Computer Algebra 2

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1 Notations and conventions

Unless otherwise mentioned, we use the following notations:

- $k, K, \mathbb{K}$ are (commutative) fields
- R is a (commutative, with 1) ring

Given a ring R, R^* is the group of its invertible elements.

We assume that algebraic computations (sum, inverse, test of 0, test of 1, inverse where applicable) can be performed.

For a vector v in a vector space V of dimension n, we denote its coordinates by $(v_0, \ldots, v_{n-1})$. If f is a polynomial of degree $\deg(f) = d$, its coefficients are denoted $f_0, \ldots, f_d$, such that

$$f(X) = f_0 + f_1 X + \cdots + f_d X^d = \sum_{i=0}^{d} f_i X^i.$$

In order to simplify notations, we may at times use the convention that $f_i = 0$ if $i < 0$ or $i > \deg(f)$, so that

$$f = \sum_{i \in \mathbb{Z}} f_i X^i.$$

By convention, the degree of the 0 polynomial is $-\infty$.

The logarithm log, without a base, is in base 2.

Definition 1.1. Given two functions $f, g : \mathbb{N} \to \mathbb{R}_{>0}$

$$\begin{aligned} f = O(g) &\iff \frac{f(n)}{g(n)} \text{ is bounded when } n \to \infty \\ &\iff \exists c \in \mathbb{R}_{>0}, n_0 \in \mathbb{N}, \forall n \geq n_0, f(n) \leq c g(n); \end{aligned}$$

$$f = \tilde{O}(g) \iff \exists l \in \mathbb{N}, f = O(g \log(g)^l).$$

1.1 Exercises

Exercise 1.1. Show that the "when $n \to \infty$" clause in the definition of O can be left out. In

other words, given $f, g : \mathbb{N} \to \mathbb{R}_{>0}$, show that

$$\begin{aligned} f = O(g) &\iff \frac{f(n)}{g(n)} \text{ is bounded} \\ &\iff \exists c \in \mathbb{R}_{>0}, \forall n \in \mathbb{N}, f(n) \leq cg(n) \end{aligned}$$

2 Semi-fast multiplication

In this chapter, let R be any ring.

Given $f, g \in R[X]$ with degree less than n, we want to compute the coefficients of $h = f \cdot g$.

The complexity of the algorithm will be evaluated in number of multiplications and additions in R. Typically, multiplications are more expensive!

2.1 Naive algorithm

Each coefficient h_k $(0 \leq k < 2n)$ can be computed with

$$h_k = \sum_{i=0}^{k} f_i g_{k-i},$$

each costing $O(n)$ multiplications and additions.

The total complexity of the naive algorithm is $O(n^2)$ multiplications and $O(n^2)$ additions.

2.2 Karatsuba's algorithm

Remark 2.1. Linear polynomials can be multiplied using 3 multiplications instead of 4 :

$$(a + bX)(c + dX) = ac + (ad + bc)X + bdX^2$$

with

$$ad + bc = ad + bc + ac + bd - ac - bd = (a + b)(c + d) - ac - bd.$$

This can be used recursively to compute polynomial multiplication faster.

Algorithm 1 Karatsuba

Input: $f = f_0 + \cdots + f_{n-1}X^{n-1}$, $g = g_0 + \cdots + g_{n-1}X^{n-1}$

Output: $h = h_0 + \cdots + h_{2n-1}X^{2n-1}$ such that $h = fg$

1. If $n = 1$, then return $f_0 g_0$
2. Write $f = A + BX^{\lceil n/2 \rceil}$, $g = C + DX^{\lceil n/2 \rceil}$ where all of A, B, C, D have degree $< \lceil \frac{n}{2} \rceil$.
3. Compute recursively:
   - $P = AC$
   - $Q = BD$
   - $R = (A + B)(C + D)$
4. Return $P + (R - P - Q)X^{\lceil n/2 \rceil} + RX^{2\lceil n/2 \rceil}$

Theorem 2.2. *Karatsuba's algorithm multiplies polynomials with $O(n^{\log_2(3)}) = O(n^{1.585})$ multiplications and additions.*

Proof. Let $M(n)$ (resp. $A(n)$) be the number of multiplications (resp. additions) in a run of Algo. 1 on an input with size n. Then:

$$M(n) = 3M(n/2)$$

and

$$A(n) = 3A(n/2) + O(n)$$

so $M(n) = O(n^{\log_2(3)})$ and $A(n) = O(n^{\log_2(3)})$. □

Remark 2.3. Karatsuba's algorithm hides an evaluation/interpolation mechanism:

$$\begin{aligned} a &= (a + bX)_{X=0} \\ a + b &= (a + bX)_{X=1} \\ b &= \left(\frac{a + bX}{X}\right)_{X=\infty} \end{aligned}$$

and for two linear polynomials f, g, if $fg = h = h_0 + h_1X + h_2X^2$, we have

$$\begin{aligned} &f(0)g(0) = h(0) = h_0 \\ &f(1)g(1) = h(X = 1) = h_0 + h_1 + h_2 \\ &\left(\frac{f}{X}\right)_{X=\infty}\left(\frac{g}{X}\right)_{X=\infty} = \left(\frac{h}{X^2}\right)_{X=\infty} = h_2 \end{aligned}$$

2.3 Toom-k algorithm

For the remainder of this section, assume that the ring R is an infinite field.

In general the coefficients of h can be obtained as a linear combination of $f(i)g(i)$ for $i \in \{0, \ldots, 2n-1\}$ via

$$\begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 1 & 1 & 1 & \ldots \\ 1 & 2 & 4 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} \left[\begin{pmatrix} 1 & 0 & 0 & \ldots \\ 1 & 1 & 1 & \ldots \\ 1 & 2 & 4 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{pmatrix} \odot \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 1 & 1 & 1 & \ldots \\ 1 & 2 & 4 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \end{pmatrix} \right]$$

where $\odot$ is the component-wise multiplication of two vectors.

This suggests the following generalization of Algo. 1 for any fixed $k \geq 2$. First, let $V = \left(i^j\right)_{i,j=0}^{2k-1}$ (Vandermonde matrix), and precompute V^{-1}.

Algorithm 2 Toom-k

Input: $f = f_0 + \cdots + f_{n-1}X^{n-1}$, $g = g_0 + \cdots + g_{n-1}X^{n-1}$

Output: $h = h_0 + \cdots + h_{2n-1}X^{2n-1}$ such that $h = fg$

1. If $n < \max(k, 16)$, compute h naively and stop # Forget the "16" until Sec. 2.4
2. Write $f = F_0 + F_1X^{\lceil n/k \rceil} + \cdots + F_{k-1}X^{(k-1)\lceil n/k \rceil}$ and $g = G_0 + G_1X^{\lceil n/k \rceil} + \cdots + G_{k-1}X^{(k-1)\lceil n/k \rceil}$ where $\deg(F_i)$ and $\deg(G_i) < \frac{n}{k}$
3. Define $F_i = G_i = 0$ for $\frac{n}{k} < i \leq 2k-1$
4. Compute $\bar{f} = V\begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ F_{2k-1} \end{pmatrix}$ and $\bar{g} = V\begin{pmatrix} G_0 \\ G_1 \\ \vdots \\ F_{2k-1} \end{pmatrix}$
5. Compute $\bar{h} = \bar{f} \odot \bar{g}$ recursively
6. Return $V^{-1}\bar{h}$

Remark 2.4. If we write $F_i = f_0^{(i)} + \cdots + f_d^{(i)}X^d$ for $i \in \{0, \ldots, k-1\}$, one can compute the product $V \cdot (F_i)$ as

$$V \cdot \begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ F_{k-1} \end{pmatrix} = V \cdot \left[\begin{pmatrix} f_0^0 \\ f_0^{(1)} \\ \vdots \\ f_0^{(k-1)} \end{pmatrix} + \begin{pmatrix} f_1^0 \\ f_1^{(1)} \\ \vdots \\ f_1^{(k-1)} \end{pmatrix} X + \cdots + \begin{pmatrix} f_d^0 \\ f_d^{(1)} \\ \vdots \\ f_d^{(k-1)} \end{pmatrix} X^d \right]$$

$$= V \cdot \begin{pmatrix} f_0^0 \\ f_0^{(1)} \\ \vdots \\ f_0^{(k-1)} \end{pmatrix} + V \cdot \begin{pmatrix} f_1^0 \\ f_1^{(1)} \\ \vdots \\ f_1^{(k-1)} \end{pmatrix} X + \cdots + V \cdot \begin{pmatrix} f_d^0 \\ f_d^{(1)} \\ \vdots \\ f_d^{(k-1)} \end{pmatrix} X^d$$

so the cost of computing that product is $O(dk^2)$.

Theorem 2.5. *A run of Algorithm 2 requires $O(n^{\log_k(2k-1)})$ operations. In particular, for any fixed $\varepsilon > 0$, there exists a multiplication algorithm for $R[X]$ which requires $O(n^{1+\varepsilon})$ operations in R.*

Proof. See Exercise 2.2. □

Remark 2.6. For fixed k, the cost of precomputing V and V^{-1} can be neglected, since it is a fixed cost of $O(k^2)$ and $O(k^3)$ respectively.

2.4 Toom-Cook algorithm

Theorem 2.7 (Toom-Cook). *There exists a multiplication algorithm for $R[X]$ that requires $O(n^{1+2/\sqrt{\log(n)}})$ operations in R. This algorithm is obtained by adapting Algo. 2 to choose at each recursion level $k = \left\lfloor 2^{2\sqrt{\log(n)}} \right\rfloor$.*

Proof. See Exercise 2.3. □

Remark 2.8. This complexity is better than that of Toom-k, since it is better than $O(2^{1+\varepsilon})$ for *all* $\varepsilon > 0$.

Remark 2.9. Strassen's algorithm for matrix multiplication is based on the same idea as Karatsuba's algorithm, and runs in time $O(n^{\log_2(7)}) \leq O(n^{2.82})$. Is there a Toom-Cook style algorithm for matrix multiplication, with complexity better than $O(2^{2+\varepsilon})$ for all $\varepsilon > 0$?

For even k, we can multiply $k \times k$ matrices with $\frac{1}{3}k^3 + 6k^2 - \frac{4}{3}k$ operations, so there are matrix multiplication algorithms with complexity $O(n^{\log_k(\frac{1}{3}k^3+6k^2-\frac{4}{3}k)})$. But $\log_k(\frac{1}{3}k^3 + 6k^2 - \frac{4}{3}k)$ tends to 3 when k tends to ∞. Its minimum (over $2\mathbb{N}$) is reached at $k = 70$, leading to a complexity $O(n^{2.796})$ (Pan's algorithm).

The current record is $O(n^{2.372\,863\,9})$ (Le Gall 2014), and yes, that many decimal points are necessary! It is conjectured that a complexity of $O(2^{1+\varepsilon})$ for all ε is realizable.

Remark 2.10. It is conjectured that polynomial multiplication in $O(n)$ operations is not possible.

2.5 Exercises

Exercise 2.1. Is it possible to use the ideas of the Algorithm of Toom-k with evaluation at $\{0, 1, \ldots, k-2, \infty\}$? Describe the matrices V and V^{-1}.

Exercise 2.2. Prove Theorem 2.5.

Exercise 2.3. Prove Theorem 2.7.

Exercise 2.4. Show that there is no algorithm which can multiply two linear polynomials (over any ring) in 2 multiplications.

3 Fast multiplication in $\bar{k}[X]$

In this chapter, let k be an *algebraically closed* field. The problem to solve is the same as previously, but this time, we assume that $\deg(f) + \deg(g) < n$.

We will be considering evaluation/interpolation methods.

Algorithm 3 Evaluation/interpolation

Input: $f = f_0 + \cdots + f_{k-1}X^k$, $g = g_0 + \cdots + g_{l-1}X^l$ with $k + l < n$

Output: $h = h_0 + \cdots + h_{n-1}X^{n-1}$ such that $h = fg$

1. Fix $(x_0, \ldots, x_{n-1}) \in k^n$
2. Compute $f(x_i)$, $g(x_i)$ for $i = 0, \ldots, n-1$
3. Compute $h(x_i) = f(x_i)g(x_i)$ for $i = 0, \ldots, n-1$
4. Compute h by interpolating $h(x_i)$ for $i = 0, \ldots, n-1$

Remark 3.1. In general, Algo. 3 requires $O(n^2) + O(n) + O(n^2) = O(n^2)$ operations in k, like the classical algorithm. The idea is to choose specific values of $x_0, \ldots, x_{n-1}$ so that steps 2 and 4 can be done faster.

3.1 Roots of unity and discrete Fourier transform

Definition 3.2. An element $\omega \in k$ is called a *n'th root of unity* if $\omega^n = 1$. It is a *primitive* n'th root of unity if additionally $\omega^i \neq 1$ for $0 < i < n$.

Example 3.3. In $\mathbb{C}$, -1 is a primitive second root of unity. i is a primitive 4th root of unity.

In $\mathbb{F}_{17}$, 2 is a primitive 8th root of unity.

Definition 3.4. The matrix

$$\mathrm{DFT}_n := \mathrm{DFT}_n^{(\omega)} := \left(\omega^{ij}\right)_{i,j=0}^{n-1} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \ldots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)^2} \end{pmatrix} \in k^{n \times n}$$

is called the *discrete Fourier transform* (wrt ω).

Example 3.5. In $\mathbb{C}$, the discrete Fourier transform wrt i is

$$\mathrm{DFT}_4^{(i)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}.$$

Remark 3.6. The DFT is a Vandermonde matrix. In particular, if $f = f_0 + f_1 X + \cdots + f_{n-1}X^{n-1}$,

$$\mathrm{DFT}_n^{(\omega)} \cdot \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{pmatrix} = \begin{pmatrix} f(\omega^0) \\ f(\omega^1) \\ \vdots \\ f(\omega^{n-1}) \end{pmatrix}.$$

Definition 3.7. Let $f, g \in k^n$. The *product* $f \odot g$ is the vector whose i'th coordinate is given by $f_i g_i$. The *convolution* $f * g$ is the vector whose i'th coordinate is given by

$$\sum_{k=0}^{n-1} f_k g_{(i-k) \bmod n}.$$

Lemma 3.8. *Let ω be a primitive n'th root of unity. Then*

1. *there is a factorization*

$$X^n - 1 = (X - \omega)(X - \omega^2) \cdots (X - \omega^n);$$

2. *for any $j \in \{1, \ldots, n-1\}$,*

$$\sum_{i=0}^{n-1} \omega^{ij} = 0.$$

3. *there is a group isomorphism*

$$\left(\{\omega^i : i \in \mathbb{Z}\}, \cdot\right) \simeq (\mathbb{Z}/n\mathbb{Z}, +)$$

4. *the DFT matrix is easy to invert:*

$$\left(\mathrm{DFT}_n^{(\omega)}\right)^{-1} = \frac{1}{n}\mathrm{DFT}_n^{(1/\omega)}$$

5. *if $m \mid n$, then ω^m is a primitive (n/m)'th root of unity*
6. *the DFT is compatible with convolution*

$$\mathrm{DFT}_n(f * g) = \mathrm{DFT}_n(f) \odot \mathrm{DFT}_n(g)$$

Proof. 1. All ω^i are distinct: if $\omega^i = \omega^j$ with $1 \le i < j \le n$, then $\omega^{j-i} = 1$ with $0 < j - i < n$, which is a contradiction because ω is a primitive root of unity. All ω^i are roots of $X^n - 1$, since $(\omega^i)^n = (\omega^n)^i = 1$, so the $X - \omega^i$ are n distinct factors of $X^n - 1$. By comparing the degree and leading coefficient, we get the wanted factorization.

2. Use the formula

$$\left(\sum_{i=0}^{n-1} X^i\right)(X-1) = X^n - 1$$

Evaluated at $X = \omega^j$ for $0 < j < n$, the right hand side is 0, the factor $(\omega^j - 1)$ is non-zero, so the sum has to be zero.

3. Clear.

4. Evaluate the product:

$$\begin{aligned}
\mathrm{DFT}_n^{(\omega)}\mathrm{DFT}_n^{(1/\omega)} &= \left(\omega^{ij}\right)_{i,j=0}^{n-1} \cdot \left(\omega^{-ij}\right)_{i,j=0}^{n-1} \\
&= \left(\sum_{k=0}^{n-1} \omega^{ik}\omega^{-kj}\right)_{i,j=0}^{n-1} \\
&= \left(\sum_{k=0}^{n-1} \omega^{k(i-j)}\right)_{i,j=0}^{n-1} \\
&= \left(n\delta_{ij}\right)_{i,j=0}^{n-1}.
\end{aligned}$$

5. Clear.

6. If we associate the vector $f = (f_0, \ldots, f_{n-1})$ with the polynomial $f(X) = f_0 + \cdots + f_{n-1}X^{n-1}$, convolution is equivalent to multiplication in $k[X]/\langle X^n - 1\rangle$, that is

$$(f * g)(X) = f(X)g(X) + q(X) \cdot (X^n - 1)$$

for some $q \in k[X]$. Indeed, write

$$\begin{aligned}
f(X)g(X) &= \sum_{i,j=0}^{n-1} f_i g_j X^{i+j} \\
&= \sum_{i+j<n} f_i g_j X^{i+j} + \sum_{n\le i+j<2n} f_i g_j X^{i+j} \\
&= \underbrace{\sum_{i+j<n} f_i g_j X^{i+j} + \sum_{n\le i+j<2n} f_i g_j X^{i+j-n}}_{(f*g)(X)} - \underbrace{\sum_{n\le i+j<2n} f_i g_j X^{i+j-n} + \sum_{n\le i+j<2n} f_i g_j X^{i+j}}_{(\sum_{n\le i+j<2n} f_i g_j X^{i+j-n})(X^n-1)}
\end{aligned}$$

The claim follows by evaluation at ω^i. □

The remark, together with property 4, makes powers of ω a good choice for evaluation and interpolation: if we can just find a fast way to evaluate $\mathrm{DFT}_n \cdot f$, we can perform both steps in a fast way.

3.2 Fast Fourier transform

Given $f = \begin{pmatrix} f_0 \\ \vdots \\ f_{2n-1} \end{pmatrix}$, we want to compute $\bar{f} = \mathrm{DFT}_{2n} \cdot f$.

Let's expand the j'th coefficient:

$$\begin{aligned}
\left(\mathrm{DFT}_{2n}^{\omega} f\right)_j &= \sum_{i=0}^{2n-1} \omega^{ij} f_i \\
&= \sum_{i=0}^{n-1} \omega^{2ij} f_{2i} + \sum_{i=0}^{n-1} \omega^{(2i+1)j} f_{2i+1} \\
&= \sum_{i=0}^{n-1} (\omega^2)^{ij} f_{2i} + \omega^j \sum_{i=0}^{n-1} (\omega^2)^{ij} f_{2i+1} \\
&= \begin{cases} \left(\mathrm{DFT}_n^{(\omega^2)} f_{\mathrm{even}}\right)_j + \omega^j \left(\mathrm{DFT}_n^{(\omega^2)} f_{\mathrm{odd}}\right)_j & \text{for } 0 \le j < n \\ \left(\mathrm{DFT}_n^{(\omega^2)} f_{\mathrm{even}}\right)_{j-n} + \omega^j \left(\mathrm{DFT}_n^{(\omega^2)} f_{\mathrm{odd}}\right)_{j-n} & \text{for } n \le j < 2n \end{cases} \\
&= \begin{cases} \left(\mathrm{DFT}_n^{(\omega^2)} f_{\mathrm{even}}\right)_j + \omega^j \left(\mathrm{DFT}_n^{(\omega^2)} f_{\mathrm{odd}}\right)_j & \text{for } 0 \le j < n \\ \left(\mathrm{DFT}_n^{(\omega^2)} f_{\mathrm{even}}\right)_{j-n} - \omega^{j-n} \left(\mathrm{DFT}_n^{(\omega^2)} f_{\mathrm{odd}}\right)_{j-n} & \text{for } n \le j < 2n \end{cases}
\end{aligned}$$

We can use this property to perform the evaluation and interpolation steps.

Algorithm 4 Fast Fourier Transform

Input: $f \in k^n$, ω a primitive n'th root of unity, $n = 2^k$

Output: $\bar{f} = \mathrm{DFT}_n^{(\omega)} f$

1. If $n = 1$ then return (f_0)
2. $u \leftarrow \mathsf{FFT}([f_0, f_2, \ldots,], \omega^2, n/2)$, $v \leftarrow \mathsf{FFT}([f_1, f_3, \ldots,], \omega^2, n/2)$
3. Return $[u_0 + v_0, u_1 + \omega v_1, u_2 + \omega^2 v_2, \ldots, u_{n/2-1} + \omega^{n/2-1} v_{n/2-1},$
   $u_0 - v_0, u_1 - \omega v_1, u_2 - \omega^2 v_2, \ldots, u_{n/2-1} - \omega^{n/2-1} v_{n/2-1}]$

Theorem 3.9. *Algo. 4 requires $O(n \log(n))$ operations in k.*

Proof. Similar to before, with the recurrence

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n).$$

□

This allows us to rewrite Algo. 3 with the FFT.

Algorithm 5 Evaluation/interpolation multiplication using FFT

Input: $f = f_0 + \cdots + f_{k-1}X^k$, $g = g_0 + \cdots + g_{l-1}X^l$ with $k + l < n$

Output: $h = h_0 + \cdots + h_{n-1}X^{n-1}$ such that $h = fg$

1. $\omega \leftarrow$ primitive n'th root of unity
2. $\bar{f} \leftarrow \mathsf{FFT}(f, \omega)$, $\bar{g} \leftarrow \mathsf{FFT}(g, \omega)$
3. $\bar{h} \leftarrow \bar{f} \odot \bar{g}$
4. Return $\frac{1}{n}\mathsf{FFT}(\bar{h}, \omega^{-1})$

Theorem 3.10. *Multiplication in $k[X]$ can be done with $O(n \log n)$ operations in k if k is algebraically closed.*

Remark 3.11. This complexity is currently the best known complexity for polynomial multiplication.

Remark 3.12. Let P be the permutation matrix such that

$$P \cdot f = \begin{pmatrix} f_{\text{even}} \\ f_{\text{odd}} \end{pmatrix}$$

and Δ be the diagonal matrix

$$\Delta = \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \omega^2 & \\ & & & \ddots \end{pmatrix}.$$

Then the computations above yield that

$$\begin{aligned} \mathrm{DFT}_{2n} &= \begin{pmatrix} \mathrm{DFT}_n & \Delta\mathrm{DFT}_n \\ \mathrm{DFT}_n & -\Delta\mathrm{DFT}_n \end{pmatrix} \cdot P \\ &= \begin{pmatrix} I & \Delta \\ I & -\Delta \end{pmatrix} \cdot \begin{pmatrix} \mathrm{DFT}_n & \\ & \mathrm{DFT}_n \end{pmatrix} \cdot P \\ &= \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \cdot \begin{pmatrix} I & \\ & \Delta \end{pmatrix} \cdot \begin{pmatrix} \mathrm{DFT}_n & \\ & \mathrm{DFT}_n \end{pmatrix} \cdot P \end{aligned}$$

This can be generalized to divisions by m instead of 2. Skipping over the details, this gives

$$\mathrm{DFT}_{mn} = \begin{pmatrix} I & I & I & \dots \\ I & \omega^n I & \omega^{2n} I & \dots \\ I & \omega^{2n} I & \omega^{4n} I & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdot \begin{pmatrix} I & & & \\ & \Delta & & \\ & & \Delta^2 & \\ & & & \ddots \end{pmatrix} \cdot \begin{pmatrix} \mathrm{DFT}_n & & & \\ & \mathrm{DFT}_n & & \\ & & \mathrm{DFT}_n & \\ & & & \ddots \end{pmatrix} \cdot P.$$

This is a result due to Cooley and Tuckey, which can be used to refine Algo. 4 so that it reduces an FFT of *any* size quickly to FFT's of prime size.

4 Fast multiplication in $R[X]$

Remark 4.1. Algo. 4 does not require that the base ring R be an algebraically closed field, but that:

1. R contains a primitive n'th root of unity ω;
2. $n = 1 + 1 + \cdots + 1$ is invertible.

In this chapter, we will see how to perform FFT without those hypotheses.

4.1 If 2 is invertible

In this section, assume that R has no n'th root of unity, but that $2 \in R^*$.

Let $f, g \in R[X]$, with $\deg f + \deg g < n = 2^k$, as before we want to compute $h = fg$. Write $n = pq$ where $p = 2^{\lceil k/2 \rceil}$ and $q = 2^{\lfloor k/2 \rfloor}$, so $p \simeq q \simeq \sqrt{n}$.

Write

$$f = F_0 + F_1 X^q + F_2 X^{2q} + \dots$$
$$g = G_0 + G_1 X^q + G_2 X^{2q} + \dots$$

with $\deg F_i < q$, $\deg G_i < q$, and define two polynomials in $R[X, Y]$

$$\bar{f} = F_0 + F_1 Y + F_2 Y^2 + \dots$$
$$\bar{g} = G_0 + G_1 Y + G_2 Y^2 + \dots.$$

Then $\deg_X \bar{f}, \deg_X \bar{g} < q$, $\deg_Y \bar{f}, \deg_Y \bar{g} < p$, and $f = \bar{f}(X, X^q)$, $g = \bar{g}(X, X^q)$. Let $\bar{h} = \bar{f}\bar{g}$, then $\deg_X \bar{h} < 2q$ and $\deg_Y \bar{h} < 2p$.

Note 4.2. It suffices to compute $\bar{h} \bmod Y^p + 1$ because

$$\deg h = \deg \bar{h}(X, X^q) < pq = n.$$

Note 4.3. Since $\deg_X \bar{h} < 2q$,

$$\bar{h}(X, Y) = \bar{h}(X, Y) \bmod X^{2q} + 1.$$

Hence, together with the previous note, we can compute in

$$\left(R[X]/\left\langle X^{2q} + 1\right\rangle\right)[Y]/\langle Y^p + 1\rangle.$$

We denote by D the ring

$$D := R[X]/\left\langle X^{2q} + 1\right\rangle.$$

Proposition 4.4. *In the ring D, X is a $4q$'th primitive root of unity. Furthermore, let*

$$\omega = \begin{cases} X^2 & \text{if } p = q \\ X & \text{if } p = 2q. \end{cases}$$

Then $\omega = X$ is a $2p$'th primitive root of unity in D.

With this setting, if

$$\bar{f}(Y) \cdot \bar{g}(Y) = \bar{h}(Y) \bmod Y^p + 1$$

then

$$\bar{f}(\omega Y) \cdot \bar{g}(\omega Y) = \bar{h}(\omega Y) \bmod (\omega Y)^p + 1 = 1 - Y^p$$

Algorithm 6 Schönhage-Strassen

Input: $f, g \in R[X]$ with $\deg f, \deg g < n = 2^k$

Output: $h = fg \bmod X^n + 1$

1. If $k \leq 2$ then compute h directly
2. Define $p, q \in \mathbb{N}$, $\bar{f}, \bar{g} \in D[Y]$ and $\omega \in D$ as above
3. Use Algo. 5 to compute $\bar{h} \in D[y]$ with

   $$\bar{h}(\omega Y) = \bar{f}(\omega Y)\bar{g}(\omega Y) \bmod Y^p - 1$$

   using ω^2 as a p'th root of unity in D and Algo. 6 recursively for multiplications in D
4. Return $h = \bar{h}(X, X^q) \bmod X^n + 1$

Remark 4.5. The algorithm requires that 2 be invertible for the FFT step: each call to the FFT multiplication algorithm is with a power of 2 as n.

Theorem 4.6. *Algo. 6 requires $O(n \log(n) \log(\log(n)))$ operations in R.*

Remark 4.7. For all practical purposes, $\log \log n \leq 6$.

Proof. Let $n \gg 1$ and suppose that

$$T(m) \leq c_1 m \log m \log \log m$$

for all $m < n$ and some constant c_1. Recall that $n = 2^k$, $p = 2^{\lceil k/2 \rceil} \leq 2\sqrt{n}$, $q = 2^{\lfloor k/2 \rfloor} \leq \sqrt{n}$.

The runtime function satisfies the recurrence

$$T(n) \leq pT(2q) + O(n \log n)$$

where $pT(2q)$ is the cost of p component-wise multiplication of polynomials of degree at most $2q$, and the trailing $O(n \log n)$ is the cost of the FFT.

Let $T_l(k) = T(2^k)$, and expand in terms of k :

$$\begin{aligned}
T_l(k) &\leq 2^{\lceil k/2 \rceil} T_l\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) + c_2 2^k k \\
&\leq c_1 2^{\lceil k/2 \rceil} 2^{\lfloor k/2 \rfloor + 1} \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) \log\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) + c_2 2^k k \\
&\leq c_1 \underbrace{2^{\lceil k/2 \rceil + \lfloor k/2 \rfloor}}_{=2^k} \cdot \underbrace{2\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right)}_{\leq k+2} \underbrace{\log\left(\underbrace{\left\lfloor \frac{k}{2} \right\rfloor + 1}_{\leq \frac{3}{4}k}\right)}_{\leq \log k - \log(4/3)} + c_2 2^k k \\
&\leq c_1 2^k k \log(k) + c_1 2^k \underbrace{\left(2 \log k - 2 \log(\frac{4}{3})\right)}_{\leq \frac{1/2}{k} \log(4/3)} + (c_2 - c_1 \log(\frac{4}{3})) 2^k k \\
&\qquad \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\leq (c_2 - \frac{1}{2}\log(4/3)) 2^k k}
\end{aligned}$$

Without loss of generality we can assume that $c_1 \geq 2c_2/\log 4/3$, so

$$T_l(k) \leq c_1 2^k k \log(k)$$

and indeed

$$T(n) = O(n \log n \log\log n).$$

□

4.2 If 2 is not invertible

The previous algorithm requires 2 to be invertible in order to divide the reverse DFT by 2^k. Without this assumption, we can skip that division, and Algo. 5 returns $2^k fg$. Analogously, we can compute $3^l fg$ using a 3-adic FFT. Then, Euclid's extended algorithm yields $u, v \in \mathbb{Z}$ such that

$$u \cdot 2^k + v \cdot 3^l = 1,$$

so

$$u \cdot 2^k fg + v \cdot 3^l fg = fg.$$

Theorem 4.8. *Polynomials in $R[X]$ of degree less than n can be multiplied using $O(n \log n \log\log n)$ operations in R, for any commutative ring R with a unity.*

Remark 4.9. This is the current world record.

4.3 Multiplication time function

Definition 4.10. Let R be a ring. A function $M : R \to \mathbb{N}$ is called *multiplication time* for $R[X]$ if there exists an algorithm that multiplies $f, g \in R[X]$ with $\deg f$, $\deg g < n$ using no more than $M(n)$ operations in R.

Finding the best possible M for various rings in an active field of research.

Proposition 4.11. *We can assume that:*

1. *if R is infinite, M is worse than linear:*

$$\frac{M(n)}{n} > \frac{M(m)}{m} \text{ if } n > m;$$

2. *in particular,*

$$M(mn) \geq mM(n)$$

and

$$M(m+n) \geq M(m) + M(n);$$

3. *M is at most quadratic:*

$$M(nm) \leq m^2 M(n)$$

4. *M is at most the complexity of the general algorithm by Schönhage and Strassen:*

$$M(n) = O(n \log n \log\log n).$$

5 Fast multiplication in $\mathbb{Z}$

Here, we are given two *integers* $f, g \in \mathbb{Z}$ with at most n digits (in base 2), and we want to compute $h = fg$.

5.1 Integer multiplication in theory

Remark that if

$$f = f_0 + 2f_1 + \cdots + 2^{n-1}f_{n-1},$$

f is the evaluation of the polynomial

$$\bar{f} = f_0 + f_1 X + \cdots + f_{n-1}X^{n-1}$$

at $X = 2$.

This reduces integer multiplication to polynomial multiplication, with similar complexity results.

Theorem 5.1 (Schönhage-Strassen)**.** *Integers of length n can be multiplied in time* $O(n \log n \log\log n)$.

Remark 5.2. It is conjectured that the lower bound for the complexity of integer multiplication is $cn \log n$.

The current best results are the following.

Definition 5.3. For $x \in \mathbb{R}_{>1}$, the *iterated logarithm* of x is

$$\log^*(x) = \max\{k \in \mathbb{N} : \log^k(x) \leq 1\}.$$

Remark 5.4. For all practical purposes, $\log^*(n) \leq 4$.

Theorem 5.5 (Fürer, 2007)**.** *Integers of length n can be multiplied in time*

$$n \log n 2^{O(\log^*(n))}.$$

Remark 5.6. Beware of constants! In general,

$$2^{O(f(n))} \neq O(2^{f(n)})$$

Indeed

$$2^{cf(n)} = (2^{f(n)})^c$$

which will in general grow faster than $2^{f(n)}$.

In the recent years, researchers have focused on improving that constant, the current best result is the following:

Theorem 5.7 (Harvey, van der Hoeven, 2018)**.** *Integers of length n can be multiplied in time*

$$O(n \log n 2^{2 \log^*(n)}).$$

Remark 5.8. Forgetting the constants, we have

$$\log \log n \geq 2^{2 \log^* n} \iff n \geq 2^{2^{2^{12}}}.$$

Remember that n is the *number of digits* of the integers we want to multiply!

5.2 Integer multiplication in practice

Those algorithms are only of theoretical interest. The following algorithm follows a more pragmatic approach, which is usually superior.

Write $F = (f_{n-1} \dots f_1 f_0)_w$ and $G = (g_{n-1} \dots g_1 g_0)_w$ in base w with w as large as possible. In practice, one can for example choose w to be the largest possible processor word.

Define

$$\bar{f} = f_0 + f_1 X + \cdots + f_{n-1} X^{n-1}$$
$$\bar{g} = g_0 + g_1 X + \cdots + g_{n-1} X^{n-1}$$

so that $\bar{f}(w) = f$ and $\bar{g}(w) = g$. Let

$$\bar{h} = \bar{f}\bar{g} = \bar{h}_0 + \bar{h}_1 X + \dots$$

Note that $0 \leq \bar{h}_i \leq nw^2$ for all i.

Assume that $n < w/8$ and fix three primes p_1, p_2, p_3 between $w/2$ and w, for which the field $\mathbb{F}_{p_i}$ contains a 2^t'th root of unity for some large t. Then compute $\bar{f}\bar{g}$ in $\mathbb{F}_{p_i}[X]$ for $i = 1, 2, 3$, and reconstruct the coefficients of $\bar{h}$ with the Chinese remainder theorem. Finally compute $h = \bar{h}(w)$.

Example 5.9. On a 64-bits processor, let's choose $w = 2^{64}$. Then

$$p_1 = 95 \cdot 2^{57} - 1$$
$$p_2 = 108 \cdot 2^{57} - 1$$
$$p_3 = 123 \cdot 2^{57} - 1$$

are suitable primes, with $t = 57$ and $55, 65$ and 493 the respective 57'th roots of unity.

This is the method of choice for multiplying integers up to $\simeq 500$ millions of bits on a 64-bits architecture.

6 Fast multiplication in $R[X, Y]$

Given $f, g \in R[X, Y]$, we want to compute $h = fg$.

6.1 Isolating a variable

We can use Algo. 6 in $R[X][Y]$. But the complexity is bounded in number of operations in $R[X]$, not in R. In order to get a complete bound, we need an estimate for the degree growth in X.

If we define

$$d_X := \deg_X(h) = \deg_X(f) + \deg_X(g)$$
$$d_Y := \deg_Y(h) = \deg_Y(f) + \deg_Y(g)$$

it suffices to compute the product in

$$R[X]/\langle X^{d_X+1} - 1\rangle[Y]/\langle Y^{d_Y+1} - 1\rangle.$$

Let

$$D := R[X]/\langle X^{d_X+1} - 1\rangle.$$

If we use for example Algo. 6 to compute the multiplication in $D[Y]/\langle Y^{d_Y+1} - 1\rangle$, it requires $M(d_Y)$ operations in D, each of them requires at most $M(d_X)$ operations in R.

Theorem 6.1. *Polynomials $f, g \in R[X, Y]$, with $\deg_X(f), \deg_X(g) \leq n$ and $\deg_Y(f), \deg_Y(g) \leq m$, can be multiplied with $M(n)M(m)$ operations in R.*

6.2 Kronecker substitution

Algorithm 7 Multiplication using Kronecker substitution

Input: $f, g \in R[X]$ with $\deg_X(fg) < n$, $\deg_Y(fg) < m$

Output: $h = fg$

1. $\bar{f} \leftarrow f(X, X^n)$, $\bar{g} \leftarrow g(X, X^N) \in R[X]$
2. Compute $\bar{h} = \bar{f} \cdot \bar{g} \in R[X]$ with a fast algorithm
3. Write $\bar{h} = h^{(0)} + h^{(1)}X^n + h^{(2)}X^{2n} + \cdots + h^{(m-1)}X^{(m-1)n}$ with $\deg(h^{(i)}) < n$
4. Return $h = h^{(0)} + h^{(1)}Y + h^{(2)}Y^2 + \cdots + h^{(m-1)}Y^{m-1}$

Theorem 6.2. *Algo. 7 requires $M(mn)$ operations in R.*

Proof. The only multiplication computed involves polynomials in $R[X]$ with degree at most nm. □

Remark 6.3. $M(mn)$ may not be strictly less than $M(m)M(n)$.

7 Fast division

Let K be a field. The task is, given $f, g \in K[X]$, to find $q, r \in K[X]$ such that $f = qg + r$ and $\deg(r) < \deg(g)$.

7.1 Horner's rule

Horner's rule is a technique for evaluating a polynomial f with degree m at some value v with $O(m)$ multiplications, instead of the naive m^2. It avoids computing successive powers of v, and instead relies on the following rewriting of f:

$$\begin{aligned} f &= a_0 + a_1 X + \cdots + a_m X^m \\ &= a_0 + X\Big(a_1 + X\big(\cdots + X(a_m)\ldots\big)\Big). \end{aligned}$$

The resulting algorithm is actually the naive Euclidean algorithm used to compute f divided by $g = X - v$. The remainder of that division is $f(v)$.

The same algorithm can be used for a polynomial g with degree n, and it then uses $O(nm)$ operations in K.

7.2 A Karatsuba-style algorithm: Jebelean's algorithm

There is also a Karatsuba-style division algorithm. Assume that $\deg f < 2 \deg g$ and $\deg g$ is a power of 2.

Algorithm 8 Jebelean's algorithm (1993)

Input: $f, g \in K[X]$, $k \in \mathbb{N}$, with $\deg g = n = 2^i$, $\deg f < 2n + k$.

Output: q, r such that $f = gX^k q + r$ and $\deg(r) < n + k$

1. If $\deg f < \deg g + k$, then return $q = 0$, $r = f$
2. If $\deg g = 1$, then use Horner's algorithm
3. Write $g = g^{(0)} + g^{(1)}X^{n/2}$ with $\deg g^{(0)} < \frac{n}{2}$ # $\deg g^{(1)} = \frac{n}{2}$

Compute $q^{(1)}, r^{(1)}$ such that $f = q^{(1)}X^{n+k}g^{(1)} + r^{(1)}$ with $\deg r^{(1)} < \frac{3n}{2} + k$

4. Find $q^{(1)}, r^{(1)}$ by calling Algo. 8 with $f, g = g^{(1)}$ and $k = n + k$

Compute the true remainder $u = f - q^{(1)}X^{n+k}g$

5. Compute $u = r^{(1)} - X^{n/2+k}g^{(0)}q^{(1)}$ using Algo. 1

Compute $q^{(0)}, r^{(0)}$ such that $u = q^{(0)}X^{n/2+k}g^{(1)} + r^{(0)}$ with $\deg r^{(0)} < \frac{n}{2} + k$

6. Find $q^{(0)}, r^{(0)}$ by calling Algo. 8 with $f = u, g = g^{(1)}$ and $k = \frac{n}{2} + k$

Compute the true remainder $r = u - q^{(0)}X^k g$

7. Compute $r = r^{(0)} - g^{(0)}q^{(0)}X^k$ using Algo. 1
8. Return $q = q^{(0)} + q^{(1)}X^{n/2}$ and r

Theorem 7.1. *Algo. 8 is correct.*

Proof. We prove it by induction on n, then on k. The case $n = 1$ is clear, as is the case $\deg f < n + k$. Now assume that the algorithm is correct for all input of size $< n$ or third argument $> k$. Consider $f, g \in R[X]$, $k \in \mathbb{N}$ with $\deg g = n$ and $\deg f < 2n + k$. In particular, $\deg g^{(1)} = \frac{n}{2}$.

So the call to Algo. 8 with $f = f$, $g = g^{(1)}$ and $k = n + k$ is correct, and the results are $q^{(1)}, r^{(1)}$ such that $f = g^{(1)}X^{n+k}q^{(1)} + r^{(1)}$, $\deg(r^{(1)}) < n + k$, and

$$\deg(q^{(1)}) = \deg(f) - \deg(X^{n+k}g^{(1)}) < \frac{n}{2}.$$

The polynomial u satisfies

$$\begin{aligned} u &= r^{(1)} - X^{n/2+k}g^{(0)}q^{(1)} \\ &= f - X^{n+k}g^{(1)}q^{(1)} - X^{n/2+k}g^{(0)}q^{(1)} \\ &= f - X^{n/2+k}q^{(1)}g, \end{aligned} \tag{7.1}$$

and it has degree

$$\deg(u) < \max\left(n + k, \frac{n}{2} + k + \frac{n}{2} + \frac{n}{2}\right) < \frac{3n}{2} + k.$$

The call to Algo. 8 with $f = u$, $g = g^{(1)}$ and $k = \frac{n}{2} + k$ is correct, and $\deg r^{(0)} < n + k$ and

$\deg(q^{(0)}) = \frac{n}{2} + k$. So we get

$$\begin{aligned} u &= X^{n/2+k} g^{(1)} q^{(0)} + r^{(0)} \\ &= X^{n/2+k} g^{(1)} q^{(0)} + X^k g^{(0)} q^{(0)} + r \quad \text{(by definition of } r\text{)} \\ &= g X^k q^{(0)} + r. \end{aligned}$$

The polynomial r has degree

$$\deg(r) < \max\left(n + k, \frac{n}{2} + \frac{n}{2} + k\right) < n + k,$$

and putting it all together using Eq. (7.1), we find

$$f = X^{n/2+k} q^{(1)} g + X^k q^{(0)} g + r = X^k \left(X^{n/2} q^{(1)} + q^{(0)}\right) g + r.$$

□

Theorem 7.2 (Jebelean, 1993)**.** *Algo. 8 requires at most $2M_K(n)$ multiplications in K where $M_K(n)$ is the number of multiplications performed by Algo. 1 (Karatsuba).*

Remark 7.3. There is no O in that result.

Proof. Recall the recurrence relation

$$M_K(2n) = 3M_K(n).$$

If we proceed by induction, the number of multiplications $T(n)$ performed by Algo. 8 satisfies the recurrence relation

$$\begin{aligned} T(n) &= 2T\left(\frac{n}{2}\right) + 2M_K\left(\frac{n}{2}\right) \\ &= 2 \cdot 2M_K\left(\frac{n}{2}\right) + 2M_K\left(\frac{n}{2}\right) \\ &= 6M_K\left(\frac{n}{2}\right) \\ &= 2M_K(n). \end{aligned}$$

□

Remark 7.4. The integer version of Algo. 8 is the best-performing division algorithm for integers of a certain size.

Remark 7.5. The total number of operations (including additions) is $O(M_K(n)\log(n))$.

7.3 Division with the cost of multiplication

We now want to perform division in time $O(M(n))$.

Definition 7.6. Let $f \in K[X]$ and $k \in \mathbb{N}$, the *k'th reversal* of f is

$$\mathrm{rev}_k(f) := X^k f\left(\frac{1}{X}\right).$$

Example 7.7. If $f = f_0 + f_1 X + \cdots + f_n X^n$, then $\mathrm{rev}_n(f) = f_n + f_{n-1}X + \cdots + f_0 X^n$.

Remark 7.8. In general, $\mathrm{rev}_k(f) \in K[X]$ if $k \geq n$.

Let $f, g \in K[X]$ with $\deg(f) = m$, $\deg g = n < m$, and q, r be the quotient and remainder respectively of the division of f by g. Performing the change of variable $X \mapsto 1/X$ and multiplying by X^m the equality $f = qg + r$ gives

$$\begin{aligned} X^m f\left(\frac{1}{X}\right) &= X^n g\left(\frac{1}{X}\right) X^{m-n} q\left(\frac{1}{X}\right) + X^{m-n+1} X^{n-1} r\left(\frac{1}{X}\right) \\ \mathrm{rev}_m f &= \mathrm{rev}_n g \cdot r_{m-n} q + X^{m-n+1} \mathrm{rev}_{n-1} r \end{aligned}$$

so

$$\mathrm{rev}_m f = \mathrm{rev}_n g \cdot \mathrm{rev}_{m-n} q \bmod X^{m-n+1}.$$

Furthermore, since $\deg g = n$, we have $(\mathrm{rev}_n g)_0 \neq 0$, so $\mathrm{rev}_n g$ is invertible modulo X^{m-n+1}. Therefore

$$\mathrm{rev}_{m-n} q = \mathrm{rev}_m f \cdot (\mathrm{rev}_n g)^{-1} \bmod X^{m-n+1}.$$

So what we need is a fast algorithm for inversion modulo X^l: an algorithm which, given $u \in K[X]$ with $u_0 \neq 0$ and $l \in \mathbb{N}$, computes $v \in K[X]$ such that $uv = 1 \bmod X^l$.

Regard $u \in K[X] \subset K[[X]]$ as a formal power series, and consider the map

$$\begin{array}{rccc} \varphi : & K[[X]]^* & \to & K[[X]] \\ & s & \mapsto & u - \frac{1}{s}. \end{array}$$

Let v be a root of φ, we can write

$$v = w + X^l r$$

with $w \in K[X]_{l-1}$, and w, seen as a power series, is invertible. Then

$$\begin{aligned} 0 = \varphi(v) &= u - \frac{1}{w + X^l r} = u - \frac{1}{w}\frac{1}{1 + X^l r/w} \\ &= u - \frac{1}{w} + X^l \frac{r}{v_0^2} - O(X^{l+1}) \end{aligned}$$

so

$$uw = 1 + X^l \frac{r}{w} + O(X^{l+1}) = 1 \bmod X^l.$$

So we have to find an approximation of order l of a root v of φ. For this purpose, we use Newton iteration: we compute successive approximations of the root, starting with

$$v^{(0)} = \frac{1}{u_0}$$

and iterating with

$$\begin{aligned} v^{(k+1)} &= v^{(k)} - \frac{\varphi(v^{(k)})}{\varphi'(v^{(k)})} = v^{(k)} - \frac{u - \frac{1}{v^{(k)}}}{\left(\frac{1}{v^{(k)}}\right)^2} \\ &= 2v^{(k)} - u \cdot (v^{(k)})^2. \end{aligned}$$

This would give us an algorithm, if only we knew when to stop!

Theorem 7.9. *For all $k \geq 0$, $u \cdot v^{(k)} = 1 \bmod X^{2^k}$.*

Proof. Proof by induction: for $k = 0$, we have

$$u \cdot v^{(0)} = u_0 \cdot \frac{1}{u_0} + O(X) = 1 \bmod X.$$

If it is true for $k \geq 0$, then

$$\begin{aligned} 1 - uv^{(k+1)} &= 1 - u(2v^{(k)} - u \cdot (v^{(k)})^2) = 1 - 2uv^{(k)} + (uv^{(k)})^2 = \left(1 - u \cdot v^{(k)}\right)^2 \\ &= O(X^{2^{k+1}}) = 0 \bmod X^{2^{k+1}}. \end{aligned}$$

□

Remark 7.10. This theorem is a particular case of a more general fact: with a starting point sufficiently close to a root, Newton iteration converges quadratically fast.

Algorithm 9 Inversion using Newton iteration

Input: $u \in K[X]$ with $u_0 \neq 0$, $n \in \mathbb{N}$

Output: $v \in K[X]$ with $u \cdot v = 1 \bmod X^n$

1. $v \leftarrow \frac{1}{u_0}$
2. For i from 1 to $\lceil \log(n) \rceil$, do
3. $\quad v \leftarrow 2v - uv^2 \text{ rem } X^l$
4. Return v

Theorem 7.11. *Algo. 9 requires $O(M(n))$ operations in K.*

Proof. Let $T(n)$ be the number of operations required. Then

$$\begin{aligned} T(n) &\leq \sum_{i=1}^{\lceil \log(n) \rceil} 2M(2^i) + c2^i \\ &\leq c2^{\lceil \log(n) \rceil + 1} + 2 \sum_{i=1}^{\lceil \log(n) \rceil} \underbrace{M(2^i)}_{\leq \frac{M(n)}{n/2^i}} \\ &\leq 4cn + 2\frac{M(n)}{n} \underbrace{\sum_{i=1}^{\lceil \log(n) \rceil} 2^i}_{\leq 4n} \\ &\leq 4cn + 8M(n) = O(M(n)). \end{aligned}$$

□

With this taken care of, we can now write down all the steps required to perform a fast division.

Algorithm 10 Fast division

Input: $f, g \in K[X]$, $k \in \mathbb{N}$, with $\deg f = m$, $\deg g < n$, $g \neq 0$

Output: q, r such that $f = qg + r$ and $\deg(r) < \deg(g)$

1. If $m < n$ then return $q = 0$, $r = f$
2. Compute $h = \mathrm{rev}_n(g)^{-1} \bmod X^{m-n+1}$ with Algo. 9
3. $\bar{q} \leftarrow \mathrm{rev}_m(f)h$
4. Return $q = \mathrm{rev}_{m-n}(\bar{q})$ and $r = f - gq$

Theorem 7.12. *Algo. 10 requires $O(M(m))$ operations in K.*

Remark 7.13. This result is the current world record for polynomial division.

Remark 7.14. In particular, if $f, g, q \in K[X]$ with $\deg(f), \deg(g), \deg(q) \leq n$, then we can compute (and reduce) $f, g \in K[X]/\langle q \rangle$ with $O(M(n))$ operations in K.

If $\gcd(f, g) = 1$, then we will see that $f^{-1} \bmod q$ can be computed using $O(M(n)\log(n))$ operation in K (using the fast GCD algorithm).

7.4 Exercises

Exercise 7.1. Assume that the field K is algebraically closed. Find a bound for the complexity of Algo. 8 if we use FFT instead of Karatsuba's algorithm for the multiplication. Is it better?

Exercise 7.2. How would you adapt Algo. 8 to work with any polynomial g (even if its degree is not a power of 2)?

Exercise 7.3.

1. Write an analogue of Algo. 8 for polynomials such that $\deg(f) \leq 3\deg(g)$. What is its complexity?
2. Generalize to any $f, g \in K[X]$. What is the resulting complexity?

8 Computing with homomorphic images

This chapter does not introduce fast algorithms, but serves as a motivation for algorithms in later chapters, namely multipoint evaluation, interpolation and half-GCD.

8.1 The problem of coefficient explosion

Let $A \in \mathbb{Q}^{n \times n}$, and assume that you want to solve the system

$$A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0.$$

This can be done in $O(n^\omega)$ operations in $\mathbb{Q}$ using fast linear algebra. However, in practice, for large n, the computation will take very long.

The reason is *expression swell*: the algorithms multiplies and adds rational numbers which become larger and larger. Additions and multiplications for rationals are defined as

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$
$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

so after each operation, the coefficients size is roughly doubled. Reducing the fractions helps, but not by a significant factor.

The typical situation is that the input is small (because it comes from actual data) and the output is small (because frequently in applications, meaningful data tends to not be overcomplicated), but intermediate expressions will be meaningless and huge.

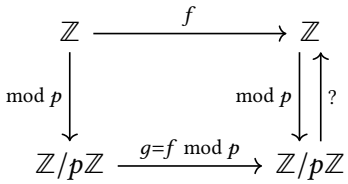
The idea to mitigate this problem is to reduce the problem to domains where objects have a fixed size, so that the actual complexity does not deviate from the predicted number of operations.

8.2 Computations in $\mathbb{Z}$ using modular arithmetic

When doing operations in $\mathbb{Z}$, multiplication doubles the size of the output.

For simplicity, consider a ring homomorphism $f : \mathbb{Z} \to \mathbb{Z}$, and assume that we want to compute $f(x)$, avoiding expression swell inside f.

We want to do the computations in $\mathbb{Z}/p\mathbb{Z}$, for $p \in \mathbb{Z} \setminus \{0\}$.



The diagram commutes, which means that given $x \in \mathbb{Z}$, $g(x \bmod p) = f(x) \bmod p$. But what we want is $f(x)$, not its equivalence class modulo p.

So we want to choose p such that $f(x)$ can be recovered from $f(x) \bmod p$.

There are two interesting scenarios:

1. We know an *a priori* bound $M(x)$ with $|M(x)| \leq M(x)$. Then taking $p > 2M(x)$ will ensure that

$$(f(x) \bmod p) \cap \{-M(x), -M(x)+1, \ldots, M(x)\} = \{f(x)\}.$$

2. We can efficiently check, given $y \in \mathbb{Z}$, whether $f(x) = y$. Then repeat the computation with increasing p until y, defined as

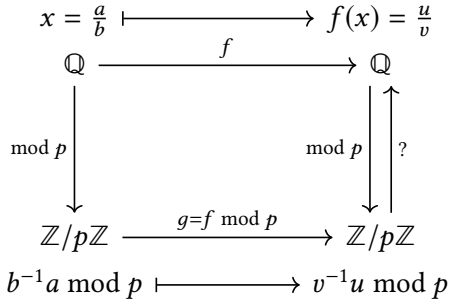
$$\{y\} = (f(x) \bmod p) \cap \left\{-\frac{p}{2}, -\frac{p}{2}+1, \ldots, \frac{p}{2}\right\},$$

is the solution.

8.3 Computations in $\mathbb{Q}$ using rational reconstruction

We now turn back to the problem of $\mathbb{Q}$, where both additions and multiplications double the size of the output. We can do the same thing for a morphism $f : \mathbb{Q} \to \mathbb{Q}$, using modular inverses.

Assume that b and v are coprime to p, we have a commutative diagram:



As in the case of $\mathbb{Z}$, we want to choose p such that $f(a/b)$ can be recovered from $g(b^{-1}a \bmod p)$.

For sufficiency, there are again two scenarios:

1. We know a bound $M(x)$ such that $u^2 + v^2 \leq M(x)^2$. Then taking $p > M(x)^2$ will ensure that

$$\{(w, z) \in \mathbb{Z} \times \mathbb{N} : z^{-1}w = v^{-1}u \bmod p\} \cap \{(w, z) : w^2 + z^2 \leq M(x)\} = \{(u, v)\}.$$

2. We can efficiently check for a given $y \in \mathbb{Q}$ whether $f(x) = y$. Then as in the case of $\mathbb{Z}$, we try increasing values of p until the result is found.

In both cases, in order to determine the intersection point, we need a way, given $y \in \mathbb{Z}/p\mathbb{Z}$ to compute $(u, v) \in \mathbb{Z} \times \mathbb{N}$ such that $v^{-1}u = y \bmod p$ and $u^2 + v^2$ is minimal.

Proposition 8.1. *Consider a run of the Extended Euclid's Algorithm on p and y. Let $(u, v) \in \mathbb{Z}\times\mathbb{N}$ such that $v^{-1}u = y \bmod p$ and $u^2 + v^2$ is minimal. Let g_i, s_i, t_i be values computed at each step of the algorithm, for $i = 1, \ldots, l$:*

$$\begin{aligned} p = g_1 &= s_1 p + t_1 y = 1 \cdot p + 0 \cdot y \\ y = g_2 &= s_2 p + t_2 y = 0 \cdot p + 1 \cdot y \\ g_3 &= s_3 p + t_3 y \\ &\vdots \\ 1 = g_l &= s_l p + t_l y \end{aligned}$$

Then

$$(v, u) \in \{(g_i, t_i) : i \in \{2, \ldots, l\}\} \ni (v, u).$$

Example 8.2. Consider the case $p = 65521$, $y = 29771$, and compute an inverse t of 29771 modulo 65521 using the Extended Euclid's Algorithm, or in other words, a pair $s, t \in \mathbb{Z}$ such that

$$1 = 65521s + 29771t.$$

Here are the intermediate values:

g	s	t
65521	0	1
29771	1	0
5979	-2	1
5855	9	-4
124	-11	5
27	526	-239
16	-2115	961
11	2641	-1200
5	-4756	2161
1	12153	-5522

Then, modulo 65521,

$$29771 = \frac{29771}{1} = -\frac{5979}{2} = \frac{5855}{9} = -\frac{124}{11} = \cdots = \frac{1}{12153}$$

and the minimal pair of numerator and denominator for 29771 is given halfway through the algorithm: it is $(-124, 11)$.

Remark 8.3. It means that the Extended Euclid's Algorithm is useful beyond returning the Bézout coefficients. If we are looking for a rational fraction equal to $x \bmod p$, given a bound on the size of the coefficients, we can find it by examining all lines in the algorithm. And, for the particular case where we want both coefficients to have roughly the same size, the relevant line will be roughly halfway through the algorithm, and can be found using half-GCD algorithms.

8.4 Computation with large moduli using Chinese Remaindering

We saw that computations in $\mathbb{Z}$ and $\mathbb{Q}$ can be done in $\mathbb{Z}/p\mathbb{Z}$, for $p \in \mathbb{N}$ large enough compared to a bound $M(x)$ on the wanted result. But if $M(x)$ is large, p will need to be large, again making the computations expensive.

It is possible to mitigate this problem using the Chinese Remainder Theorem:

$$(n \bmod p) \cap (n \bmod q) = n \bmod \operatorname{lcm}(p, q).$$

So by running the computations modulo p and q, we can reconstruct the result modulo $\operatorname{lcm}(p, q)$.

We still need to be able to find the canonical (small) representative of n modulo $\operatorname{lcm}(p, q)$, given the representatives modulo p and q.

For simplicity, assume that p and q are coprime, so that $\operatorname{lcm}(p, q) = pq$. We are given $n_p, n_q \in \mathbb{Z}$, and we want to find $n \in \mathbb{Z}$ such that

$$n \bmod p = n_p \bmod p \text{ and } n \bmod q = n_q \bmod q.$$

Since $\gcd(p, q) = 1$, there exists $s, t \in \mathbb{Z}$ such that

$$sq + tq = 1.$$

Let

$$n = n_p + (n_q - n_p)sp \in \mathbb{Z}$$

it is congruent to n_p modulo p and to $n_p + (n_q - n_p) = n_q$ modulo q. So we can just take the canonical representative of n modulo pq.

This can be generalized to more moduli.

Algorithm 11 Chinese Remainder reconstruction

Input:

- $u_1, \ldots, u_n \in \mathbb{Z}$
- $p_1, \ldots, p_n \in \mathbb{Z}$, pairwise coprime

Output: $u \in \mathbb{Z}$ such that $u \bmod p_i = u_i \bmod p_i$ for $i = 1, \ldots, n$

1. $u \leftarrow u_1$
2. $m \leftarrow 1$
3. For k from 2 to n do
4. $\quad m \leftarrow m \cdot p_{k-1}$
5. $\quad s \leftarrow m^{-1} \bmod p_k$
6. $\quad u \leftarrow ((u_k - u)s \text{ rem } p_k)\, m$
7. Return u

Remark 8.4. It means that in the second scenarios, both for $\mathbb{Z}$ and $\mathbb{Q}$, when computing modulo p for increasing values of p, we do not have to throw away results for values of p which were too small. We can use them to reconstruct larger moduli.

Remark 8.5. For example, if we take $p_1, \ldots, p_{20}$ to be the first 20 primes, we can reconstruct results modulo

$$p_1 \cdots p_{20} = 2 \cdot 3 \cdots 71 \simeq 5.6 \cdot 10^{26} \simeq 1.8 \cdot 2^{88}$$

8.5 Computations in $K[X]$ and $K(X)$

In $K[X]$ and $K(X)$, we face the same problem as in $\mathbb{Z}$ and $\mathbb{Q}$ respectively. We can use the same techniques as in the case of integers to reduce to problems over $K[X]/\langle P \rangle$ for some small irreducible polynomial P.

A good choice for P is $X - a$, with $a \in K$, and then $K[X]/\langle P \rangle = K$. In that case, the operations of reducing modulo $X - a_i$, $a_i \in K$, $i \in \{0, \ldots, n\}$, running the computations in K and reconstructing the resulting polynomial constitute the evaluation/interpolation method seen before.

Remark 8.6. Algo. 11 for polynomials is Newton's interpolation.

9 Fast evaluation and interpolation

Fast multiplication and fast division algorithms are useful because those operations are heavily used in many higher-level algorithms. However, it is frequently not enough, in order to obtain a speed-up, to replace the operations with their fast counterparts.

Example 9.1. Given $n \in \mathbb{N}$ (with n smaller than a machine word), how to compute $n!$?

The usual algorithm uses the formula

$$n! = n \cdot (n-1)!.$$

This algorithm is recursively called linearly-many times, and at each step does one multiplication with a small integer (with a linear cost). Its complexity satisfies

$$T(n) = T(n-1) + O(n),$$

so

$$T(n) = O(n^2).$$

On the other hand, an algorithm using the following formula

$$n! = \left(\frac{n}{2}\right)! \cdot \left(\prod_{k=\frac{n}{2}+1}^{n} k \right)$$

is recursively called log-many times, and at each step adds one large multiplication. Its complexity satisfies

$$T(n) = 2T\left(\frac{n}{2}\right) + M\left(\frac{n}{2}\right)$$

so

$$T(n) = O(M(n)\log(n)).$$

If $M(n) = O(n^2)$, it's worse. If $M(n) = \tilde{O}(n)$, it's better.

The lesson is that in order to take advantage of fast multiplication, algorithms need to be adjusted. It is usually not sufficient to plug fast multiplication into a standard algorithm.

We want to do two things in this chapter:

Evaluation Given $f \in K[X]$ with $\deg(f) < n$ and $a = (a_0, \ldots, a_{n-1}) \in K^n$, compute the multipoint evaluation $f(a_0), \ldots, f(a_{n-1}) \in K$

Interpolation Given $a = (a_0, \dots, a_{n-1}) \in K^n$ with $a_i \neq a_j$ for $i \neq j$, and $b = (b_0, \dots, b_{n-1}) \in K^n$, compute $f \in K[X]$ with $\deg(f) < n$ such that $f(a_i) = b_i$ for all i.

Remark 9.2. If ω is a n'th root of unity in K and $a_i = \omega^i$ $(i = 0, \dots, n-1)$, then we can accomplish both tasks with $O(n \log(n))$ operations in K. But this doesn't work with arbitrary a_i.

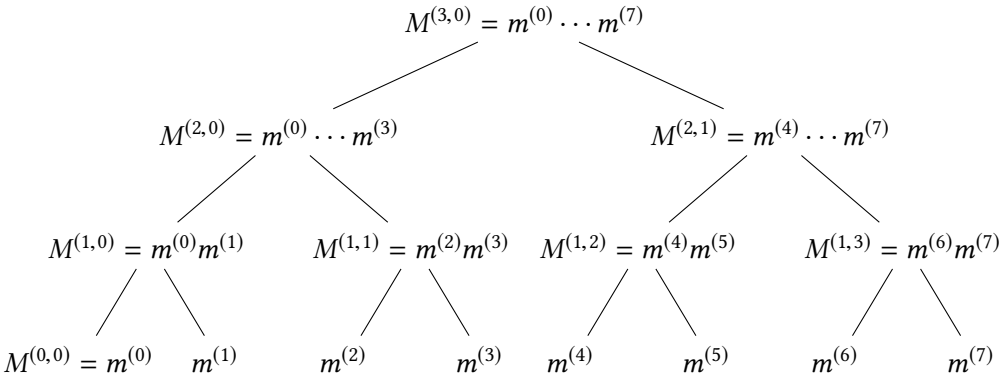
Remark 9.3. The standard algorithms (Horner rule called n times, Newton interpolation) require $O(n^2)$ operations, with no improvements with fast multiplications.

The goal is to find algorithms for both operations using $O(M(n)\log(n))$ operations in K. The main idea, similar to the example of the factorial, is to split the problem into parts of equal size, instead of proceeding point by point.

9.1 Evaluation

Let $m^{(i)} = X - a_i$ and define, for $0 \leq i < k = \log(n)$, $O \leq j < 2^{k-i}$,

$$M^{(i,j)} = m^{(j2^i)} m^{(j2^i+1)} \cdots m^{(j2^i+2^i-1)} = \prod_{l=0}^{2^i-1} m^{(j2^i+l)}.$$



Algorithm 12 Splitting subroutine

Input: $a = (a_0, \dots, a_{n-1}) \in K^n$, $n = 2^k$

Output: $M^{(i,j)}$ as defined above

1. $M^{(0,j)} \leftarrow X - a_j$ $(j \in \{0, \dots, n-1\})$
2. For i from 1 to k do
3. $\quad M^{(i,j)} \leftarrow M^{(i-1,2j)} \cdot M^{(i-1,2j+1)}$ $(j \in \{0, \dots, 2^{k-i}-1\})$

Theorem 9.4. *Algorithm 12 requires $O(M(n)\log(n))$ operations in K.*

Proof. Let $T(n)$ be the number of operations required in a run of Algorithm 12. Note that

$\deg(M^{(i,j)}) = 2^i$. Then $T(n)$ satisfies

$$\begin{aligned} T(n) &= \sum_{i=1}^{k} \sum_{j=0}^{2^{k-i}-1} M(2^{i-1}) \\ &\leq \sum_{i=1}^{k} M\left(\sum_{j=0}^{2^{k-i}-1} 2^{i-1} \right) \\ &\leq kM(2^{k-1}) \leq kM\left(\frac{n}{2}\right) = O(M(n)\log(n)). \end{aligned}$$

□

Note that $X - a_j$ divides $M^{(k-1,0)}$ for $j \in \{0, \ldots, \frac{n}{2} - 1\}$. So, if we write

$$f = qM^{(k-1,0)} + r$$

with $r = f \text{ rem } M^{(k-1,0)}$, then

$$f(a_j) = r(a_j)$$

for $j \in \{0, \ldots, \frac{n}{2} - 1\}$. Likewise, for $j \in \{\frac{n}{2}, \ldots, n-1\}$,

$$f(a_j) = \left(f \text{ rem } M^{(k-1,1)}\right)(a_j).$$

This suggests the following algorithm for evaluation.

Algorithm 13 Multipoint evaluation

Input: $f \in K[X]$, $\deg(f) < n = 2^k$, $a = (a_0, \ldots, a_{n-1}) \in K^n$

Output: $(f(a_0), \ldots, f(a_{n-1})) \in K^n$

1. If $n = 1$ then return $f(a_0)$
2. Compute $(M^{(i,j)})$ with Algorithm 12 and cache the result
3. $r^{(0)} \leftarrow f \text{ rem } M^{(k-1,0)}$, $r^{(1)} \leftarrow f \text{ rem } M^{(k-1,1)}$
4. Compute recursively $\left(r^0(a_0), \ldots, r^{(0)}(a_{\frac{n}{2}-1})\right)$
5. Compute recursively $\left(r^1(a_{\frac{n}{2}}), \ldots, r^{(1)}(a_{n-1})\right)$
6. Return $\left(r^0(a_0), \ldots, r^{(0)}(a_{\frac{n}{2}-1}), r^1(a_{\frac{n}{2}}), \ldots, r^{(1)}(a_{n-1})\right)$

Theorem 9.5. *Algorithm 13 requires $O(M(n)\log(n))$ operations in K.*

Remark 9.6. This is the best known complexity for multipoint evaluation.

Proof. We only need to compute the $M^{(i,j)}$ once, for a fixed cost of $O(M(n)\log(n))$. Let $T(n)$ be the number of operations required for the rest of the computations, we will prove by induction

that $T(n) \leq cM(n)\log(n)$. The complexity $T(n)$ satisfies the recurrence

$$\begin{aligned} T(n) &= 2T\left(\frac{n}{2}\right) + \overbrace{O(M(n))}^{r^{(0)} \text{ and } r^{(1)}} \\ &\leq 2cM\left(\frac{n}{2}\right)\log\left(\frac{n}{2}\right) + c_2 M(n) \\ &\leq cM(n)\log\left(\frac{n}{2}\right) + c_2 M(n) \\ &\leq cM(n)\log(n) + (c_2 - 2c)M(n) \\ &\leq cM(n)\log(n) \quad \text{by choosing } c \text{ large enough for the second term to be negative.} \end{aligned}$$

□

9.2 Interpolation

Recall that given $a = (a_0, \ldots, a_{n-1}) \in K^n$ with $a_i \neq a_j$ for $i \neq j$, and $b = (b_0, \ldots, b_{n-1}) \in K^n$, we want to compute $f \in K[X]$ with $\deg(f) < n$ such that $f(a_i) = b_i$ for all i.

Recall (Lagrange interpolation).

$$f = \sum_{j=0}^{n-1} b_j L_j$$

where

$$L_j = \prod_{i \neq j} \frac{X - a_i}{a_j - a_i} = \begin{cases} 1 \text{ at } X = a_j \\ 0 \text{ at } X = a_i,\ i \neq j. \end{cases}$$

It can be rewritten as

$$L_j = \frac{M^{(k,0)}}{(X - a_j)S^{(j)}} \text{ where } S^{(j)} = \prod_{i \neq j} a_j - a_i.$$

Observe that

$$\frac{\mathrm{d}M^{(k,0)}}{\mathrm{d}X} = \frac{\mathrm{d}}{\mathrm{d}X}\prod_{i=0}^{n-1}(X - a_i) = \sum_{j=0}^{n-1}\prod_{i \neq j}(X - a_i) = \sum_{j=0}^{n-1}\frac{M^{(k,0)}}{X - a_j},$$

so that $S^{(j)} = \frac{\mathrm{d}}{\mathrm{d}X}M^{(k,0)}|_{X=a_j}$ can be obtained by fast multipoint evaluation applied to $\frac{\mathrm{d}}{\mathrm{d}X}M^{(k,0)}$.

Next we need a fast way to compute linear combinations $\sum_{j=0}^{n-1} c_j \frac{M^{(k,0)}}{X - a_j}$. This is the purpose of the next subroutine.

Algorithm 14 Linear combination subroutine

Input:

- $a = (a_0, \dots, a_{n-1}) \in K^n$ with $a_i \neq a_j$ $(i \neq j)$
- $n = 2^k$
- $c = (c_0, \dots, c_{n-1}) \in K^n$
- $M^{(i,j)}$ as computed by Algo. 12

Output: $\displaystyle\sum_{j=0}^{n-1} c_j \frac{M^{(k,0)}}{X - a_j}$

1. If $n = 1$ then return c_0
2. Compute $r^0 \leftarrow \displaystyle\sum_{j=0}^{n/2-1} c_j \frac{M^{(k-1,0)}}{X - a_j}$ recursively
3. Compute $r^1 \leftarrow \displaystyle\sum_{j=n/2}^{n-1} c_j \frac{M^{(k-1,1)}}{X - a_j}$ recursively
4. Return $M^{(k-1,1)} r^{(0)} + M^{(k-1,0)} r^{(1)}$

Theorem 9.7. *Algo. 14 requires $O(M(n)\log(n))$ operations in K.*

Proof. As usual by induction, with the complexity satisfying the recurrence formula

$$T(n) = 2T\left(\frac{n}{2}\right) + O\left(M\left(\frac{n}{2}\right)\right)$$

□

Algorithm 15 Fast interpolation

Input:

- $a = (a_0, \dots, a_{n-1}) \in K^n$ with $a_i \neq a_j$ $(i \neq j)$
- $n = 2^k$
- $b = (b_0, \dots, b_{n-1}) \in K^n$

Output: $f \in K[X]$ with $\deg(f) < n$ and $f(a_i) = b_i$ for $i = 0, \dots, n-1$

1. Compute $M^{(i,j)}$ using Algo. 12
2. $g \leftarrow \dfrac{\mathrm{d}}{\mathrm{d}X} M^{(k,0)}$
3. Compute $(S^{(0)}, S^{(1)}, \dots) \leftarrow (g(a_0), g(a_1), \dots)$ using Algo. 13
4. Compute $f \leftarrow \displaystyle\sum_{j=0}^{n-1} \frac{b_j}{S^{(j)}} \frac{M^{(k,0)}}{X - a_j}$ using Algo. 14
5. Return f

Theorem 9.8. *Algo. 14 requires $O(M(n)\log(n))$ operations in K.*

Remark 9.9. This is the best known complexity for polynomial interpolation.

Remark 9.10. The algorithms above carry over from $K[X]$ to $R[X]$ provided that $a_i - a_j \in R^*$ for $i \neq j$. Without this condition, the Vandermonde matrix needs not be invertible and the interpolation polynomial may not exist or be unique.

Remark 9.11. There are integer versions of Algo. 13 and 15 (fast simultaneous modular reduction / fast Chinese remaindering), also running in time $O(M(n)\log(n))$.

Remark 9.12. The algorithms presented in this chapter are not faster than Algo. 3 if classical multiplication is used.

10 Fast GCD

In this chapter, the task is, given $f, g \in K[X]$ with degree $< n$, to find $h = \gcd(f, g) \in K[X]$.

10.1 "Slow" GCD: Euclid's algorithm

Algorithm 16 Euclid's algorithm

Input: $f, g \in K[X]$

Output: $\gcd(f, g)$

1. $h \leftarrow f, \bar{h} \leftarrow g$
2. While $\bar{h} \neq 0$, do
3. $\quad \begin{pmatrix} h \\ \bar{h} \end{pmatrix} \leftarrow \begin{pmatrix} \bar{h} \\ h \text{ rem } \bar{h} \end{pmatrix}$
4. Return h

Let r_i, for $i \in \mathbb{N}$, be the value of h in the i'th iteration of the loop, so $r_0 = f, r_1 = g, r_2 = f \text{ rem } g$... Let $q_i = r_{i-1} \text{ quo } r_i$ for $i \in \mathbb{N}$. Then $r_{i+1} = r_{i-1} \text{ rem } r_i = r_{i-1} - q_i r_i$, so

$$\begin{pmatrix} r_i \\ r_{i+1} \end{pmatrix} = \begin{pmatrix} r_{i+1} \\ r_{i-1} - q_i r_i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -q_i \end{pmatrix} \cdot \begin{pmatrix} r_{i-1} \\ r_i \end{pmatrix} = Q_i Q_{i-1} \cdots Q_2 Q_1 \cdot \begin{pmatrix} f \\ g \end{pmatrix}$$

where $Q_i = \left(\begin{smallmatrix} 0 & 1 \\ 1 & -q_i \end{smallmatrix}\right)$.

Eventually,

$$\begin{pmatrix} \gcd(f, g) \\ 0 \end{pmatrix} = Q_k Q_{k-1} \cdots Q_2 Q_1 \begin{pmatrix} f \\ g \end{pmatrix}$$

where the first row of $Q_k \cdots Q_1$ contains the Bézout coefficients s, t such that

$$\gcd(f, g) = sf + tg.$$

Remark 10.1. Euclid's algorithm can be extended to keep track of the q_i's and return the Bézout coefficients as well as the GCD.

10.2 Fast GCD

Theorem 10.2. *There exists an algorithm computing the GCD of two polynomials f and g with degree $< n$, together with their Bézout coefficients, using $O(M(n)\log(n))$ operations in K.*

Note that in order to compute $\gcd(f,g)$, it is enough to compute $R := Q_k \cdots Q_1$, and we even get s, t for free in the process. In order to get a fast algorithm for the GCD, we will try to split this product in the middle.

To see how, observe that the trailing coefficients of r_0, r_1 have no influence on the first quotient q_1: if $\deg(r_0) - \deg(r_1) = k$, the degree of q_1 is k and only depends on the first coefficient of r_1 and the first k coefficients of r_0.

Remark 10.3. Generically, and except maybe at the first step, $\deg(r_i) - \deg(r_{i+1}) = 1$.

Definition 10.4. For $f \in K[X]$ with $\deg(f) = n$ and $k \in \mathbb{Z}$, we denote

$$f \upharpoonright k := f \text{ quo } X^{n-k} = f_{n-k} + f_{n-k+1}X + \ldots + f_n X^k$$

Proposition 10.5.

- *If* $k < 0$, $f \upharpoonright k = 0$.
- *If* $k \geq n$, $f \upharpoonright k = f$.
- *If* $i \in \mathbb{N}$, $(X^i f) \upharpoonright k = f \upharpoonright k$.
- *Assume that* $\deg(f) = \deg(g)$. *Then* $f \upharpoonright k = g \upharpoonright k$ *if and only if* $\deg(f - g) < \deg(f) - k$.

Definition 10.6. Let $(f,g), (\bar{f},\bar{g}) \in K[X]^2$ with $\deg(f) \geq \deg(g)$, $\deg(\bar{f}) \geq \deg(\bar{g})$ and $k \in \mathbb{Z}$. Then (f,g) and $(\bar{f},\bar{g})$ are said to *coincide up to* k, written $(f,g) \sim_k (\bar{f},\bar{g})$, if

$$\begin{cases} f \upharpoonright k = \bar{f} \upharpoonright k \\ g \upharpoonright (k - (\deg(f) - \deg(g))) = \bar{g} \upharpoonright (k - (\deg(\bar{f}) - \deg(\bar{g}))). \end{cases}$$

Remark 10.7. The relation $\sim_k$ is an equivalence relation (Exercise 10.1).

Proposition 10.8. *If* $(f,g) \sim_k (\bar{f},\bar{g})$ *and* $k \geq \deg(f) - \deg(g)$, *then* $\deg(f) - \deg(g) = \deg(\bar{f}) - \deg(\bar{g})$.

Proof. If $k \geq \deg(f) - \deg(g)$, $g \upharpoonright (k - (\deg(f) - \deg(g)))$ has degree $k - (\deg(f) - \deg(g))$ which is non-negative, and so is non-zero. By hypothesis, it is equal to $\bar{g} \upharpoonright (k - (\deg(\bar{f}) - \deg(\bar{g})))$, and in particular their degrees are equal, hence $\deg(f) - \deg(g) = \deg(\bar{f}) - \deg(\bar{g})$. □

Theorem 10.9. *Let* $(f,g), (\bar{f},\bar{g}) \in (K[X] \setminus \{0\})^2$ *and* $k \in \mathbb{Z}$ *be such that* $(f,g) \sim_{2k} (\bar{f},\bar{g})$ *and* $k \geq \deg(f) - \deg(g) \geq 0$. *Let* $q, \bar{q}, r, \bar{r}$ *be such that*

$$f = qg + r \text{ with } \deg(r) < \deg(g)$$
$$\bar{f} = \bar{q}\bar{g} + \bar{r} \text{ with } \deg(\bar{r}) < \deg(\bar{g}).$$

Then

$$q = \bar{q} \text{ and } \begin{cases} (g,r) \sim_{2(k-\deg(q))} (\bar{g},\bar{r}) \\ \text{or } r = 0 \\ \text{or } k - \deg(q) < \deg(g) - \deg(r) \end{cases}$$

Proof. Without loss of generality we can assume that $\deg(f) = \deg(\bar{f}) > 2k$: otherwise, one can multiply $(f, \bar{f})$ and $(g, \bar{g})$ by suitable powers of X and all hypotheses are still satisfied. Then, by Prop. 10.8, $\deg(g) = \deg(\bar{g})$ and $k \geq \deg(q) = \deg(f) - \deg(g) = \deg(\bar{f}) - \deg(\bar{g}) = \deg(\bar{q})$. Furthermore,

$$\begin{aligned}
\deg(f - \bar{f}) &< \deg(f) - 2k \leq \deg(g) - k \\
\deg(g - \bar{g}) &< \deg(g) - (2k - (\deg(f) - \deg(g))) = \deg(f) - k \\
&\leq \deg(g) - k \leq \deg(g) - \deg(q) \\
\deg(r - \bar{r}) &\leq \max(\deg(r), \deg(\bar{r})) < \deg(g).
\end{aligned}$$

Since

$$f - \bar{f} = q(g - \bar{g}) + (q - \bar{q})\bar{g} + (r - \bar{r}) \tag{10.1}$$

$(q - \bar{q})\bar{g}$ is a sum of terms with degree $< \deg(g)$, hence it also has degree $< \deg(g)$, hence $q = \bar{q}$.

Assume now that $r \neq 0$ and $k - \deg(q) \geq \deg(g) - \deg(r)$. We have to show that

$$g \upharpoonright 2(k - \deg(q)) = \bar{g} \upharpoonright 2(k - \deg(q)) \tag{10.2}$$

$$r \upharpoonright 2(k - \deg(q)) - (\deg(g) - \deg(r)) = \bar{r} \upharpoonright 2(k - \deg(q)) - (\deg(\bar{g}) - \deg(\bar{r})) \tag{10.3}$$

Since (f, g) and $(\bar{f}, \bar{g})$ coincide up to $2k$,

$$g \upharpoonright (2k - \deg(q)) = \bar{g} \upharpoonright (2k - \deg(q))$$

and Eq. (10.2) follows from the fact that $\deg(q) > 0$.

For Eq. (10.3), by Eq. (10.1) we have

$$\begin{aligned}
\deg(r - \bar{r}) &\leq \max(\deg(f - \bar{f}), \deg(q) + \deg(g - \bar{g})) < \deg(q) + \deg(f) - 2k \\
&= \deg(g) - 2(k - \deg(q)) = \deg(r) - 2(k - \deg(q)) - (\deg(g) - \deg(r)).
\end{aligned}$$

Furthermore by assumption

$$\deg(r) \geq \deg(q) + \deg(g) - k \geq \deg(q) + \deg(f) - 2k > \deg(r - \bar{r})$$

so $\deg(r) = \deg(\bar{r})$ and Eq. (10.3) follows from the above bound on $\deg(r - \bar{r})$. □

Theorem 10.9 gives a sufficient condition for two Euclidean quotients to be equal. We now do the same for a sequence of reductions as they happen in the Euclidean algorithm.

Let r_0, r_1 be two polynomials in $K[X]$ such that $\deg(r_0) > \deg(r_1)$, and consider as before a sequence of reductions of length l:

$$\begin{aligned}
r_{i-1} &= q_i r_i + r_{i+1} \text{ for } i = 1, \ldots, l-1 \\
r_{l-1} &= q_l r_l.
\end{aligned}$$

Let $m_i = \deg(q_i)$, $n_i = \deg(r_i)$. For $k \in \mathbb{N}$, define $\eta(k) \in \mathbb{N}$ by

$$\eta(k) = \max\left\{ j \in \{0, \dots, l\} : \sum_{i=1}^{j} m_i \leq k \right\}.$$

Note that if $0 \leq i \leq l$, then $n_i = n_0 - m_1 - \cdots - m_i$, and so

$$n_0 - n_{\eta(k)} = \sum_{i=1}^{\eta(k)} m_i \leq k < \sum_{i=1}^{\eta(k)+1} m_i = n_0 - n_{\eta(k)+1}.$$

Let $\bar{r}_0, \bar{r}_1$ be two polynomials in $K[X]$ such that $\deg(\bar{r}_0) > \deg(\bar{r}_1)$, we define analogously $\bar{l}$, $\bar{m}_i$, $\bar{n}_i$ and $\bar{\eta}(k)$.

Theorem 10.10. *Let $k \in \mathbb{N}$, $h = \eta(k)$ and $\bar{h} = e\bar{t}a(k)$. If (r_0, r_1) and $(\bar{r}_0, \bar{r}_1)$ coincide up to $2k$, then $h = \bar{h}$ and $q_i = \bar{q}_i$ for $1 \leq i \leq h$.*

Proof. We show by induction on j that the following holds for $0 < j \leq h$:

$$\begin{aligned}
&j \leq \bar{h} \\
&q_i = \bar{q}_i \text{ for } 1 \leq i \leq j \\
&\begin{cases} j = h \\ \text{or } (r_j, r_{j+1}) \text{ and } (\bar{r}_j, \bar{r}_{j+1}) \text{ coincide up to } 2(k - n_0 + n_j). \end{cases}
\end{aligned}$$

Then the claim follows by symmetry.

There is nothing to prove for $j = 0$. Assume that the induction hypothesis holds for $0 \leq j - 1 < h$. Then $r_{j-1} \neq 0$ and $k \geq n_0 - n_j \geq n_{j-1} - n_j = \bar{n}_{j-1} - \bar{n}_j$ and so $\bar{r}_j \neq 0$. Theorem 10.9 applied with $k \leftarrow k - n_0 + n_{j-1}$ implies that $q_j = \bar{q}_j$, and either (r_j, r_{j+1}) and $(\bar{r}_j, \bar{r}_{j+1})$ coincide up to $2(k - n_0 + n_j)$, or $r_{j+1} = 0$, or $k - n_0 + n_j < n_j - n_{j+1}$.

In the second case, $j + 1 > l$, $j \geq 1$ so $j = h = 1$. In the third case, $h = \eta(k) = j$ by definition of η.

Finally, since

$$\sum_{i=1}^{j} \deg(\bar{q}_i) = \sum_{i=1}^{j} \deg(q_i) \leq \sum_{i=1}^{h} \deg(q_i) \leq k$$

so $j \leq \bar{\eta}(j) = \bar{h}$. □

We will use this result to describe a divide-and-conquer algorithm for the GCD. For this purpose, we want to divide the problem into two subproblems of approximately the same size, taking into account the degrees of the quotients. This is the reason why we introduced $\eta(k)$ (todo...)

Algorithm 17 Fast Half-GCD (Knuth, Strassen)

Input: $f, g \in K[X]$, $n = n_0 = \deg(f) > \deg(g) = n_1$, $k \in \mathbb{N}$ such that $0 \leq k \leq n$

Output: $h = \eta(k) \in \mathbb{N}$, $q_1, \ldots, q_h \in K[X]$, $R_h = Q_h \cdots Q_1 \in K[X]^{2\times 2}$

Base cases

1. If $r_1 = 0$ or $k < n_0 - n_1$, then return 0, the empty sequence and I_2
2. If $k = 0 = n_0 - n_1$, then return 1, $\frac{\mathrm{LC}(f)}{\mathrm{LC}(g)}$ and $\begin{pmatrix} 0 & 1 \\ 1 & -\frac{\mathrm{LC}(f)}{\mathrm{LC}(g)} \end{pmatrix}$

First half of the reductions

3. $r_0 \leftarrow f$, $r_1 \leftarrow g$
4. $d \leftarrow \lceil k/2 \rceil$
5. Call the algorithm recursively with $r_0 \upharpoonright (2d-2)$, $r_1 \upharpoonright (2d-2-(n_0-n_1))$ and $d-1$, obtaining $\eta(d-1)$, $q_1, \ldots, q_{\eta(d-1)}$ and $R^{(1)} = Q_{\eta(d-1)} \cdots Q_1$

Propagating the reductions to the second half

6. $j \leftarrow \eta(d-1)+1$, $\delta \leftarrow \deg(R^{(1)}_{2,2})$
7. $\begin{pmatrix} r_{j-1} \\ r_j \end{pmatrix} \leftarrow R^{(1)} \cdot \begin{pmatrix} r_0 \\ r_1 \end{pmatrix}$
8. If $r_j = 0$ or $k < \delta + \deg(r_{j-1}) - \deg(r_j)$, then return $j-1$, $q_1, \ldots, q_{j-1}$ and $R^{(1)}$

Second half of the reductions

9. $q_j \leftarrow r_{j-1}$ quo r_j
10. $r_{j+1} \leftarrow r_{j-1}$ rem r_j
11. $d \leftarrow k - \delta - (\deg(r_{j-1}) - \deg(r_j))$
12. Call the algorithm recursively with input $r_j \upharpoonright 2d$, $r_{j+1} \upharpoonright (2d - (\deg(r_j) - \deg(r_{j+1})))$ and d, obtaining $\eta(d)$, $q_{j+1}, \ldots, q_h$ and $R^{(2)} = Q_{\eta(d)+j} \ldots Q_{j+1}$
13. Return $\eta(d)+j$, $q_1, \ldots, q_{\eta(d)+1}$ and $R^{(2)} \begin{pmatrix} 0 & 1 \\ 1 & -q_j \end{pmatrix} R^{(1)}$.

Remark 10.11. To compute the GCD of f and g, call the algorithm with $k = n$.

Theorem 10.12. *Algo. 17 requires $O(M(n)\log(n))$ operations in K if $n \leq 2k$.*

Remark 10.13. This is currently the best known complexity for computing the GCD of two polynomials.

Remark 10.14. The algorithm may be optimized further, by dropping trailing coefficients of the polynomials before Step 7. This improves the constant in the complexity, and it eliminates the requirement that $n \leq 2k$. See [1, Algo. 11.6] for details.

Remark 10.15. Generically, $\deg(r_{i+1}) = \deg(r_i) - 1$ and $\eta(k) = k$, and the algorithm really splits the problem into first the first $l/2$ reductions, and then the last $l/2$. In this case, the constant in the complexity can be further improved.

Remark 10.16. The algorithm can return any line in the sequence of reductions of the Extended Euclid's Algorithm, but not all of them. The version presented above return the h'th line, with

$h = \eta(k)$, which corresponds to the line where the sum of the degrees of the quotients is roughly k. This means that it can directly be used for rational reconstruction purposes.

Remark 10.17. If the field is infinite, expression swell is to be expected. This can be mitigated by clearing constants at every step and using homomorphic images.

Remark 10.18. If $h = \gcd(f, g)$ then $\mathrm{rev}_{\deg(h)} h = \gcd(\mathrm{rev}_{\deg(f)} f, \mathrm{rev}_{\deg(g)} g)$? This may be used to gain an extra speed when the trailing coefficients are in some way simpler than the leading coefficients.

Remark 10.19. There is an integer analog of Algo. 17, also running in $O(M(n)\log(n))$ time.

Remark 10.20. Algo. 17 can be used for fast modular inversion, but it is still slower than Newton inversion.

Remark 10.21. There is no speed-up if classical multiplication is used.

Remark 10.22. Kronecker substitution can be used to obtain a fast GCD algorithm over $K[X_1, \ldots, X_n]$.

10.3 Exercises

Exercise 10.1. Show that the relation $\sim_k$ is an equivalence relation.

11 Fast squarefree decomposition

The task in this chapter is, given $f \in K[X]$, find $g_1, \dots, g_m \in K[X]$ such that the g_i's are squarefree and pairwise coprime, and $f = g_1 g_2^2 \cdots g_m^m$.

11.1 Definitions and naive algorithm

Definition 11.1. $(g_1, \dots, g_m)$ is called a *squarefree decomposition* of f. Its components are uniquely determined up to multiplication by elements of K.

The product $\bar{f} = g_1 \cdots g_m$ is called the *squarefree part* of f.

f is called *squarefree* if $f = \bar{f}$, or equivalently if for all $g \in K[X] \setminus K$, $g^2 \nmid f$, or equivalently if f has no multiple factor in its decomposition as a product of primes.

For the moment, we assume that $\mathbb{Q} \subset K$, *i.e.* $n = 1 + \cdots + 1 \neq 0$ for all $n \in \mathbb{N}$.

Proposition 11.2. *Let $f \in K[X]$, write $f' = \frac{\mathrm{d}}{\mathrm{d}X} f$. Then f is squarefree if and only if* $\gcd(f, f') = 1$. *Furthermore,*

$$\gcd(f, f') \cdot \bar{f} = f.$$

Proof. If f is squarefree, write its prime decomposition $f = p_1 p_2 \dots p_m$ where all p_i's are distinct. Then

$$f' = \sum_{i=1}^{m} p_i' \prod_{j \neq i} p_j,$$

therefore

$$p_i \mid f' \iff p_i \mid p_i' \prod_{j \neq i} p_j \iff p_i \mid p_i'.$$

So $p_i \nmid f'$ and therefore $\gcd(f, f') = 1$.

Now suppose that $f = g_1 g_2^2 \dots g_m^m$ with pairwise coprime squarefree g_i's. Then

$$f' = \sum_{i=1}^{m} i g_i' g_i^{i-1} \prod_{j \neq i} g_j^j.$$

So $g_k^{k-1} \mid f'$, but

$$g_k^k \mid f' \iff g_k \mid i g_k' \prod_{j \neq i} g_j^j \iff g_k \mid g_k'$$

so g_k^k does not divide f'. Therefore

$$\gcd(f, f') = g_2 g_3^2 \dots g_m^{m-1}.$$

□

Algorithm 18 Squarefree decomposition

Input: $f \in K[X]$

Output: $(g_1, \dots, g_m)$ the squarefree decomposition of f

1. $u \leftarrow \gcd(f, f')$
2. If $u = 1$ then return (f)
3. Else
4. Recursively compute the squarefree decomposition $(g_2, g_3, \dots, g_m)$ of u
5. Return $\left(\frac{f}{u g_1 \dots g_m}, g_2, \dots, g_m\right)$

Theorem 11.3. *Algorithm 18 requires $O(mM(n)\log(n))$ operations in K.*

11.2 Fast squarefree decomposition

We now want an algorithm running in time $O(M(n)\log(n))$.

Theorem 11.4. *Let $g_1, \dots, g_m \in K[X]$ be squarefree and pairwise coprime, $g = g_1 \dots g_m$ and*

$$h = \sum_{i=1}^{m} c_i g_i' \frac{g}{g_i} \in K[X] \text{ for some } c_i \in K.$$

Then

$$\gcd(g, h - cg') = \prod_{c_j = c} g_j \text{ for all } c \in K.$$

Proof. Since $g' = \sum_{i=1}^m g_i' \frac{g}{g_i}$, we have

$$h - cg' = \sum_{i=1}^{m} (c_i - c) g_i' \frac{g}{g_i}.$$

$\gcd(g_i, g_i' \frac{g}{g_i}) = 1$, because $g_i' \frac{g}{g_i}$ is the product of g_i' and the g_j's with $j \neq i$, each of them being coprime to g_i. Furthermore, for $j \neq i$, $g_j \mid (c_i - c) g_i' \frac{g}{g_i}$. Therefore

$$\gcd(g_i, h - cg') = \gcd(g_i, (c_i - c) g_i' \frac{g}{g_i}) = \gcd(g_i, c_i - c) = \begin{cases} g_i \text{ if } c = c_i \\ 0 \text{ otherwise,} \end{cases}$$

which concludes the proof. □

Now let $(g_1, \dots, g_m)$ be the squarefree decomposition of $f \in K[X]$. For $k = 1, \dots, m$, let

$$V_k = g_k g_{k+1} \dots g_m$$
$$W_k = \sum_{i=k}^{m} (i-k+1) g_i' \frac{V_k}{g_i}.$$

Then:

- $V_k' = \sum_{i=k}^{m} g_i' \frac{V_k}{g_i}$
- By Th. 11.4, $\gcd(V_k, W_k - V_k') = g_k$
- $\frac{W_k - V_k'}{g_k} = \sum_{i=k}^{m} (i-k+1-1) g_i' \frac{V_k}{g_i g_k} = \sum_{i=k+1}^{m} (i-(k+1)+1) g_i' \frac{V_{k+1}}{g_i} = W_{k+1}.$
- $V_1 = \bar{f} = \frac{f}{u}$
- $\frac{f'}{u} = \sum_{i=1}^{m} i g_i' \frac{f}{g_i u} = \sum_{i=1}^{m} (i-1+1) g_i' \frac{V_1}{g_i} = W_1.$

This motivates the following algorithm.

Algorithm 19 Squarefree decomposition (Yun)

Input: $f \in K[X]$

Output: $(g_1, \dots, g_m)$ the squarefree decomposition of f

1. $u \leftarrow \gcd(f, f')$
2. $k \leftarrow 1, V_1 \leftarrow \frac{f}{u}, W_1 \leftarrow \frac{f'}{u}$
3. Repeat
4. $\quad g_k \leftarrow \gcd(V_k, W_k - V_k')$
5. $\quad V_{k+1} \leftarrow \frac{V_k}{g_k}$
6. $\quad W_{k+1} \leftarrow \frac{W_k - V_k'}{g_k}$
7. $\quad k \leftarrow k+1$
8. Until $V_k = 1$
9. Return $(g_1, \dots, g_k)$

Theorem 11.5. *Algo. 19 requires $O(M(n)\log(n))$ operations in K.*

Remark 11.6. This is the best known complexity for finding the square-free decomposition of a polynomial.

Proof. Let $d_k = \deg(g_k)$ for $k = 1, \dots, m$. Then $\deg(V_k) = \sum_{i=k}^{m} d_i$ and $\deg(W_k) \leq \deg(V_k) - 1 <$

$\deg(V_k)$. Let $T(n)$ be the number of operations required by Algo. 19. We have

$$\begin{aligned} T(n) &= O(M(n)\log(n)) + \sum_{k=1}^{m} O(M(\deg(V_k))\log(\underbrace{\deg(V_k)}_{\leq n})) \\ &= O(M(n)\log(n)) + O\left(M\left(\sum_{k=1}^{m}\deg(V_k)\right)\log(n)\right). \end{aligned}$$

Since

$$\sum_{k=1}^{m}\deg(V_k) = \sum_{k=1}^{m}\sum_{i=k}^{m} d_i = \sum_{i=1}^{m} i d_i = n,$$

we conclude that $T(n) = O(M(n)\log(n))$. □

11.3 Fast squarefree decomposition in $\mathbb{Z}/p\mathbb{Z}$

Algorithms 18 and 19 do not work if K has positive characteristic, for example if $K = \mathbb{F}_p$ for p prime.

Example 11.7. Consider $f = X^3 + 1 \in \mathbb{F}_3[X]$, with $f' = 3X^2 = 0$, so $\gcd(f, f') = f$. Algo. 18 would not terminate and Algo. 19 would return 1.

In fact f is not squarefree: $f = X^3 + 1 = (X+1)^3$.

Theorem 11.8. *For all $f \in \mathbb{F}_p[X]$,*

$$f' = 0 \iff \exists g \in \mathbb{F}_p[X] \text{ such that } g^p = f.$$

Proof. Write $f = f_0 + f_1 X + \ldots$, and assume that $f' = 0$. So $f_1 + 2f_2 X + \cdots = 0$, so for all $k > 0$, $k f_k = 0$. So for all k, p divides k or $f_k = 0$. Then

$$f = f_0 + f_p X^p + f_{2p} X^{2p} + \cdots = g(X^p) = g(X)^p$$

for $g = f_0 + f_p X + f_{2p} X^2 + \ldots$.

Conversely, if $f = g^p$, $f' = p g' g^{p-1} = 0$. □

Let us examine the output of Algo. 19 more closely. It does terminate also for $K = \mathbb{F}_p$, but as we have seen above the output might be incorrect. Actually, if $(g_1, \ldots, g_m)$ is the squarefree decomposition of $f \in K[X]$, Algo. 18 will return $(h_1, \ldots, h_{p-1})$ where for all $i = 1, \ldots, p-1$,

$$h_i = \prod_{j = i \bmod p} g_j.$$

It is not a problem if $m < p$, and in particular if $\deg(f) < p$.

Otherwise, we still have that

$$\begin{aligned} b &:= \frac{f}{h_1 h_2^2 \dots h_{p-1}^{p-1}} \\ &= \frac{g_1 g_2^2 \dots g_m^m}{g_1 g_2^2 \dots g_{p-1}^{p-1} g_{p+1} g_{p+2}^2 \dots g_{2p-1}^{p-1} \dots} \\ &= (g_p g_{p+1} \dots g_{2p-1})^p (g_{2p} g_{2p+1} \dots g_{3p-1})^{2p} \dots \end{aligned}$$

and it is a p'th power. If $(s_1, \dots, s_l)$ is a squarefree decomposition of $b^{1/p}$, then for all $j = 1, \dots, l$

$$s_j = \prod_{i=jp}^{(j+1)p-1} g_i,$$

therefore

$$\begin{aligned} g_i \mid h_j &\iff i = j \bmod p \\ g_i \mid s_j &\iff j = \left\lfloor \frac{i}{p} \right\rfloor. \end{aligned}$$

Algorithm 20 Squarefree decomposition in $\mathbb{F}_p$

Input: $f \in \mathbb{F}_p[X]$

Output: $(g_1, \dots, g_m)$ the squarefree decomposition of f

1. Call Algo. 19 on f, obtaining $(h_1, \dots, h_{p-1}) = (h_1, \dots, h_k, 1, \dots, 1)$
2. $b \leftarrow \dfrac{f}{h_1 h_2^2 \dots h_{p-1}^{p-1}}$
3. If $b = 1$ then return $(h_1, \dots, h_k)$
4. Recursively compute the squarefree decomposition $(s_1, \dots, s_l)$ of $b^{1/p}$
5. $g_{jp+i} \leftarrow \gcd(h_i, s_j)$ $(i = 1, \dots, p-1, j = 1, \dots, l)$
6. $g_{jp} \leftarrow \dfrac{s_j}{g_{jp+1} \dots g_{jp+p-1}}$ $(j = 1, \dots, l)$
7. $g_i \leftarrow \dfrac{h_i}{g_{p+i} \dots g_{lp+i}}$ $(i = 1, \dots, p-1)$
8. Return $(g_1, g_2, \dots)$