Journal of Statistical Planning and Inference 82 (1999) 25–34

journal of statistical planning and inference

www.elsevier.com/locate/jspi

Some remarks on repeated significance tests for linear contrasts

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Received 1 November 1996; accepted 1 November 1997

Abstract

We review some results obtained in an earlier paper on the relation between some standard multidimensional test statistics and their corresponding one-dimensional versions in a sequential setup. One conclusion is that, in general, no one-dimensional (marginal) null hypothesis can be rejected even though the multidimensional null hypothesis is. An alternative method based on a multiple maximum modulus test is proposed with a somewhat better conclusion. The results are extended to the case of general linear contrasts, in particular to pairwise comparisons.

MSC: primary: 62J15; 62L10; secondary: 60F05; 60F15; 60G40; 60G50

Keywords: Log-likelihood; Marginal test statistic; Multiple test; Linear contrast; Pairwise comparison; Stopping time; Law of large numbers; Central limit theorem

1. Introduction

The use of repeated significance tests aims at stopping a sequence of statistical experiments as soon as a significant deviation from the proposed general null hypothesis can be detected. However, if one is interested in particular subhypotheses which describe the deviation more properly, it is very likely that an insufficient amount of information has been collected on those marginal hypotheses at the time of rejection of the global hypothesis and, hence, at the time of stopping. The main problem is that it may, in fact, happen rather frequently that, even though a composite null hypothesis, such as “all means are equal”, is rejected, one cannot detect any significant effect or any significant difference between pairs (of treatments).

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PII: S0378-3758(99)00029-4

The present paper is a continuation of Gut and Schwabe (1996), where a sequential setup was investigated. For the readers convenience we also include a brief review of some results obtained there. For a general background on multiple test procedures and related topics we refer to Miller (1981) and Hochberg and Tamhane (1987). For a further background to the present paper, see Woodroofe (1982, 1989), Siegmund (1977, 1985, 1993) and Gut (1992).

In Gut and Schwabe (1996) we focus on two main problems. The first one is to try to determine to what extent, if at all, one may detect influential components at the time of rejection of H_0. More precisely, under the above assumptions, for testing H_0: $\boldsymbol{\theta}=\mathbf{0}$ against the alternative H_1: $\boldsymbol{\theta}\neq\mathbf{0}$, the log-likelihood ratio is

$$T_n = \frac{1}{2n}||\boldsymbol{S}_n||^2 = \frac{n}{2}||\overline{\boldsymbol{Y}}_n||^2,$$

where $\boldsymbol{S}_n = \sum_{k=1}^n \boldsymbol{Y}_k = (S_{1,n}, S_{2,n}, \ldots, S_{d,n})$ and $S_{i,n} = \sum_{k=1}^n Y_{i,k}$, where $\overline{\boldsymbol{Y}}_n = (1/n)\boldsymbol{S}_n$ and where $||\cdot||$ denotes Euclidean distance in $\mathbb{R}^d$, which implies that

$$\tau(t) = \min\{n\colon T_n > t\}, \quad t > 0,$$

becomes the relevant stopping time for the test; this is sometimes called *repeated χ^2-test*, cf. Sen (1989, p. 176). Similarly, $T_{i,n} = (1/2n)(\sum_{k=1}^n Y_{i,k})^2 = (1/2n)S_{i,n}^2$ is the one-dimensional log-likelihood ratio for the (marginal) test H_0: $\theta_i = 0$ against the alternative H_1: $\theta_i \neq 0$, and the relevant stopping times are $\tau_i(t_i) = \min\{n\colon T_{i,n} > t_i\}$, $t_i > 0$, for $i = 1, 2, \ldots, d$. The quantities of interest in this problem thus are $T_{i,\tau(t)}$, for $i = 1, 2, \ldots, d$, and the question is whether or not they fall into their respective (marginal) critical regions.

Secondly, we compared the stopping time $\tau(t)$ of the multivariate test with the stopping times $\tau_i(t_i)$, $i = 1, 2, \ldots, d$, of the marginal tests. In particular, we could determine $P_{\boldsymbol{\theta}}(\tau_i(t_i) \leqslant \tau(t))$ asymptotically, that is, as t and t_i tend to infinity simultaneously; see Gut and Schwabe (1996, Section 3).

The theoretical basis for our considerations in Gut and Schwabe (1996) and in the present paper are the following general results; cf. Gut and Schwabe (1996, Theorems 1 and 2), and Gut (1997, Theorems 4.3 and 4.4).

Theorem 1. *Suppose that* $\{(\boldsymbol{X}_k, \boldsymbol{Y}_k), k \geqslant 1\}$ *are* (*arbitrary*) *i.i.d. random vectors with mean vector* $(\boldsymbol{\theta}_x, \boldsymbol{\theta}_y)$, *that* g_x *is a real-valued function, which is continuous* (*in a neighbourhood of* (*at*) $\boldsymbol{\theta}_x$), *that* g_y *is real-valued, positive and continuous* (*in a neighbourhood of* (*at*) $\boldsymbol{\theta}_y$). *Further, let* $(Z_n^{(x)}, Z_n^{(y)}) = (n \cdot g_x(\overline{\boldsymbol{X}}_n), n \cdot g_y(\overline{\boldsymbol{Y}}_n))$, $n \geqslant 1$, *set* $\tau(t) = \min\{n\colon Z_n^{(y)} > t\}, t \geqslant 0$, *and consider the family*

$$\{Z_{\tau(t)}^{(x)},\ t \geqslant 0\}.$$

Then,

$$\frac{Z_{\tau(t)}^{(x)}}{t} \overset{\text{a.s.}}{\to} \frac{g_x(\boldsymbol{\theta}_x)}{g_y(\boldsymbol{\theta}_y)} \quad \textit{as } t \to \infty.$$

Suppose, in addition, that the partial derivatives ∇g_x and ∇g_y of g_x and g_y are continuous at $\boldsymbol{\theta}_x$ and $\boldsymbol{\theta}_y$, respectively. Then

$$\frac{Z_{\tau(t)}^{(x)} - (g_x(\boldsymbol{\theta}_x)/g_y(\boldsymbol{\theta}_y))t}{\sqrt{(g_y(\boldsymbol{\theta}_y))^{-3}\gamma^2 t}} \xrightarrow{\text{d}} \mathrm{N}(0,1) \quad \text{as } t \to \infty,$$

where

$$\gamma^2 = \mathrm{Var}(g_y(\boldsymbol{\theta}_y)\nabla g_x(\boldsymbol{\theta}_x)'\boldsymbol{X}_1 - g_x(\boldsymbol{\theta}_x)\nabla g_y(\boldsymbol{\theta}_y)'\boldsymbol{Y}_1)$$

is assumed to be positive.

The case of interest in the present context is the following special version, which we formulate separately for the readers convenience.

Theorem 1′. *Suppose that $\{\boldsymbol{Y}_k,\ k \geqslant 1\}$ are (arbitrary) i.i.d. random vectors with mean vector $\boldsymbol{\theta}$. Suppose further that g and $\tilde{g}$ are real-valued functions, that are continuous (in a neighbourhood of (at) $\boldsymbol{\theta}$) and that $g(\boldsymbol{\theta}) > 0$. Further, let $(\tilde{Z}_n, Z_n) = (n \cdot \tilde{g}(\overline{\boldsymbol{Y}}_n), n \cdot g(\overline{\boldsymbol{Y}}_n))$, $n \geqslant 1$, set $\tau(t) = \min\{n\colon Z_n > t\}$, $t \geqslant 0$, and consider the family*

$$\{\tilde{Z}_{\tau(t)},\ t \geqslant 0\}.$$

Then,

$$\frac{\tilde{Z}_{\tau(t)}}{t} \xrightarrow{\text{a.s.}} \frac{\tilde{g}(\boldsymbol{\theta})}{g(\boldsymbol{\theta})} \quad \text{as } t \to \infty. \tag{1.1}$$

Suppose, in addition, that the partial derivatives ∇g and $\nabla \tilde{g}$ of g and $\tilde{g}$ are continuous at $\boldsymbol{\theta}$. Then

$$\frac{\tilde{Z}_{\tau(t)} - (\tilde{g}(\boldsymbol{\theta})/g(\boldsymbol{\theta}))t}{\sqrt{(g(\boldsymbol{\theta}))^{-3}\gamma^2\, t}} \xrightarrow{\text{d}} \mathrm{N}(0,1) \quad \text{as } t \to \infty, \tag{1.2}$$

where

$$\gamma^2 = (g(\boldsymbol{\theta})\nabla\tilde{g}(\boldsymbol{\theta}) - \tilde{g}(\boldsymbol{\theta})\nabla g(\boldsymbol{\theta}))' \,\mathrm{Cov}(\boldsymbol{Y}_1)(g(\boldsymbol{\theta})\nabla\tilde{g}(\boldsymbol{\theta}) - \tilde{g}(\boldsymbol{\theta})\nabla g(\boldsymbol{\theta}))$$

is assumed to be positive.

It follows from the first part of Theorem 1′, with $g = \frac{1}{2}||\cdot||^2$ and $\tilde{g} = 1$, that (under the relevant alternatives $\boldsymbol{\theta} \neq \boldsymbol{0}$ and $\theta_i \neq 0$, respectively),

$$\frac{\tau(t)}{t} \xrightarrow{\text{a.s.}} \frac{2}{||\boldsymbol{\theta}||^2} \quad \text{as } t \to \infty, \tag{1.3}$$

that (cf. Gut, 1992)

$$\frac{\tau_i(t_i)}{t_i} \xrightarrow{\text{a.s.}} \frac{2}{\theta_i^2} \quad \text{as } t_i \to \infty,\ i = 1,2,\ldots,d, \tag{1.4}$$

where τ_i, $i = 1,2,\ldots,d$, are the one-dimensional stopping times corresponding to the one-dimensional tests, and, with $\tilde{g}(y) = \frac{1}{2}y_i^2$, that

$$\frac{T_{i,\tau(t)}}{t} \xrightarrow{\text{a.s.}} \frac{\theta_i^2}{||\boldsymbol{\theta}||^2} \quad \text{as } t \to \infty,\ i = 1,2,\ldots,d. \tag{1.5}$$

As for the first problem discussed above, we may compare (1.5) with the relation

$$\frac{T_{i,\tau_i(t_i)}}{t_i} = \frac{T_{i,\tau_i(t_i)}}{\tau_i(t_i)} \cdot \frac{\tau_i(t_i)}{t_i} \xrightarrow{\text{a.s.}} \frac{1}{2}\theta_i^2 \frac{2}{\theta_i^2} = 1 \quad \text{as } t_i \to \infty,\ i = 1, 2, \ldots, d, \tag{1.6}$$

which follows from the law of large numbers and (1.4). Since the limit in (1.5) is (at most equal to one and) strictly less than one as soon as (at least) one $\theta_j \neq 0$ for some $j \neq i$, it follows that, asymptotically, the marginal test statistic does not fall into the (marginal) critical region at the time of rejection of the multivariate hypothesis if $t_i/t \to 1$ for the critical value t_i in the marginal test as $t, t_i \to \infty$. Indeed, if we consider a sequence of repeated significance tests at prespecified levels of significance, with finite horizons tending to infinity, it can be shown that the ratio t_i/t of the rejection bounds tends to $\eta_i = 1$; see Gut and Schwabe (1996, Section 4).

However, since $\{T_{i,\tau(t)} > t_i\} \subset \{\tau_i(t_i) \leqslant \tau(t)\}$, it is possible that the marginal test statistic has visited the critical region at an earlier time point, and then escaped from there at time $\tau(t)$. It is therefore (also) of interest to compare the stopping times themselves. Toward this end we let t and t_i tend to infinity in (1.3) and (1.4) in such a way that $t_i/t \to \eta_i > 0$. Then

$$\frac{\tau_i(t_i)}{\tau(t)} = \frac{\tau_i(t_i)}{t_i} \cdot \frac{t_i}{t} \cdot \frac{t}{\tau(t)} \xrightarrow{\text{a.s.}} \eta_i \frac{||\boldsymbol{\theta}||^2}{\theta_i^2} \quad \text{as } t \to \infty, \tag{1.7}$$

from which we conclude that the limit is greater than one whenever $\eta_i > \theta_i^2/||\boldsymbol{\theta}||^2$, in which case the marginal stopping time is strictly larger than $\tau(t)$, and, in view of the fact that $\eta_i = 1$ (see above), we conclude that, asymptotically, no marginal hypothesis can be rejected; the rejection of H_0: $\boldsymbol{\theta} = \mathbf{0}$ only leads to the conclusion that "something is wrong" (again, under the proviso that (at least) two different components $\theta_i, \theta_j \neq 0$ for some $i \neq j$). For further details, see Gut and Schwabe (1996).

As an alternative procedure a sequential version of a multiple maximum modulus test was investigated; see Gut and Schwabe (1996, Section 5). For this procedure the relevant test statistic is

$$T_n = \max\{T_{1,n}, T_{2,n}, \ldots, T_{d,n}\} = \frac{n}{2} \max\{\overline{Y}_{1,n}^2, \overline{Y}_{2,n}^2, \ldots, \overline{Y}_{d,n}^2\} = n \cdot g(\overline{\boldsymbol{Y}}_n),$$

with $g(\boldsymbol{y}) = \frac{1}{2}\max\{y_1^2, y_2^2, \ldots, y_d^2\} = \frac{1}{2}||\boldsymbol{y}||_\infty^2$, where $||\cdot||_\infty$ denotes the supremum norm (the marginal log-likelihoods are of course as before). It follows that the corresponding stopping rule is

$$\begin{aligned} \tau(t) &= \min\{n\colon n \cdot g(\overline{\boldsymbol{Y}}_n) > t\} = \min\{n\colon \max\{T_{1,n}, T_{2,n}, \ldots, T_{d,n}\} > t\} \\ &= \min\{\tau_1(t), \tau_2(t), \ldots, \tau_d(t)\}. \end{aligned} \tag{1.8}$$

This shows that, at least, one marginal hypothesis will be rejected as, typically, $t_i \leqslant t$ for the critical values. For this test situation the first part of Theorem 1′ shows that, under H_1,

$$\frac{\tau(t)}{t} \xrightarrow{\text{a.s.}} \frac{1}{g(\boldsymbol{\theta})} = \frac{2}{||\boldsymbol{\theta}||_\infty^2} \quad \text{as } t \to \infty. \tag{1.9}$$

Moreover, we conclude that, asymptotically, the marginal null hypothesis corresponding to the largest of $|\theta_1|, |\theta_2|, \ldots, |\theta_d|$ is, (with a very high probability) rejected when the multidimensional hypothesis is rejected.

The aim of the present note is to investigate general linear contrasts or pairwise comparisons along the same lines. Extensions to general linear aspects are straightforward.

2. A first example; one linear contrast

As a first example we consider the single linear contrast $\boldsymbol{a}'\boldsymbol{\theta} = \sum_{i=1}^{d} a_i\theta_i$ (where thus $\boldsymbol{a}'\mathbf{1} = \sum_{i=1}^{d} a_i = 0$), and we wish to test

$$\mathrm{H}_0\colon \boldsymbol{a}'\boldsymbol{\theta} = 0 \quad \text{versus} \quad \mathrm{H}_1\colon \boldsymbol{a}'\boldsymbol{\theta} \neq 0.$$

W.l.o.g. we also assume the normalization $||\boldsymbol{a}|| = 1$. Derivations analogous to those of Gut and Schwabe (1996) and an application of Theorem 1′ show that the log-likelihood ratio is

$$T_n = n \cdot g(\overline{\boldsymbol{Y}}_n),$$

where

$$g(\boldsymbol{y}) = \frac{1}{2}(\boldsymbol{a}'\boldsymbol{y})^2 \quad \left(= \frac{1}{2}\left(\sum_{i=1}^{d} a_i y_i\right)^2 \right),$$

and

$$\gamma^2 = ||\nabla g(\boldsymbol{\theta})||^2 = (\boldsymbol{a}'\boldsymbol{\theta})^2 \quad \left(= \left(\sum_{i=1}^{d} a_i\theta_i\right)^2 \right)$$

in case $\tilde{g} = 1$. It follows that, under H_1,

$$\frac{\tau(t)}{t} \xrightarrow{\text{a.s.}} \frac{2}{(\boldsymbol{a}'\boldsymbol{\theta})^2} \quad \text{as } t \to \infty, \tag{2.1}$$

and that

$$\frac{\tau(t) - 2t/(\boldsymbol{a}'\boldsymbol{\theta})^2}{(1/(\boldsymbol{a}'\boldsymbol{\theta})^2)\sqrt{8t}} \xrightarrow{\text{d}} \mathrm{N}(0,1) \quad \text{as } t \to \infty, \tag{2.2}$$

where, of course, $\tau(t) = \min\{n\colon T_n > t\}$, $t > 0$.

Example. For $d=2$ the normalized single linear contrast is given by $\boldsymbol{a}=(1/\sqrt{2})(1,-1)'$, up to sign change, which implies that $\gamma^2 = (\boldsymbol{a}'\boldsymbol{\theta})^2 = \frac{1}{2}(\theta_1 - \theta_2)^2$. Hence,

$$\frac{\tau(t)}{t} \xrightarrow{\text{a.s.}} \frac{4}{(\theta_1 - \theta_2)^2} \quad \text{and} \quad \frac{\tau(t) - 4t/(\theta_1 - \theta_2)^2}{(4/(\theta_1 - \theta_2)^2)\sqrt{2t}} \xrightarrow{\text{d}} \mathrm{N}(0,1) \quad \text{as } t \to \infty.$$

3. Several linear contrasts

Now, we have several linear contrasts, $\boldsymbol{a}_i'\boldsymbol{\theta}$, $i=1,2,\dots,k$, which are assumed to be linearly independent. More precisely, we consider $L\boldsymbol{\theta}$, where L is an orthonormal $k\times d$ matrix, whose rows are the transposed $\boldsymbol{a}$-vectors;

$$L' = (\boldsymbol{a}_1\ \boldsymbol{a}_2\ \dots\ \boldsymbol{a}_k).$$

We wish to test

$$\mathrm{H}_0\colon L\boldsymbol{\theta}=\mathbf{0} \quad \text{versus} \quad \mathrm{H}_1\colon L\boldsymbol{\theta}\neq\mathbf{0}.$$

Note that for testing hypotheses every set of linear contrasts can equivalently be replaced by a minimal set which is described by a matrix L with orthonormal rows. By computations like those above it follows that the log-likelihood ratio equals

$$T_n = n\cdot g(\overline{\boldsymbol{Y}}_n) = \frac{n}{2}||L\overline{\boldsymbol{Y}}_n||^2 = \frac{1}{2n}||L\boldsymbol{S}_n||^2,$$

that, in particular, $g(\boldsymbol{\theta})=\frac{1}{2}||L\boldsymbol{\theta}||^2$, and that

$$\gamma^2 = ||L'L\boldsymbol{\theta}||^2 = \boldsymbol{\theta}'L'LL'L\boldsymbol{\theta} = ||L\boldsymbol{\theta}||^2$$

in case $\tilde{g}=1$, since $LL'=\boldsymbol{I}$, the identity matrix. (Note that for $k=1$ we rediscover the results of the previous section.)

An application of Theorem 1′ now shows that

$$\frac{\tau(t)}{t} \xrightarrow{\text{a.s.}} \frac{2}{||L\boldsymbol{\theta}||^2} \quad \text{as } t\to\infty, \tag{3.1}$$

and that

$$\frac{\tau(t)-2t/||L\boldsymbol{\theta}||^2}{(1/||L\boldsymbol{\theta}||^2)\sqrt{8t}} \xrightarrow{\text{d}} \mathrm{N}(0,1) \quad \text{as } t\to\infty, \tag{3.2}$$

under H_1.

Example. For testing the null hypothesis of overall equality H_0: $\theta_1=\theta_2=\dots=\theta_d$ the quantity $||L\boldsymbol{\theta}||^2=[1/(d-1)]\sum_{i=1}^d(\theta_i-\bar{\theta})^2$ is the actual variation of the components θ_i of $\boldsymbol{\theta}$, where $\bar{\theta}=(1/d)\sum_{j=1}^d\theta_j$ is their mean.

4. Comparing linear contrasts with subsets of linear contrasts

In this section we consider linear contrasts $\tilde{L}\boldsymbol{\theta}$, where $\tilde{L}$ is a submatrix of L of size $k_1\times d$, $k_1<k$. A derivation analogous to that in connection with (1.5) and (1.7) shows that, under the alternative $\tilde{L}\boldsymbol{\theta}\neq\mathbf{0}$,

$$\frac{\tilde{T}_{\tau(t)}}{t} \xrightarrow{\text{a.s.}} \frac{||\tilde{L}\boldsymbol{\theta}||^2}{||L\boldsymbol{\theta}||^2} = \left(1-\frac{||\overline{L}\boldsymbol{\theta}||^2}{||L\boldsymbol{\theta}||^2}\right) \quad \text{as } t\to\infty, \tag{4.1}$$

and

$$\frac{\tilde{\tau}(\tilde{t})}{\tau(t)} \xrightarrow{\text{a.s.}} \tilde{\eta}\frac{||L\boldsymbol{\theta}||^2}{||\tilde{L}\boldsymbol{\theta}||^2} = \tilde{\eta}\left(1+\frac{||\overline{L}\boldsymbol{\theta}||^2}{||\tilde{L}\boldsymbol{\theta}||^2}\right) \quad \text{as } t,\tilde{t}\to\infty,\ \tilde{t}/t\to\tilde{\eta}, \tag{4.2}$$

where $\tilde{T}_n = n\tilde{g}(\overline{\boldsymbol{Y}}_n) = (1/2n)||\tilde{L}\boldsymbol{S}_n||^2$, $\tilde{g}(\boldsymbol{y}) = \frac{1}{2}||\tilde{L}\boldsymbol{y}||^2$, and $\tilde{\tau}(\tilde{t}) = \min\{n\colon \tilde{T}_n > \tilde{t}\}$ are defined in the obvious way, and where $\overline{L}$ is the orthogonal complement of $\tilde{L}$ with respect to L.

Next, we recall that the probability of rejecting a particular subhypothesis at the moment of rejecting a global null hypothesis is given by $P_{\boldsymbol{\theta}}(\tilde{T}_{\tau(t)} > \tilde{t})$ or, alternatively, by $P_{\boldsymbol{\theta}}(\tilde{\tau}(\tilde{t}) \leqslant \tau(t))$, depending on the chosen strategy. If the bounds t and $\tilde{t}$ tend to infinity simultaneously in such a way that $\tilde{t}/t \to \tilde{\eta}$, it follows from relations (4.1) and (4.2) that both probabilities of rejection tend either to 0 or to 1, depending on whether $(||\tilde{L}\boldsymbol{\theta}||^2/||L\boldsymbol{\theta}||^2) < \tilde{\eta}$ or $(||\tilde{L}\boldsymbol{\theta}||^2/||L\boldsymbol{\theta}||^2) > \tilde{\eta}$. Therefore, the size of the limiting ratio $\tilde{\eta}$ of the bounds $\tilde{t}$ and t plays an important role.

Now, let us consider the situation of a finite horizon in which the test procedure terminates with an acception of the null hypothesis H_0 if $\tau(t) > m$, that is, the boundary has not been reached before or at time m. Then $t = t(m, \alpha)$ can be chosen in such a way that a prespecified significance level α is attained. By the obvious analog of Theorem 4.1 of Gut and Schwabe (1996) it follows that $t(m, \alpha)/\log\log m \to 1$ as $m \to \infty$ for orthonormal L. Similarly, for a subhypothesis the bound $\tilde{t} = \tilde{t}(m, \alpha)$ shows the same limiting behaviour; $\tilde{t}(m, \alpha)/\log\log m \to 1$. As a consequence, the limiting ratio $\tilde{\eta} = 1$. However, since the limit in (4.2) is strictly greater than one whenever at least one of the linear contrasts in $\overline{L}$ differs from zero, we conclude that, asymptotically, we are in the indifference position of the kind described in the Introduction, which means that, in general, no subhypothesis can be rejected (asymptotically).

Furthermore, with $\overline{g}(\boldsymbol{y}) = \frac{1}{2}||\overline{L}\boldsymbol{y}||^2$ one has the decomposition $g = \tilde{g} + \overline{g}$, and it follows that

$$\gamma^2 = \tfrac{1}{4}||L\boldsymbol{\theta}||^2||\tilde{L}\boldsymbol{\theta}||^2||\overline{L}\boldsymbol{\theta}||^2,$$

with the aid of which one may formulate a central limit theorem.

Example. Consider again the overall null hypothesis H_0: $\theta_1 = \cdots = \theta_d$ of equality of all means described by an appropriate linear contrast matrix L, and a marginal null hypothesis $\tilde{H}_0$: $\theta_i = \theta_j, i, j \in \mathscr{I}$ of equal means in a subset $\mathscr{I} \subset \{1, 2, \ldots, d\}$ described by the submatrix $\tilde{L}$. Then

$$||\tilde{L}\boldsymbol{\theta}||^2 = \frac{1}{|\mathscr{I}| - 1} \sum_{i \in \mathscr{I}} \left(\theta_i - \frac{1}{|\mathscr{I}|} \sum_{j \in \mathscr{I}} \theta_j \right)^2$$

and

$$||\overline{L}\boldsymbol{\theta}||^2 = \frac{1}{d - |\mathscr{I}| - 1} \sum_{i \notin \mathscr{I}} \left(\theta_i - \frac{1}{d - |\mathscr{I}|} \sum_{j \notin \mathscr{I}} \theta_j \right)^2,$$

where $|\mathscr{I}|$ is the cardinality of $\mathscr{I}$, and appropriate laws of large numbers and central limit theorems can be formulated. Moreover, the null hypothesis H_0 may be equivalently expressed as the infinite intersection of all univariate null hypotheses based on arbitrary linear contrasts. Denote by $\mathscr{L} = \mathscr{L}(\boldsymbol{\theta})$ the null space of linear contrasts given $\boldsymbol{\theta}$, i.e.

$\mathscr{L}$ is the set of all linear contrasts $\boldsymbol{a}$ satisfying $\boldsymbol{a}'\boldsymbol{\theta}=0$. Under the alternative, the dimension of $\mathscr{L}$ is equal to $d-2$. In that case, asymptotically, at most, the single null hypothesis $\mathrm{H}_{0,\boldsymbol{\theta}}$: $\boldsymbol{a}_{\boldsymbol{\theta}}'\boldsymbol{\theta}=0$ may be rejected; here $\boldsymbol{a}_{\boldsymbol{\theta}}=(\boldsymbol{\theta}-\bar{\theta}\mathbf{1})/||\boldsymbol{\theta}-\bar{\theta}\mathbf{1}||$ is the standardized direction in which $\boldsymbol{\theta}$ deviates from the mean $\bar{\theta}$ of its components. None of the null hypotheses $\tilde{\mathrm{H}}_0$: $\boldsymbol{a}'\boldsymbol{\theta}=0$ can be rejected for any other linear contrast $\boldsymbol{a}$ after the observations are stopped when detecting a deviation from the overall null hypothesis H_0: $\theta_1=\cdots=\theta_d$.

5. Maximum modulus

Recall (Gut and Schwabe, 1996, Section 5) that in the maximum modulus case we could (asymptotically, with a very high probability) reject the marginal null hypothesis corresponding to the unique maximum $\theta_{\max}=\max\{|\theta_1|,|\theta_2|,\ldots,|\theta_d|\}$ at the time of rejection of the multidimensional hypothesis H_0: $\boldsymbol{\theta}=\mathbf{0}$. In this section we study the analog for pairwise comparisons. We thus consider the global hypothesis H_0: $\theta_1=\theta_2=\cdots=\theta_d$ of equal means and the associated marginal hypotheses H_0: $\theta_i=\theta_j$ of pairwise equality. Hence

$$g(\boldsymbol{y})=\tfrac{1}{2}\max_{i,j}(y_i-y_j)^2=\tfrac{1}{2}r^2, \tag{5.1}$$

that is, our g-function equals the squared range $r=\max_{i,j}|y_i-y_j|$ of $\boldsymbol{y}$, normalized by $\frac{1}{2}$. With L defined as the $\frac{1}{2}d(d-1)\times d$ matrix of pairwise comparisons, that is, the matrix which contains exactly one entry equal to 1, one subsequent entry equal to -1 and all other entries equal to zero, we may rewrite (5.1) as

$$g(\boldsymbol{y})=\tfrac{1}{2}||L\boldsymbol{y}||_\infty^2 \tag{5.2}$$

with $g(\boldsymbol{\theta})=\frac{1}{2}||L\boldsymbol{\theta}||_\infty^2=\frac{1}{2}\max_{i,j}(\theta_i-\theta_j)^2$. For the case $d=3$ we may, for example, use

$$L=\begin{pmatrix}1 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & -1\end{pmatrix}.$$

We remark, in passing, that, in contrast to the matrices L above, the present one is not of full rank (for $d>2$), let alone orthonormal.

Differentiation in (5.1) yields $(\partial g(\boldsymbol{y}))/\partial y_j=r$ when $j=j_{\mathrm{M}}$, say, is the index corresponding to the maximal component $y_{j_{\mathrm{M}}}=\max_j y_j$, $\partial g(\boldsymbol{y})/\partial y_j=-r$ when $j=j_{\mathrm{m}}$, say, is the index corresponding to the minimal component $y_{j_{\mathrm{m}}}=\min_j y_j$ and 0 otherwise, as long as the maximal and minimal indices j_{M} and j_{m} are unique, i.e., as long as there are no multiple maxima $\max_j y_j$ or multiple minima $\min_j y_j$ in $\boldsymbol{y}$. Hence, computations along the usual lines, with $\tilde{g}=1$, yield

$$\gamma^2=2(\theta_{\max}-\theta_{\min})^2=2||L\boldsymbol{\theta}||_\infty^2=4g(\boldsymbol{\theta}), \tag{5.3}$$

where $\theta_{\max} = \max_j \theta_j$ and $\theta_{\min} = \min_j \theta_j$ are the (unique) maximum and minimum, respectively, in $\boldsymbol{\theta}$. It follows that, under H_1, the appropriate stopping time satisfies

$$\frac{\tau(t)}{t} \xrightarrow{\text{a.s.}} \frac{2}{||L\boldsymbol{\theta}||_\infty^2} \quad \text{as } t \to \infty, \tag{5.4}$$

and, for $\boldsymbol{\theta}$ with a simple maximum $\theta_{\max}$ and a simple minimum $\theta_{\min}$,

$$\frac{\tau(t) - 2t/||L\boldsymbol{\theta}||_\infty^2}{(4/||L\boldsymbol{\theta}||_\infty^2)\sqrt{t}} \xrightarrow{\text{d}} \mathrm{N}(0,1) \quad \text{as } t \to \infty. \tag{5.5}$$

Further, for a subset $\mathscr{I} \subset \{1, 2, \ldots, d\}$, which is described by the corresponding submatrix $\tilde{L}$ of pairwise comparisons within the subset $\mathscr{I}$ (cf. Section 4), one can show e.g. that $\tilde{g}(\boldsymbol{\theta}) = \frac{1}{2}||\tilde{L}\boldsymbol{\theta}||_\infty^2$, which coincides with $g(\boldsymbol{\theta})$ precisely when the indices responsible for $\theta_{\max}$ and $\theta_{\min}$, respectively, both are in the subset under consideration. This means that for pairwise comparisons only the difference between $\theta_{\max}$ and $\theta_{\min}$ can (and will) be detected, since $\tilde{\eta} = 1$ by an analog to the finite horizon argument in Gut and Schwabe (1996). We omit the details.

6. Some remarks on further results

We close by mentioning that our results obviously hold in more general contexts. For example, for general normal distributions it is well known that the covariance matrix can be diagonalized, from which it follows immediately that any linear contrast in this case corresponds to some other linear contrast for the model of Section 2.

Another case is unknown variances. This situation can, of course, also be dealt with, although the technicalities become harder.

The above results can also be extended to non-normal distributions. In this case, however, the T_n are not log-likelihoods, just test statistics. If the variances are unknown one must, in addition, assume higher order moments. Moreover, central third and fourth moments (which are 0 and $3\sigma^4$ for normal variables) enter in the expressions for the asymptotic variance.

Acknowledgements

The present work was finally completed while the second author was visiting the University of Technology at Darmstadt. He wishes to express his sincere thanks for the hospitality he met during his stay. The research was partly supported by the grant Ku719/2-2 of the Deutsche Forschungsgemeinschaft.

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