Physical Modelling of Complex Systems: Assignment 1

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February 27, 2021

1 Logistic growth

We solve the logistic equation

$$\dot{N} = \alpha N \left(1 - \frac{N}{K}\right) , \tag{1.1}$$

via the change of variables $y = 1/N$, with initial condition $N = N_0$. We have $\dot{y} = -1/N^2 \dot{N}$ and $y(0) = 1/N_0$, so that

$$\dot{y} = -\frac{1}{N^2}\dot{N} = -\alpha y + \frac{\alpha}{K} \, . \tag{1.2}$$

It is easy to see that a solution to this equation, with the correct initial condition, is given by

$$y(t) = \left(\frac{1}{N_0} - \frac{1}{K}\right) e^{-\alpha t} + \frac{1}{K} \, . \tag{1.3}$$

From this, we can find the solution for N, by using $y = 1/N$. This results in

$$N(t) = \frac{N_0 K}{N_0 + (K - N_0)e^{-\alpha t}} = \frac{K}{1 + \left(\frac{K-N_0}{N_0}\right) e^{-\alpha t}} \, . \tag{1.4}$$

We now solve the above differential equation numerically. For this, a function in Python is written which discretizes time into N time steps of equal length Δt, such that the differential equation $\dot{N} = f(N)$ is solved by iteratively solving $N(t + \Delta t) \approx N(t) + f(N(t))\Delta t$. Below in Figure 1.1, we plot the exact solution, given by equation (1.4), along with the solution obtained by the numerical method for $\alpha = 3$ and $K = 10^3$. One solution has initial condition $N_0 = 10$, while the other has $N_0 = 10^5$. The two graphs clearly coincide in both cases.

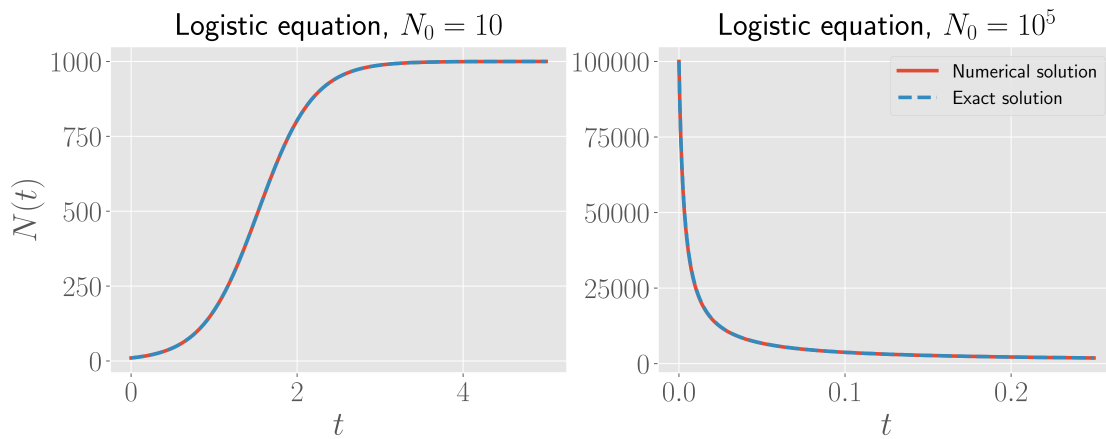

Figure 1.1: Exact and numerically obtained solutions to the logistic differential equation, for $\alpha = 3$ and $K = 10^3$, for $N_0 = 10$ and $N_0 = 10^5$.

The two fixed points of the logistic equation (1.1) are

$$N_1^* = 0, \quad N_2^* = K\,. \tag{1.5}$$

We now perform a linear stability analysis of this differential equation by expanding around the fixed point $N^* = K$

$$\dot{N} = f(N) \approx f'(N^*)(N - N^*) + \ldots \tag{1.6}$$

We define $\eta(t) = N(t) - N^*$, such that the solution to the linearised equation is given by

$$\eta(t) = Ae^{\pm t/\tau}, \quad \tau = \frac{1}{|f'(N^*)|}\,, \tag{1.7}$$

where the sign in the exponent is the sign of $f'(N^*)$ and τ is a characteristic relaxation time. A short computation shows

$$f'(N) = \alpha - \frac{2\alpha N}{K}\,, \tag{1.8}$$

such that $f'(N^*) = -\alpha < 0$. Hence the fixed point N^* is a stable fixed point, and the characteristic relaxation time is $\tau = 1/\alpha$. This was already clear from the two specific examples discussed in Figure 1.1: the solutions tend towards the value K, as is clear by letting $t \to +\infty$ in the exact solution (1.4). The long time behaviour we expect to see is an exponential in time as given by equation (1.7). Indeed, after a large amount of time, the system should be close to the fixed point, at which the linear approximation should be correct, of which the solution is an exponential.

We verify the above statements graphically. We plot $\log|\eta(t)| = \log|N(t) - N^*|$ as a function of t. According to our earlier remarks, we expect the plot to represent a straight line for large enough time periods. In Figure 1.2 below, this is indeed the case, for both the exact as well as the numerically obtained solutions shown in Figure 1.1. The long time behaviour indeed follows a straight line with slope $-\frac{1}{\tau} = -3$.

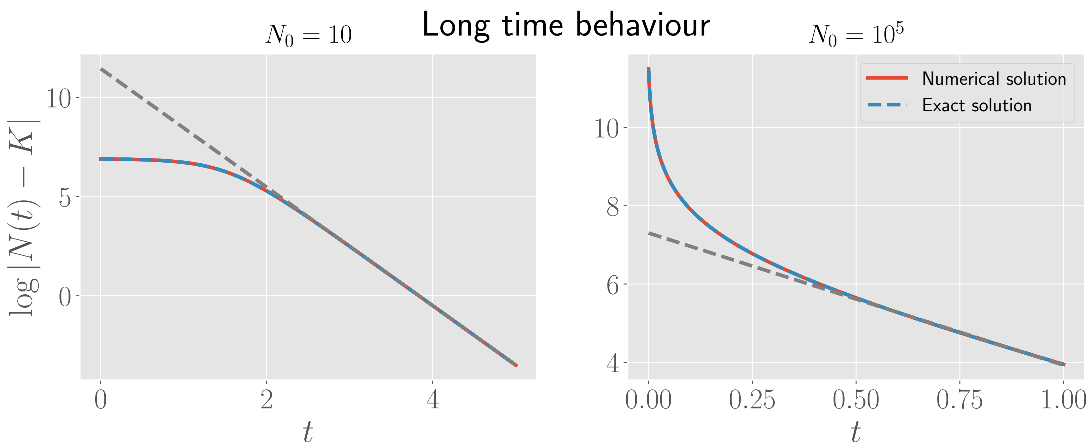

Figure 1.2: Long time behaviour of the solutions shown in Figure 1.1. As discussed in the text, $N(t) - K$ is described by an exponential distribution.

2 Sustained Harvesting

In this section, we consider a population evolving following a logistic growth. The population is subject to harvesting, for which we consider two types of models:

$$\dot{N} = \alpha N \left(1 - \frac{N}{K}\right) - EN \tag{2.1}$$

$$\dot{N} = \alpha N \left(1 - \frac{N}{K}\right) - Y_0\,, \tag{2.2}$$

where α and K are fixed, while E and Y_0 can vary. We refer to equation (2.1) as *constant effort* harvesting, while equation (2.2) is called *constant yield* harvesting. We say that there is a *sustained harvesting* if a stable fixed point with $N^* > 0$ exists. The *yield* is the fraction of harvested population per unit time, which is $Y = EN$ and $Y = Y_0$ for case (2.1) and (2.2) respectively.

We now show that for E and Y_0 sufficiently large, the above equations have no stable fixed point $N^* > 0$, i.e., there is no sustained harvesting. Moreover, we will show that for large E and Y_0, there are no fixed points at all. This can be shown graphically, but perhaps it is even simpler (and also more exact) if we just solve for the fixed points.

To find the fixed points, we should look for solutions of $\dot{N} = 0$. These equations are

$$0 = -\frac{\alpha}{K}N^2 + (\alpha - E)N \tag{2.3}$$

$$0 = -\frac{\alpha}{K}N^2 + \alpha N - Y_0\,, \tag{2.4}$$

for the constant effort and constant yield case, respectively. These are quadratic equations, and hence both equations have at most two fixed points. Let us first look at the solutions of equation (2.3). The solutions are

$$N_1^* = 0, \quad N_2^* = \frac{K}{\alpha}(\alpha - E)\,, \tag{2.5}$$

and it is clear that for $E \geq \alpha$, the above solutions are both negative, assuming α and K are positive numbers as in the previous problem. We also note that for $E = 0$, i.e. the ordinary logistic equation, these two solutions are the fixed points found the previous section. For the constant yield harvesting, the solutions are

$$N_\pm^* = \frac{K\left(\alpha \pm \sqrt{\alpha^2 - 4\frac{\alpha Y_0}{K}}\right)}{2\alpha}\,. \tag{2.6}$$

From equation (2.6), we see that there are fixed points appearing if

$$\alpha^2 - 4\frac{\alpha Y_0}{K} \geq 0\,, \tag{2.7}$$

which translates into the condition

$$Y_0 \leq \frac{\alpha K}{4} \tag{2.8}$$

for the yield parameter. If Y_0 is larger than the right hand side of the above inequality, there are no fixed points. If $Y_0 = \frac{\alpha K}{4}$, then there is a single fixed point $N^* = K/2$. Again we note that for $Y_0 = 0$, i.e. the ordinary logistic equation, this gives us the two fixed points of the logistic equation.

Now we look at values for E and Y_0 low enough such that fixed points are present and determine the stability of these fixed points. Let us first look at the case of constant effort harvesting. As discussed above, strictly positive fixed points appear if $E < \alpha$, and the fixed points are given in equation (2.5). If we write (2.1) as $\dot{N} = f_e(N)$, then the stability of a fixed point N^* is determined by the sign of $f_e'(N^*)$. A short computation shows

$$f_e'(N) = \alpha - E - \frac{2\alpha}{K}N\,, \tag{2.9}$$

such that

$$f_e'(N_1^*) = \alpha - E > 0, \quad f_e'(N_2^*) = -(\alpha - E) < 0\,, \tag{2.10}$$

where we used that $E < \alpha$. Hence, N_1^* is an unstable fixed point, while N_2^* is a stable fixed point. This agrees with the results from the previous section, if we let E tend to zero.

Now consider the constant yield case. Denoting equation (2.2) as $\dot{N} = f_y(N)$, we find

$$f_y'(N) = \alpha - \frac{2\alpha}{K} N \,, \tag{2.11}$$

and evaluating at the fixed points given in equation (2.6), this gives

$$f_y'(N_\pm^*) = \mp \sqrt{\alpha^2 - 4\frac{\alpha Y_0}{K}} \,. \tag{2.12}$$

For $Y_0 < \frac{\alpha K}{4}$, there is an unstable fixed point N_-^* and a stable solution N_+^*. If $Y_0 = \frac{\alpha K}{4}$, there is a single fixed point $N^* = K/2$, and $f_y'(K/2) = 0$. For this specific value of Y_0, we can rewrite $f_y(N)$ as

$$f_y(N) = -\frac{\alpha}{K} N^2 + \alpha N - \frac{\alpha K}{4} = -\frac{\alpha}{K}\left(N - \frac{K}{2}\right)^2 \,. \tag{2.13}$$

Hence $f_y(N) \leq 0$ for all values of N, which implies the flow is always to the left for this value of Y_0. Hence the fixed point $N^* = K/2$ is a half-stable fixed point.

Now we consider the range of parameters in which a stable fixed point N^* is present in both cases, which is $E < \alpha$ and $Y_0 < \frac{\alpha K}{4}$. If we keep α and K fixed, the maximum yield possible is to do

Harvesting at a constant yield is in general not a good strategy. To justify this claim, we look at the right hand sides of the two differential equations (2.1) and (2.4). We plot both right hand sides as a function of N in Figure 2.1 below for the parameters $\alpha = 3$, $K = 10^3$, $E = 2$ and $Y_0 = 500$.

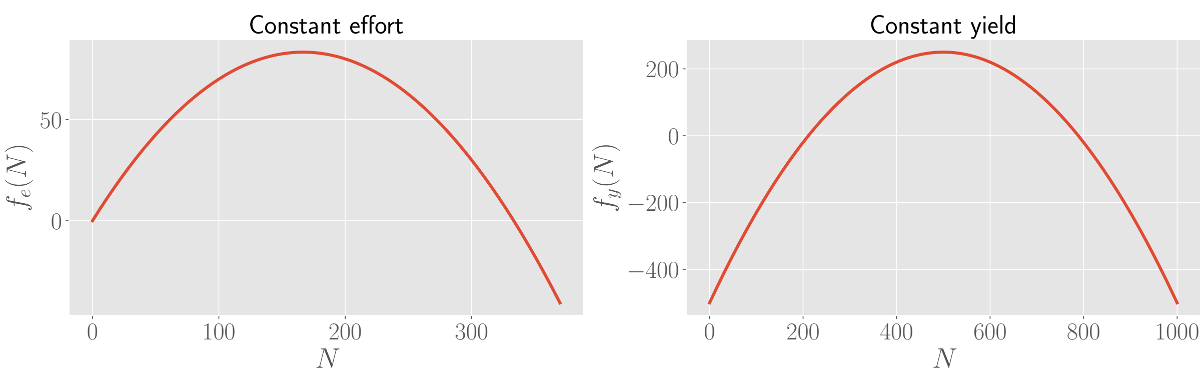

Figure 2.1: Right hand sides of the two differential equations (2.1) and (2.4) as functions of N, for $\alpha = 3$ and $K = 10^3$ fixed. *Left:* constant effort, with $E = 2$. *Right:* constant yield, with $Y_0 = 500$.

The problem with the constant yield method is the unstable fixed point N_-^*, which has a value of $N_-^* \approx 200$ in the figure above. If the initial value population size N_0 lies below this fixed point, then the population will decrease in size until it goes extinct, and harvest is no longer possible. Since N_0 is likely unknown to us, the method of constant yield has the risk of letting the population go extinct if the value Y_0 is too high. The constant effort method is therefore preferred over the constant yield method. As shown above, the unstable fixed point is always located at $N = 0$, such that for any value of $E < \alpha$ and any N_0, the population will tend to the stable fixed point after long time. Hence this method guarantees that the population will not go extinct, as long as $E < \alpha$.

3 Gompertz Law

According to the Gompertz model for population growth the individuals of a population follow the differential equation

$$\dot{N} = \alpha N \log\left(\frac{K}{N}\right) . \tag{3.1}$$

Note that the fixed points of the Gompertz equation are $N^* = 0$ and $N^* = K$, which are identical to the fixed points of the logistic equation.

Consider the change of variable $y = \log N$. Then $\dot{y} = \frac{1}{N}\dot{N}$, and hence the above differential equation becomes

$$\dot{y} = \frac{1}{N}\dot{N} = \alpha\left(\log K - y\right) . \tag{3.2}$$

This closely resembles the differential equation (1.2); the solution with initial condition $N(0) = N_0$ (and hence $y(0) = \log N_0$) is given by

$$y(t) = \left(\log N_0 - \log K\right) e^{-\alpha t} + \log K . \tag{3.3}$$

From this, we find that the solution to equation (3.1) with initial condition $N(0) = N_0$ is

$$N(t) = K\left(\frac{N_0}{K}\right)^{e^{-\alpha t}} . \tag{3.4}$$

In Figure 3.1 below, we plot the exact solutions to the logistic equation and the Gompertz equation for the parameters $\alpha = 3$, $K = 10^3$, and for initial conditions $N_0 = 10$, $N_0 = 10^5$, respectively. For the initial condition of the plot on the left hand side, the Gompertz law gives a greater rate of growth of the population, while for the initial condition of the plot on the right hand side, the converse is true. Moreover, for the plot on the left hand side, the curve for the Gompertz model converges quicker towards the stable fixed point compared to the logistic curve, while for the right hand side again the converse holds. to do: explain why

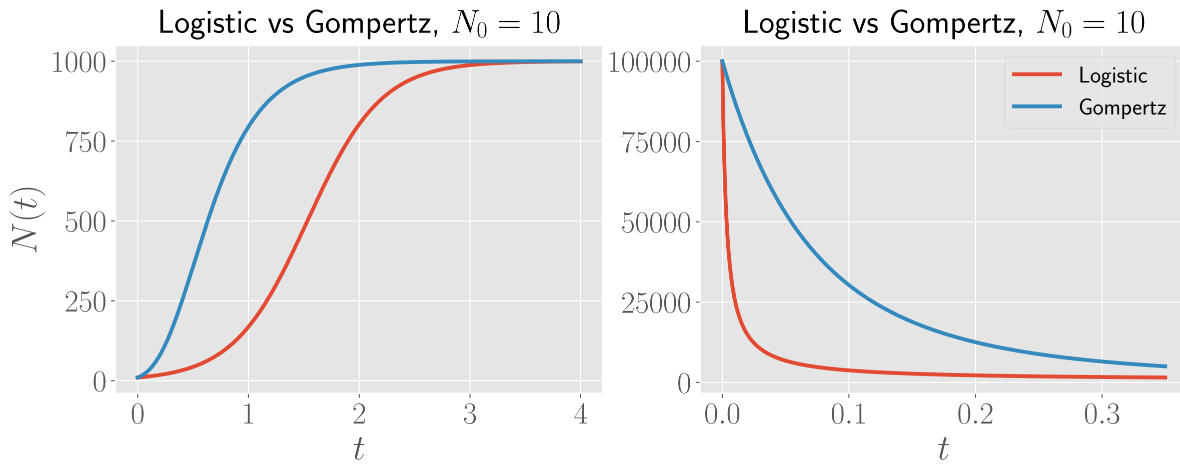

Figure 3.1: Exact solutions of the logistic and the Gompertz equation for $\alpha = 3$ and $K = 10^3$. *Left:* initial condition $N_0 = 10$. *Right:* initial condition $N_0 = 10^5$

To understand this behaviour, we need to compare the right hand sides of both differential equations as a function of N. These functions are shown in Figure 3.2. For small N, i.e. $N \ll K$, the right hand side is larger for the Gompertz model, while for large N, i.e. $N \gg K$, it is larger for the logistic model, which agrees with the observed behaviour of the solutions in Figure 3.1. Around the stable fixed point, $N \approx K$, the two models are almost identical.

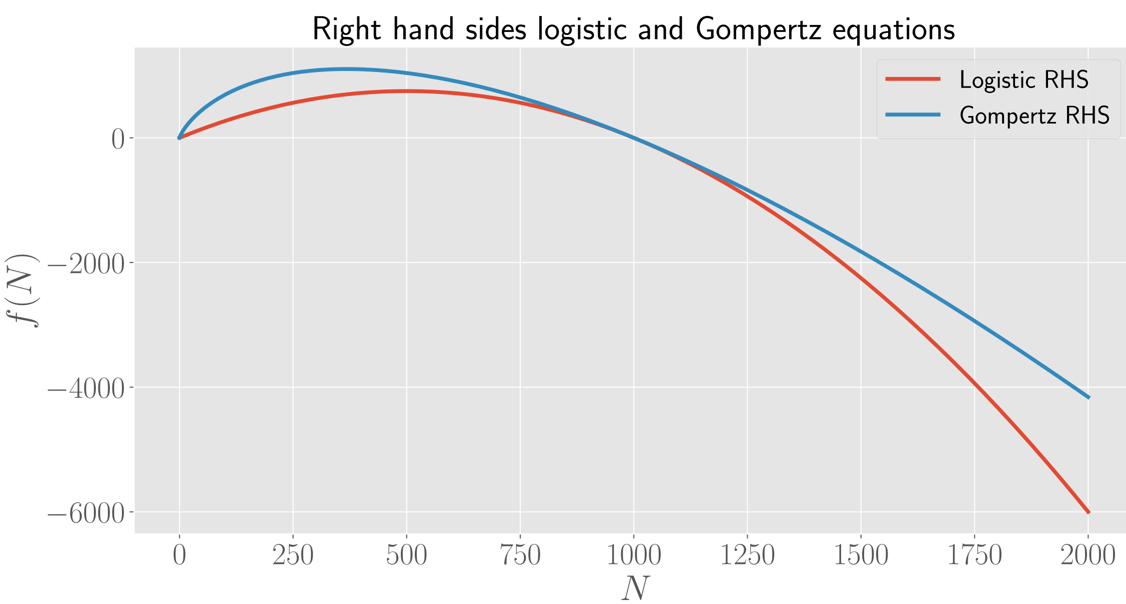

Figure 3.2: Right hand sides of the logistic and the Gompertz equation as functions of N, for $\alpha = 3$ and $K = 10^3$.

The two curves intersect in $N = 0$ and $N = K$. To derive this, we equate the right hand sides of the logistic and the Gompertz equation. It is easy to see that $N = 0$ satisfies this equation. Dividing by N to get the secon point of intersection, this gives

$$1 - \frac{N}{K} = -\log\left(\frac{N}{K}\right) , \tag{3.5}$$

and since the solution to the equation $x - \log x = 1$ is 1, the second point of intersection is indeed located at $N = K$.

4 Delays

In general, first order non-linear differential equations cannot have oscillating solutions. However, oscillating solutions are possible if the equation contains some delay. To justify this claim, we consider a delay in the logistic growth

$$\dot{N}(t) = \alpha N(t)\left(1 - \frac{N(t-T)}{K}\right) , \tag{4.1}$$

where we fix $T = 3$, $\alpha = 1$ and $K = 10^3$.

We solve the above equation numerically with initial conditions $N(t) = 10$ for $t \leq 0$, and $N(t) = 10^5$, for $t \leq 0$. The two solutions are shown in Figure 4.1. For the first initial condition, oscillations already appear around $t \approx 10$, while for the second initial condition, the oscillations appear much later, around $t \approx 300$.

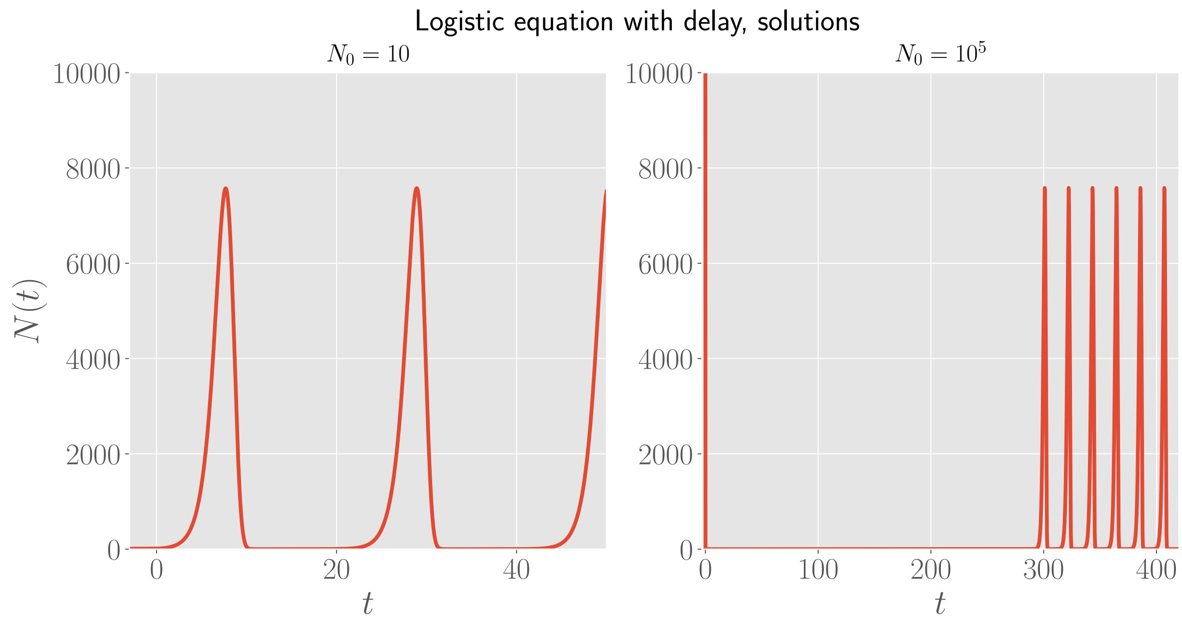

Figure 4.1: Solutions to the logistic equation with a delay, equation (4.1), with parameters mentioned in the text. *Left:* $N_0 = 10$. *Right:* $N_0 = 10^5$.

At long times, both the solutions are identical, except for possibly a phase shift. Indeed, once the oscillations are present for both of the solutions, they are identical to each other, up to a translation on the time axis. This is verified by plotting both solutions in the same figure, and restricting the range in time to an interval in which both solutions are oscillating. This is shown in Figure 4.2. Indeed, both oscillations are identical, up to a translation along the time axis.

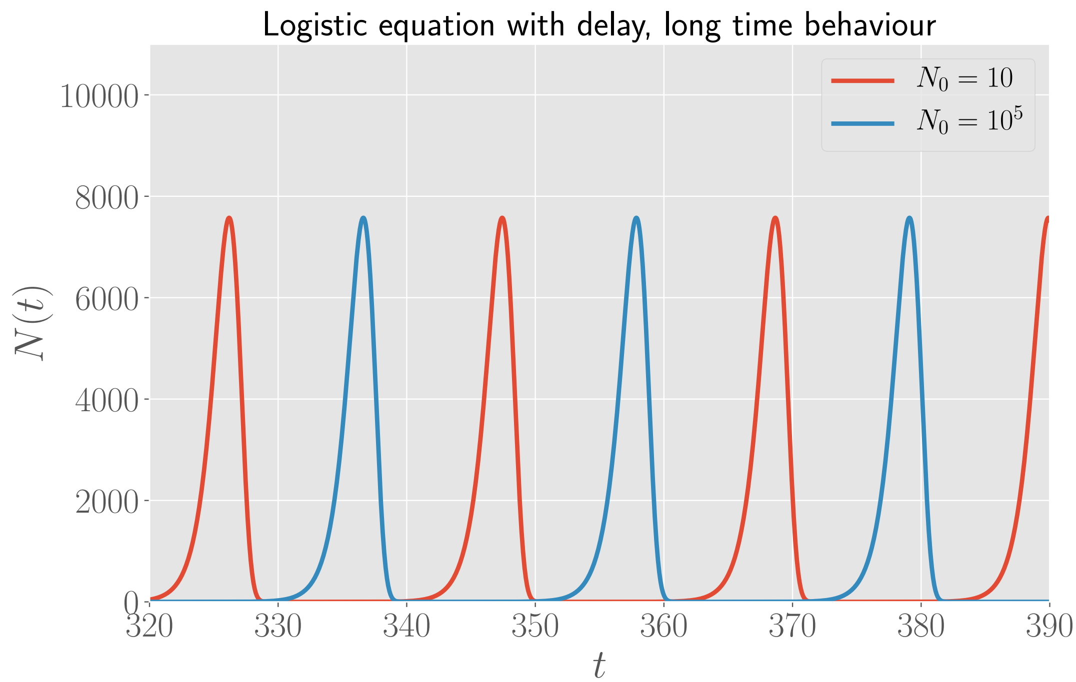

Figure 4.2: Long time behaviour of the two solutions shown in Figure 4.1, for $320 \le t \le 390$.