Leveraging Differentiable Programming in the Inverse Problem of Neutron Stars

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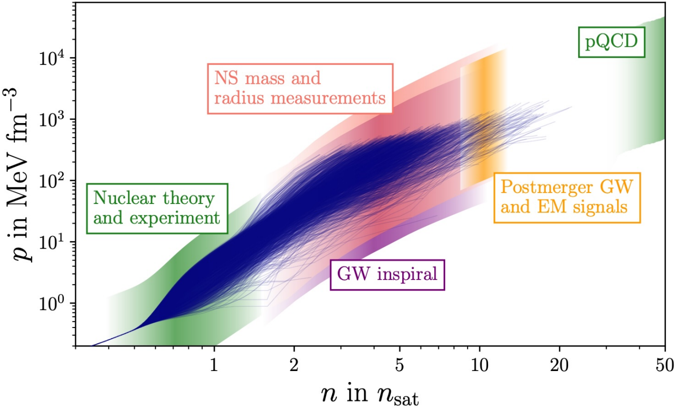
Introduction

Neutron stars (NSs) offer unique probes of the high-density regime of the equation of state (EOS) of dense nuclear matter.

Inverse problem of NSs: infer the EOS from observations of NSs (masses, radii, tidal deformabilities, . . .)



Inverting neutron stars with differentiable programming

Bottleneck: solving Tolman-Oppenheimer-Volkoff (TOV) equations

Our solution: differentiable programming with JAX [1]

- GPU accelerators – no emulators required!
- Automatic differentiation to compute gradients of functions

We invert neutron stars using

1. **Bayesian inference**: ~ 0.24 ms per TOV call, ~ 1 h for MCMC inference
2. **Gradient-based optimizers**: efficient inversion of mass-radius curves

Available open source: JESTER (tsunhopang/jester)

Methods – Equation of state parametrization (1)

- At low density, we use the metamodel: Taylor expansion of energy per nucleon E/A [2, 3] in[1] $x = (n - n_{\rm sat})/3n_{\rm sat}$
- Nuclear empirical parameters (NEPs): coefficients of the expansion

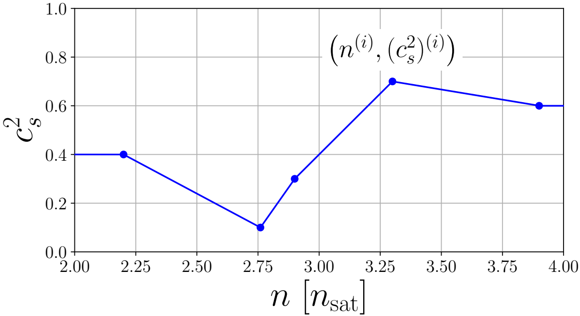
$$E/A(n, \delta) = e_{\rm sat}(n) + e_{\rm sym}(n)\delta^2 + \mathcal{O}(\delta^4)\,, \quad \delta = (n_n - n_p)/n$$

$$\begin{aligned} e_{\rm sat}(n) &= E_{\rm sat} + \frac{1}{2}K_{\rm sat}x^2 + \frac{1}{3!}Q_{\rm sat}x^3 + \frac{1}{4!}Z_{\rm sat}x^4 + \mathcal{O}(x^5) \\ e_{\rm sym}(n) &= E_{\rm sym} + L_{\rm sym}x + \frac{1}{2}K_{\rm sym}x^2 \\ &\quad + \frac{1}{3!}Q_{\rm sym}x^3 + \frac{1}{4!}Z_{\rm sym}x^4 + \mathcal{O}(x^5) \end{aligned}$$

[1] $n_{\rm sat} \equiv 0.16\ {\rm fm}^{-3}$

Methods – Equation of state parametrization (2)

- The metamodel description breaks down at high density [4]
- Happens at $n_{\mathrm{break}} \sim [1, 2]\ n_{\mathrm{sat}}$
- Higher density: parametrize the EOS with $c_s^2(n)$ grid points and linear interpolation [5]
- By default, 8 grid points



Methods – Bayesian inference

Bayesian inference: get posterior of EOS parameters θ_{EOS} with Markov chain Monte Carlo (MCMC) and NS data d

$$p(\theta_{\mathrm{EOS}}|d) \propto p(d|\theta_{\mathrm{EOS}})p(\theta_{\mathrm{EOS}})$$

Computationally expensive: solve TOV equations for many θ_{EOS}!

- `JAX`: compiles code, runs on GPU
- `flowMC` [6, 7]: MCMC with normalizing flows as proposal distributions

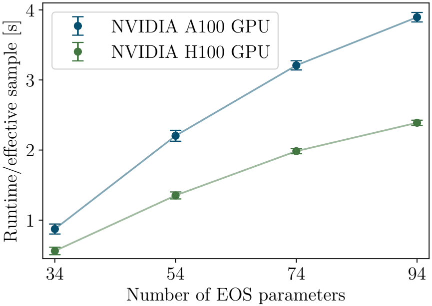
With this, we achieve (on NVIDIA H100 GPU)

- $\sim$ 0.24 ms per TOV call
- Complete MCMC run in $\sim$ 1 h

Results – Validation and scaling

- EOS constraints: nuclear theory (χ_{EFT}), NS observations (heavy PSRs, NICER, GW170817)
- Extend Koehn+ [8]: directly sample θ_{EOS}
- Scales well with number of parameters (more $c_s^2(n)$ grid points)

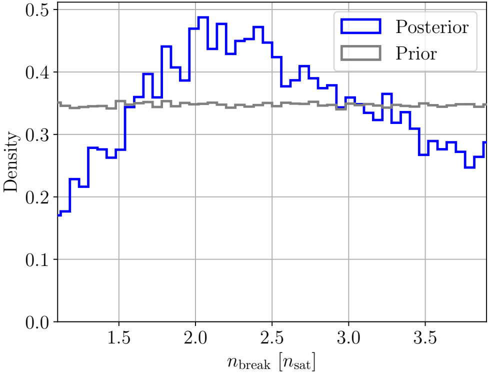
	$R_{1.4}$ [km]	
Constraint	Koehn+	This work
χ_{EFT}	$12.11^{+1.69}_{-3.39}$	$12.59^{+2.24}_{-3.51}$
Radio timing	$13.70^{+1.41}_{-2.17}$	$13.71^{+1.19}_{-1.88}$
PSR J0030+0451	$13.17^{+1.65}_{-2.24}$	$13.48^{+1.42}_{-2.15}$
PSR J0740+6620	$13.39^{+1.57}_{-1.72}$	$13.79^{+1.26}_{-1.73}$
GW170817†	$11.98^{+1.08}_{-1.09}$	$12.40^{+1.33}_{-1.49}$
All	$12.26^{+0.80}_{-0.91}$	$12.62^{+1.04}_{-1.11}$



Results – Measuring n_{break}

χ_{EFT} predicts metamodel to break down at a density $n_{\mathrm{break}} \sim 1 - 2\ n_{\mathrm{sat}}$
Can we determine this with NSs?

- Wide, agnostic prior on n_{break}: $U(1, 4)\ n_{\mathrm{sat}}$
- Only consider heavy PSRs, NICER, GW170817

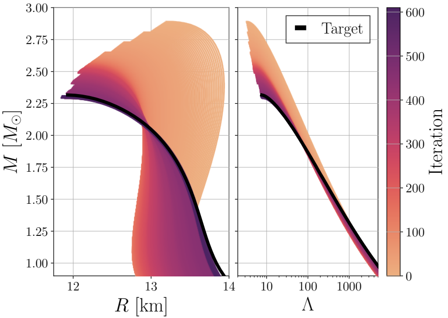


Methods – Variational inference

Alternative to Bayesian inference: optimization with gradients

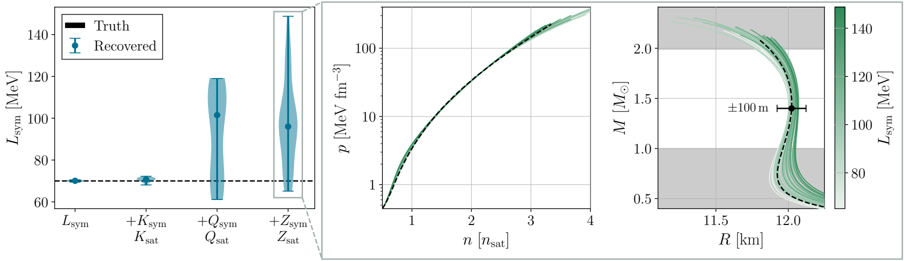
- $\hat{R}_i, \hat{\Lambda}_i$: "target" tidal deformabilities at masses M_i
- Loss function $L(\boldsymbol{\theta}_{\rm EOS})$: relative error in tidal deformability Λ
- Gradient descent: $\boldsymbol{\theta}^{(i+1)} \leftarrow \boldsymbol{\theta}^{(i)} - \gamma \nabla L(\boldsymbol{\theta}^{(i)})$
- Efficiently invert a complete NS mass-radius curve

$$L(\boldsymbol{\theta}_{\rm EOS}) = \frac{1}{N} \sum_{i=1}^{N} \left| \frac{\Lambda_i(\boldsymbol{\theta}_{\rm EOS}) - \hat{\Lambda}_i}{\hat{\Lambda}_i} \right|$$



Results – recovery of metamodel parameters

- Consider only metamodel NEPs – no $c_s^2(n)$ grid points
- Recover with increasingly more NEPs being varied
- If more NEPs are varied, $L_{\rm sym}$ is no longer constrained
- Recovered EOS deviate < 100 meters, < 10 in Λ from the true EOS in $[1, 2]\ M_\odot$ range



Conclusion

- We have developed JESTER, a differentiable programming framework for the inverse problem of NSs
- We demonstrate accurate and scalable Bayesian inference on NS data
- Gradient-based optimizers can efficiently invert a complete NS family
- New tool to gain insights into how NSs probe the EOS

Available open source: JESTER (tsunhopang/jester)

More validation results

Constraint	$M_{\rm TOV}$ $[M_\odot]$		$p(3n_{\rm sat})$ [MeV fm^{-3}]		$n_{\rm TOV}$ $[n_{\rm sat}]$	
	Koehn+	This work	Koehn+	This work	Koehn+	This work
$\chi_{\rm EFT}$	$2.05^{+1.08}_{-1.16}$	$2.03^{+1.03}_{-0.97}$	69^{+186}_{-53}	72^{+165}_{-65}	$6.51^{+10.7}_{-3.11}$	$6.53^{+9.26}_{-4.09}$
Radio timing	$2.35^{+0.73}_{-0.29}$	$2.20^{+0.39}_{-0.26}$	111^{+140}_{-49}	97^{+65}_{-49}	$5.51^{+1.89}_{-1.66}$	$5.68^{+1.63}_{-1.74}$
PSR J0030+0451	$2.16^{+0.83}_{-0.71}$	$2.19^{+0.78}_{-0.76}$	89^{+143}_{-46}	94^{+135}_{-62}	$5.62^{+4.44}_{-1.91}$	$5.60^{+3.65}_{-2.59}$
PSR J0740+6620	$2.34^{+0.65}_{-0.32}$	$2.38^{+0.70}_{-0.42}$	107^{+125}_{-40}	118^{+150}_{-59}	$5.34^{+1.63}_{-1.61}$	$5.16^{+1.72}_{-2.04}$
GW170817	$2.21^{+0.45}_{-0.18}$	$2.17^{+0.39}_{-0.23}$	80^{+81}_{-32}	80^{+70}_{-48}	$6.25^{+1.40}_{-1.70}$	$6.35^{+1.65}_{-1.88}$
All	$2.25^{+0.42}_{-0.22}$	$2.24^{+0.36}_{-0.23}$	90^{+71}_{-31}	93^{+66}_{-37}	$5.92^{+1.35}_{-1.38}$	$5.91^{+1.36}_{-1.45}$