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Pythagorean Theorem Before and After Pythagoras

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Abstract: Following the footpaths of Lakshmikantham, et. al. [15], and succeeding by Agarwal et. al. [3], in this article a sincere effort has been made to report the origin of the Pythagorean Theorem. Out of about 500 known different proofs of this theorem, we select five which have historical importance. We also discuss several generalizations of this theorem, and list some antique enduring problems. We genuinely hope students and teachers of mathematics will appreciate this article.

In (two-dimensional) Euclidean geometry (after Euclid of Alexandria, around 325-265 BC) *Pythagorean Theorem*, also known as *Pythagoras' Theorem* (after Pythagoras, around 582-481 BC) states that: If a and b are the lengths of the two legs of a right triangle, and c is the length of the hypothenuse (Greek word with meaning: The side opposite to the right angle), then the sum of the areas of the two squares on the legs equals the area of the square on the hypotenuse, i.e.,

$$a^2 + b^2 \ = \ c^2 \tag{1}$$

(see Figure 1).

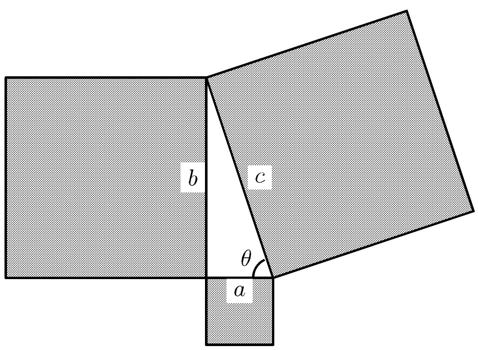

Figure 1

Pythagorean Theorem is inherent union between geometry and arithmetic, and serves as the cornerstone of the Euclidean distance formula: If (x_1, y_1) and (x_2, y_2) are the Cartesian coordinates (due to René Descartes, 1596-1650) of two points p and q in a plane, then the Euclidean distance between these points is the length of the line segment given by

$$d(p,q) \ = \ \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

It is known that the Pythagorean Theorem is equivalent to *Parallel Postulate* (Euclid's 5th Axiom in his *Elements*, about 300 BC): If a straight line meets two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles. A Postulate is something whose truth is assumed as part of the study of a science, and that an Axiom (Greek word meaning something worthy) is a principle assumed to be true, but cannot be demonstrated. There are signs that Euclid was not satisfied with this postulate, in the sense that he suspected it might not be necessary. In fact, this postulate is not self-evident, and hundreds of later attempts by renowned mathematicians to prove the fifth postulate from the earlier four axioms: 1. A straight line segment may be drawn from any given point to any other, 2. A straight line may be extended to any finite length, 3. A circle may be described with any given point as its center and any distance as its radius, 4. All right angles are congruent; turned out to be wrong or inconclusive. However, many of them were successful in proving its equivalent, e.g., Proclus Diadochus (410-485): A line parallel to a given line has a constant distance from it; John Wallis (1616-1703): There exist similar (but not equal) triangles, whose angles are equal but whose sides are unequal; The Italian Jesuit Girolamo Saccheri (1667-1733): There exists at least one rectangle, a quadrangle whose angles are all right angles; John Playfair (1748-1819): In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point (known as Playfair's axiom); Adrien-Marie Legendre (1752-1833): A line perpendicular in one arm of an acute angle also intersects the other arm, also the sum of the angles of a triangle is equal to two right angles; Karl Friedrich Gauss (1777-1855): There exist triangles of arbitrarily large area; and the list goes on. Finally, the Parallel Postulate out of three choices: impossible, meaningless, and improperly posed, was avoided by the third possibility. This led to a self-consistent geometry, known as non-Euclidean geometry. In literature the fundamental/elegant relation (equation) (1) is often called *Pythagorean relation* (equation). From this relation it is immediate that in any right triangle, the hypotenuse is greater than any of the other sides, but less than their sum. Further, if the length of any two sides is known the length of the third side can be calculated. For a given complex number $z = x + iy$ the absolute value (modulus) is given by $r = |z| = \sqrt{x^2 + y^2}$, and hence the numbers x, y, and r satisfy the Pythagorean relation $r^2 = x^2 + y^2$. Here r is a nonnegative number, representing the distance from the origin to z in the complex plane, but x and y can be negative numbers. Pythagorean Theorem is familiar (known by heart) to many people who studied it in High School. According to Johannes Kepler (1571-1630), "Geometry has two great treasures: one is the theorem of Pythagoras, the other the division of a line into extreme and mean ratio (*golden section/ratio*). The first we may compare to a measure of gold; the second to a precious jewel," whereas Charles Lutwidge Dodgson who used the pen name Lewis Carroll (1832-1898) commented in 1895 "It is as dazzlingly beautiful now as it was in the day when Pythagoras first discovered it." Further appreciating (1), Tobias Dantzig (1884-1956) in 1955 wrote "No other proposition of geometry has exerted so much influence on so many branches of mathematics as has the simple quadratic formula known as the Pythagorean Theorem. Indeed, much of the history of classical mathematics, and of modern mathematics, too, could be written around that proposition," and Jacob Bronowski (1908-1974) commented "To this day, theorem of Pythagoras remains the most important single theorem in the whole of mathematics." Michio Kaku (born 1947) has reported "The Pythagorean Theorem, of course, is the foundation of all architecture: every structure built on this planet is based on it." In recent years Pythagorean Theorem has been successfully applied in various branches of mathematics such as discrete, combinatorial, and computational geometry, e.g., in combinatorics to prove the famous Sylvester-Gallai-Erdös theorem (after James Joseph Sylvester, 1814-1897; Tibor Gallai, 1912-1992; Paul Erdös, 1913-1996) Pythagorean Theorem has been used: "Let n points be given in a plane, not all on a line. Join every pair of points by a line. At least n distinct lines are obtained in this way." Nicaragua issued a series of ten stamps commemorating mathematical formulas, including the Pythagorean Theorem. In a survey in 2004, in the Journal Physical World, (1) ranked fourth place among the twenty most beautiful equations in science. Undoubtedly, if one has to select a mathematical theorem which enjoys "perpetual youth" which has a very long history as well as deep significance unto this day, the Pythagorean Theorem is a robust nominee. It is of vital importance in problems ranging from carpentry and navigation to astronomy. Pythagorean Theorem also finds various applications in discrete, combinatorial, and computational geometry. However, Pythagorean Theorem horribly mangled by the Scarecrow in *The Wizard of Oz.* One cannot think trigonometry without Pythagorean Theorem, since trigonometric (circular, angle, or goniometric) functions are rather easily defined based on the sides of a right angle triangle. Further, from the relation (1), the trigonometric identities so called *Pythagorean identities* $\cos^2\theta+\sin^2\theta=1,\ 1+\tan^2\theta=\sec^2\theta$ and $\cot^2\theta+1=\csc^2\theta$ are immediate. (The English word sine comes from a series of mistranslations of the Sanskrit jyā-ardha (chord-half). Aryabhatta (born 2765 BC) frequently abbreviated this term to jyā or its synonym Jīvā. When some of the Hindu works were later translated into Arabic, the word was simply transcribed phonetically into an otherwise meaningless Arabic word *jiba*. However, since Arabic is written without vowels, later writers interpreted the consonants *jb* as *jaib*, which means bosom or breast. In the twelfth century when an Arabic work of trigonometry was translated into Latin, the translator used the equivalent Latin word *sinus*, which means almost meant bosom, and by extension, fold (as a toga over a breast), or a bay or gulf. This Latin word has now become our English sine. The first abbreviation of sine to sin is due to Edmund Gunter (1581-1626) in 1624. Similarly, the Sanskrit word kotijyā in English has become cosine. The tangent, cotangent, secant, and cosecant functions made their appearance in Islamic works in the ninth century, perhaps earliest in the works of Ahmad ibn Ábdallāh al-Marwazī Habas al-Hāsib (around 770-870) and

Al-Battani (around 858-929), although the tangent function had already been used in China in the eighth century. An extensive discussion of these functions is available in the work of Abu Arrayhan Muhammad ibn Ahmad al-Biruni (973-1048).) The pictorial representation, Figure 1, of the Pythagorean Theorem is known under many names, for example bride's chair, Franciscan's cowl, the goose foot, the peacock's tail, the windmill, and the chase of the little married women.

According to one of the fables "Pythagoras discovered *his* theorem while waiting in a palace hall to be received by Polycrates. Being bored, Pythagoras studied the stone square tiling of the floor and imagined the right triangles (half-squares) *hidden* in the tiling together with the squares erected over its sides. Having *seen* that the area of a square over the hypotenuse is equal to the sum of areas of squares over the legs, Pythagoras came to think that the same might also be true when the legs have unequal lengths". Throughout the history of mathematics it has been claimed that Pythagoras (which made him immortal) gave first proof of Pythagoras Theorem by deductive method. However, the earliest known mention of Pythagoras's name in connection with the theorem occurred five centuries after his death, in the writings of Marcus Tullius Cicero (106-43 BC), well known as "Cicerone", and Plutarch (Lucius Mestrius Plutarchus, around 46-120). It is very likely that one of the Pythagoreans (Pythagoras follower) proved the theorem, and as it was common in the ancient world, particularly in the Asian culture, out of respect for their leader, credited the proof to his famous teacher. This result has been recorded as the Proposition 47 in Book I of Euclid's *Elements* (A systematic and logical compilation of the works based on his experience and achievements of his predecessors in the three centuries just past, consisting 13 books (chapters or parts) with 465 propositions on plane and solid geometry, and number theory. This work set the trends how mathematics is written and studied even today. Since 1482, Elements has appeared in more editions than any work other than Bible, and it has been translated into countless languages.). In Elements the proposition reads: "In right-angled triangles the square on the side subtending the right angle is equal to the sum of the squares on the sides containing the right angle." Euclid provides two proofs of this proposition, first in Book I and second in Book VI. Its first proof uses knowledge about congruent triangles, and although it is not too demanding, many readers are puzzled by the strangeness of the acquired relations. For this proof philosopher Arthur Schopenhauer (1788-1860) wrote "the same uncomfortable feeling that we experience after a juggling trick." But the well-known 17-th century English philosopher Thomas Hobbes (1588-1679) who never studied geometry admired this proof. At the age of 40, Hobbes came across this theorem quite by chance on the page of an opened book while waiting at his friend's study. He wondered how it could that be possible. The proof, however, referred to a previous proposition whose proof in turn refereed to more preceding propositions. After several hours' of detailed investigation, he was finally convinced of the truth of Proposition 47. Hobbes not only finished Book I of Euclid's Elements, but started his life-long love for geometry. Denis Henrion (1580-1632) in 1615, comments: "Now it is said that this celebrated and very famous theorem was discovered by Pythagoras, who was so full of joy at his discovery that, as some say, he showed his gratitude to the Gods by sacrificing a Hecatomb of oxen. Others say he only sacrificed one ox..." But this is nonsense because being a follower of Lord Gautama Buddha (1887-1807 BC), Pythagoras must have been very scrupulous about shedding the blood of animals. In fact, Eudoxus of Cnidus (around 400-347 BC) writes "Pythagoras was distinguished by such purity and so avoided killing and killers that he not only abstained from animal foods, but even kept his distance from cooks and hunters".

However, Pythagorean Theorem was certainly known before 4th century BC. Sulbasutras are extant, named for the sages who wrote them: Baudhayana (born 3200 BC) contains one of the earliest references to this theorem (with a convincing valid proof): a rope that is stretched across the diagonal of a square produces an area double the size of the original square. This is a special case of the Pythagorean Theorem for a 45^0 right triangle. Egyptian civilizations around 2500 BC used ropes to measure out distances to form right triangles that were in whole number ratios (Berlin Papyrus 6619, and pyramids). However, some prominent historians of mathematics: Bartel Leendert van der Waerden (1903-1996), Dirk Jan Struik (1894-2000), and Sir Thomas Little Heath (1861-1940) [14] have suggested that Egyptians had no knowledge of Pythagorean Theorem. There is a sufficient evidence that Pythagorean Theorem was known to Mesopotamian (tablet number 7289 in the Babylonian Collection of Yale University famous as "YBC 7289", and tablet number 322 in the Babylonian Collection of Columbia University popular as "Plimpton 322", written between 1790 and 1750 BC, during the time of the Babylonian king Hammurabi (around 1811-1750 BC), which was discovered by Edgar James Banks (1866-1945) shortly after 1900, and sold to George Arthur Plimpton (1855-1936) in 1922, for $10). Of more mathematical interest is a group of tablets uncovered by the French at Susa in 1936. These provide some of the oldest Babylonian examples of the use of the theorem of Pythagoras. One tablet computes the radius r of a circle that circumscribes an isosceles triangle of sides, $50, 50$, and 60. The Apastamba Sulbautra (one of the oldest Dharma-related texts of Hinduism) gives a general statement of Pythagoras's theorem: The diagonal of a rectangle produces the sum of what the largest and the smallest side produce separately. Apastamba was also familiar with the result that as a special case of this theorem, the diagonal of a square is the side of a square with twice the area of the original one. The Katyayana, written later, gives a more general version of the Pythagorean Theorem: a rope that is stretched along the length of the diagonal of a rectangle produces an area which the vertical and horizontal sides make together. In other words, the square of the hypotenuse equals the sum of the squares of the sides. Chinese mathematician Tschou-Gun who lived in 1100 BC knew the characteristics of the right angle. In Chinese literature the Pythagorean Theorem is known as Gougu theorem (in Chinese gou means base, gu stands for shorter leg, and hypotenuse is called xian), and Shang Gao theorem (named after the Duke of Zhou's astronomer and mathematician). The theorem was also known to the Caldeans more than a thousand years before Pythagoras. Before and after Pythagoras this theorem has been given numerous logically correct different proofs (almost 500) – possibly the most for any mathematical theorem (several false, and with little or no variations, proofs have also been published). These proofs are very diverse, including both geometric and algebraic proofs, some make use of vectors, while others are demonstrations based on physical devices. Some of these proofs are extremely complicated, while others are astonishingly simple. A life long project of Elisha Scott Loomis (1852-1940) [16], a mathematics teacher, was to publish all available demonstrations of Pythagorean Theorem in his book *Pythagorean Proposition* in 1927, which was written in 1907 and revised in 1940, the year of his death. The revised edition contains 371 proofs, a "Pythagorean Curiosity," "five Pythagorean magic squares" and an extensive bibliography. National Council of Teachers of Mathematics (Washington, D.C.) republished this book in 1968. According to him "The Pythagorean Theorem is regarded as the most fascinating Theorem of all of Euclid, so much so, that thinkers from all classes and nationalities, from the aged philosopher in his armchair to the young soldier in the trenches next to no-man's land have whiled away hours seeking a new proof of its truth." This book includes proofs of those of Leonardo da Vinci, a blind girl Miss E.A. Coolidge in 1888, a sixteen year old high school student Miss Ann Condit in 1938, and by the United States Representative James Abram Garfield (1831-1881), 5 years before he became the 20th President of the United States in 1881. In his book, Loomis remarked that in the Middle Ages (5th to the 15th century), it was required that a student taking Master's degree in mathematics offer a new and original proof of the Pythagorean Theorem; this, he asserts that has resulted several new proofs. In the Foreword, the author rightly asserts that the number of algebraic proofs is limitless as is also the number of geometric proofs, but that the proposition admits no trigonometric proof. However, in 2009, Jason Zimba gave a very clever trigonometric proof, which is followed by more trigonometric proofs by David Houston and Luc Gheysens. Several Web sites (e.g., see [26]) deal with Pythagorean Theorem and give fairly decent update of this theorem; however, https://www.cut-the-knot.org/pythagoras/ found by Alexander Bogomolny (1948-2018) in 1996, is particularly interesting as it provides 118 different proofs. We also refer to the additional information provided in the monograph of Agarwal and Sen [3], and papers of Siu [22], [23], and Veljan [24].

Converse of Pythagorean Theorem. The converse of Pythagorean Theorem also holds. Euclid's Elements (Book I, Proposition 48) reads "If in a triangle the square on one of the sides equals the sum of the squares on the remaining two sides of the triangle, then the angle contained by the remaining two sides of the triangle is right." Thus, for any three positive numbers a, b and c such that $a^2 + b^2 = c^2$, there exists a triangle with sides a, b and c, and this triangle has a right angle between the sides of lengths a and b. The proof is by contradiction. Assume that the triangle has sides a, b, c such that $c^2 = a^2 + b^2$. We construct a right triangle with sides a and b and assume its hypotenuse to be d. But then by the Pythagoras Theorem $a^2 + b^2 = d^2$, and this implies $a^2 + b^2 = c^2 = d^2$, and hence $d = c$. Thus, for both the triangles all the three sides are equal, and therefore these triangles are congruent. Since (a, b, d) is a right triangle, the triangle (a, b, c) must also be a right triangle.

The above proof requires Pythagorean Theorem; however, using several known results from geometry the converse has also been proved by Stephen Casey [7] in 2008 (also see the work of Macro [18] in 1973) without employing the Pythagorean

Theorem. Here we give an ingenious such proof which is due to Douglas Mitchell [19] in 2009.

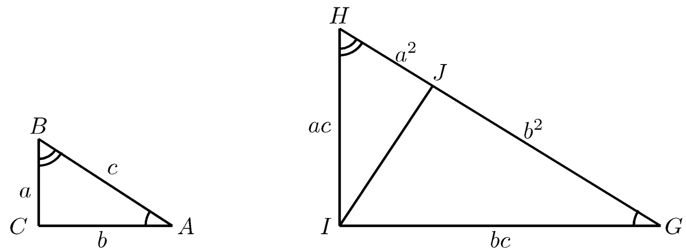

Figure 2

We multiply each side of the triangle ABC by c and use $a^2 + b^2 = c^2$, to obtain a similar triangle GHI (see Figure 2). Now by SAS (side angle side) theorem $\triangle IHJ$ is congruent to $\triangle ABC$ scaled by the factor a, thus $\angle HJI = \angle BCA$. Similarly, by SAS theorem $\triangle GIJ$ is congruent to $\triangle ABC$ scaled up by the factor b, so $\angle IJG = \angle BCA$. This leads to $\angle HJI = \angle BCA = \angle IJG$ and since HJG is a side of $\triangle GHI$ it follows that $\angle HJI + \angle IJG = \pi$. But, then $\angle BCA = \pi/2$.

As a consequence of the Pythagorean Theorem's converse we can determine whether a triangle is acute, right, or obtuse, as follows: Let c be chosen to be the longest of the three sides a, b, c and $a + b > c$. Then, the following Ernest Julius Wilczynski's (1876-1932) statements of 1914 hold:

If $a^2 + b^2 > c^2$, then the triangle is acute.
If $a^2 + b^2 = c^2$, then the triangle is right.
If $a^2 + b^2 < c^2$, then the triangle is obtuse.

Edsger Wybe Dijkstra (1930-2002) in [10] combined these statements in the following relation

$$\mathrm{sgn}\ (\alpha + \beta - \gamma) \;=\; \mathrm{sgn}\ (a^2 + b^2 - c^2),$$

where α is the angle opposite to side a, β is the angle opposite to side b, γ is the angle opposite to side c, and $\mathrm{sgn}\ (x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0. \end{cases}$

Hippocrates's Generalization of Pythagorean Theorem. A nontrivial generalization of Pythagorean Theorem far off the areas of squares on the three sides (see Figure 1) to similar figures (figures which are of the same shape, but not necessarily of the same size, for example, two N-sided polygons are similar if the ratios of their corresponding sides are all equal) was known to Hippocrates of Chios (about 470 BC), see Figures 3 and 4 (multiplying relation (1) by $\pi/2$, Figure 3 immediately follows). Recall that the area A of a regular polygon is

$$A \;=\; \frac{S^2 N}{4\tan(180/N)},$$

where S is the length of any side, N is the number of sides, and tan is the tangent function calculated in degrees. Further, the length of a side s_{2n} of a $2n$-sided regular polygon circumscribing a circle of radius 1 in terms of the length of a side s_n of an n-sided circumscribing polygon is

$$s_{2n} = \frac{2\sqrt{4+s_n^2}-4}{s_n}.$$

Archimedes (287-212 BC) used Pythagorean Theorem to obtain this formula, and employed it to a series of inscribed and circumscribing polygons to compute an approximate value of π, he showed that $3\frac{10}{71} < \pi < 3\frac{10}{70}$ (R.P. Agarwal, H. Agarwal, and Sen [2]). Hippocrates's result is included in Euclid's Elements in Book VI as Proposition VI 31. It reads "In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle," (see, Heath 1956). This extension presumes that the sides of the original triangle are the corresponding sides of the three similar figures (so the common ratios of sides between the similar figures are $a : b : c$). Euclid's proof applies only to convex polygons; however, in 2003, John Frank Putz and Timothy Sipka [21] have shown that the result also applies to concave polygons and even to similar figures that have curved boundaries. To show this result for a simple case we recall that the area of a plane figure is proportional to the square of any linear dimension, and in particular is proportional to the square of the length of any side. Now we erect similar figures with areas A, B and C on sides with corresponding lengths a, b and c. Then, it follows that

$$\frac{A}{a^2} = \frac{B}{b^2} = \frac{C}{c^2},$$

which implies

$$A + B = \frac{a^2}{c^2}C + \frac{b^2}{c^2}C = \frac{a^2+b^2}{c^2}C = C.$$

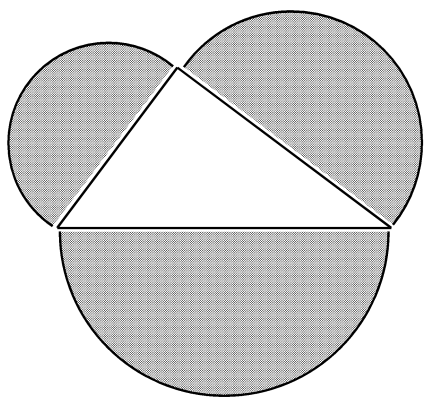

Figure 3

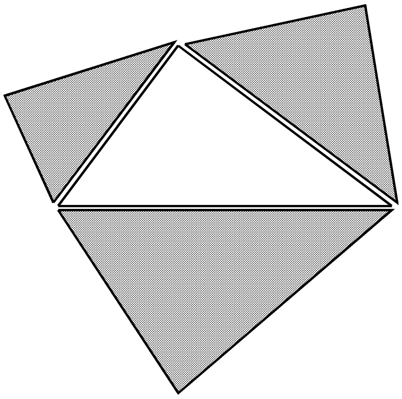

Figure 4

Conversely, as for the converse of Pythagorean Theorem, if the sides of a triangle are corresponding parts in three similar figures such that the area of one is the sum of the areas of the other two, then the triangle is a right triangle.

Reconstructing Figure 3 as Figure 5, we find Hippocrates's famous result: The sum of the areas of two lunes is equal to the area of the triangle, i.e., Area of $I+$ Area of II = Area of A. Encouraged with this result, Hippocrates unsuccessfully tried to square the circle.

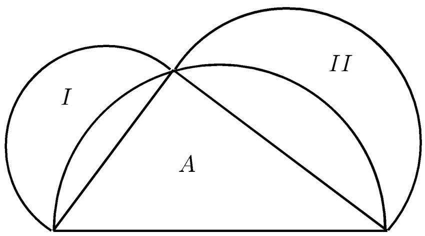

Figure 5

Now we shall give five different proofs (in modern terminology with necessary changes) of the Pythagorean Theorem, which are fairly easy and have some historical importance.

Proof 1. Euclid perhaps acknowledging the complications in the proof given in Book I (Proposition 47), he himself derived (according to Proclus) a simpler proof in Book VI (Proposition 31). A semi-algebraic version of Euclid's proof appeared in Legendre's textbook *Eléments de géométrie* in 1794. His book was translated into English in 1858 by Charles Davis (1798-1876), which became very popular in America. Davis's proof is now very popular all over the world: In Figure 6, $\angle ACB$ is the right angle. We draw the perpendicular CD from C on the hypotenuse AB, so that $\angle ADC$ and $\angle BDC$ are right angles. We also note that $\angle DAC = \angle DCB$ ($\angle ACD = \angle DBC$). Thus, ΔADC and ΔCDB are similar to each other, and both are similar to ΔACB. Hence, it follows that $AC/AB = AD/AC$ and $BC/AB =$

BD/BC, and therefore $AC^2 = AB \times AD$ and $BC^2 = AB \times BD$. Finally, adding these relations, we get

$$AC^2 + BC^2 \;=\; AB(AD + BD) \;=\; AB \times AB \;=\; AB^2. \tag{2}$$

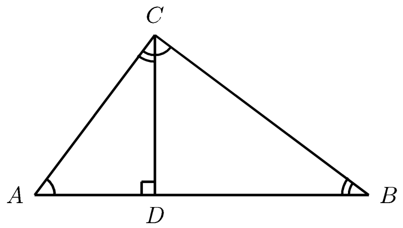

Figure 6

Albert Einstein (1879-1955) when he was 12-year-old succeeded in "proving" Pythagorean Theorem (without claiming its originality) by using the similarity of the triangles. However, unfortunately, he left no such record of his childhood proof. The general consensus among Einstein's biographers is that he probably rediscovered Euclid's proof, or found one of its variants. However, Walter Isaacson (born 1952), Jeremy Bernstein (born 1929), and Banesh Hoffman (1906-1986) showed some resistance to this conclusion. Ten years later Einstein discovered four dimensional form of Pythagorean Theorem and used it in his special theory of relativity. After few years he expanded this theorem further and used it in his study of general relativity.

In Figure 6, we also note that

$$A_{ABC} \;=\; \frac{1}{2}AC \times BC \;=\; \frac{1}{2}AB \times CD,$$

and hence $AB = AC \times BC/CD$. Using this relation in (2), we find

$$\frac{1}{AC^2} + \frac{1}{BC^2} \;=\; \frac{1}{CD^2}. \tag{3}$$

Eli Maor (born 1937) in his book [17] of 2007 calls the relation (3) as the *Little Pythagorean Theorem*, but in the literature it is better known as the *Reciprocal Pythagorean Theorem*.

Proof 2. In the ancient Chinese text *Zhou Bi Suan Jing* (The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven) of Zhou dynasty (1046-256 BC), there is a passage that gives the following dissection proof of Pythagorean Theorem: Rotate the given right-angled triangle (ABC) about the center of the square on the hypotenuse to form triangles FCY, GYX and EXB as in Figure 7.

Then, it is easy to see that

$$A_{AFGE} \;=\; A_{CDHF} + A_{BEMD} + 2A_{ABDC} \quad \text{and} \quad A_{AFGE} \;=\; A_{BXYC} + 4A_{ABC}.$$

From this, one sees that the area of the square on the hypotenuse (BC) is the sum of the areas of the squares on the other two sides (AB and CA) of the right-angled triangle (ABC).

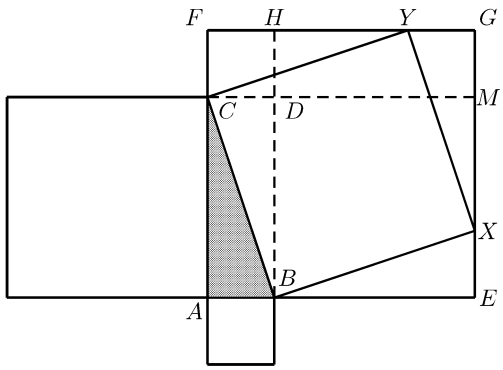

Figure 7

During the time of the Three Kingdoms (3rd century AD) in China, the Wu mathematician Zhao Shuang provided a similar proof in his annotation of Zhou Bi Suan Jing (Figure 8).

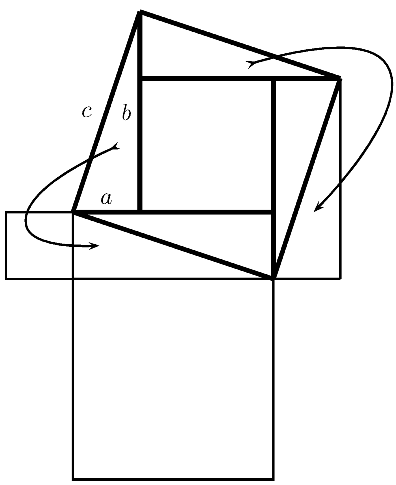

Figure 8

Proclus conjectures that the following variation of the Chinese proof by *dissection* is due to Pythagoras:

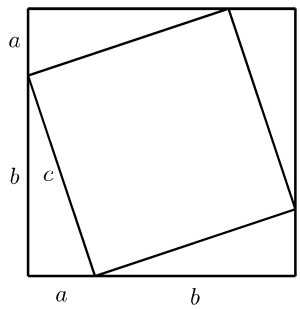

Figure 9

From the Figure 9 it follows that

$$c^2 + 4\frac{ab}{2} \;=\; (a+b)^2 \;=\; a^2 + b^2 + 2ab,$$

and hence (1) holds.

Another similar idea was proposed by the Indian Mathematician Bhaskara II or Bhaskaracharya (working 486) (Figure 10(a)). It is amusing to note that, besides the diagram, Bhaskaras proof consists only of a single exclamation: "Behold"! This is perhaps the first *visual proof* (a proof without words of an identity or mathematical statement which can be demonstrated as self-evident by a diagram without any accompanying explanatory text), for more details of such proofs, see (Nelsen, [20]). Coolidge's proof is similar to Bhaskara's proof.

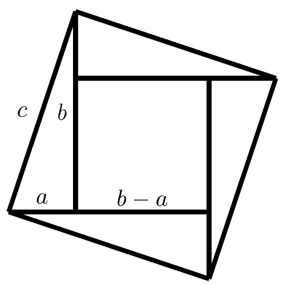

Figure 10(a)

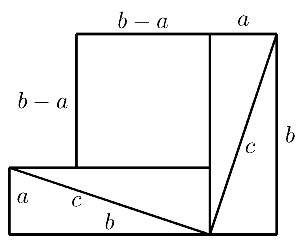

Figure 10(b)

Algebraically, from Figures 10 (a) and (b) it follows that

$$c^2 \;=\; 2ab + (b-a)^2 \;=\; a^2 + b^2.$$

Proof 3. From Figure 11 and Hippocrates's Generalization of Pythagorean Theorem, (1) is immediate.

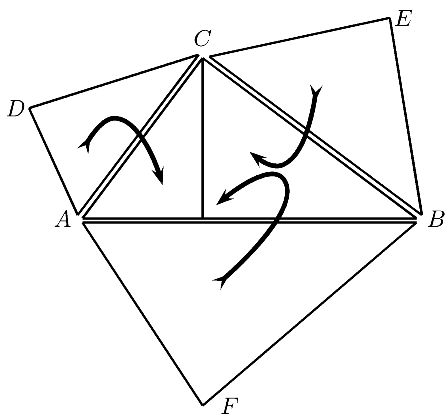

Figure 11

This proof was originally given by R.P. Lamy in 1685, which was rediscovered by Stanley Jashemski in 1934 at the age of nineteen, and then after seventy years by Eli Maor who calls it as the *Folding Bag* in his book of 2007.

Proof 4. The following direct proof is due to Garfield. It appeared in 1876 in the *New England Journal of Education.* Figure 12 shows three triangles forming half of a square with sides of length $a+b$. The angles A, B and D satisfy the relations

$$A+B \ = \ 90^0 \quad \text{and} \quad A+B+D \ = \ 180^0.$$

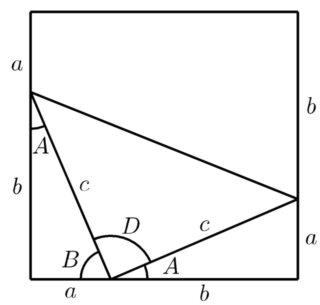

Figure 12

Thus, $D = 90^0$, and hence all the three triangles are right triangles. The area of the half square is

$$\frac{1}{2}(a+b)^2 \ = \ \frac{1}{2}(a^2+2ab+b^2),$$

while the equivalent total area of the three triangles is

$$\frac{1}{2}ab+\frac{1}{2}c^2+\frac{1}{2}ab.$$

Equating these two expressions, we get

$$a^2+2ab+b^2 \ = \ ab+c^2+ab \quad \text{or} \quad a^2+b^2 \ = \ c^2.$$

Proof 5. We note that Daivajna Varahamihira (working 123 BC) gave several trigonometric formulae that correspond to $\sin x = \cos(\pi/2 - x)$, $(1 - \cos 2x)/2 = \sin^2 x$, and $\sin^2 x + \cos^2 x = 1$, which is the same as the Pythagorean Theorem (Identity). Here we shall give one of the easiest trigonometric proofs of the Pythagorean Theorem which was deduced by Edmund Georg Hermann (Yehezkel) Landau (1877-1938) from the Cosine addition formula

$$\cos(x+y) = \cos x \cos y - \sin x \sin y. \tag{4}$$

This formula was known to Bhaskara II. Landau's proof of (4) is based on infinite series representations of $\sin x$ and $\cos x$. From Figure 13, in which $\angle AFD = 90^0$, we have

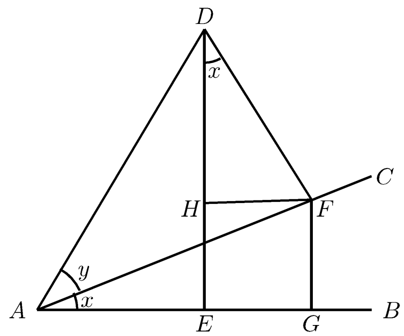

Figure 13

$$\begin{aligned} \cos(x+y) &= \frac{AE}{AD} = \frac{AG}{AD} - \frac{HF}{AD} \\ &= \frac{AG}{AF}\frac{AF}{AD} - \frac{HF}{FD}\frac{FD}{AD} \\ &= \cos x \cos y - \sin x \sin y. \end{aligned}$$

In (4) we let $y = -x$, to obtain

$$\cos 0 = \cos x \cos(-x) - \sin x \sin(-x),$$

which is the same as $1 = \cos^2 x + \sin^2 x$. Thus, Pythagorean Identity is hidden in (4).

Finally, we note that the relation (4) immediately follows from the formula $e^{i\theta} = \cos\theta + i\sin\theta$ of Leonhard Euler (1707-1783). In fact, from this formula, we have

$$e^{i(x+y)} = \cos(x+y) + i\sin(x+y)$$

and

$$\begin{aligned} e^{i(x+y)} = e^{ix}e^{iy} &= (\cos x + i\sin x)(\cos y + i\sin y) \\ &= (\cos x\cos y - \sin x\sin y) + i(\sin x\cos y + \cos x\sin y). \end{aligned}$$

Now comparing the real parts of the above two equations (4) follows.

Now we shall provide various generalizations of the Pythagorean Theorem.

Ptolemy's generalization of the Pythagorean Theorem. Alexandrian Claudius Ptolemaeus (around 90-168 AD) known in English as Ptolemy proved that in any cyclic quadrilateral (vertices all lie on a single circle) $ABCD$ (see Figure 14)

$$AB \times CD + BC \times DA \;=\; AC \times BD. \tag{5}$$

This result appears in his great work *Almagest*.

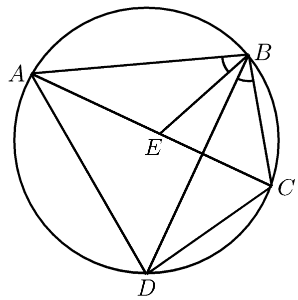

Figure 14

In Figure 14, E on AC is such that $\angle ABE = \angle CBD$. On BC the angles $\angle BAC = \angle BDC$ and on AB, $\angle ADB = \angle ACB$. Thus, $\triangle ABE$ is similar to $\triangle DBC$, and $\triangle EBC$ is similar to $\triangle ABD$. Hence, it follows that

$$\frac{AE}{AB} \;=\; \frac{DC}{DB} \quad \text{and} \quad \frac{EC}{BC} \;=\; \frac{AD}{BD},$$

which are the same as

$$AE \times DB \;=\; AB \times DC \quad \text{and} \quad EC \times BD \;=\; BC \times AD.$$

An addition of these relations gives

$$(AE + EC) \times BD \;=\; AB \times DC + BC \times AD.$$

But, since $AE + EC = AC$ the above relation is the same as (5)

Pythagorean Theorem follows as a special case when $ABCD$ is a rectangle.

Pappus's generalization of the Pythagorean Theorem. Pappus of Alexandria (around 290-350) showed that for an arbitrary triangle with arbitrary parallelograms drawn to its two sides how to construct a parallelogram on the third side whose area is equal to the sum of the areas of the other two parallelograms. This extension of Pythagorean Theorem has been of considerable interest, e.g., see Howard Whitley Eves (1911-2004) in 1958, Eli Maor in 2007, and Claudi Alsina (born 1952) and Roger Nelsen (born 1942) in 2010.

Let ABC be any triangle, and let $ABDE$ and $ACFG$ be two parallelograms built on the sides AB and AC, respectively (see Figure 15). Extend DE and FG until they intersect at H. Draw $BL = CM$, each parallel and equal to AH. This produces the parallelogram $BLMC$. Pappus's construction says that

$$A_{ABDE} + A_{ACFG} \;=\; A_{BLMC}. \tag{6}$$

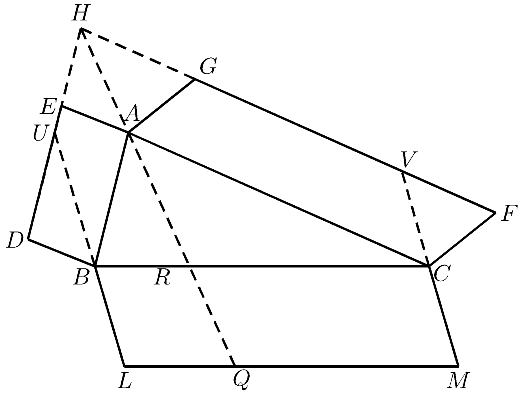

Figure 15

To show (6) it suffices to notice that the parallelograms $ABDE$ and $ABUH$ have the same base length and height, and hence have the same area. A similar argument holds for the parallelograms $ACFG$ and $ACVH$, $ABUH$ and $BLQR$, and $ACVH$ and $RCMQ$. This gives

$$A_{ABDE} + A_{ACFG} = A_{ABUH} + A_{ACVH} = A_{BLQR} + A_{RCMQ} = A_{BLMC}.$$

ibn Qurra's generalization of the Pythagorean Theorem. Thabit ibn Qurra (826-901) in an arbitrary triangle ABC with $\angle BAC \geq 90^0$ drew two straight lines AP and AQ so that $\angle APB = \angle AQC = \angle BAC$, (see Figure 16). Thus, the triangles ABC, PBA and QAC are similar. Hence, it follows that

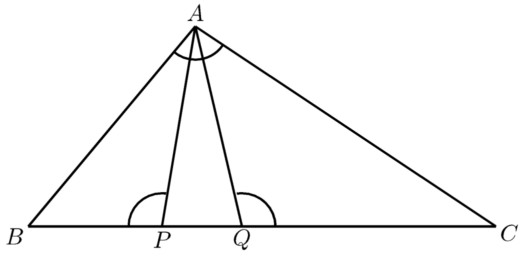

Figure 16

$$\frac{AB}{BC} \;=\; \frac{PB}{AB} \quad \text{and} \quad \frac{AC}{BC} \;=\; \frac{QC}{AC},$$

which are the same as

$$AB^2 \;=\; BC \times PB \quad \text{and} \quad AC^2 \;=\; BC \times QC.$$

An addition of these relations give

$$AB^2 + AC^2 \ = \ BC(PB + QC). \tag{7}$$

If $\angle BAC = 90^0$, then the points P and Q are the same, and thus $PB + QC = BC$. Hence, in this case (7) reduces to the Pythagorean Theorem.

The Law of Cosines. The law of cosines states that for any triangle ABC, with sides a, b, c (see Figure 17)

$$c^2 \ = \ a^2 + b^2 - 2ab\cos C. \tag{8}$$

The nontrigonometric form of the Law of Cosines is available in Euclid's Book II (Propositions 12 and 13): In any triangle, the sum of squares of two sides is equal to the square of the third side increased by twice the product of the first side with orthogonal projection of the second to the first side.

If $C = \pi/2$, then the cosines law (8) reduces to Pythagorean relation (1).

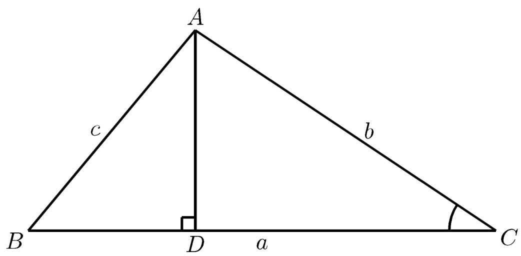

Figure 17

In the triangle ACD, we have

$$DC \ = \ b\cos C \quad \text{and} \quad AD \ = \ b\sin C,$$

and hence

$$BD \ = \ a - DC \ = \ a - b\cos C.$$

Now in the triangle ABD, Pythagorean Theorem and the above relations give

$$\begin{aligned} c^2 \ = \ BD^2 + AD^2 \ &= \ (a - b\cos C)^2 + (b\sin C)^2 \\ &= \ a^2 + b^2(\cos^2 C + \sin^2 C) - 2ab\cos C \\ &= \ a^2 + b^2 - 2ab\cos C. \end{aligned}$$

If in (8), $\angle C > 90^0$, then $\cos C < 0$ which implies $c^2 > a^2 + b^2$, and if $\angle C < 90^0$, then $\cos C > 0$ which gives $c^2 < a^2 + b^2$. Hence, the converse of the Pythagorean Theorem also follows from the Law of Cosines.

Jemshid al-Kashi (around 1380-1429), a Persian mathematician and astronomer, provided the first explicit statement of the Law of Cosines in a form suitable for triangulation. Francois Viéte (1540-1603) popularized this law in the Western world. Finally, at the beginning of the 19th century, this law was written in its current symbolic form.

Pythagorean Theorem in Vector Spaces. We need the following definitions, see Agarwal and Flaut [4].

A *field* is a set of scalars, denoted by F, in which two binary operations, addition $(+)$, and multiplication $(\cdot)$ are defined so that the following axioms hold:

A1. *Closure property of addition:* If $a, b \in F$, then $a + b \in F$.

A2. *Commutative property of addition:* If $a, b \in F$, then $a + b = b + a$.

A3. *Associative property of addition:* If $a, b, c \in F$, then $(a + b) + c = a + (b + c)$.

A4. *Additive identity:* There exists a zero element, denoted by 0, in F such that for all $a \in F$, $a + 0 = 0 + a = a$.

A5. *Additive inverse:* For each $a \in F$, there is an unique element $(-a) \in F$ such that $a + (-a) = (-a) + a = 0$.

A6. *Closure property of multiplication:* If $a, b \in F$, then $a \cdot b \in F$.

A7. *Commutative property of multiplication:* If $a, b \in F$, then $a \cdot b = b \cdot a$.

A8. *Associative property of multiplication:* If $a, b, c \in F$, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

A9. *Multiplicative identity:* There exists a unit element, denoted by 1, in F such that for all $a \in F$, $a \cdot 1 = 1 \cdot a = a$.

A10. *Multiplicative inverse:* For each $a \in F$, $a \neq 0$ there is an unique element $a^{-1} \in F$ such that $a \cdot a^{-1} = a^{-1}a = 1$.

A11. *Left distributivity:* If $a, b, c \in F$, then $a \cdot (b + c) = a \cdot b + a \cdot c$.

A12. *Right distributivity:* If $a, b, c \in F$, then $(a + b) \cdot c = a \cdot c + b \cdot c$.

The set of rational numbers Q, the set of real numbers R, and the set of complex numbers C, with the usual definitions of addition and multiplication, are fields. The set of natural numbers $N = \{1, 2, \cdots\}$, and the set of all integers $Z = \{\cdots, -2, -1, 0, 1, 2 \cdots\}$ are not fields.

A *vector space* V over a field F denoted as (V, F) is a nonempty set of elements called *vectors* together with two binary operations, addition of vectors and multiplication of vectors by scalars so that the following axioms hold:

B1. *Closure property of addition:* If $u, v \in V$, then $u + v \in V$.

B2. *Commutative property of addition:* If $u, v \in V$, then $u + v = v + u$.

B3. *Associativity property of addition:* If $u, v, w \in V$, then $(u+v)+w = u+(v+w)$.

B4. *Additive identity:* There exists a zero vector, denoted by 0, in V such that for all $u \in V$, $u + 0 = 0 + u = u$.

B5. *Additive inverse:* For each $u \in V$, there exists a vector v in V such that $u + v = v + u = 0$. Such a vector v is usually written as $-u$.

B6. *Closure property of multiplication:* If $u \in V$ and $a \in F$, then the product $a \cdot u = au \in V$.

B7. If $u, v \in V$ and $a \in F$, then $a(u + v) = au + av$.

B8. If $u \in V$ and $a, b \in F$, then $(a + b)u = au + bu$.

B9. If $u \in V$ and $a, b \in F$, then $ab(u) = a(bu)$.

B10. *Multiplication of a vector by a unit scalar:* If $u \in V$ and $1 \in F$, then $1u = u$.

The spaces (V, R) and (V, C) will be called *real* and *complex vector spaces*, respectively.

The n-tuple space. Let F be a given field. We consider the set V of all ordered n-tuples $u = (a_1, \cdots, a_n)$ of scalars (known as *components*) $a_i \in F$. If $v = (b_1, \cdots, b_n)$ is in V, the addition of u and v is defined by $u + v = (a_1 + b_1, \cdots, a_n + b_n)$, and the product of a scalar $c \in F$ and vector $u \in V$ is defined by $cu = (ca_1, \cdots, ca_n)$. It is to be remembered that $u = v$, if and only if, their corresponding components are equal, i.e., $a_i = b_i,\ i = 1, \cdots, n$. With this definition of addition and scalar multiplication it is easy to verify all the axioms B1 - B10, and hence this (V, F) is a vector space. If $F = R$, then V is denoted as R^n, which for $n = 2$ and 3 reduces, respectively, to the two and three dimensional usual vector spaces. Similarly, if $F = C$, then V is written as C^n.

The space of polynomials. Let F be a given field. We consider the set $\mathcal{P}_n,\ n \geq 1$ of all polynomials of degree *at most* $n - 1$, i.e.,

$$\mathcal{P}_n \ = \ \left\{ a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} = \sum_{i=0}^{n-1} a_i x^i : a_i \in F, \quad x \in R \right\}.$$

If $u = \sum_{i=0}^{n-1} a_i x^i,\ v = \sum_{i=0}^{n-1} b_i x^i \in \mathcal{P}_n$, then the addition of vectors u and v is defined by

$$u + v \ = \ \sum_{i=0}^{n-1} a_i x^i + \sum_{i=0}^{n-1} b_i x^i \ = \ \sum_{i=0}^{n-1} (a_i + b_i) x^i,$$

and the product of a scalar $c \in F$ and vector $u \in \mathcal{P}_n$ is defined by

$$cu \ = \ c \sum_{i=0}^{n-1} a_i x^i \ = \ \sum_{i=0}^{n-1} (ca_i) x^i.$$

This $(\mathcal{P}_n, F)$ is a vector space. We remark that the set of all polynomials of degree exactly $n - 1$ is not a vector space. In fact, if we choose $b_{n-1} = -a_{n-1}$, then $u + v$ is a polynomial of degree $n - 2$.

The space of functions. Let F be a field of complex numbers, and $X \subseteq F$. We consider the set V of all functions from the set X to F. The sum of two functions $u, v \in V$ is defined by $(u+v)$, i.e., $(u+v)(x) = u(x)+v(x),\ x \in X$, and the product of a scalar $c \in F$ and function $u \in V$ is defined by cu, i.e., $(cu)(x) = cu(x)$. This (V, F) is a vector space. In particular, $(C[X], F)$, where $C[X]$ is the set of all continuous functions from X to F, with the same vector addition and scalar multiplication is a vector space.

An *inner product* on (V, C) is a function that assigns to each pair of vectors $u, v \in V$ a complex number, denoted as (u, v), or simply by $u \cdot v$, which satisfies the following axioms:

C1. *Positive definite property:* $(u, u) > 0$ if $u \neq 0$, and $(u, u) = 0$ if and only if $u = 0$.

C2. *Conjugate symmetric property:* $(u, v) = \overline{(v, u)}$.

C3. *Linear property:* $(c_1 u + c_2 v, w) = c_1(u, w) + c_2(v, w)$ for all $u, v, w \in V$ and $c_1, c_2 \in C$.

The vector space (V, C) with an inner product is called a *complex inner product space*. From C2 we have $(u, u) = \overline{(u, u)}$ and hence (u, u) must be real. Further, from C2 and C3 it immediately follows that $(w, c_1 u + c_2 v) = \overline{c}_1(w, u) + \overline{c}_2(w, v)$. The definition of a *real inner product space* (V, R) remains the same as above except now for each pair $u, v \in V$, (u, v) is real, and hence in C2 complex conjugates is omitted. In (V, C) the angle between the vectors u, v is defined by the relation

$$\cos\theta = \frac{Re(u, v)}{(u, u)^{1/2}(v, v)^{1/2}}, \tag{9}$$

where $Re(u, v)$ is the real part of (u, v). In (9), the right-hand side lies between -1 and 1 (Cauchy-Schwarz inequality).

Inner Product in C^n and R^n. Let $u = (z_1, \cdots, z_n),\ v = (w_1, \cdots, w_n) \in C^n$. The standard inner product in C^n is defined as

$$(u, v) = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n = \sum_{i=1}^{n} z_i\overline{w}_i.$$

The vector space C^n with the above inner product is called a *unitary space*. Similarly, for $u = (a_1, \cdots, a_n),\ v = (b_1, \cdots, b_n) \in R^n$, the inner product in R^n is defined as

$$(u, v) = a_1 b_1 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i.$$

The inner product in R^n is also called *dot product* and sometimes denoted as $u \cdot v$. The vector space R^n with the above inner product is simply called an inner product, or dot product, or *Euclidean n-space.*

Inner Product in $C_C[a, b]$ and $C_R[a, b]$. For the functions $u(t) = f(t) + ig(t)$ and $v(t) = p(t) + iq(t),\ t \in [a, b]$ in the vector space of complex-valued continuous functions $C_C[a, b]$ an inner product is defined as

$$(u, v) = \int_a^b (f(t) + ig(t))(p(t) - iq(t))dt.$$

Similarly, for the functions $u(t) = \phi(t)$ and $v(t) = \psi(t),\ t \in [a, b]$ in the vector space of real-valued continuous functions $C_R[a, b]$ an inner product is defined as

$$(u, v) = \int_a^b \phi(t)\psi(t)dt.$$

A subset S of an inner product space (V, F) is said to be *orthogonal* if and only if for every pair of vectors $u, v \in S,\ u \neq v$ the inner product $(u, v) = 0$. From (9) two vectors $u, v \in (V, R)$ are orthogonal if and only if $(u, v) = 0$, i.e., $\theta = \pi/2$. Thus, orthogonality naturally generalizes the geometric concept perpendicular in R^2.

A *norm* (or *length*) on a vector space (V, F) is a function that assigns to each vector $u \in V$ a nonnegative real number, denoted as $\|u\|$, which satisfies the following axioms:

D1. *Positive definite property:* $\|u\| \geq 0$, and $\|u\| = 0$ if and only if $u = 0$,

D2. *Homogeneity property:* $\|cu\| = |c|\|u\|$ for each scalar c,

D3. *Triangle inequality:* $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$.

A vector space (V, F) with a norm $\|\cdot\|$ is called a *normed linear space*, and is denoted as $(V, F, \|\cdot\|)$. In what follows we shall use only the Euclidean norm defined as $\|u\| = (u, u)^{1/2}$. In the vector space C^n for two vectors $u = (z_1, \cdots, z_n)$, $v = (w_1, \cdots, w_n)$ the Euclidean distances is denoted and defined as

$$\|u - v\| \;=\; (|z_1 - w_1|^2 + \cdots + |z_n - w_n|^2)^{1/2} \;=\; \left(\sum_{j=1}^{n} |z_j - w_j|^2\right)^{1/2}.$$

Similarly, in $C_C[a, b]$ for two functions $u(t) = f(t) + ig(t)$ and $v(t) = p(t) + iq(t)$ the Euclidean distances is defined as

$$\|u - v\| \;=\; \left(\int_a^b (|f(t) - p(t)|^2 + |g(t) - q(t)|^2)dt\right)^{1/2}.$$

The subset $\hat{S}$ is called *orthonormal* if $\hat{S}$ is orthogonal and for every $\hat{u} \in \hat{S}$, $\|\hat{u}\|^2 = (\hat{u}, \hat{u}) = 1$.

The subset $S_1 = \{u^1, u^2, u^3\} = \{(1, 2, 0, -1), (5, 2, 4, 9), (-2, 2, -3, 2)\}$ of R^4 is orthogonal. The subset $S_2 = \{w^1, w^2, w^3, w^4\} = \{(1+i, 1, 1-i, i), (1+5i, 6+5i, -7-i, 1-6i), (-7+34i, -8-23i, -10+22i, 30+13i), (-2-4i, 6+i, 4+3i, 6-i)\}$ of C^4 is orthogonal. The set $S_3 = \left\{\frac{1}{\sqrt{\pi}}, \sqrt{\frac{2}{\pi}}\cos nx,\ n = 1, 2, \cdots\right\}$ is orthonormal on $0 < x < \pi$. The set $S_4 = \left\{\sqrt{\frac{2}{\pi}}\sin nx,\ n = 1, 2, \cdots\right\}$ is orthonormal on $0 < x < \pi$. The set $S_5 = \left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos nx, \frac{1}{\sqrt{\pi}}\sin nx,\ n = 1, 2, \cdots\right\}$ is orthonormal on $-\pi < x < \pi$. The set $S_6 = \{P_n(x),\ n = 0, 1, 2, \cdots\}$, where $P_n(x)$ is the Legendre polynomial of degree n defined by (see Agarwal and O'Regan [1])

$$P_n(x) \;=\; \sqrt{\frac{2n+1}{2}} \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{(2n - 2k)!}{2^n\ k!\ (n-k)!\ (n-2k)!} x^{n-2k}$$

is orthonormal on $-1 < x < 1$. In particular, we have

$$P_0(x) \;=\; \frac{1}{\sqrt{2}}, \quad P_1(x) \;=\; \sqrt{\frac{3}{2}}x, \quad P_2(x) = \sqrt{\frac{5}{2}} \cdot \frac{1}{2}(3x^2 - 1),$$

$$P_3(x) \;=\; \sqrt{\frac{7}{2}} \cdot \frac{1}{2}(5x^3 - 3x), \quad P_4(x) \;=\; \sqrt{\frac{9}{2}} \cdot \frac{1}{8}(35x^4 - 30x^2 + 3).$$

We are now in the position to state the following *Generalized Pythagorean Theorem*: Let $\{u^1, \cdots, u^r\}$ be an orthogonal subset of an inner product space (V, F). Then, the following holds

$$\|u^1 + \cdots + u^m\|^2 \;=\; \|u^1\|^2 + \cdots + \|u^m\|^2. \tag{10}$$

Indeed, from the definition of inner product and orthogonality of $\{u^1, \cdots, u^m\}$, it follows that

$$\begin{aligned}\|u^1 + \cdots + u^m\|^2 &= ((u^1 + \cdots + u^m), (u^1 + \cdots + u^m))\\ &= (u^1, u^1) + (u^1, u^2) + \cdots + (u^1, u^m)\\ &\quad + \cdots\\ &\quad + (u^m, u^1) + (u^m, u^2) + \cdots + (u^m, u^m)\\ &= (u^1, u^1) + \cdots + (u^m, u^m)\\ &= \|u^1\|^2 + \cdots + \|u^m\|^2.\end{aligned}$$

For $m = 3$ equation (10) immediately extends Pythagorean relation (1) to rectangular solids. Indeed from Figure 18 and Pythagorean Theorem twice it follows that

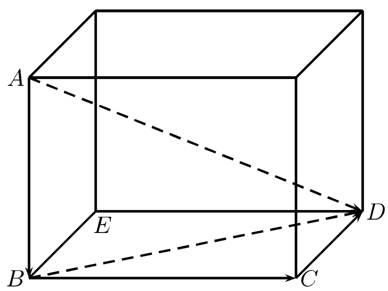

Figure 18

$$|\vec{BD}|^2 \;=\; |\vec{BC}|^2 + |\vec{CD}|^2$$

and

$$|\vec{AD}|^2 \;=\; |\vec{AB}|^2 + |\vec{BD}|^2.$$

Thus, it follows that

$$|\vec{AD}|^2 \;=\; |\vec{AB}|^2 + |\vec{BC}|^2 + |\vec{CD}|^2. \tag{11}$$

In particular for the vectors $\vec{BC} = (6, 0, 0)$, $\vec{BE} = \vec{CD} = (0, 2, 0)$ and $\vec{BA} = (0, 0, 3)$, we have $7^2 = 3^2 + 6^2 + 2^2$.

As further examples, for the vectors in the sets S_1 and S_2, respectively, we have

$$\|(1,2,0,-1) + (5,2,4,9) + (-2,2,-3,2)\|^2 = \|(4,6,1,10)\|^2 = 153$$
$$= \|(1,2,0,-1)\|^2 + \|(5,2,4,9)\|^2 + \|(-2,2,-3,2)\|^2 = 6 + 126 + 21 = 153$$

and

$$\|w^1 + w^2 + w^3 + w^4\|^2 \;=\; \|(-7+36i, 5-17i, -12+23i, 37+7i)\|_2 = 3750$$
$$= \|w^1\|^2 + \|w^2\|^2 + \|w^3\|^2 + \|w^4\|^2 \;=\; 6 + 174 + 3451 + 119 = 3750.$$

Clearly in (10) if the set $\{u^1, \cdots, u^m\}$ is orthonormal, then it becomes

$$\|u^1 + \cdots + u^m\|^2 \;=\; \|u^1\|^2 + \cdots + \|u^m\|^2 \;=\; m. \tag{12}$$

Thus, for the vectors in the sets S_3 and S_6, respectively, we have

$$\int_0^\pi \left(\frac{1}{\sqrt{\pi}} + \sum_{k=1}^{m-1}\sqrt{\frac{2}{\pi}}\cos kx\right)^2 dx = m$$
$$= \int_0^\pi \left(\frac{1}{\sqrt{\pi}}\right)^2 dx + \sum_{k=1}^{m-1}\int_0^\pi \left(\sqrt{\frac{2}{\pi}}\cos kx\right)^2 dx = m$$

and

$$\int_{-1}^1 \left(\sum_{k=0}^{m-1} P_k(x)\right)^2 dx \;=\; \sum_{k=0}^{m-1}\int_{-1}^1 P_k^2(x)dx \;=\; m.$$

We note that in (10), m can be infinite provided $\sum_{k=1}^\infty \|u^k\|^2$ converges (finite). For this, as an example we note that the set $\bar{S}_4 = \left\{\sqrt{\frac{2}{\pi}}\frac{\sin nx}{n},\ n = 1, 2, \cdots\right\}$ is orthogonal on $0 < x < \pi$, and we have

$$\int_0^\pi \left(\sum_{k=1}^\infty \sqrt{\frac{2}{\pi}}\frac{\sin kx}{k}\right)^2 dx \;=\; \sum_{k=1}^\infty \frac{1}{k^2}\int_0^\pi \left(\sqrt{\frac{2}{\pi}}\sin kx\right)^2 dx \;=\; \sum_{k=1}^\infty \frac{1}{k^2} \;=\; \frac{\pi^2}{6}.$$

In the literature finding the sum of the series $\sum_{k=1}^\infty 1/k^2$ is known as Basel problem. It was posed by Pietro Mengoli (1625-1686) in 1650 and solved by Euler in 1734.

Another generalization of Pythagorean Theorem in inner product spaces is known as *Parallelogram Law*: For any pair of vectors u, v in an inner product space (V, F),

$$\|u+v\|_2^2 + \|u-v\|_2^2 \;=\; 2\|u\|_2^2 + 2\|v\|_2^2, \tag{13}$$

i.e., the sum of the squares of the lengths of the four sides of a parallelogram equals the sum of the squares of the lengths of the two diagonals.

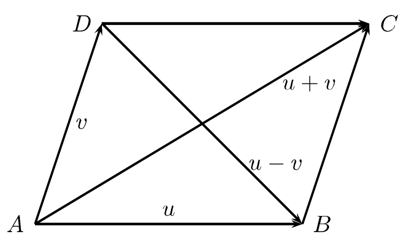

Figure 19

In R^2 from Figure 19, the relation (13) is the same as

$$|\vec{AC}|^2 + |\vec{DB}|^2 \;=\; 2|\vec{AB}|^2 + 2|\vec{BC}|^2,$$

which for a rectangle is the same as (1).

De Gua's Theorem. In the year 1783, Jean Paul de Gua de Malves (1713-1785) showed that given three right triangles with leg lengths such that we can form a tetrahedron, the sum of the squares of the areas of the three right triangles is equal to the square of the area of the base. From Figure 20 it means that

$$A^2_{ABC} \;=\; A^2_{ABO} + A^2_{ACO} + A^2_{BCO}. \tag{14}$$

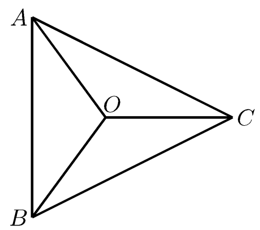

Figure 20

To show the relation (14) algebraically, in 2017 Hartzer [13] cleverly used the formula $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = (a+b+c)/2$ and A is the area of the triangle with sides a, b, and c. This formula is originally due to Bhaskara I (before 123 BC), but known in the literature as Heron's formula (about 75 AD). Substituting s in the formula of A and squaring, we find

$$A^2 \;=\; \frac{2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4)}{16}.$$

Now let $d = OA,\ e = OB,\ f = OC$, and $a = AB,\ b = BC,\ c = CA$. Then by the Pythagorean Theorem three times, it follows that

$$a^2 \;=\; d^2 + e^2, \quad b^2 \;=\; e^2 + f^2, \quad c^2 = f^2 + d^2$$

and now these relations give

$$A^2 \;=\; \frac{d^2e^2 + f^2d^2 + e^2f^2}{4} \;=\; \left(\frac{de}{2}\right)^2 + \left(\frac{fd}{2}\right)^2 + \left(\frac{ef}{2}\right)^2,$$

which is the same as (14).

Around the same year 1783, De Gua proved his theorem, a slightly more general version was published by Charles de Tinseau d'Amondans (1746-1818). This theorem was also known much earlier to Johann Faulhaber (1580-1635) and René Descartes (1596-1650). We now state a result which was proved in 1974 which is a far reaching generalization of De Gua's theorem.

Conant and Beyer's Theorem. [9] Let U be a measurable subset of an n-dimensional affine subset of $\mathbb{R}^m,\ n \le m$. For any subset $I \subseteq \{1, \cdots, m\}$ with exactly n elements, let U_I be the orthogonal projection of U onto the linear span of $e^{i_1}, \cdots, e^{i_n}$, where $I = \{i_1, \cdots, i_n\}$ and $e^1, \cdots, e^m$ is the standard basis for $\mathbb{R}^m$. Then,

$$\mathrm{vol}_n^2(U) \;=\; \sum_I \mathrm{vol}_n^2(U_I), \tag{15}$$

where $\mathrm{vol}_n(U)$ is the n-dimensional volume of U and the sum is over all subsets $I \subseteq \{1, \cdots, m\}$ with exactly n elements.

Next, we state a theorem of Eisso Atzema [6] which he proved in 2000 for $m \times n$ matrices, where $n \leq m$. In the year 2010 Charles Frohman [11] used a different method to prove Atzema's result, which is more transparent. Related results also established by Sergio Alvarez [5] in 2018 and Willie Wong [25] in 2002.

Atzema's Theorem. For an $m \times n$, $n \leq m$ matrix A the following relation holds

$$\det(A^t A) \;=\; \sum_{I \subset \{1,\cdots,m\}\ |I|=n} \det(A_I)^2, \tag{16}$$

where A^t is the transpose of A, $|I|$ represents the cardinality of I, A_I denotes the $n \times n$ matrix made from the rows of A corresponding to the subset I, and the summation is taken on all possible combinations I.

As (15) the relation (16) geometrically can be interpreted as follows: the square of the content of the parallelepiped spanned by A is equal to the sum of the squares of the orthogonal projections of the parallelepiped into the n-dimensional coordinate hyperplanes.

To illustrate the relation (16) we consider the matrix

$$A \;=\; \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 9 \end{pmatrix}$$

for which the left hand side is

$$|A^t A| \;=\; \begin{vmatrix} 12 & 18 & 36 \\ 18 & 30 & 56 \\ 36 & 56 & 110 \end{vmatrix} \;=\; 24$$

and the right side is

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix}^2 + \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 9 \end{vmatrix}^2 + \begin{vmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 3 & 4 & 9 \end{vmatrix}^2 + \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 4 & 9 \end{vmatrix}^2 = 0^2 + 2^2 + 4^2 + 2^2 = 24.$$

Relation Between Cross and Inner Products. Recall that in R^3 the cross product between two vectors u and v is a vector w denoted and defined as

$$w \;=\; u \times v \;=\; \|u\|\|v\|(n)\sin\theta; \tag{17}$$

here, $0^0 \leq \theta \leq 180^0$ is the angle between u and v in the plane P containing them, and (n) is a unit vector perpendicular to the plane P in the direction given by the right-hand rule. It is clear that w is orthogonal to both u and v, and if u and v are parallel, then the angle θ is either 0^0 or 180^0.

Now inner product between u and v from (9) can be written as

$$(u, v) \;=\; \|u\|\|v\| \cos\theta. \tag{18}$$

Squaring and adding both sides of (17) and (18), and using the Pythagorean Identity $\sin^2\theta + \cos^2\theta = 1$, we have the following relation

$$\|u \times v\|^2 + (u, v)^2 \;=\; \|u\|^2\|v\|^2. \tag{19}$$

Inner and cross products were introduced in 1881 by Josiah Willard Gibbs (1839-1903), and independently by Oliver Heaviside (1850-1925).

Pythagorean Theorem in Non-Euclidean Geometry. To draw parallel lines from a point P not on a line ℓ there are several possibilities:

1. There is one and only one parallel line through P. This statement is equivalent to the Parallel Postulate and leads to the Euclidean geometry.

2. There is no parallel line through P. This possibility leads to a non-Euclidean geometry known as *spherical geometry* (which is crucial in navigation by sea). As an example, we can consider the geometry of the surface of the earth or the celestial sphere. A line on a sphere is the shortest distance between two points on the sphere. If a line is extended, it forms a *great circle.* A great circle, is the end of the lines path. A great circle revolves around the entire sphere with its radius as the radius of the sphere. There are infinitely many great circles on a sphere. The points which are exactly opposite of each other on the sphere, such as poles, are called *antipodal points.* Thus, two great circles will always cross paths at antipodal points. In spherical geometry, angles are defined between great circles, and a triangle is formed by three great circles intersecting. It is clear that in spherical geometry the sum of the interior angles of a triangle always lies between 180^0 and 540^0. Further, the size of an angle increases according as the size of the triangle increases. In antiquity, in India, several astronomical rules for spherical triangles were discovered that are scattered all over ancient astronomical texts such as *Surya Siddhanta* and its commentaries. For example, on a sphere of radius R the spherical law of cosine is given as

$$\cos\left(\frac{c}{R}\right) \;=\; \cos\left(\frac{a}{R}\right)\cos\left(\frac{b}{R}\right) + \sin\left(\frac{a}{R}\right)\sin\left(\frac{b}{R}\right)\cos C, \tag{20}$$

where A, B, C are the angles of a spherical triangle, of which the opposite sides are $a, b,$ and c, respectively (see Figure 21).

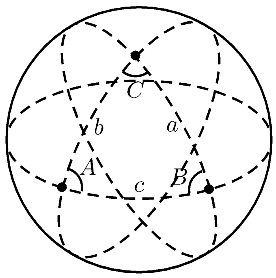

Figure 21

If $\angle C$ is a right angle, then spherical law of cosine (20) reduces to

$$\cos\left(\frac{c}{R}\right) = \cos\left(\frac{a}{R}\right)\cos\left(\frac{b}{R}\right), \tag{21}$$

which is the same as

$$\sin^2\left(\frac{c}{2R}\right) = \sin^2\left(\frac{a}{2R}\right) + \sin^2\left(\frac{b}{2R}\right) - 2\sin^2\left(\frac{a}{2R}\right)\sin^2\left(\frac{b}{2R}\right).$$

Thus, in view of $\sin^2\theta = \theta^2 - O(\theta^4)$ for small θ; here, the symbol O (called "big-O") is due to Edmund Georg Hermann (Yehezkel) Landau (1877-1938), it follows for large R that

$$\left(\frac{c}{2R}\right)^2 - O\left(\left(\frac{c}{2R}\right)^4\right) = \left(\frac{a}{2R}\right)^2 - O\left(\left(\frac{a}{2R}\right)^4\right) + \left(\frac{b}{2R}\right)^2 - O\left(\left(\frac{b}{2R}\right)^4\right)$$
$$-2\left[\left(\frac{a}{2R}\right)^2 - O\left(\left(\frac{a}{2R}\right)^4\right)\right]\left[\left(\frac{b}{2R}\right)^2 - O\left(\left(\frac{b}{2R}\right)^4\right)\right],$$

which is the same as

$$\left(\frac{c}{2R}\right)^2 = \left(\frac{a}{2R}\right)^2 + \left(\frac{b}{2R}\right)^2 + O\left(\left(\frac{1}{2R}\right)^4\right) \quad \text{as} \quad R \to \infty.$$

Hence, we find

$$c^2 = a^2 + b^2 + O\left(\frac{1}{R^2}\right) \quad \text{as} \quad R \to \infty. \tag{22}$$

Thus, in the limit, we get back the Pythagorean Relation (1) as the radius R of the sphere tends to infinity.

3. There are more than one parallel lines through P. This possibility leads to the sum of angles in a triangle less than 180 degrees. This branch of geometry was invented by Nicolai Ivanovich Lobachevsky (1792-1856) in 1826, and later in the year 1831 by Janos Bolyai (1802-1860). However, Karl Friedrich Gauss (1777-1855) had anticipated this non-Euclidean geometry more than 30 years before Bolyai, but

Gauss withheld it from publication. A modern use of this geometry known as *hyperbolic geometry* is in the theory of special relativity. Hyperbolic law of cosines was first known to Franz Adolph Taurinus (1794-1874) in 1826, and then Lobachevsky in 1830. Here for this law we shall need the following representation which Jane Gilman [12] has presented in 1995.

$$\cosh\left(\frac{c}{K}\right) = \cosh\left(\frac{a}{K}\right)\cosh\left(\frac{b}{K}\right) - \sinh\left(\frac{a}{K}\right)\sinh\left(\frac{b}{K}\right)\cos C. \tag{23}$$

Here A, B, C are the angles of a hyperbolic triangle, of which the opposite sides are a, b, and c, respectively, and $-1/K^2$ is the *Gaussian curvature* (see Figure 22).

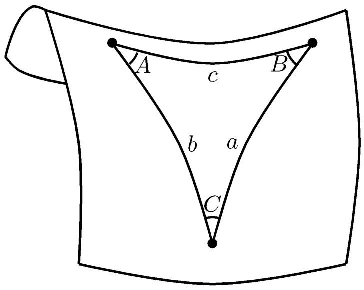

Figure 22

If $\angle C$ is a right angle, then hyperbolic law of cosine (23) reduces to

$$\cosh\left(\frac{c}{K}\right) = \cosh\left(\frac{a}{K}\right)\cosh\left(\frac{b}{K}\right), \tag{24}$$

which is the same as

$$\sinh^2\left(\frac{c}{2K}\right) = \sinh^2\left(\frac{a}{2K}\right) + \sinh^2\left(\frac{b}{2K}\right) + 2\sinh^2\left(\frac{a}{2K}\right)\sinh^2\left(\frac{b}{2K}\right).$$

Thus, in view of $\sinh^2\theta = \theta^2 + O(\theta^4)$ for small θ, it follows for large K that

$$c^2 = a^2 + b^2 + O\left(\frac{1}{K^2}\right) \quad \text{as} \quad K \to \infty. \tag{25}$$

Thus, in the limit, we get back the Pythagorean Relation (1) as K tends to infinity.

4. *Elliptic geometry* is an another example of non-Euclidian geometry. In elliptic geometry all lines perpendicular to one side of a given line intersect at a single point called the *absolute pole* of that line. The perpendiculars on the other side of the given line also intersect at a point. However, unlike in spherical geometry, the poles on either side are the same. Elliptic geometry is also sometimes called *Riemannian geometry* after George Friedrich Bernhard Riemann (1826-1866). In elliptic geometry also the sum of the interior angles of a triangle is greater than 180^0. The Pythagorean Theorem fails in elliptic geometry. For this, on a sphere of radius R consider a spherical triangle with three right angles $A = B = C = \pi/2$,

and sides a, b, c, as in Figure 23. Since the arc length of each side is $L = R\theta$, where θ is the angle from the origin to each endpoint of the arc, it follows that $L_a = RA = R\pi/2, L_b = RB = R\pi/2, L_c = RC = R\pi/2$, where L_a represents the length of side a. If we assume that (1) holds, and as usual let a and b be the sides of the right triangle and c be the hypotenuse, then we must have $(R\pi/2)^2 + (R\pi/2)^2 = (R\pi/2)^2$, which leads to a contradiction.

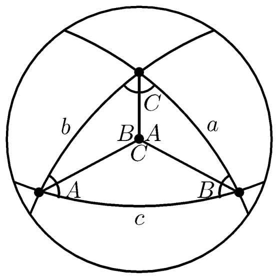

Figure 23

5. *Riemannian geometry* is a very broad and abstract generalization of the *differential geometry* of surfaces. It enabled the formulation of Einstein's general theory of relativity. In particular, in n-dimensional space V on an infinitesimal level Pythagorean Theorem takes the following quadratic form (Riemann introduced this in his doctoral address in 1854, also see Tai Chow [8])

$$ds^2 = \sum_{ij}^{n} g_{ij} dx_i dx_j; \tag{26}$$

here, ds is the line element (the differential of arc length) in V, g_{ij} is the matrix tensor, and $(dx_1, \cdots, dx_n)$ are the components of the vector separating the two pints. For the rectangular coordinates, we have $(dx_1, dx_2, dx_3) = (dx, dy, dz)$, and

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that (26) reduces to

$$ds^2 = dx^2 + dy^2 + dz^2.$$

Similarly, for the spherical coordinates we have $(dx_1, dx_2, dx_3) = (dr, d\theta, d\phi)$, and

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

so that (26) becomes

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

Pythagorean Theorem Problems. We conclude this article with the following interesting problems which require Pythagorean Theorem. These problems are of historical importance.

1. A bamboo 36 cubits tall is broken (bent) by the wind so that the top touches the ground 12 cubits from the stem. Tell the height of the break. (Babylonia and China) [The height of the break is 16 cubits.]

2. In a pond, the flower of a water lily is 2 cubits (cubit was a linear measurement from one's elbow to the tip of the longest (middle) finger, usually 17 to 21 inches) above the water. When it is bent by a gentle breeze, it touches the water at a distance of 4 cubits. Tell the depth of the water. (China) [The depth of the water is 3 cubits.]

3. A chain suspended from an upright post has a length of 9 cubits lying on the ground. When stretched out to its full length so as to just touch the ground, the end is found to be 21 cubits from the post. What is the length of the chain? (China) [The length of the chain is 29 cubits.]

4. A snake's hole is at the foot of a pillar which is 24 cubits high with a peacock perched on its summit. Seeing the snake at a distance of 48 cubits gliding toward its hole, the peacock pounces on it. Say quickly (perhaps means mentally) now at how many cubits from the snake's hole they meet, both proceeding an equal distance. (India) [They meet 18 cubits from the hole.]

5. Two magicians live on a cliff of height 40 cubits. There is a stream at a distance of 120 cubits from the foot of the cliff. One magician climbs down and walks to the stream. The other levitates directly up a short distance and then directly to the stream. If both magicians travel the same distance, tell how high the second one flies. (India) [The magician flies 24 cubits high.]

6. The height of a door (say, x) is 6 *chi* 8 *cun* (say, a) greater than its width (say, y) and that the opposite corners are 1 *zhang* (say, d) apart. Find the height and the width of the door. (China) [$x = \frac{1}{2}(a + \sqrt{2d^2 - a^2}),\ y = \frac{1}{2}(-a + \sqrt{2d^2 - a^2})$.]

7. Find the area of the following pointed field whose sides and one diagonal are labeled as in Figure 24.

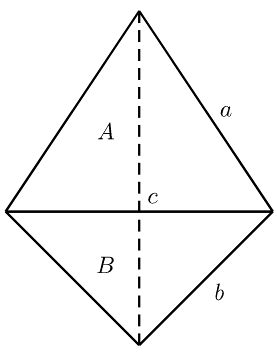

Figure 24

Using Pythagorean Theorem the area of the lower triangle is given by $B = (c/2) \times \sqrt{b^2 - (c/2)^2}$ and that of the upper triangle by $A = (c/2)\sqrt{a^2 - (c/2)^2}$. Then the area x of the entire field is given by $x = A + B$. It follows that x satisfies the fourth degree polynomial equation $-x^4 + 2(A^2 + B^2)x^2 - (A^2 - B^2)^2 = 0$. If $a = 39,\ b = 25,$

and $c = 30$ this equation becomes

$$-x^4 + 763,200x^2 - 40,642,560,000 \ = \ 0. \tag{27}$$

Ch'in Chiu-Shao (around 1202-1261) in his book *Mathematical Treatise in Nine Sections* (1247), solved polynomial equations up to tenth degree, particularly, he found a root of (27) as $x = 840$ by using the method *fan fa* which is now known as Horner's method (William George Horner 1786-1837, this method was also known to Viéte in 1600). The other three roots of (27) are $-840, 240, -240$, but for this geometric problem only the solution 840 is meaningful.

8. Chinese mathematician Chu Shih-Chieh (1249-1314) in his book *Precious Mirror of the Four Elements* (1303) considered the following problem: "Given that the length of the diameter of a circle inscribed in a right triangle multiplied by the product of the lengths of the two legs equals 24, and the length of the vertical leg added to the length of the hypotenuse equals 9, what is the length of the horizontal leg?" For this, let a stand for the vertical leg, b the horizontal leg, c the hypotenuse, and d the diameter of the circle (see Figure 25). The problem can be translated into the two equations

$$dab \ = \ 24 \quad \text{and} \quad a + c \ = \ 9.$$

Chu in addition assumed the two known equations

$$a^2 + b^2 \ = \ c^2 \quad \text{and} \quad d = b - (c - a),$$

where the second gives the relationship between the diameter of the inscribed circle and the lengths of the sides of the triangle. From $b^2 = c^2 - a^2 = (c - a)(c + a)$ and $c + a = 9$, we conclude that $b^2 = 9(c - a)$. Next, we multiply the equation $(c + a) - (c - a) = 2a$ by 9 to get $9(c + a) - 9(c - a) = 18a$. Thus, it follows that $81 - b^2 = 18a$ and

$$18ab \ = \ 81b - b^3. \tag{28}$$

Now we multiply $d = b - (c - a)$ by 9 to get $9d = 9b - 9(c - a)$, or

$$9d \ = \ 9b - b^2. \tag{29}$$

Multiplying together equations (28) and (29) gives

$$162dab \ = \ 729b^2 - 81b^3 - 9b^4 + b^5.$$

Because $dab = 24$, Chu had to solve the fifth degree equation in b:

$$b^5 - 9b^4 - 81b^3 + 729b^2 - 3888 \ = \ 0.$$

However, he did not illustrate his method of solution, Chu merely wrote that $b = 3$. The other approximate values of b are $10.367, \ 6.6143, \ -8.8439$, and -2.1372.

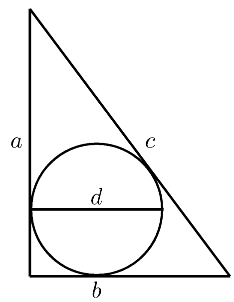

Figure 25

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