Exercise Session 1: Kinematics and Dynamics

Robotics (B-KUL-H02A4A)

Academic Year 2024-2025

Exercise 1

(a) Three rigid bodies move in space independently. How many degrees of freedom does this system of three bodies have?

(b) Now you constrain them so that each body must make contact with at least one of the other two bodies. (The bodies are allowed to slide and roll relative to each other, but they must remain in contact.) How many degrees of freedom does this system of three bodies have?

Exercise 2 Assume your arm has 7 dof and you constrain your hand to be at a fixed configuration (e.g., your palm is flat against a table).

(a) What is an explicit representation of the arm's configuration?

(b) What is an implicit representation?

(c) What does the set of feasible configurations look like in the 7-dimensional configuration space of the unconstrained arm?

Exercise 3 The experimental surgical manipulator shown in Figure 1, developed at the National University of Singapore, is a parallel mechanism with three identical legs, each with a prismatic joint and two universal joints (the joints are marked for one of the legs). Use Grübler's formula to calculate the number of degrees of freedom of this mechanism.

Exercise 4 Consider the 2D quadcopter and rod shown in Figure 2. The rod is attached to the quadcopter by a revolute joint, and you are given the task of balancing the rod upright (a flying version of the classic cart pendulum problem). Assume the configuration of the quadcopter center is described by (x_q, y_q, θ_q) and the configuration of the rod center is described as (x_r, y_r, θ_r), where θ_q and θ_r are measured with respect to the world x axis. The length of the rod is $2l$ and the height and width of the quadcopter body are $2h$ and $2w$ respectively.

(a) Solve for the configuration constraints that keep the rod and quadcopter connected.

(b) Express these as a Pfaffian constraint where $q = [x_q\ y_q\ \theta_q\ x_r\ y_r\ \theta_r]^T$.

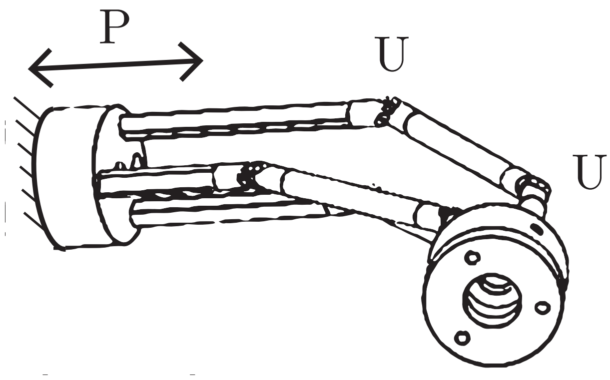

Figure 1: A miniature parallel surgical manipulator with three PUU legs.

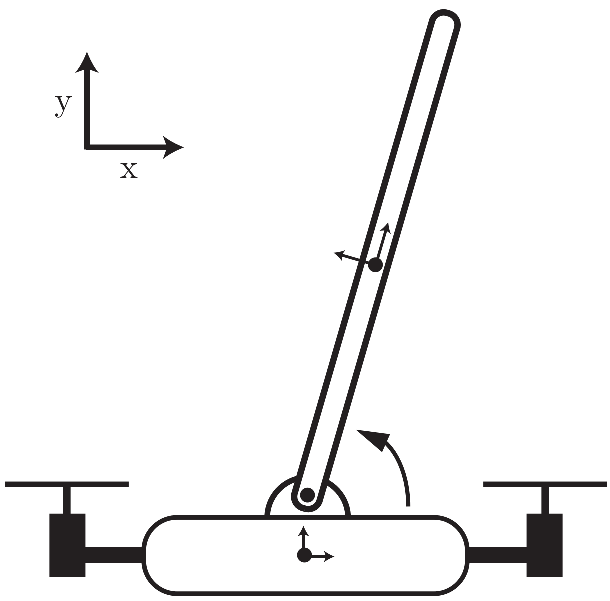

Figure 2: 2D quadcopter balancing a rod.

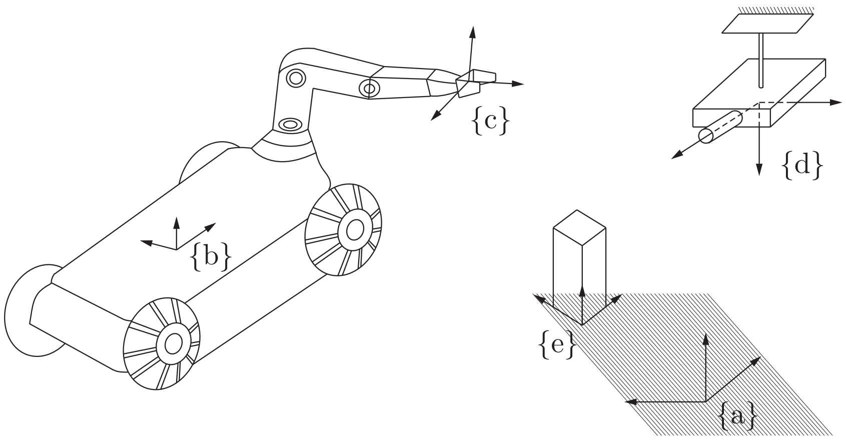

Figure 3: The fixed world frame {a}, the mobile robot's chassis frame {b}, the gripper frame {c}, the RGBD camera frame {d}, and the object frame {e}.

Exercise 5 The mobile manipulator in Figure 3 needs to orient its gripper to grasp the block. For subsequent placement of the block, we have decided that the orientation of the gripper relative to the block, when the gripper grasps the block, should be R_{eg} . Our job is to determine the rotation operator to apply to the gripper to achieve this orientation relative to the block. Figure 3 shows the fixed world frame {a}, the mobile robot's chassis frame {b}, the gripper frame {c}, the RGBD camera (color vision plus depth, like the Kinect) frame {d}, and the object frame {e}. Because we put the camera at a known location in space, we know R_{ad}. The camera reports the configuration of {e} relative to {d}, so we know R_{de}. From the mobile robot's localization procedure (e.g., vision-based localization or odometry) we know R_{ab}. From the robot arm's forward kinematics we know R_{bc}.

(a) In terms of the four known rotation matrices R_{ad}, R_{de}, R_{ab}, and R_{bc}, and using only matrix multiplication and the transpose operation, express the current orientation of the gripper relative to the block, R_{ec}.

(b) To align the gripper properly, you could apply to it a rotation R_1 expressed in terms of axes in the gripper's {c} frame. What is R_1, in terms of the five known rotation matrices ($R_{ad}, R_{de}, R_{ab}, R_{bc}, R_{eg}$), matrix multiplication, and transpose?

(c) The same rotation could be written R_2, in terms of the axes of the frame of the mobile base {b}. What is R_2?

Exercise 6 Figure 4 shows a screw, a frame {b}, and a frame {s}. The $\hat{x}_b$-axis of {b} is along the axis of the screw, and the origin of the frame s is displaced by 2 cm, along the $\hat{y}_b$-axis, from the b frame. The $\hat{z}_s$-axis is aligned with $\hat{x}_b$ and the $\hat{x}_s$-axis is aligned with $\hat{z}_b$. Taking note of the direction of the screw's threads, as the machine screw goes into a tapped hole driven by a screwdriver rotating at 3 radians per second, what is the screw's twist expressed in {b}, $\mathcal{V}_b$? What is the screw axis expressed in {b}, $\mathcal{S}_b$? What is $\mathcal{V}_s$? What is $\mathcal{S}_s$? Give units as appropriate.

(extra) Exercise 7 A wrench F and a twist V are represented in {a} as $\mathcal{F}_a$ and $\mathcal{V}_a$, respectively, and they are represented in b as $\mathcal{F}_b$ and $\mathcal{V}_b$. Without consulting any other

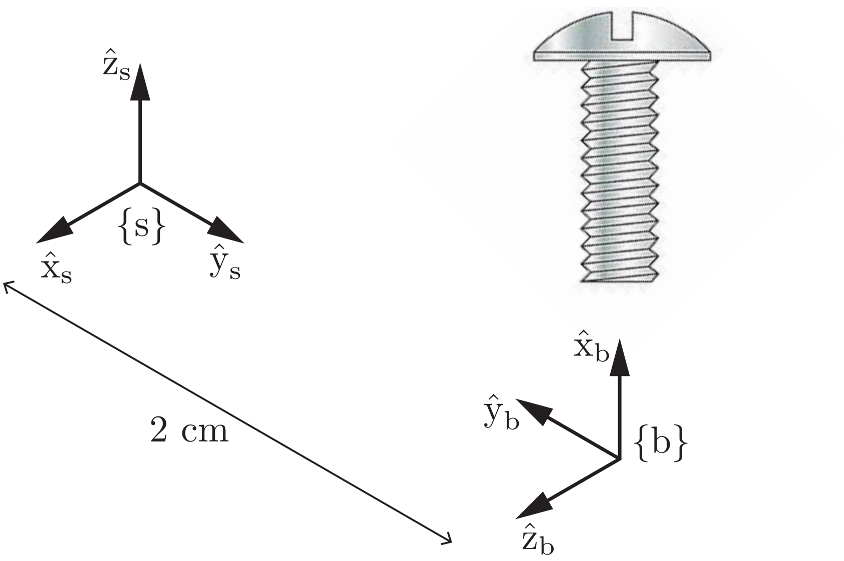

Figure 4: As the machine screw goes into a tapped hole, it advances linearly by $4\,\pi$mm every full rotation of the screw.

source, and using the facts that $(AB)^T = B^T A^T$, that the adjoint of the transformation matrix T_{ab} can be used to change the frame of representation of a twist from the a frame to the b frame, and that the scalar power generated (or dissipated) by applying a wrench F along a twist V is independent of the frame of reference, show that $\mathcal{F}_a = [Ad_{T_{ba}}]^T \mathcal{F}_b$.

Exercise 8 Figure 5 shows a screw axis in the $(\hat{y}_c, \hat{z}_c)$ plane, at a 45° angle with respect to the $\hat{y}_c$-axis. (The $\hat{x}_c$-axis points out of the page.) The screw axis passes through the point (0,3,0).

(a) If the pitch of the screw is $h = 10$ linear units per radian, what is the screw axis $\mathcal{S}_c$? Make sure you can also write this in its $se(3)$ form $[\mathcal{S}_c]$, too.

(b) Using your answer to (a), if the speed of rotation about the screw axis is $\dot{\theta} = \sqrt{2}$ rad/s, what is the twist $\mathcal{V}_c$?

(c) Using your answer to (a), if a frame initially at c rotates by $\theta = \pi/2$ about the screw axis, yielding a new frame {c'}, what are the exponential coordinates describing the configuration of {c'} relative to {c}?

(d) What is $T_{cc'}$, corresponding to the motion in part (c)? Hint: You can use Pinocchio to do the calculation. Note that the rotational and linear components are swapped in Pinocchio.

(e) Now imagine that the axis in Figure 5 represents a wrench: a linear force along the axis and a moment about the axis (according to the right-hand rule). The linear force in the direction of the axis is 20 and the moment about the axis is 10. What is the wrench $\mathcal{F}_c$?

Exercise 9 Figure 6 shows a machine screw. As it advances into a tapped hole, it moves 5 mm linearly for every radian of rotation. A frame {a} has its $\hat{z}_a$-axis along the axis of the screw and its $\hat{x}_a$-axis out of the page. The frame {b} has its origin at $p_a =$

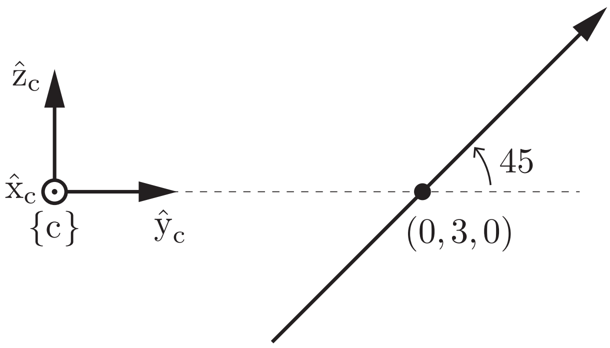

Figure 5: A screw axis in the $(\hat{y}_c, \hat{z}_c)$ plane.

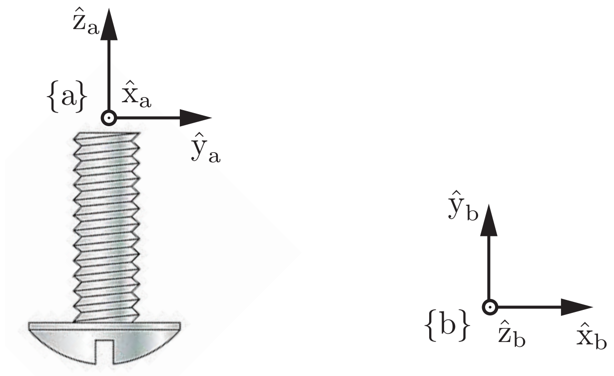

Figure 6: A machine screw. Notice the direction of the threads.

(0, 3,-2) mm and its orientation is shown in the figure ($\hat{z}_a$ is out of the page). Use mm as your linear units and radians as your angular units.

(a) What is the screw axis $\mathcal{S}_a$ corresponding to advancing into a tapped hole? Give a numerical 6-vector.

(b) What is the screw axis $\mathcal{S}_b$? Give a numerical 6-vector.

(c) What is $[\mathcal{S}_b]$?

(d) From the initial configuration T_{ab} shown in the figure, the {b} frame follows the screw an angle θ, ending at the final configuration $T_{ab'}$. If we write $T_{ab'} = TT_{ab}$, what is T? Express this symbolically (don't write numbers), using any of $\mathcal{S}_a$, $\mathcal{S}_b$, θ and any math operations you need.

(e) Referring to the previous question, if we instead write $T_{ab'} = T_{ab}T$, what is T? Again, express this symbolically (don't write numbers), using any of $\mathcal{S}_a$, $\mathcal{S}_b$, θ and any math operations you need.

Exercise 10 Figure 7 shows a KINOVA ultra lightweight 4-dof robot arm at its home configuration. An {s} frame is at its base and a {b} frame is at its end-effector. All the relevant dimensions are shown. The $\hat{y}_b$-axis is displaced from the $\hat{y}_s$-axis by 9.8 mm, as shown in the image. Positive rotation about joint axis 1 is about the $\hat{y}_s$-axis (by the right-hand rule, as always) and joint axis 4 is about the $\hat{y}_b$-axis. Joint axes 2 and 3 are also illustrated.

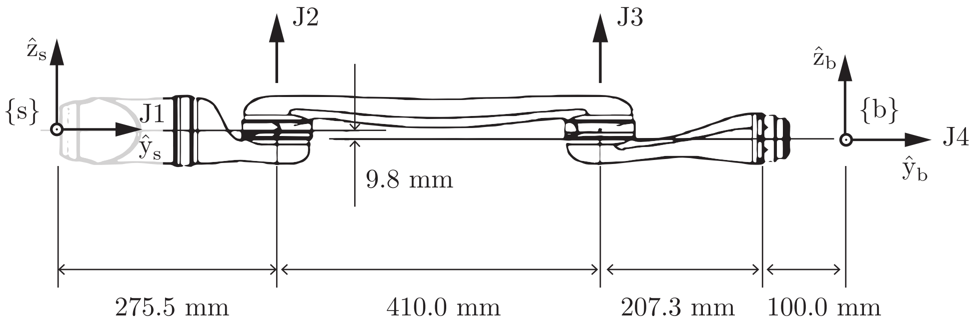

Figure 7: The KINOVA ultra lightweight 4-dof robot arm at its home configuration.

1. Write M (i.e., T_{sb} when the robot is at its home configuration). All entries should be numerical (no symbols or math).
2. Write the space-frame screw axes $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$. All entries should be numerical (no symbols or math).
3. Give the product of exponentials formula for $T_{sb}(\theta)$ for arbitrary joint angles $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$. Your answer should be purely symbolic (no numbers), using only the symbols $M, \mathcal{S}_1, \ldots, \mathcal{S}_4, \theta_1, \ldots, \theta_4$, and the matrix exponential.

Exercise 11 Figure 8 shows a da Vinci Xi, used in several types of robot-assisted surgery. Though it is mechanically constrained to have only 3 degrees of freedom per arm, for the sake of this exercise assume each arm is a simple serial chain with 6 degrees of freedom.

(a) Write the M matrix for the arm if its home configuration is shown in Figure 9.

(b) Find the space frame screw axes for this system.

(c) Determine the position of the end-effector if the joints are at $(0, \frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{2})$ rad.

Exercise 12 Figure 10 shows a KUKA LBR iiwa 7R robot arm. The figure defines an {s} frame at the base with the $\hat{y}_s$-axis pointing out of the page and a {b} frame aligned with {s} at the end-effector. The robot is at its home configuration. The screw axes for the seven joints are illustrated (positive rotation about these axes is by the right-hand rule). The axes for joints 2, 4, and 6 are aligned, and the axes for joints 1, 3, 5, and 7 are identical at the home configuration. The dimensions are $L_1 = 0.34\,\text{m}$, $L_2 = 0.4\,\text{m}$, $L_3 = 0.4\,\text{m}$, and $L_4 = 0.15\,\text{m}$.

(a) What is the space Jacobian when the robot is at its home configuration?

(b) What is the body Jacobian when the robot is at its home configuration?

(c) What is the rank of the space and body Jacobian at the home configuration? (It is always the same.) Is the home configuration a singularity? What is the dimension of the space of feasible twists at the home configuration?

Figure 8: Da Vinci Xi surgical robot.

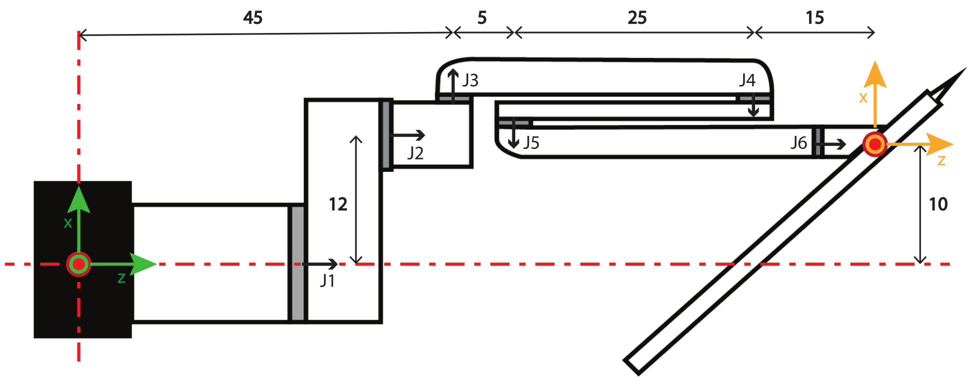

Figure 9: Top view of one da Vinci Xi surgical robot arm. Note that the grey regions represent R joints, green indicates the {s} frame, and yellow represents the end-effector frame {b} in this exercise. Dimensions are in cm.

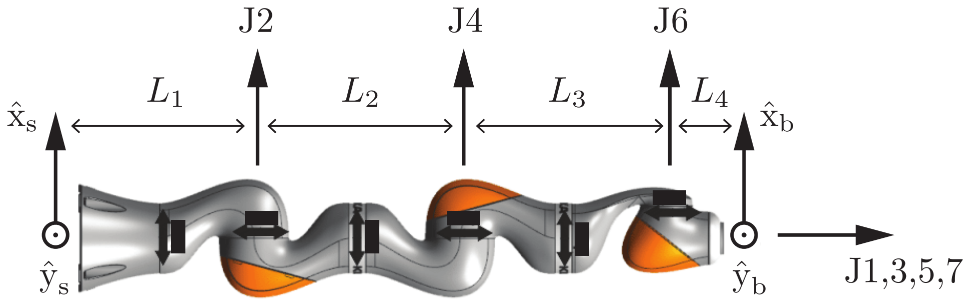

Figure 10: The KUKA LBR iiwa 7-dof robot.

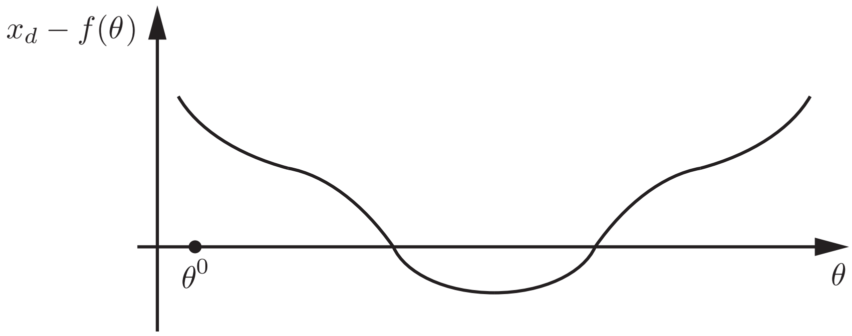

Figure 11: A scalar function $x_d - f(\theta)$ of θ.

For the remaining questions, assume the angles of the joints are $i\pi/16$ for joints $i = 1, \ldots, 7$.

(d) What is the space Jacobian? What joint torques are needed to generate the wrench $\mathcal{F}_s = (1\text{Nm}, 1\text{Nm}, 1\text{Nm}, 1\text{N}, 1\text{N}, 1\text{N})$? What is the manipulability measure μ_2 for the angular velocity manipulability ellipsoid in the space frame? What is the manipulability measure μ_2 for the linear manipulability ellipsoid in the space frame?

(e) What is the body Jacobian? What joint torques are needed to generate the wrench $\mathcal{F}_b = (1\text{Nm}, 1\text{Nm}, 1\text{Nm}, 1\text{N}, 1\text{N}, 1\text{N})$? What is the manipulability measure μ_2 for the angular velocity manipulability ellipsoid in the body frame? What is the manipulability measure μ_2 for the linear manipulability ellipsoid in the body frame?

Exercise 13 Perform three iterations of (approximate) iterative Newton-Raphson root finding on the scalar function $x_d - f(\theta)$ in Figure 11, starting from θ^0. (A general vector function $f(\theta)$ could represent the forward kinematics of a robot, and x_d could represent the desired configuration in coordinates. The roots of $x_d - f(\theta)$ are the joint vectors θ satisfying $x_d - f(\theta) = 0$, i.e., solutions to the inverse kinematics problem.) Draw the iterates θ^1, θ^2, and θ^3 on the θ axis and illustrate clearly how you obtain these points.

Exercise 14 Figure 12 illustrates an RP robot moving in a vertical plane. The mass of link 1 is $\mathfrak{m}_1$ and the center of mass is a distance L_1 from joint 1. The scalar inertia of link 1 about an axis through the center of mass and out of the plane is $\mathcal{I}_1$. The mass of link 2 is $\mathfrak{m}_2$, the center of mass is a distance θ_2 from joint 1, and the scalar inertia of link 2 about its center of mass is $\mathcal{I}_2$. Gravity g acts downward on the page.

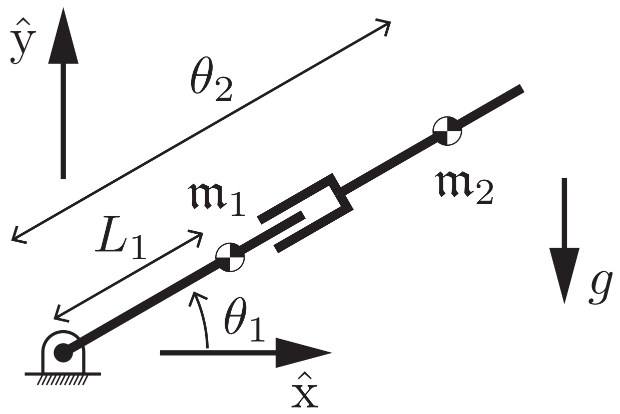

Figure 12: An RP robot operating in a vertical plane.

(a) Let the location of the center of the mass of link i be (x_i, y_i). Find (x_i, y_i) for $i = 1, 2$, and their time derivatives, in terms of θ and $\dot{\theta}$.

(b) Write the potential energy of each of the two links, $\mathcal{P}_1$ and $\mathcal{P}_2$, using the joint variables θ.

(c) Write the kinetic energy of each of the two links, $\mathcal{K}_1$ and $\mathcal{K}_2$. (Recall that the kinetic energy of a rigid body moving in the plane is $\mathcal{K} = (1/2)\mathfrak{m}v^2 + (1/2)\mathcal{I}w^2$, where $\mathfrak{m}$ is the mass, v is the scalar linear velocity at the center of mass, w is the scalar angular velocity, and $\mathcal{I}$ is the scalar inertia of the rigid body about its center of mass.)

(d) What is the Lagrangian in terms of $\mathcal{K}_1, \mathcal{K}_2, \mathcal{P}_1, \mathcal{P}_2$?

(e) One of the terms in the Lagrangian can be expressed as

$$\frac{1}{2}\mathfrak{m}_2\theta_2^2\dot{\theta}_1^2.$$

If this were the complete Lagrangian, what would the equations of motion be? Derive these by hand (no symbolic math software assistance). Indicate which of the terms in your equations are a function of $\ddot{\theta}$, which are Coriolis terms, which are centripetal terms, and which are gravity terms, if any.

(f) Now derive the equations of motion (either by hand or using symbolic math software for assistance) for the full Lagrangian and put them in the form

$$\tau = M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta).$$

Identify which of the terms in $c(\theta, \dot{\theta})$ are Coriolis and which are centripetal. Explain as if to someone who is unfamiliar with dynamics why these terms contribute to the joint forces and torques.

(g) Consider the configuration-dependent mass matrix $M(\theta)$ from your previous answer. When the robot is at rest (and ignoring gravity), the mass matrix can be visualized as the ellipse of joint forces/torques that are required to generate the unit circle of joint accelerations in $\ddot{\theta}$ space. As θ_2 increases, how does this ellipse change? Describe it in text and provide a drawing.

Exercise 15 The mass matrix of the 2R robot of Figure 13 is

$$M(\theta) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) \\ \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2 L_2^2 \end{bmatrix},$$

where each link is modeled as a point mass at the end of the link.

(a) Explain in text and/or figures why each of the entries makes sense, for example using the joint accelerations $\ddot{\theta} = (1, 0)$ and $(0, 1)$.

Let $\mathfrak{m}_1 = 1$ kg, $\mathfrak{m}_2 = 1$ kg, $L_1 = 1$ m, $L_2 = 1$ m, $\theta_1 = \pi/4$ rad and $\theta_2 = \pi/8$ rad. Assume the robot is at rest.

(b) Give the forward kinematics $T_{sb}(\theta)$.

(c) Give the space Jacobian J_s and the body Jacobian J_b.

(d) Give $c(\theta, \dot{\theta})$ and $g(\theta)$ of the equations of motion in the form $\tau = M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta)$.

(e) Given an input torque $\tau = (1, 1)$ Nm, what are the joint accelerations $\ddot{\theta}$?

(f) Make a URDF file for the 2R robot and solve the previous questions via Pinocchio.

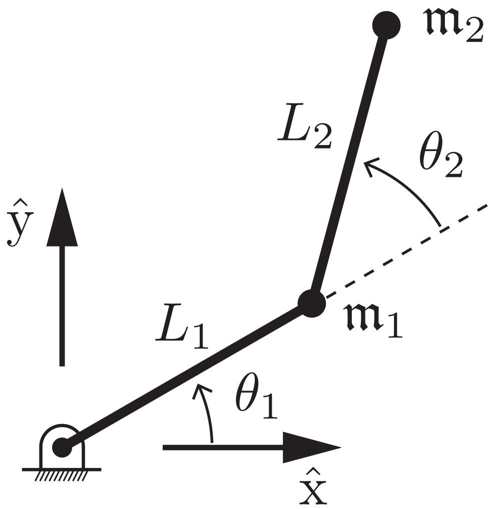

Figure 13: A 2R robot with all mass concentrated at the ends of the links.

Exercise 16 For a given configuration θ of a two-joint robot, the mass matrix is

$$M(\theta) = \begin{bmatrix} 3 & a \\ b & 2 \end{bmatrix}.$$

What constraints must a and b satisfy for this to be a valid mass matrix?

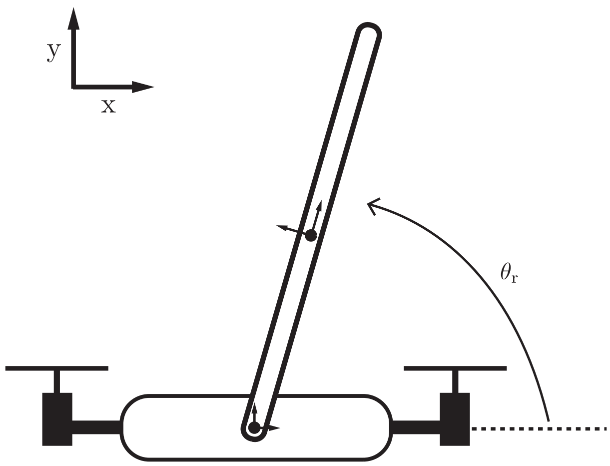

Figure 14: 2D quadcopter and attached pendulum.

Exercise 17 Consider the quadcopter and attached pendulum shown in Figure 14. The rod is attached to the quadcopter by a revolute joint, and you are given the task of balancing the rod upright (a flying version of the classic cart pendulum problem). Assume the configuration of the quadcopter center is described by (x_q, y_q, θ_q) and the configuration of the rod center is described as (x_r, y_r, θ_r), where θ_q and θ_r are measured with respect to the world x axis. The length of the rod is $2l$, the masses are m_q and m_r, and the rotational inertias are I_q and I_r.

Solve for the kinetic and potential energy terms and the Lagrangian for the generalized coordinates $(x_q, y_q, \theta_q, \theta_r)$.