = ku_k$, $(S \cdot u)_k = u_{k+1}$

If $(u_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is a solution, then for all $n \in \mathbb{Z}$:

- $(n-1)u_{n+3} = -(n-3)u_{n+1} - (n-1)(n+1)u_n$

| $\cdots$ | $-1$ | 0 | 1 | 2 | 3 | 4 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| | | 1 | 0 | 0 | $-1$ | $\times$ | $0 = -2$ |
| | | 0 | 1 | 0 | 0 | 0 | $0 = 0$ |
| | | 0 | 0 | 1 | $-3$ | $\times$ | $0 = -6$ |
| | | 0 | 0 | 0 | 0 | 1 | |

# Finding solutions of P-recursive sequences

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3 \in \mathbb{C}[n]\langle S\rangle$ with $Sn = (n+1)S$

- Natural action: $L$ acts on $\mathbb{C}^{\mathbb{Z}}$ via $(n \cdot u)_k = ku_k$, $(S \cdot u)_k = u_{k+1}$

If $(u_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is a solution, then for all $n \in \mathbb{Z}$:

- $(n-1)u_{n+3} = -(n-3)u_{n+1} - (n-1)(n+1)u_n$

| $\cdots$ | $-1$ | 0 | 1 | 2 | 3 | 4 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| | | 1 | 0 | 0 | $-1$ | $\times$ | $0 = -2$ |
| | | 0 | 1 | 0 | 0 | 0 | $0 = 0$ |
| | | 0 | 0 | 1 | $-3$ | $\times$ | $0 = -6$ |
| | | 0 | 0 | 0 | 0 | 1 | |
| | | 0 | 3 | $-1$ | 0 | 0 | $0 = 0$ |

# Finding solutions of P-recursive sequences

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3 \in \mathbb{C}[n]\langle S\rangle$ with $Sn = (n+1)S$

- Natural action: $L$ acts on $\mathbb{C}^{\mathbb{Z}}$ via $(n \cdot u)_k = ku_k$, $(S \cdot u)_k = u_{k+1}$

If $(u_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is a solution, then for all $n \in \mathbb{Z}$:

- $(n-1)u_{n+3} = -(n-3)u_{n+1} - (n-1)(n+1)u_n$

| $\cdots$ | $-1$ | 0 | 1 | 2 | 3 | 4 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| | | 1 | 0 | 0 | $-1$ | $\times$ | $0 = -2$ |
| | | 0 | 1 | 0 | 0 | 0 | $\cdots$ |
| | | 0 | 0 | 1 | $-3$ | $\times$ | $0 = -6$ |
| | | 0 | 0 | 0 | 0 | 1 | $\cdots$ |
| | | 0 | 3 | $-1$ | 0 | 0 | $\cdots$ |

# Finding solutions of P-recursive sequences

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3 \in \mathbb{C}[n]\langle S\rangle$ with $Sn = (n+1)S$

- Natural action: $L$ acts on $\mathbb{C}^{\mathbb{Z}}$ via $(n \cdot u)_k = k u_k$, $(S \cdot u)_k = u_{k+1}$

If $(u_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is a solution, then for all $n \in \mathbb{Z}$:

- $(n-1)u_{n+3} = -(n-3)u_{n+1} - (n-1)(n+1)u_n$
- $(n-1)(n+1)u_n = -(n-3)u_{n+1} - (n-1)u_{n+3}$

| $\cdots$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $\cdots$ |
|---|---|---|---|---|---|---|---|
| | | $1$ | $0$ | $0$ | $-1$ | $\times$ | $0=-2$ |
| $0=4$ | $\times$ | $0$ | $1$ | $0$ | $0$ | $0$ | $\cdots$ |
| | | $0$ | $0$ | $1$ | $-3$ | $\times$ | $0=-6$ |
| | $0$ | $0$ | $0$ | $0$ | $0$ | $1$ | $\cdots$ |
| $0=10$ | $\times$ | $0$ | $3$ | $-1$ | $0$ | $0$ | $\cdots$ |

# Finding solutions of P-recursive sequences

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3 \in \mathbb{C}[n]\langle S\rangle$ with $Sn = (n+1)S$

- Natural action: $L$ acts on $\mathbb{C}^{\mathbb{Z}}$ via $(n \cdot u)_k = ku_k$, $(S \cdot u)_k = u_{k+1}$

If $(u_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is a solution, then for all $n \in \mathbb{Z}$:

- $(n-1)u_{n+3} = -(n-3)u_{n+1} - (n-1)(n+1)u_n$
- $(n-1)(n+1)u_n = -(n-3)u_{n+1} - (n-1)u_{n+3}$

| $\cdots$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $\cdots$ |
|---|---|---|---|---|---|---|---|
| | | $1$ | $0$ | $0$ | $-1$ | $\times$ | $0=-2$ |
| $0=4$ | $\times$ | $0$ | $1$ | $0$ | $0$ | $0$ | $\cdots$ |
| | | $0$ | $0$ | $1$ | $-3$ | $\times$ | $0=-6$ |
| | $0$ | $0$ | $0$ | $0$ | $0$ | $1$ | $\cdots$ |
| $0=10$ | $\times$ | $0$ | $3$ | $-1$ | $0$ | $0$ | $\cdots$ |
| | $1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\cdots$ |
| $0=0$ | $0$ | $5$ | $-6$ | $2$ | $0$ | $0$ | $\cdots$ |

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3 \in \mathbb{C}[n]\langle S \rangle$ with $Sn = (n+1)S$

- Natural action: $L$ acts on $\mathbb{C}^{\mathbb{Z}}$ via $(n \cdot u)_k = ku_k$, $(S \cdot u)_k = u_{k+1}$

If $(u_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is a solution, then for all $n \in \mathbb{Z}$:

- $(n-1)u_{n+3} = -(n-3)u_{n+1} - (n-1)(n+1)u_n$
- $(n-1)(n+1)u_n = -(n-3)u_{n+1} - (n-1)u_{n+3}$

| $\cdots$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $\cdots$ |
|---|---|---|---|---|---|---|---|
| | | $1$ | $0$ | $0$ | $-1$ | $\times$ | $0=-2$ |
| $0=4$ | $\times$ | $0$ | $1$ | $0$ | $0$ | $0$ | $\cdots$ |
| | | $0$ | $0$ | $1$ | $-3$ | $\times$ | $0=-6$ |
| $\cdots$ | $0$ | $0$ | $0$ | $0$ | $0$ | $1$ | $\cdots$ |
| $0=10$ | $\times$ | $0$ | $3$ | $-1$ | $0$ | $0$ | $\cdots$ |
| $\cdots$ | $1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\cdots$ |
| $\cdots$ | $0$ | $5$ | $-6$ | $2$ | $0$ | $0$ | $\cdots$ |

# Finding robust solutions of P-recursive sequences

$L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3 \in \mathbb{C}[n]\langle S \rangle$ with $Sn = (n+1)S$

- Deformed action: $L$ acts on $\mathbb{C}(q)^{\mathbb{Z}}$ or $\mathbb{C}((q))^{\mathbb{Z}}$ via $(n \cdot u)_k = (k+q)u_k$
- Recover usual solutions by setting $q = 0$

| 0 | 1 | 2 | 3 | 4 | 5 | $\cdots$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | | | | |
| 0 | 1 | 0 | | | | |
| 0 | 0 | 1 | | | | |

# Finding robust solutions of P-recursive sequences

$L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3 \in \mathbb{C}[n]\langle S \rangle$ with $Sn = (n+1)S$

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- Recover usual solutions by setting $q = 0$

| 0 | 1 | 2 | 3 | 4 | 5 | $\cdots$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | $-q-1$ | | | |
| 0 | 1 | 0 | 0 | | | |
| 0 | 0 | 1 | $\dfrac{3-q}{q-1}$ | | | |

## Finding robust solutions of P-recursive sequences

$L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3 \in \mathbb{C}[n]\langle S\rangle$ with $Sn = (n+1)S$

- Deformed action: $L$ acts on $\mathbb{C}(q)^{\mathbb{Z}}$ or $\mathbb{C}((q))^{\mathbb{Z}}$ via $(n \cdot u)_k = (k+q)u_k$
- Recover usual solutions by setting $q = 0$

| 0 | 1 | 2 | 3 | 4 | 5 | $\cdots$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | $-q-1$ | $\dfrac{(q+1)(q-2)}{q}$ | | |
| 0 | 1 | 0 | 0 | $-q-2$ | | |
| 0 | 0 | 1 | $\dfrac{3-q}{q-1}$ | $\dfrac{(q-3)(q-2)}{q(q-1)}$ | | |

# Finding robust solutions of P-recursive sequences

$L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3 \in \mathbb{C}[n]\langle S\rangle$ with $Sn = (n+1)S$

- Deformed action: $L$ acts on $\mathbb{C}(q)^{\mathbb{Z}}$ or $\mathbb{C}((q))^{\mathbb{Z}}$ via $(n \cdot u)_k = (k+q)u_k$
- Recover usual solutions by setting $q = 0$

| 0 | 1 | 2 | 3 | 4 | 5 | $\cdots$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | $-q-1$ | $\frac{(q+1)(q-2)}{q}$ | $\frac{(q-2)(1-q)}{q}$ | |
| 0 | 1 | 0 | 0 | $-q-2$ | $\frac{(q-1)(q+2)}{q+1}$ | |
| 0 | 0 | 1 | $\frac{3-q}{q-1}$ | $\frac{(q-3)(q-2)}{q(q-1)}$ | $\frac{6+\ldots}{q(q+1)}$ | |

# Finding robust solutions of P-recursive sequences

$L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3 \in \mathbb{C}[n]\langle S\rangle$ with $Sn = (n+1)S$

- Deformed action: $L$ acts on $\mathbb{C}(q)^{\mathbb{Z}}$ or $\mathbb{C}((q))^{\mathbb{Z}}$ via $(n \cdot u)_k = (k+q)u_k$
- Recover usual solutions by setting $q = 0$

| 0 | 1 | 2 | 3 | 4 | 5 | $\cdots$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | $-q-1$ | $\frac{(q+1)(q-2)}{q}$ | $\frac{(q-2)(1-q)}{q}$ | $\cdots$ |
| 0 | 1 | 0 | 0 | $-q-2$ | $\frac{(q-1)(q+2)}{q+1}$ | $\cdots$ |
| 0 | 0 | 1 | $\frac{3-q}{q-1}$ | $\frac{(q-3)(q-2)}{q(q-1)}$ | $\frac{6+\dots}{q(q+1)}$ | $\cdots$ |

# Finding robust solutions of P-recursive sequences

$L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3 \in \mathbb{C}[n]\langle S\rangle$ with $Sn = (n+1)S$

- Deformed action: $L$ acts on $\mathbb{C}(q)^{\mathbb{Z}}$ or $\mathbb{C}((q))^{\mathbb{Z}}$ via $(n \cdot u)_k = (k+q)u_k$
- Recover usual solutions by setting $q = 0$

| 0 | 1 | 2 | 3 | 4 | 5 | … |
|---|---|---|---|---|---|---|
| $1$ | $0$ | $0$ | $-q-1$ | $\frac{(q+1)(q-2)}{q}$ | $\frac{(q-2)(1-q)}{q}$ | … |
| $0$ | $1$ | $0$ | $0$ | $-q-2$ | $\frac{(q-1)(q+2)}{q+1}$ | … |
| $0$ | $0$ | $1$ | $\frac{3-q}{q-1}$ | $\frac{(q-3)(q-2)}{q(q-1)}$ | $\frac{6+\dots}{q(q+1)}$ | … |
| $q$ | $0$ | $0$ | $-q(q+1)$ | $(q+1)(q-2)$ | $(q-2)(1-q)$ | … |
| $0$ | $0$ | $q$ | $\frac{q(3-q)}{q-1}$ | $\frac{(q-3)(q-2)}{q-1}$ | $\frac{6+\dots}{q+1}$ | … |

# Finding robust solutions of P-recursive sequences

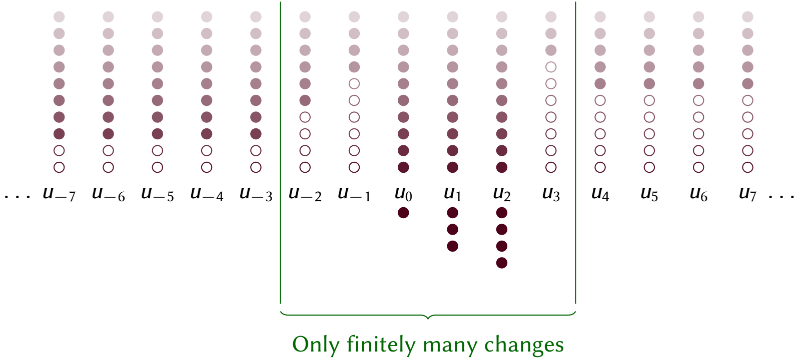
$L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3 \in \mathbb{C}[n]\langle S\rangle$ with $Sn = (n+1)S$

- Deformed action: $L$ acts on $\mathbb{C}(q)^{\mathbb{Z}}$ or $\mathbb{C}((q))^{\mathbb{Z}}$ via $(n \cdot u)_k = (k+q)u_k$
- Recover usual solutions by setting $q = 0$

| 0 | 1 | 2 | 3 | 4 | 5 | $\cdots$ |
|---|---|---|---|---|---|---|
| $1$ | $0$ | $0$ | $-q-1$ | $\frac{(q+1)(q-2)}{q}$ | $\frac{(q-2)(1-q)}{q}$ | $\cdots$ |
| $0$ | $1$ | $0$ | $0$ | $-q-2$ | $\frac{(q-1)(q+2)}{q+1}$ | $\cdots$ |
| $0$ | $0$ | $1$ | $\frac{3-q}{q-1}$ | $\frac{(q-3)(q-2)}{q(q-1)}$ | $\frac{6+\dots}{q(q+1)}$ | $\cdots$ |
| $q$ | $0$ | $0$ | $-q(q+1)$ | $(q+1)(q-2)$ | $(q-2)(1-q)$ | $\cdots$ |
| $0$ | $0$ | $q$ | $\frac{q(3-q)}{q-1}$ | $\frac{(q-3)(q-2)}{q-1}$ | $\frac{6+\dots}{q+1}$ | $\cdots$ |
| $\frac{q-3}{q-1}$ | $0$ | $-q-1$ | $0$ | $0$ | $(q+1)(q+3)$ | $\cdots$ |

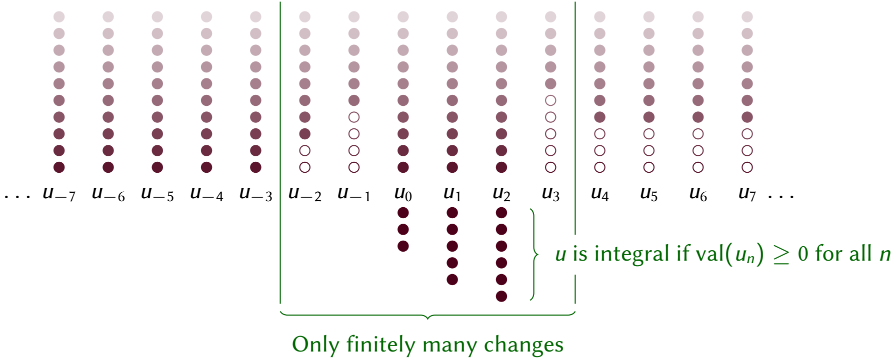
# P-recursive sequences: what are poles?

In practice, robust solutions of an operator look like this:



## P-recursive sequences: what are poles?

In practice, normalized robust solutions of an operator look like this:




- Given $L \in C[n]\langle S\rangle$ with order $r$, it has $r$ independent normalized solutions $u^{(1)}, \ldots, u^{(r)}$ in $C((q))^{\mathbb{Z}}$
- $B \in C(n)\langle S\rangle / \langle L\rangle$ acts on those solutions
- Valuation of $B$ at $\alpha \in \mathbb{Z}$ : min of the valuations of $B \cdot u^{(i)}$ at $\alpha$
- $B$ is integral iff it has non-negative valuation everywhere

# Van Hoeij's algorithm for finding integral bases (recurrence case)

Local algorithm: exactly the same!

$u^{(1)}, \dots, u^{(r)} \in C((q))^{\mathbb{Z}}$
basis of solutions of $L$

Input. $L \in C[n]\langle S\rangle$ with order $r$, $\alpha \in \mathbb{Z}$

Output. $B_1, \dots, B_r$ basis of $C(n)\langle S\rangle/\langle L\rangle$ integral at $\alpha$

1. $B_1, \dots, B_r \leftarrow$ basis of $C(n)\langle S\rangle/\langle L\rangle$ — $B_1, \dots, B_r \leftarrow 1, \dots, S^{r-1}$
2. For $d \in \{1, \dots, r\}$:
3. While $B_i$ is not integral at $\alpha$ — $B_i \cdot u^{(j)}$ has val $< 0$ at $\alpha$ for some $j$
4. $B_i \leftarrow (x-\alpha)B_i$
5. While there exists $a_1, \dots, a_{d-1} \in C$
   such that $A := \frac{1}{x-\alpha}\left(a_1B_1 + \dots + a_{d-1}B_{d-1} - B_d\right)$ is integral at $\alpha$
6. $B_d \leftarrow A$
7. Return $B_1, \dots, B_r$

$\iff \forall j, (a_1B_1 + \dots + a_{d-1}B_{d-1} - B_d)\cdot u^{(j)}$ has val $> 0$ at $\alpha$

Linear system of equations

Global algorithm: loop over all $\alpha \in \mathbb{Z}$

Nothing happens at all but finitely many of them
(roots of the leading and trailing coefficients of $L$)

# Van Hoeij's algorithm for recurrences on an example (1)

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3$

Basis of solutions in $C((q))^{\mathbb{N}}$:

| | 0 | 1 | 2 | 3 | 4 | 5 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| $u$ | 1 | 0 | 0 | $-1+O(q)$ | $-2q^{-1}+O(1)$ | $-2q^{-1}+O(1)$ | $\cdots$ |
| $v$ | 0 | 1 | 0 | 0 | $-2+O(q)$ | $-2+O(q)$ | $\cdots$ |
| $w$ | 0 | 0 | 1 | $-3+O(q)$ | $-6q^{-1}+O(1)$ | $-6q^{-1}+O(1)$ | $\cdots$ |

# Van Hoeij's algorithm for recurrences on an example (2)

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3$, $\alpha = 3$

| $B$ | $1$ | |
|---|---|---|
| $(B \cdot u)_3$ | $-1+O(q)$ | |
| $(B \cdot v)_3$ | $0$ | |
| $(B \cdot w)_3$ | $-3+O(q)$ | |

# Van Hoeij's algorithm for recurrences on an example (2)

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3$, $\alpha = 3$

| $B$ | $1$ | $S$ |
|---|---|---|
| $(B \cdot u)_3$ | $-1+O(q)$ | $-2q^{-1}+O(1)$ |
| $(B \cdot v)_3$ | $0$ | $-2+O(q)$ |
| $(B \cdot w)_3$ | $-3+O(q)$ | $-6q^{-1}+O(1)$ |

# Van Hoeij's algorithm for recurrences on an example (2)

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3$, $\alpha = 3$

| $B$ | $1$ | $S$ | $(n-3)S$ |
|---|---|---|---|
| $(B \cdot u)_3$ | $-1+O(q)$ | $-2q^{-1}+O(1)$ | $-2+O(q)$ |
| $(B \cdot v)_3$ | $0$ | $-2+O(q)$ | $-2q+O(q^2)$ |
| $(B \cdot w)_3$ | $-3+O(q)$ | $-6q^{-1}+O(1)$ | $-6+O(q)$ |

## Van Hoeij's algorithm for recurrences on an example (2)

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3$, $\alpha = 3$

| $B$ | $1$ | $S$ | $(n-3)S$ |
|---|---|---|---|
| $(B \cdot u)_3$ | $-1 + O(q)$ | $-2q^{-1} + O(1)$ | $-2 + O(q)$ |
| $(B \cdot v)_3$ | $0$ | $-2 + O(q)$ | $-2q + O(q^2)$ |
| $(B \cdot w)_3$ | $-3 + O(q)$ | $-6q^{-1} + O(1)$ | $-6 + O(q)$ |

# Van Hoeij's algorithm for recurrences on an example (2)

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3$, $\alpha = 3$

| $B$ | $1$ | $S$ | $(n-3)S$ | $(n-3)S-2$ |
|---|---|---|---|---|
| $(B\cdot u)_3$ | $-1+O(q)$ | $-2q^{-1}+O(1)$ | $-2+O(q)$ | $q+O(q^2)$ |
| $(B\cdot v)_3$ | $0$ | $-2+O(q)$ | $-2q+O(q^2)$ | $-2q+O(q^2)$ |
| $(B\cdot w)_3$ | $-3+O(q)$ | $-6q^{-1}+O(1)$ | $-6+O(q)$ | $3q+O(q^2)$ |

# Van Hoeij's algorithm for recurrences on an example (2)

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3$, $\alpha = 3$

| $B$ | $1$ | $S$ | $(n-3)S$ | $(n-3)S-2$ | $S - \frac{2}{n-3}$ |
|---|---|---|---|---|---|
| $(B \cdot u)_3$ | $-1+O(q)$ | $-2q^{-1}+O(1)$ | $-2+O(q)$ | $q+O(q^2)$ | $1+O(q)$ |
| $(B \cdot v)_3$ | $0$ | $-2+O(q)$ | $-2q+O(q^2)$ | $-2q+O(q^2)$ | $-2+O(q)$ |
| $(B \cdot w)_3$ | $-3+O(q)$ | $-6q^{-1}+O(1)$ | $-6+O(q)$ | $3q+O(q^2)$ | $3+O(q)$ |

# Van Hoeij's algorithm for recurrences on an example (2)

Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3$, $\alpha = 3$

| $B$ | $1$ | $S$ | $(n-3)S$ | $(n-3)S - 2$ | $S - \frac{2}{n-3}$ |
|---|---|---|---|---|---|
| $(B \cdot u)_3$ | $-1+O(q)$ | $-2q^{-1}+O(1)$ | $-2+O(q)$ | $q+O(q^2)$ | $1+O(q)$ |
| $(B \cdot v)_3$ | $0$ | $-2+O(q)$ | $-2q+O(q^2)$ | $-2q+O(q^2)$ | $-2+O(q)$ |
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Example: $L = (n-1)(n+1) + (n-3)S^2 + (n-1)S^3$, $\alpha = 3$

| $B$ | $1$ | $S$ | $(n-3)S$ | $(n-3)S-2$ | $S - \frac{2}{n-3}$ | $S^2$ |
|---|---|---|---|---|---|---|
| $(B \cdot u)_3$ | $-1+O(q)$ | $-2q^{-1}+O(1)$ | $-2+O(q)$ | $q+O(q^2)$ | $1+O(q)$ | $\cdots$ |
| $(B \cdot v)_3$ | $0$ | $-2+O(q)$ | $-2q+O(q^2)$ | $-2q+O(q^2)$ | $-2+O(q)$ | $\cdots$ |
| $(B \cdot w)_3$ | $-3+O(q)$ | $-6q^{-1}+O(1)$ | $-6+O(q)$ | $3q+O(q^2)$ | $3+O(q)$ | $\cdots$ |

# Application and perspectives

## Why do we care?

- In the differential case, integral bases can be used to compute integrals
- We hope that in the recurrence case, they can be used to compute sums
- Future work: /s/hope/prove/

## What if it is the wrong definition for that?

- The definitions and the algorithm generalize to valued vector spaces
- No particularly restricting hypothesis
- So if the definition is wrong, we only have to find the correct one!

## References


- Kauers and Koutschan, 'Integral D-finite Functions' (2015)
- Chen, van Hoeij, Kauers and Koutschan, 'Reduction-based Creative Telescoping for Fuchsian D-finite Functions' (2018)
- Chen, Du, Kauers and Verron, 'Integral P-Recursive Sequences' (2020) [preprint]

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# Thank you for your attention