Physical Modelling of Complex Systems: Assignment 2

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Contents

1 Rabbit versus sheep

We consider the following model for two competing populations, which belongs to the class of Lotka-Volterra models:

$$\begin{cases} \dot{N} &= N(3 - N - 2M) \\ \dot{M} &= M(2 - N - M) \end{cases} . \tag{1.1}$$

The two populations follow a logistic growth, but compete for common resources. This interaction is indicated by the negative non-linear cross terms. The system has four fixed points, which are solutions to the equations

$$\begin{cases} N(3 - N - 2M) &= 0 \\ M(2 - N - M) &= 0 \end{cases} . \tag{1.2}$$

The first obvious solution is $(N, M) = (0, 0)$. Other solutions are found by letting $N = 0$ but $M \neq 0$ and vice versa. These result in the solutions $(N, M) = (0, 2)$ and $(N, M) = (3, 0)$. If

$N \neq 0$ and $M \neq 0$, there is one more solution. Indeed, then the system of equations (1.2) becomes

$$\begin{cases} 3 - N - 2M & = 0 \\ 2 - N - M & = 0 \,. \end{cases} \tag{1.3}$$

The second equation gives $M = -N + 2$, substituting this in the first equation then gives $N = 1$ such that $M = 1$. Hence $(N, M) = (1, 1)$ is another fixed point of the system of equations.

We now determine the nature of these fixed points. For this, we will need the Jacobian matrix associated to the system of equations (1.1). If we write $\dot{N} = f(N, M)$ and $\dot{M} = g(N, M)$, then the Jacobian matrix is

$$J(N, M) = \begin{pmatrix} \partial_N f & \partial_M f \\ \partial_N g & \partial_M g \end{pmatrix} = \begin{pmatrix} 3 - 2N - 2M & -2N \\ -M & 2 - N - 2M \end{pmatrix} \,. \tag{1.4}$$

The nature of the fixed point is determined by the trace and determinant of the Jacobian, evaluated at the fixed point. Below, we will often compute the discriminant

$$\Delta = \operatorname{tr}^2 A - 4 \det A \,. \tag{1.5}$$

Let us first investigate the fixed point $(N, M) = (0, 0)$. The Jacobian evaluated at this fixed point gives

$$J(0, 0) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \,. \tag{1.6}$$

We can directly see the two eigenvalues of the matrix are $\lambda_1 = 3$ and $\lambda_2 = 2$. Hence this fixed point is an unstable node. For the second fixed point, $(N, M) = (0, 2)$ we have

$$J(0, 2) = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \,. \tag{1.7}$$

Since the matrix is triangular, we can see that its eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -2$. Hence both eigendirections are stable, and this fixed point is a stable node. For the third fixed point, $(N, M) = (0, 3)$, we have

$$J(3, 0) = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix} \,. \tag{1.8}$$

As before, we can directly read off the eigenvalues since the matrix is triangular. We find $\lambda_1 = -3$ and $\lambda_2 = -1$, such that both eigendirections are stable, and this fixed point is also a stable node. Finally, for the fourth fixed point, $(N, M) = (1, 1)$, we have

$$J(1, 1) = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \,. \tag{1.9}$$

For this matrix we find $\det J(1, 1) = -1$, such that the fourth fixed point is a saddle node. A short calculation shows that the eigenvalues are $\lambda_{\pm} = -1 \pm \sqrt{2}$, such that indeed one eigenvalue is positive, while the other eigenvalue is negative. The eigenvectors corresponding to these eigenvalues are $\boldsymbol{v}_- = (\sqrt{2}, -1)$ and $\boldsymbol{v}_+ = (\sqrt{2}, 1)$.

The full phase portrait is given below in Figure 1.1 below, from which we can deduce some general features. The two fixed points on the axes attract solutions from all directions, while the fixed point $(1, 1)$ attracts solutions from one direction, but repels solutions from another direction. When close to the origin, solutions get repelled away from the origin.

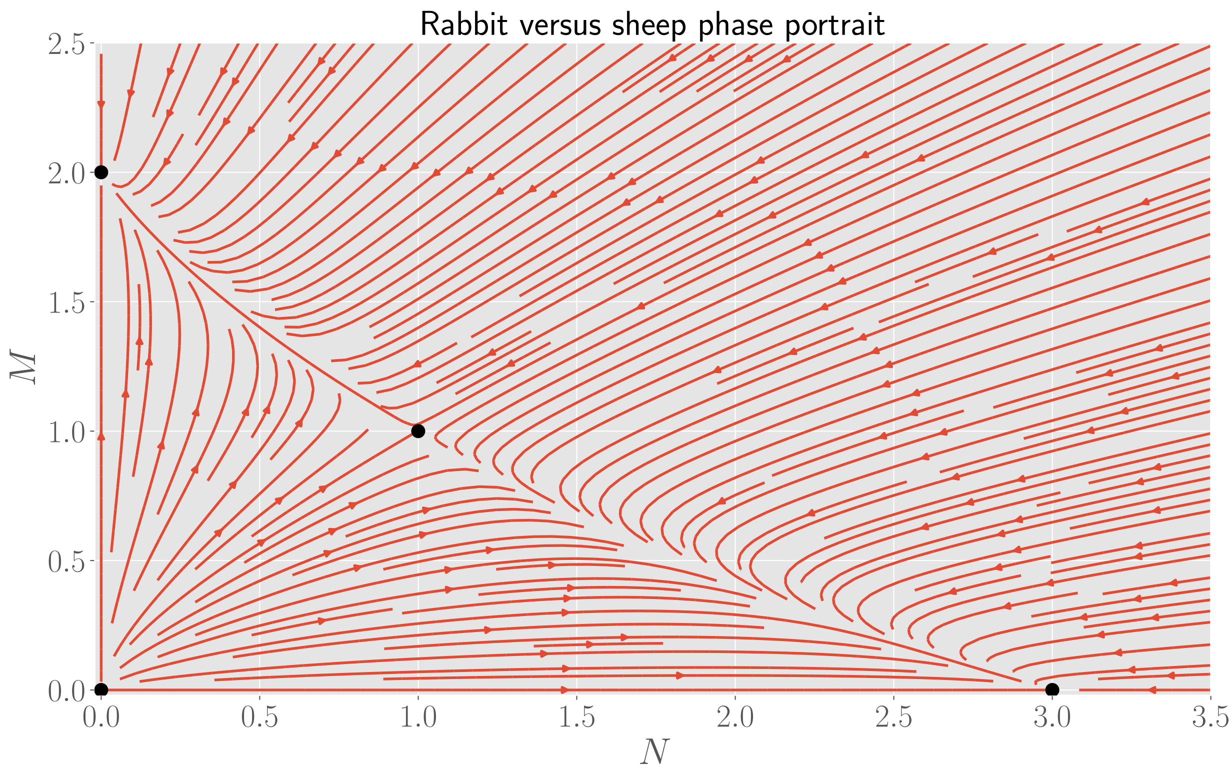

Figure 1.1: Phase portrait of the system of equations in (1.1). The black dots indicate the fixed points.

We can also verify the eigendirections of the saddle node. A close-up of the saddle node $(N, M) = (1, 1)$ is given in Figure 1.2 below. The phase portrait close to this saddle node agrees with our expectations based on the earlier analytic computations.

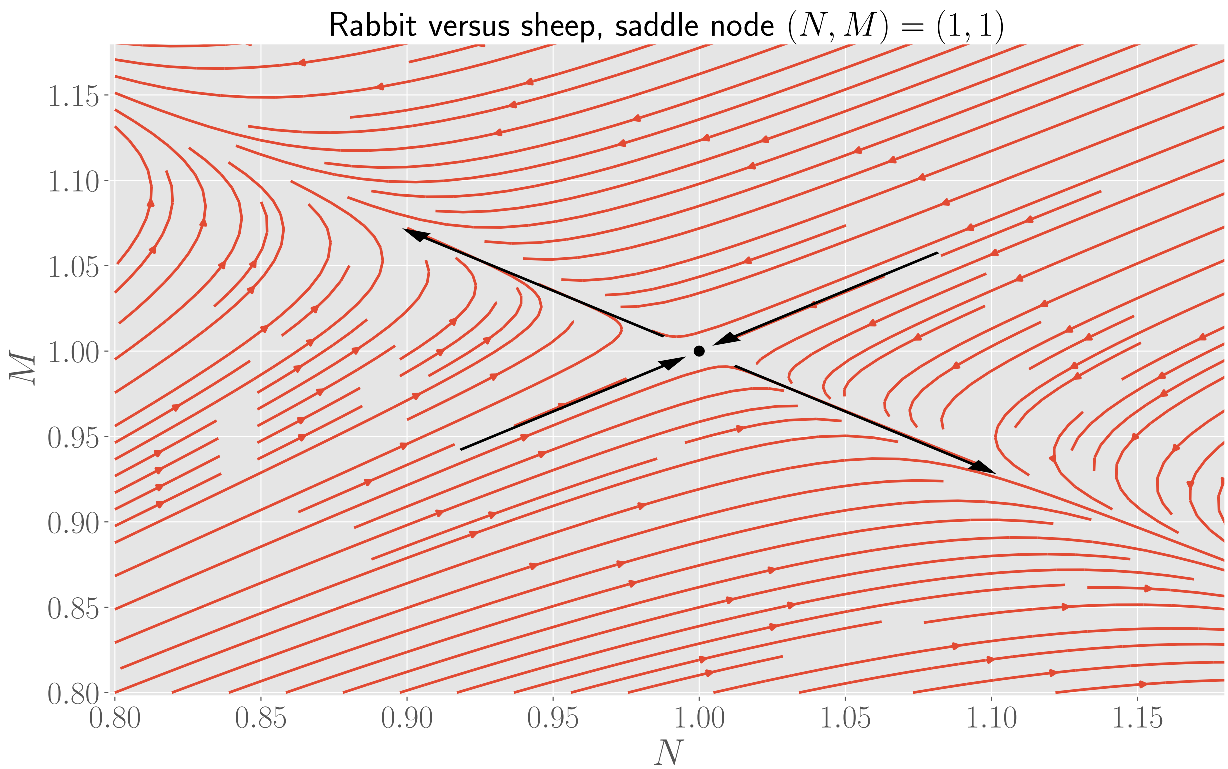

Figure 1.2: Close-up of the saddle node $(N, M) = (1, 1)$, the black dot in the plot. The two arrows denote the eigenvectors of the Jacobian matrix $J(1, 1)$, the direction of the arrow denoting attraction or repulsion.

We can also conclude from Figure 1.1 that the two populations can not coexist with each other. Indeed, for almost all initial conditions, the phase portrait shows that the solutions tend towards either the fixed point $(N, M) = (2, 0)$ or the fixed point $(N, M) = (0, 3)$. In both cases, this means that one of the species goes extinct. There is an exception to this scenario, namely for initial conditions which lie on the straight line through the fixed point $(N, M) = (1, 1)$ and with slope $1/\sqrt{2}$, i.e. along the attractive eigendirection of that fixed point. Indeed, Figure 1.2 shows that in that case, the solution is attracted towards $(N, M) = (1, 1)$, and the two populations can live together. Of course, if the initial condition is located precisely at the fixed point $(1, 1)$, the two species can also coexist with each other. However, the set of points on which coexistence is possible has measure zero, such that this is in fact an exotic scenario. For a generic initial condition, one of the populations tends towards extinction.

2 Classification of linear systems

We discuss the nature of the fixed point $(x, y) = (0, 0)$ for a few systems of linear differential equations. The first system is

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -2x - 3y \end{cases} . \tag{2.1}$$

To determine the nature of the fixed point $(x, y) = (0, 0)$, we compute the Jacobian, and evaluate it at the fixed point. However, since the differential equations are linear, the Jacobian is constant:

$$J_1(x, y) = J_1 = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} . \tag{2.2}$$

We have tr $J_1 = -3$, det $J_1 = 2$ and hence $\Delta = 1 > 0$, where Δ was defined in equation (1.5). Therefore, this is a stable node. The eigenvalues and -vectors of this matrix are

$$\lambda_1 = -1,\ \lambda_2 = -2, \qquad \boldsymbol{v}_1 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{2.3}$$

The phase portrait, as well as the eigendirections found above, are shown in Figure 2.1 below.

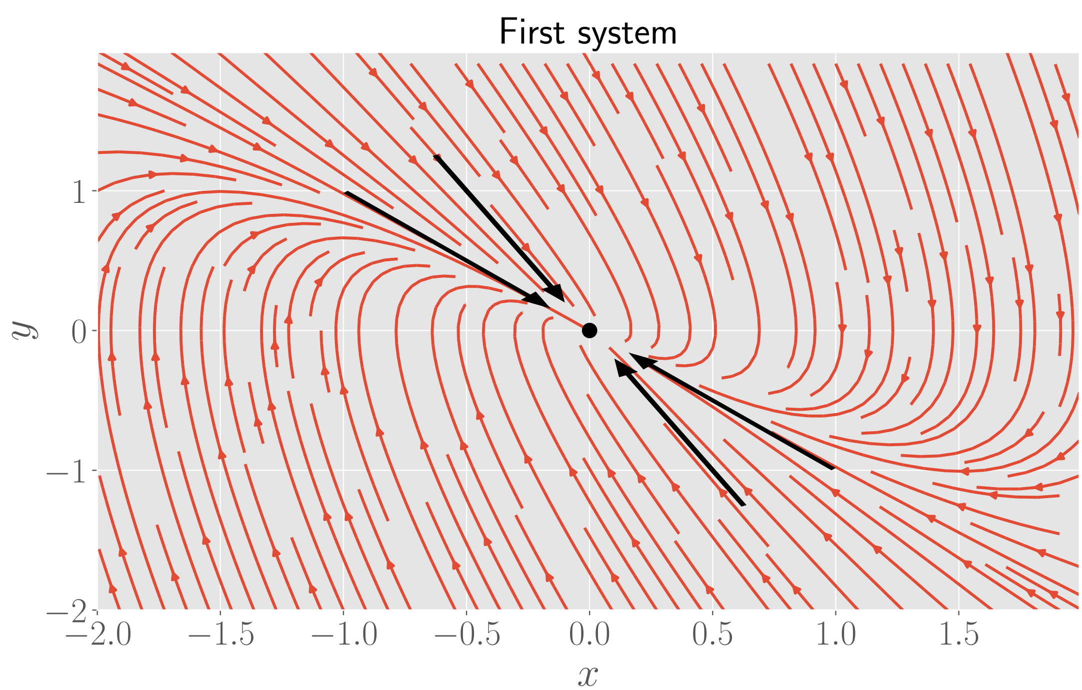

Figure 2.1: Phase portrait and eigendirections of the origin as fixed point for the first system. The direction of the arrows denotes attraction towards the origin.

The second system of equations is

$$\begin{cases} \dot{x} &= -3x + 4y \\ \dot{y} &= -x + y \end{cases} . \tag{2.4}$$

Again, the Jacobian is constant, and given by

$$J_2 = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix} , \tag{2.5}$$

such that tr $J_2 = -2$ and det $J_2 = 1$ and hence $\Delta = 0$. Hence this is a limiting case, for which the linear stability analysis is inconclusive. A short calculation shows that this matrix has a degenerate eigenvalue

$$\lambda = -1, \quad \boldsymbol{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} . \tag{2.6}$$

The phase portrait, as well as the eigendirection found above, are shown in Figure 2.2 below.

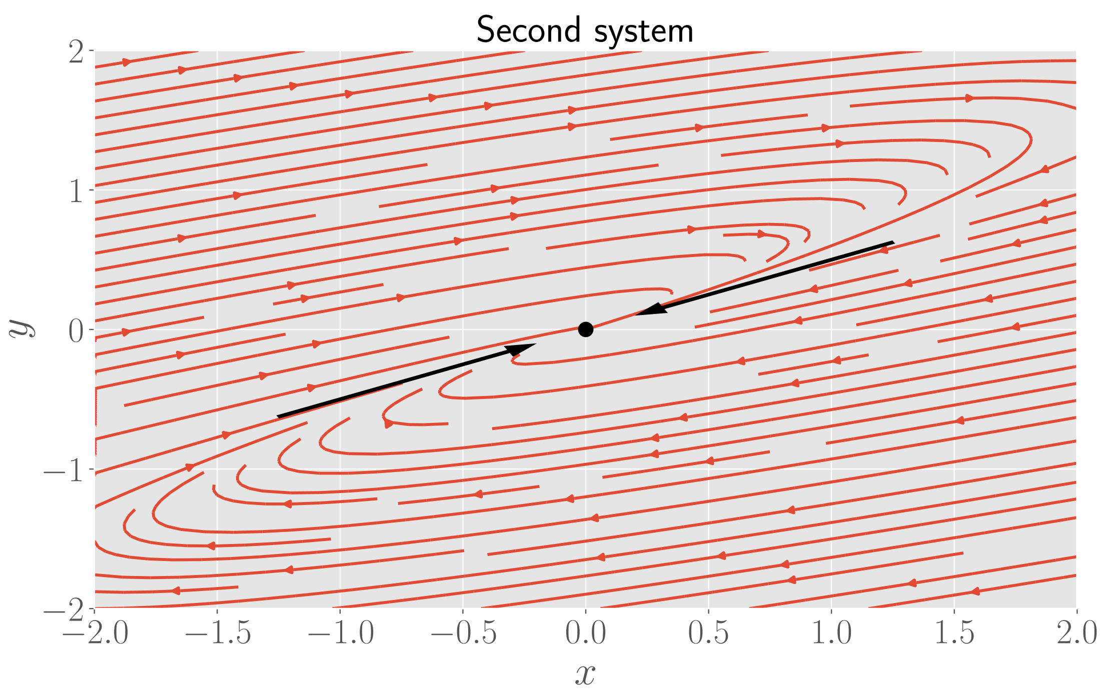

Figure 2.2: Phase portrait and eigendirection of the origin as fixed point for the second system. The direction of the arrows denotes attraction towards the origin.

The third system of equations is

$$\begin{cases} \dot{x} &= -3x + 4y \\ \dot{y} &= -2x + 3y \end{cases} . \tag{2.7}$$

Again, the Jacobian is constant, and given by

$$J_3 = \begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix} , \tag{2.8}$$

such that we have $\operatorname{tr} J_3 = 0$ and $\det J_3 = -1$. Since $\det J_3 < 0$, the origin is guaranteed to be a saddle node. A short calculation shows that the eigenvalues and -vectors of this matrix are

$$\lambda_+ = 1,\ \lambda_- = -1, \quad \boldsymbol{v}_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix} ,\ \boldsymbol{v}_- = \begin{pmatrix} 2 \\ 1 \end{pmatrix} . \tag{2.9}$$

The phase portrait, as well as the eigendirections found above, are shown in Figure 2.3 below.

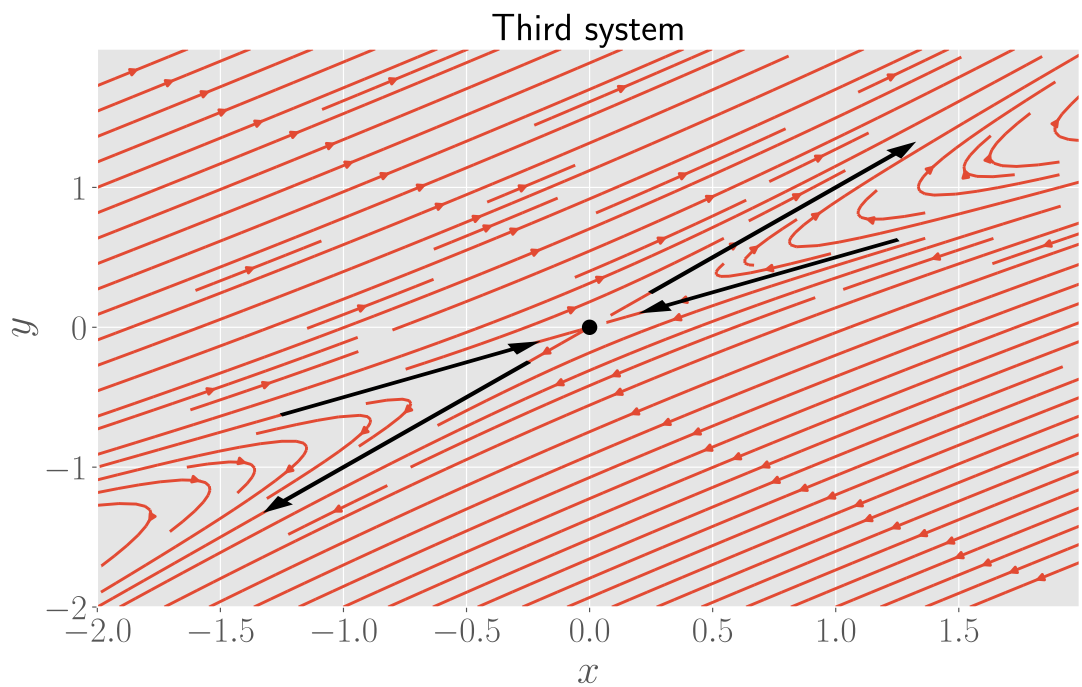

Figure 2.3: Phase portrait and eigendirections of the origin as fixed point for the third system. The direction of the arrows denotes attraction towards or repulsion away from the origin.

3 Phase portrait in two dimensions

Consider the following system of differential equations

$$\begin{cases} \dot{x} &= x - y \\ \dot{y} &= 1 - e^x \end{cases} . \tag{3.1}$$

We are interested in the nullclines, which are the curves in the plane for which $\dot{x} = 0$ or $\dot{y} = 0$. The equation $\dot{x} = 0$ gives $y = x$ and hence this nullcline is a straight line with slope one. The second equation, $\dot{y} = 0$, gives $e^x = 1$, so the solution to this equation is $x = 0$, which is the y-axis. The fixed points are the intersections between these two nullclines, and hence the origin is a fixed point for our system of equations. The nullclines divide the (x, y)-plane into regions with different signs for $\dot{x}$ and $\dot{y}$. These regions are shown in Figure 3.1 below. We have

- Region I: $\dot{x} > 0$ and $\dot{y} < 0$,
- Region II: $\dot{x} < 0$ and $\dot{y} < 0$,
- Region III: $\dot{x} < 0$ and $\dot{y} > 0$,
- Region IV: $\dot{x} > 0$ and $\dot{y} > 0$.

In Figure 3.1, we also show the expected behaviour of the phase portrait. That is, we sketch the behaviour of a few solutions (increasing or decreasing in x, increasing or decreasing in y) for a few initial conditions in the four different regions. We also made use of the fact that whenever a solution crosses a nullcline, the line tangent to the curve at that point is parallel to the x- or y-axis.

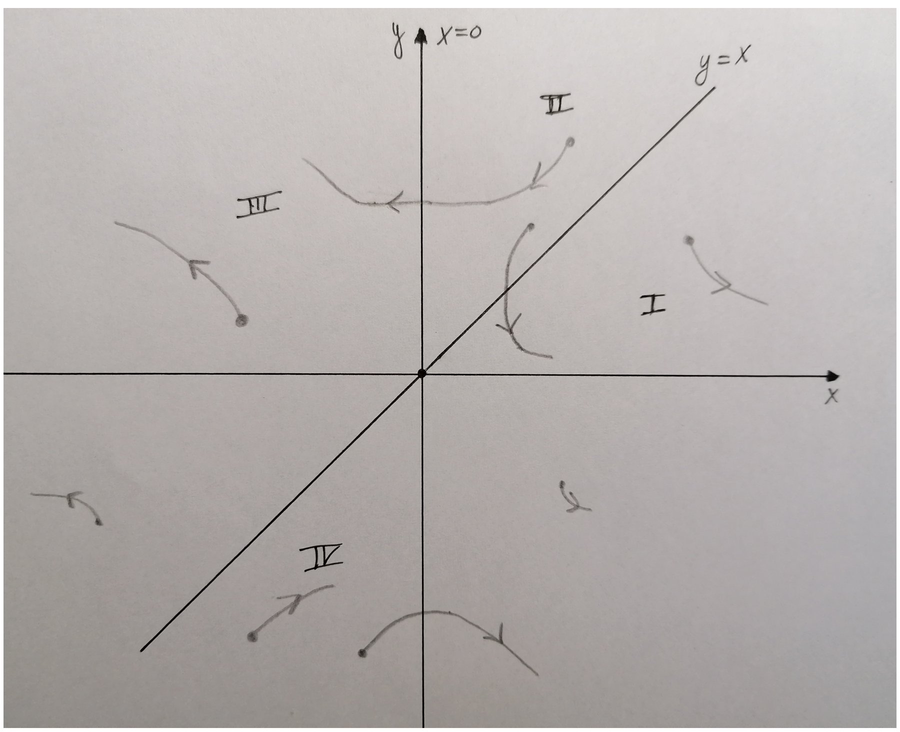

Figure 3.1: Sketch of the phase portrait of the system of equations (3.1) and the four different regions.

We now verify these findings by plotting the solutions to the system of equations obtained via a numerical computation, for the initial conditions $(1, 2)$, $(1, 1.5)$, $(-1, -2)$ and $(-1, -1.2)$. This is shown in Figure 3.2 below, where we also plot the phase portrait. Comparing with our sketch, we see that the solutions indeed agree with the expected behaviour, based on the signs of $\dot{x}$ and $\dot{y}$ in the four regions partitioned by the nullclines. We can also check that when the solutions cross the nullclines, they are parallel to the x- or y-axis.

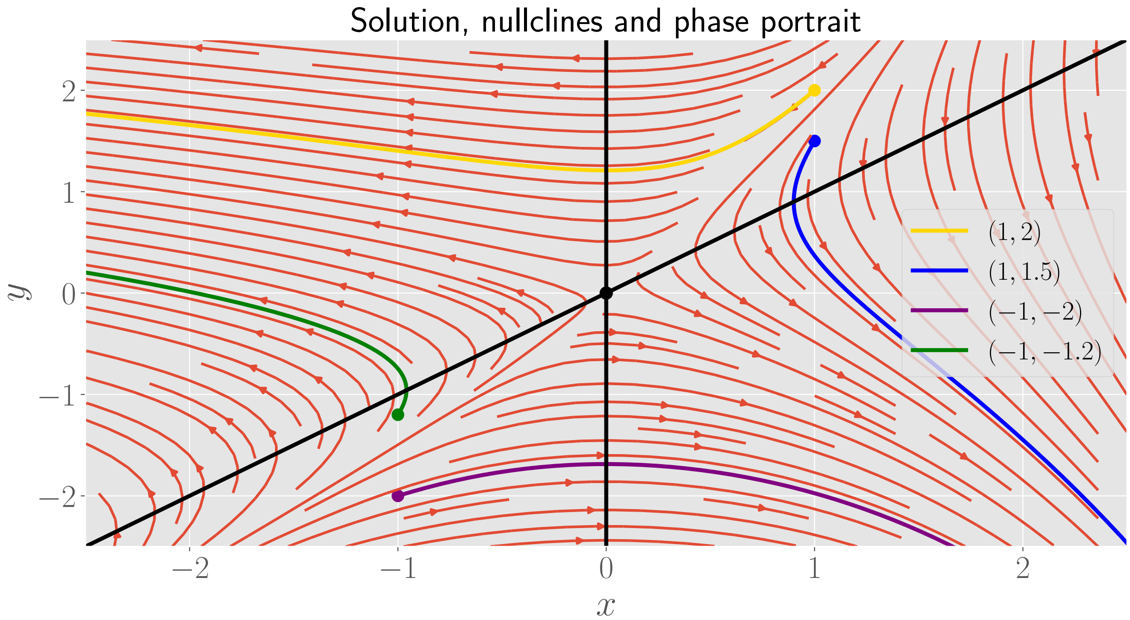

Figure 3.2: Solutions to the system of equations (3.2), with initial conditions given in the legend and the phase portrait. The black lines denote the two nullclines and their intersection, the origin, is the fixed point.