Physical Modelling of Complex Systems: Assignment 7

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1 The period of a non-uniform oscillator

The Adler equation is

$$\dot{\theta} = \omega - a \sin\theta \,, \tag{1.1}$$

where $\omega > a > 0$. The period of the oscillations of the Adler system is then given by the integral

$$T = \int \mathrm{d}t = \int_0^{2\pi} \frac{\mathrm{d}t}{\mathrm{d}\theta}\, \mathrm{d}\theta = \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{\omega - a\sin\theta} \,, \tag{1.2}$$

The goal of this exercise is to evaluate this integral and find an expression for the period. First, we make the change of variables $u = \tan\frac{\theta}{2}$. Then $\theta = 2\arctan(u)$ and the lower, upper integration bound becomes $-\infty$, $+\infty$ respectively. We have that

$$\mathrm{d}u = \frac{1}{2\cos^2\left(\frac{\theta}{2}\right)}\, \mathrm{d}\theta \,, \tag{1.3}$$

and therefore $\mathrm{d}\theta = 2\cos^2(\theta/2)\,\mathrm{d}u$. Recall that

$$\cos\left(\arctan(x)\right) = \frac{1}{\sqrt{x^2+1}} \,, \tag{1.4}$$

and therefore, we find

$$\mathrm{d}\theta = \frac{2}{u^2+1}\,\mathrm{d}u\,. \tag{1.5}$$

Now we have to rewrite $\sin\theta$ in terms of u. Note that

$$\begin{aligned}\sin\theta &= 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\\ &= 2\frac{u}{u^2+1}\,,\end{aligned} \tag{1.6}$$

where in the second equality, we made use of equation (1.4) along with

$$\sin(\arctan(x)) = \frac{x}{\sqrt{x^2+1}}\,. \tag{1.7}$$

Putting everything together, the above integral becomes

$$\begin{aligned}T &= \int_{-\infty}^{+\infty} \frac{2}{u^2+1}\frac{1}{\omega - a\frac{2u}{u^2+1}}\,\mathrm{d}u\\ &= \int_{-\infty}^{+\infty} \frac{1}{\frac{\omega}{2}u^2 - au + \frac{\omega}{2}}\,\mathrm{d}u\,.\end{aligned} \tag{1.8}$$

The denominator can be written as

$$\left(\sqrt{\frac{\omega}{2}}u - \frac{a}{\sqrt{2\omega}}\right)^2 + \frac{\omega}{2} - \frac{a^2}{2\omega}\,, \tag{1.9}$$

which is of the form $r + x^2$, with

$$x = \sqrt{\frac{\omega}{2}}u - \frac{a}{\sqrt{2\omega}}, \qquad r = \frac{\omega}{2} - \frac{a^2}{2\omega} = \frac{\omega^2 - a^2}{2\omega}\,. \tag{1.10}$$

Note that this change of variables will not affect the integration bounds. Hence we have to evaluate the integral

$$T = \sqrt{\frac{2}{\omega}}\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{r + x^2}\,. \tag{1.11}$$

We make the change of variables $x = \sqrt{r}\tan\alpha$, such that $\mathrm{d}x = (\sqrt{r}/\cos^2(\alpha))\,\mathrm{d}\alpha$, and the integral becomes

$$\sqrt{\frac{2}{\omega}}\int_{-\pi/2}^{\pi/2} \frac{1}{r(1+\tan^2\alpha)}\frac{\sqrt{r}}{\cos^2\alpha}\,\mathrm{d}\alpha\,. \tag{1.12}$$

Using the identity $1 + \tan^2\alpha = 1/\cos^2\alpha$, the cosines cancel each other and we end up with

$$\sqrt{\frac{2}{\omega}}\frac{1}{\sqrt{r}}\int_{-\pi/2}^{\pi/2} \mathrm{d}\alpha = \sqrt{\frac{2}{\omega}}\frac{\pi}{\sqrt{r}}\,. \tag{1.13}$$

After substituting r and some algebraic manipulations, we find

$$T = \frac{2\pi}{\sqrt{\omega^2 - a^2}} \, . \tag{1.14}$$

If $a = 0$, this is indeed the period of an ordinary oscillator. If $a \to \omega$, the period starts to diverge, which is the "bottleneck behaviour" of the system as discussed during the lecture on synchronization.

2 Triangle wave in firefly model

In the lecture on synchronization, we studied the oscillating flashes of fireflies, where $\theta(t)$ denoted the phase of the firefly's flashing rhythm, with $\theta = 0$ corresponding to the instant when a flash is emitted. In the absence of stimuli, we have $\dot{\theta} = \omega$, with ω the frequency of the flashing. Now we apply a periodic stimulus with phase Φ determined by $\dot{\Phi} = \Omega$ (again, $\Phi = 0$ is the instant the stimulus emits a flash). When using the Adler equation to model the behaviour of fireflies, the sinusoidal form of the firefly's response function was chosen somewhat arbitrily. In this section, we consider an alternative model, taking $\dot{\theta} = \omega + Af(\Phi - \theta)$, where $f(\phi)$ is now a triangle wave

$$f(\phi) = \begin{cases} \phi & \text{for } -\pi/2 \leq \phi \leq \pi/2 \\ \pi - \phi & \text{for } \pi/2 \leq \phi \leq 3\pi/2 \end{cases} \, , \tag{2.1}$$

and this extended periodically to the whole real line. The parameter A is called the *resetting strength* and effectively measures the strength with which the firefly modifies its instantaneous frequency. The triangle wave $f(\phi)$ is shown in Figure 2.1 below. We are interested in the behaviour of the system with this response function. For our calculations below, we let us inspire by section 4.5 in Strogatz [1].

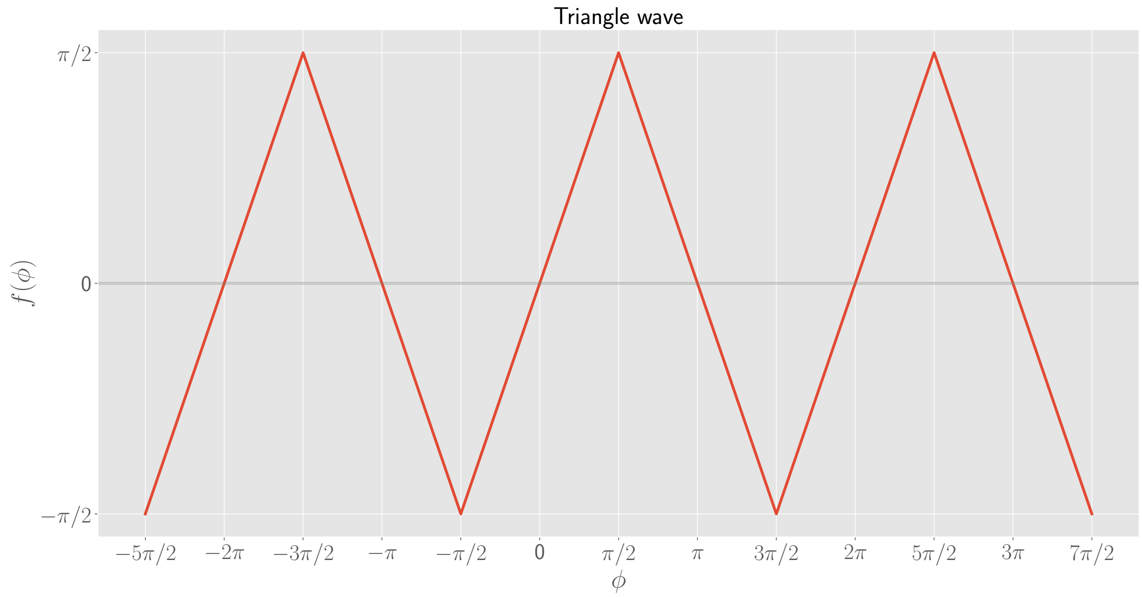

Figure 2.1: Plot of three periods of the triangle wave $f(\phi)$, given by equation (2.1).

The response of the firefly is best described by considering the phase difference $\phi \equiv \Phi - \theta$. From the differential equations defined earlier, we derive

$$\dot{\phi} = \dot{\Phi} - \dot{\theta} = \Omega - \omega - Af(\phi)\,. \tag{2.2}$$

It is convenient to study this equation by non-dimensionalizing it. To this end, introduce

$$\mu = \frac{\Omega - \omega}{A}, \quad \tau = At\,. \tag{2.3}$$

First of all, we have that

$$\dot{\phi} = A\left(\mu - f(\phi)\right)\,. \tag{2.4}$$

Furthermore, we find (letting primes denote derivatives with respect to τ)

$$\dot{\phi} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = A\phi'\,, \tag{2.5}$$

such that we end up with

$$\phi' = \mu - f(\phi)\,. \tag{2.6}$$

This differential equation allows us to study the behaviour of the phase difference ϕ between the stimulus and the firefly as a function of a single parameter μ, which can easily be done graphically. In Figure 2.2 below, we plot the right hand side of the differential equation for three different values for a positive parameter μ. We can easily see that for $0 < \mu < \pi/2$, the above equation has two fixed points ϕ^*, which can be determined graphically or analytically by solving $\phi' = 0$. This gives $f(\phi^*) = \mu$, with stability determined by the sign of $-f'(\phi^*)$.

So there exists a stable fixed point at $\phi = \mu$, and an unstable fixed point at $\phi = \pi - \mu$. As μ increases and reaches $\mu = \pi/2$, these fixed points come together in a saddle-node bifurcation.

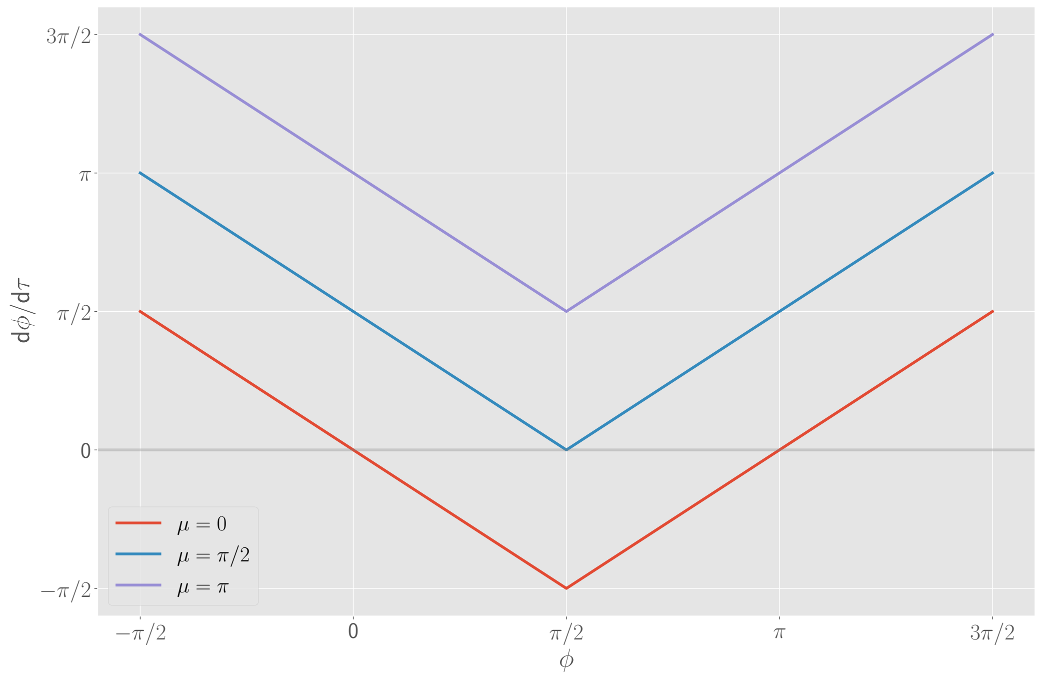

Figure 2.2: Plot of the differential equation (2.6) for $\mu = 0, \pi/2$ and π, showing the stable and unstable fixed points come together in a saddle-node bifurcation at $\mu = \pi/2$.

The important aspect is that the *range of entrainment* in the triangle wave model is $0 < \mu < \pi/2$, with phase difference during *phase locking* given by

$$\phi^* = \mu = \frac{\Omega - \omega}{A}\,. \tag{2.7}$$

Note that $\mu > 0$ means that $\Omega > \omega$, and we intuitively understand why $\phi^* > 0$. This means that the stimulus flashes ahead of the firefly, but both have the same frequency.

The above considerations and plot were for a positive μ, but we can equally well do the same analysis for negative μ. The expressions for the fixed points still hold in this regime, and due to the symmetry of the right hand side of the differential equation (2.6), we see that the range of entrainment for negative μ is $-\pi/2 < \mu < 0$. Hence we conclude that the full range of entrainment is $-\pi/2 < \mu < \pi/2$. This implies that the entrainment range for the stimulus frequency is

$$\omega - \frac{\pi}{2}A < \Omega < \omega + \frac{\pi}{2}A\,. \tag{2.8}$$

For μ (or Ω) outside the range of entrainment, the right hand side of equation (2.6) is always strictly positive (as is also clear from the example in Figure 2.2), such that the phase difference increases indefinitely, and the solution is a *phase drift*. To find the period T for these drifting solutions, we proceed as in the first section, and compute

$$T = \int \mathrm{d}t = \int_0^{2\pi} \frac{\mathrm{d}t}{\mathrm{d}\phi}\,\mathrm{d}\phi = \int_0^{2\pi} \frac{1}{\Omega - \omega - Af(\phi)}\,\mathrm{d}\phi\,. \tag{2.9}$$

Since the integrand is periodic with period 2π, we can equally well integrate in the range $-\pi/2 \leq \phi \leq 3\pi/2$ to get the same result. Using the definition of $f(\phi)$, we find

$$T = \int_{-\pi/2}^{\pi/2} \frac{1}{\Omega - \omega - A\phi}\,\mathrm{d}\phi + \int_{\pi/2}^{3\pi/2} \frac{1}{\Omega - \omega - A(\pi - \phi)}\,\mathrm{d}\phi\,. \tag{2.10}$$

We can easily compute the above integrals, and we find, after some straightforward algebraic manipulations, that the period is given by

$$T = \frac{2}{A}\ln\left|\frac{\Omega - \left(\omega - A\frac{\pi}{2}\right)}{\Omega - \left(\omega + A\frac{\pi}{2}\right)}\right|\,. \tag{2.11}$$

We verify this expression by numerically integrating equation (2.2) for the specific values[1] $A = 1$, $\omega = 1.6$ and $\Omega = 3.3$, such that $\Omega > \omega + A\pi/2$ and there are drifting solutions. Equation (2.11) predicts that $T \approx 6.4628$. In Figure 2.3 below, we plot the solution with initial condition $\phi(0) = 0$. The sudden jump in the graph is since we take into account that ϕ is a periodic variable with period 2π. Therefore, at the jump, $\phi(t)$ reaches 2π and the system returns to its initial configuration, from which we then deduce the period. This example agrees with the analytic calculation above. Note that the system spends the majority of its time getting through a bottleneck (which is the 'plateau' we see in the graph). As expected, this plateau (and hence the period) becomes larger if we let $\Omega \to \omega - A\pi/2$ or $\Omega \to \omega + A\pi/2$ (i.e., approach the bounds of the entrainment range). This is also predicted by equation (2.11).

[1] All parameters have dimensions of a frequency, but we will not use a specific unit in what follows, since we only want to briefly check via a numerical computation if the analytic calculation makes sense.

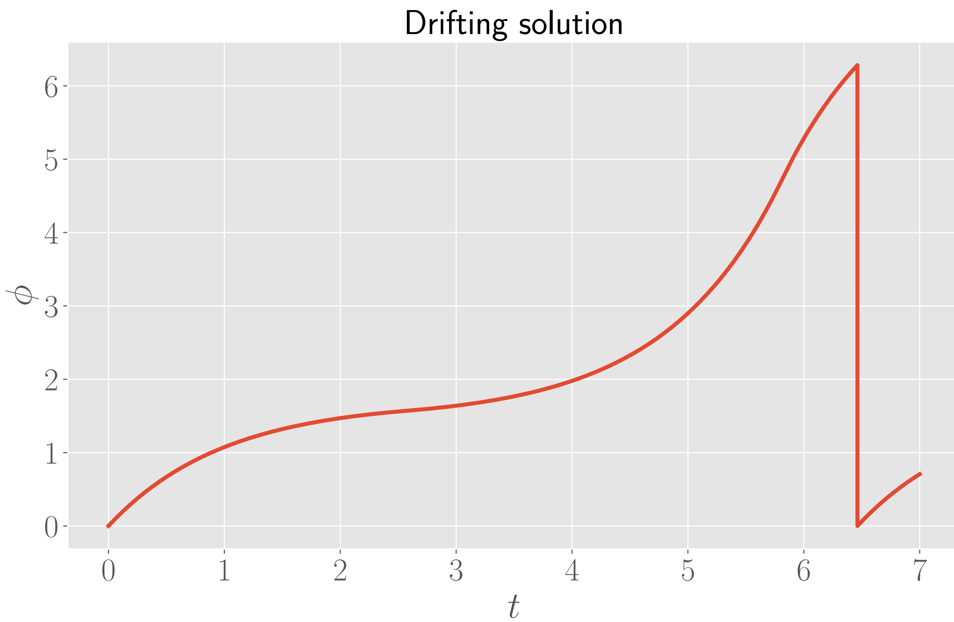

Figure 2.3: Drifting solution for $A = 1$, $\omega = 1.6$ and $\Omega = 3.3$, with period $T \approx 6.4628$.

2.1 Comparison between firefly models

We will now briefly compare the triangle wave model, which used $f(\phi)$ as defined above, with the model that was studied during the lecture (or section 4.5 in Strogatz) which used $f(\phi) = \sin\phi$. In the latter, the range of entrainment was $\omega - A \leq \Omega \leq \omega + A$, so the triangle wave model has a larger range of entrainment. However, both ranges are symmetric around ω. In Figure 2.4, we compare the behaviour of the period T of drifting solutions as a function of Ω for both models. Both models have a similar dependence on Ω for the period, especially for larger values of Ω. We conclude that the qualitative behaviour of the system is identical in both models.

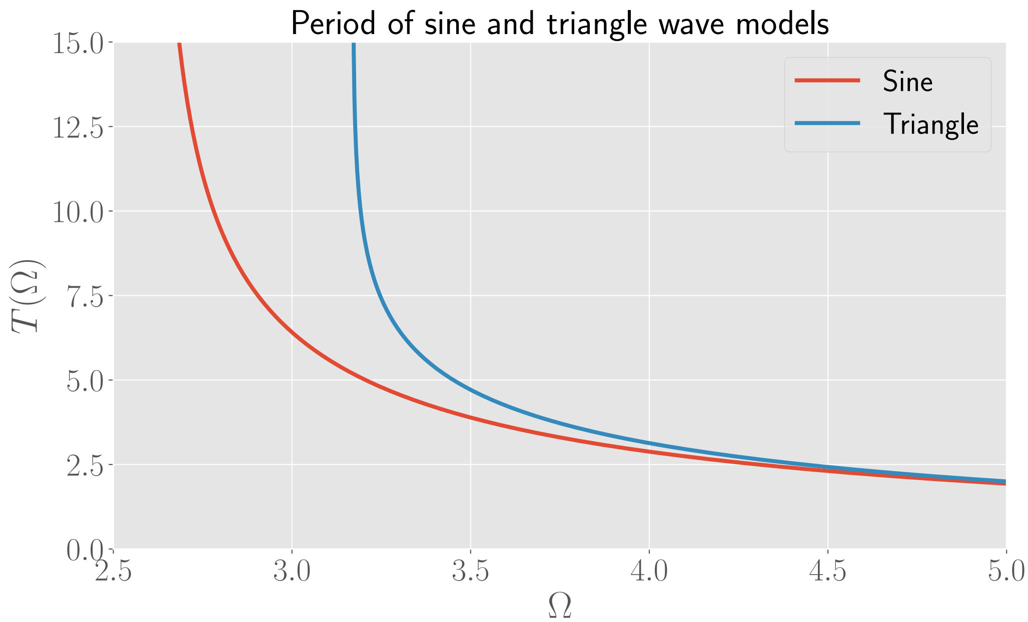

Figure 2.4: Period of drifting solutions for the sine and triangle wave models as a function of the stimulus frequency Ω, taking $A = 1$ and $\omega = 1.6$ for both graphs. Note that the range of entrainment is smaller for the sine model, which explains why the divergence of this curve occurs earlier than the triangle model.

References

[1] Steven H Strogatz. *Nonlinear dynamics and chaos: With applications to physics, biology, chemistry, and engineering.* CRC press, 2018.