Leveraging Differentiable Programming in the Inverse Problem of Neutron Stars

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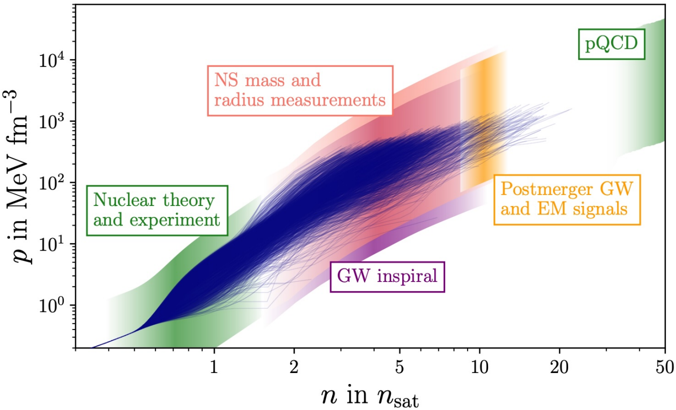
Introduction

Inverse problem of neutron stars: infer the equation of state (EOS) of dense nuclear matter from observations of neutron stars (NSs): masses, radii, tidal deformabilities, . . .

Bottleneck: solving Tolman-Oppenheimer-Volkoff (TOV) equations



Inverting neutron stars with differentiable programming

Our solution: differentiable programming with JAX [1]

- GPU accelerators – no emulators required!
- Automatic differentiation to compute gradients of code

We efficiently invert neutron stars with

1. **Bayesian inference**: using NS observations to constrain EOS parameters
2. **Gradient-based optimizers**: given $R(M)$ or $\Lambda(M)$, recover EOS parameters

Available open source: JESTER (tsunhopang/jester)

Methods – Equation of state parametrization (1)

- At low density, we use the metamodel [2, 3]: Taylor expansion of energy per nucleon E/A in[1] $x = (n - n_{\mathrm{sat}})/3n_{\mathrm{sat}}$
- Nuclear empirical parameters (NEPs): coefficients of the expansion

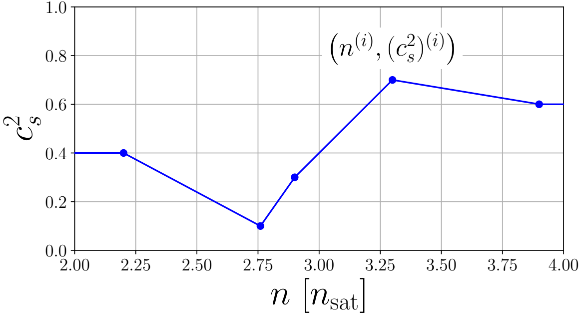
$$E/A(n,\delta) = e_{\mathrm{sat}}(n) + e_{\mathrm{sym}}(n)\delta^2 + \mathcal{O}(\delta^4)\,, \quad \delta = (n_n - n_p)/n$$

$$\begin{aligned} e_{\mathrm{sat}}(n) &= E_{\mathrm{sat}} + \tfrac{1}{2}K_{\mathrm{sat}}x^2 + \tfrac{1}{3!}Q_{\mathrm{sat}}x^3 + \tfrac{1}{4!}Z_{\mathrm{sat}}x^4 + \mathcal{O}(x^5) \\ e_{\mathrm{sym}}(n) &= E_{\mathrm{sym}} + L_{\mathrm{sym}}x + \tfrac{1}{2}K_{\mathrm{sym}}x^2 \\ &\quad + \tfrac{1}{3!}Q_{\mathrm{sym}}x^3 + \tfrac{1}{4!}Z_{\mathrm{sym}}x^4 + \mathcal{O}(x^5) \end{aligned}$$

[1] $n_{\mathrm{sat}} \equiv 0.16\ \mathrm{fm}^{-3}$

Methods – Equation of state parametrization (2)

- Metamodel description breaks down at $n_{\mathrm{break}} \sim [1, 2]\ n_{\mathrm{sat}}$
- Higher density: parametrize the EOS with $c_s^2(n)$ grid points and linear interpolation [4]

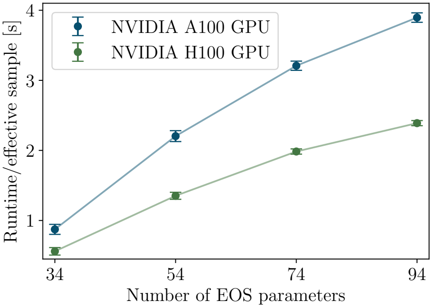


Contents

Results – Bayesian inference

- EOS constraints: nuclear theory (χ_{EFT}), NS observations (heavy PSRs, NICER, GW170817)
- Extend Koehn *et al.* [5]: directly sample $\boldsymbol{\theta}_{\mathrm{EOS}}$
- $\sim$ 0.24 ms per TOV solver call, $\sim$ 1 h for MCMC run (H100 GPU)
- Scales well with number of parameters (more $c_s^2(n)$ grid points)

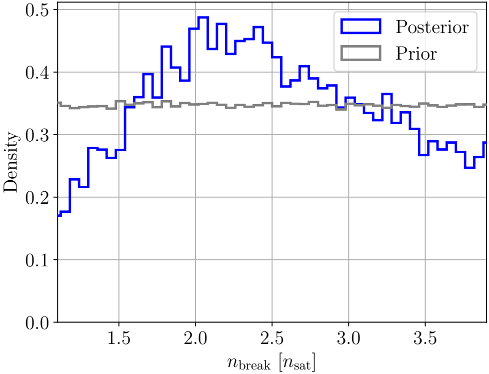
	$R_{1.4}$ [km]	
Constraint	Koehn+	This work
χ_{EFT}	$12.11^{+1.69}_{-3.39}$	$12.59^{+2.24}_{-3.51}$
Radio timing	$13.70^{+1.41}_{-2.17}$	$13.71^{+1.19}_{-1.88}$
PSR J0030+0451	$13.17^{+1.65}_{-2.24}$	$13.48^{+1.42}_{-2.15}$
PSR J0740+6620	$13.39^{+1.57}_{-1.72}$	$13.79^{+1.26}_{-1.73}$
GW170817†	$11.98^{+1.08}_{-1.09}$	$12.40^{+1.33}_{-1.49}$
All	$12.26^{+0.80}_{-0.91}$	$12.62^{+1.04}_{-1.11}$



Results – Measuring $n_{\rm break}$

$\chi_{\rm EFT}$ predicts metamodel to break down at a density $n_{\rm break} \sim 1 - 2\ n_{\rm sat}$
Can we determine this with NSs?

- Wide, agnostic prior on $n_{\rm break}$: $U(1, 4)\ n_{\rm sat}$
- Only consider heavy PSRs, NICER, GW170817

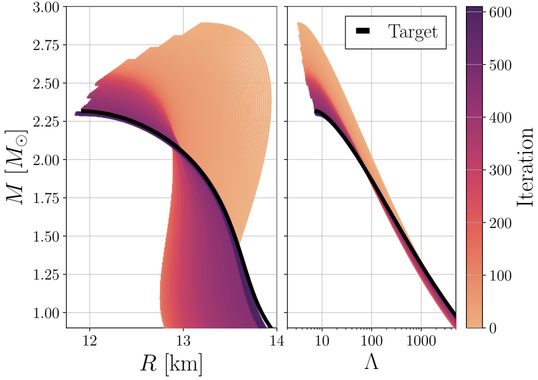


Contents

Methods – Gradient-based optimization

- Given $\hat{R}(M)$ or $\hat{\Lambda}(M)$: recover EOS parameters $\boldsymbol{\theta}_{\rm EOS}$
- Loss function $L(\boldsymbol{\theta}_{\rm EOS})$: relative error in tidal deformability Λ
- Gradient descent: $\boldsymbol{\theta}^{(i+1)} \leftarrow \boldsymbol{\theta}^{(i)} - \gamma \nabla L(\boldsymbol{\theta}^{(i)})$ with Adam
    - Automatic differentation!

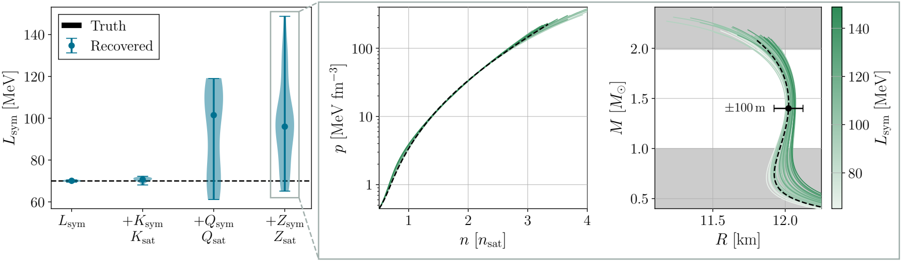
$$L(\boldsymbol{\theta}_{\rm EOS}) = \frac{1}{N} \sum_{i=1}^{N} \left| \frac{\Lambda_i(\boldsymbol{\theta}_{\rm EOS}) - \hat{\Lambda}_i}{\hat{\Lambda}_i} \right|$$



Results – Recovery of nuclear empirical parameters

What do NSs tell us about nuclear empirical parameters (NEPs)?

- Use only NEPs – no $c_s^2(n)$ grid points
- If more NEPs are varied, $L_{\rm sym}$ is no longer constrained
- Yet, recovered EOSs deviate < 100 meters, < 10 in Λ from the true EOS $\rightarrow$ observationally indistinguishable



Conclusion

- We have developed JESTER, a differentiable programming framework for the inverse problem of NSs
- We demonstrate accurate and scalable Bayesian inference on NS data
- Gradient-based optimizers can efficiently invert a given NS family
- New tool to gain insights into how NSs probe the EOS

Available open source: JESTER (tsunhopang/jester)

References I

[1] Roy Frostig, Matthew James Johnson, and Chris Leary. "Compiling machine learning programs via high-level tracing". In: *Systems for Machine Learning* 4.9 (2018). Available at: https://github.com/google/jax.

[2] Jérôme Margueron, Rudiney Hoffmann Casali, and Francesca Gulminelli. "Equation of state for dense nucleonic matter from metamodeling. I. Foundational aspects". In: *Phys. Rev. C* 97.2 (2018), p. 025805. DOI: 10.1103/PhysRevC.97.025805. arXiv: 1708.06894 [nucl-th].

[3] Jérôme Margueron, Rudiney Hoffmann Casali, and Francesca Gulminelli. "Equation of state for dense nucleonic matter from metamodeling. II. Predictions for neutron star properties". In: *Phys. Rev. C* 97.2 (2018), p. 025806. DOI: 10.1103/PhysRevC.97.025806. arXiv: 1708.06895 [nucl-th].

[4] Rahul Somasundaram, Ingo Tews, and Jérôme Margueron. "Investigating signatures of phase transitions in neutron-star cores". In: *Phys. Rev. C* 107.2 (2023), p. 025801. DOI: 10.1103/PhysRevC.107.025801. arXiv: 2112.08157 [nucl-th].

[5] Hauke Koehn et al. "From existing and new nuclear and astrophysical constraints to stringent limits on the equation of state of neutron-rich dense matter". In: (Feb. 2024). arXiv: 2402.04172 [astro-ph.HE].

More validation results

Constraint	$M_{\rm TOV}$ $[M_\odot]$		$p(3n_{\rm sat})$ [MeV fm^{-3}]		$n_{\rm TOV}$ $[n_{\rm sat}]$	
	Koehn+	This work	Koehn+	This work	Koehn+	This work
$\chi_{\rm EFT}$	$2.05^{+1.08}_{-1.16}$	$2.03^{+1.03}_{-0.97}$	69^{+186}_{-53}	72^{+165}_{-65}	$6.51^{+10.7}_{-3.11}$	$6.53^{+9.26}_{-4.09}$
Radio timing	$2.35^{+0.73}_{-0.29}$	$2.20^{+0.39}_{-0.26}$	111^{+140}_{-49}	97^{+65}_{-49}	$5.51^{+1.89}_{-1.66}$	$5.68^{+1.63}_{-1.74}$
PSR J0030+0451	$2.16^{+0.83}_{-0.71}$	$2.19^{+0.78}_{-0.76}$	89^{+143}_{-46}	94^{+135}_{-62}	$5.62^{+4.44}_{-1.91}$	$5.60^{+3.65}_{-2.59}$
PSR J0740+6620	$2.34^{+0.65}_{-0.32}$	$2.38^{+0.70}_{-0.42}$	107^{+125}_{-40}	118^{+150}_{-59}	$5.34^{+1.63}_{-1.61}$	$5.16^{+1.72}_{-2.04}$
GW170817	$2.21^{+0.45}_{-0.18}$	$2.17^{+0.39}_{-0.23}$	80^{+81}_{-32}	80^{+70}_{-48}	$6.25^{+1.40}_{-1.70}$	$6.35^{+1.65}_{-1.88}$
All	$2.25^{+0.42}_{-0.22}$	$2.24^{+0.36}_{-0.23}$	90^{+71}_{-31}	93^{+66}_{-37}	$5.92^{+1.35}_{-1.38}$	$5.91^{+1.36}_{-1.45}$