checking to control mistakes, better control of errors, and relative ease of computation." Anyone who took his courses will agree that he had a unique and ingenious way of presenting involved computational methods in a clear, concise, tabular fashion, with many built-in checking devices. Some of his ideas were published in "Basic Instructions in Statistical Computations" in *The American Statistician* 1957.

As a result of his consulting work with the Personnel Research Section, Department of the Army from 1951 to 1953, Paul wrote research reports on test selection methods in differential classification. In 1954 *Psychometrika,* he identified the Personnel Classification Problem as a special case of the linear programming problem and proposed a method of optimal regions that was detailed in 1957 *Psychometrika.* This again produced solutions that could be mechanized, making the method "specially effective with machines," where the original matrix is successively transformed to a reduced matrix to which the orthogonality condition is applicable. In 1964 *Psychometrika* he described a similar idea that applied to group assembly sums in a U.S. Air Force Project for its Crew Research Laboratory. Paul reduced the Air Force Project to the Hitchcock transportation problem and related the results to ANOVA. The method of reduced matrices for the general transportation problem was already translated to computing machines by Paul and was published with Bernard Galler in 1957 *JACM* and *NRLQ*. The direct solution to the general problem was featured in 1966 in *Management Science.* Paul typically attempted "to provide a *general* method which gives *general* solutions to the *general* problems mentioned above which is operationally practical and generally more efficient than rival methods." His *direct* method meant "one in which the basic specifications of the problem are used directly in solving the problem without replacing them, in whole or in part, with auxiliary theorems or criteria and without using the circuitous approach."

Paul and M.S. Macphail introduced matrix derivatives in statistical applications with a pioneering article "Symbolic Matrix Derivatives" in 1948 *Annals.* His interest in matrix derivatives continued, as shown by several doctoral theses that he supervised. His JASA articles, "Some Applications of Matrix Derivatives in Multivariate Analysis" (1967) and "Multivariate Maxima and Minima With Matrix Derivatives" (1969), are further evidence of that interest. On his desk, the morning after he died, were his work on the manuscript, "The Derivative of Any Element of a Matrix With Respect to Another Element of That Matrix," and his correspondence with Gerald Rogers, author of *Matrix Derivatives* (Marcel Dekker, 1980).

Some of Paul's early applied work on educational investigations, done primarily as consultant to the Bureau of Education at Michigan, appeared in various issues of the *Journal of Higher Education, Journal of Educational Psychology, Journal of Applied Psychology, Journal of Educational Research,* and *Psychometrika,* on whose editorial board he served since 1956.

For his excellence in teaching and in serving on various campus committees at Michigan, Paul received the Faculty Award for Distinguished Achievement in 1958. He served the Institute of Mathematical Statistics as Secretary-Treasurer during the difficult war and post-war period from 1943 to 1949, and he became its President in 1951. He was elected a Fellow of the Institute of Mathematical Statistics, of the American Statistical Association, of the American Association for the Advancement of Science, and a Member of the International Statistical Institute. Among honorary fraternities, he was elected to Phi Beta Kappa, Sigma Xi, Phi Kappa Phi, Sigma Pi Sigma, Alpha Chi Rho, and Omicron Kappa Upsilon.

Across the international border, in Canada, he helped build the graduate statistics program during its formative years from 1965 to 1971 at the University of Windsor, Canada. In recognition of his accomplishments, his 50 years of teaching, and his numerous scholarly articles (over 70), he was awarded an honorary Doctor of Science degree at Windsor's 16th Convocation in October 1971, when the new mathematics building was dedicated.

In addition to mathematics, Paul's other interest in retirement was vegetable gardening. He supplied fresh vegetables to many residents of Mackinaw City, bringing them cheer and goodwill. In spite of his many distinguished achievements, Paul was a modest, friendly person as well as a gifted teacher who presented his material clearly and forcefully. Careful preparation and individual attention were the hallmarks of his lectures. Paul brought out the best in every student. None of his graduate students will forget his system of assigning extra exercises—an optional set of challenging assignments for brighter students. Many theses grew out of these exercises. He always sensed his students' needs and went more than halfway to meet them. When honesty and integrity are considered, Paul had no equal.

He revered his elders. The two men he admired most were his father Edmond, a Baptist minister, and his uncle Charles Sumner, a medical doctor. Paul always treasured the watches they gave him and wrote "Two Watches," a poem with these closing stanzas:

These men who lived in prior times,
In rural and in urban climes,
Determined that, as each best can,
He'd love and serve his fellow man.

And so it is, whene'er I see
These watches, and in memory
Recall the past, 'tis ever then
My thoughts converge on two fine men.

Paul was an admiring son, a dutiful husband, a loving father, and a doting grandfather, who is missed by his family. To friends, colleagues, and students, he was a superb man—a man who lived a useful and inspiring life.

Derrick S. Tracy
Department of Mathematics
University of Windsor
Windsor, Ontario

In this section, *The American Statistician* publishes articles and notes of interest to teachers of the first mathematical statistics course and of applied statistics courses. To be suitable for this section, articles and notes should be useful to a substantial number of teachers of such a course or should have the potential for fundamentally affecting the way in which the course is taught.

Multiple Comparisons in a Mixed Model

YOSEF HOCHBERG and AJIT C. TAMHANE*

In this note we give a simple proof of the result that, in the symmetric version of Scheffé's mixed model for a balanced two-way layout, an exact T procedure for pairwise comparisons between the levels of the fixed factor can be based on the interaction mean square as the Studentizing factor. Such a proof does not seem to be available in the literature and will hopefully remove the confusion caused by some textbooks that incorrectly prescribe the error mean square as the Studentizing factor.

KEY WORDS: Multiple comparisons; Tukey procedure; Mixed two-way layout; Intraclass correlation; Studentized range distribution.

1. INTRODUCTION

We consider Sheffé's (1956,1959) mixed model for the analysis of variance of a two-way layout with one factor (say, A) fixed, the other factor (say, B) random, and with equal number of observations per cell. Scheffé (1959, p. 270) mentions that generally the ratio MS_A/MS_{AB} can be used only as an approximate F statistic for testing the null hypothesis for factor A. Scheffé gives an exact test and the associated simultaneous confidence intervals for all contrasts among the levels of factor A using Hotelling's T^2 test. The resulting confidence intervals are too conservative if the experimenter is interested only in pairwise comparisons. This leads us to consider a Tukey (T) procedure based on the Studentized range distribution. Scheffé (1959, p. 271) describes such a T procedure, which uses MS_{AB} as the Studentizing factor in the denominator; in a footnote on page 270 he also mentions that under a "symmetric" version of his model (see Section 2 for more details) the usual F test just mentioned and the T procedure are exact. No proof is given, however.

We have two objectives in presenting this note. The first is to give a simple proof of this exact result. The proof is probably known to some people but, to our knowledge, has not appeared in print. Our second objective is to remove the confusion caused by some textbooks (namely, Gibra 1973, p. 397) that incorrectly prescribe MS_E, the error mean square, as the Studentizing factor to be used in the T procedure. The correct Studentizing factor is the interaction mean square, MS_{AB}, regardless of which one of the three common models (Hocking 1973) for mixed two-way layout is employed. The idea of the proof is of some intrinsic value because it can be easily extended to higher-way balanced mixed models and as such can be introduced in a linear models course.

2. MODEL AND PRELIMINARIES

We shall not go into the details of how Scheffé arrives at his mixed model. Instead, we give the model in a final form that is most convenient for our purposes; this model is also given as Model Ia by Hocking (1973). Let Y_{ijk} be the kth observation on the ith level of factor A and the jth level of factor B ($1 \le i \le I$, $1 \le j \le J$, $1 \le k \le K$). We assume that the Y_{ijk} are jointly normally distributed with a covariance structure that depends on the error variance σ_E^2 and an $I \times I$ positive definite, symmetric matrix $\mathbf{\Sigma} = ((\sigma_{ii'}))$ both of which are assumed unknown. The specific model is

$$\begin{aligned} E(Y_{ijk}) &= \mu_i, \\ \operatorname{cov}(Y_{ijk}, Y_{i'j'k'}) &= \sigma_{ii} + \sigma_E^2 && \text{if } i = i', j = j', k = k' \\ &= \sigma_{ii} && \text{if } i = i', j = j', k \ne k' \\ &= \sigma_{ii'} && \text{if } i \ne i', j = j' \\ &= 0 && \text{if } j \ne j'. \end{aligned} \tag{2.1}$$

Scheffé's symmetric model assumes that the σ_{ii} are all equal to, say, σ^2, and the $\sigma_{ii'}$ for $i \ne i'$ are all equal to, say, $\rho\sigma^2$ with $-1/(I-1) \le \rho \le 1$. (Models II and III in Hocking (1973) are special cases of the symmetric Scheffé model for $\rho > 0$.) Henceforth we restrict our-

*Yosef Hochberg is Associate Professor, Department of Statistics, Tel-Aviv University, Tel-Aviv, Ramat-Aviv, Israel. Ajit C. Tamhane is Associate Professor, Department of Industrial Engineering and Management Sciences, Northwestern University, Evanston, IL 60201. This note was written when the first author was visiting Northwestern University in winter and spring 1982.

selves to this symmetric version of (2.1), which can be stated as follows:

$$
\begin{aligned}
E(Y_{ijk}) &= \mu_i, \\
\operatorname{var}(Y_{ijk}) &= \sigma^2 + \sigma_E^2 = \sigma_Y^2 \quad \text{(say)}, \\
\operatorname{corr}(Y_{ijk}, Y_{i'j'k'}) &= \rho_1 = \frac{\sigma^2}{\sigma_Y^2} \quad \text{if } i = i',\ j = j',\ k \neq k' \\
&= \rho_2 = \frac{\rho\sigma^2}{\sigma_Y^2} \quad \text{if } i \neq i',\ j = j' \\
&= 0 \qquad\qquad \text{if } j \neq j', \qquad (2.2)
\end{aligned}
$$

where all the parameters are unknown.

Our objective is to derive the T procedure for making all pairwise comparisons $\mu_i - \mu_{i'}$ $(1 \leq i < i' \leq I)$. Although we give the proof only for the T procedure, the same results can also be used to show the exactness of the F test. Huynh and Feldt (1970) have used a different technique to show the F distribution of certain mean square ratios in repeated measurements designs.

3. DERIVATION OF THE T PROCEDURE

The case of no replication, that is, $K = 1$, which leads to an intraclass correlation model among the Y_{ij} for fixed j, has been dealt with by Bhargava and Srivastava (1973). When we have multiple observations per cell we get an intraclass correlation model with two different correlation coefficients among the Y_{ijk}, as can be seen from (2.2). Therefore, the method of proof used by Bhargava and Srivastava must be applied iteratively in two steps.

In the following we use the usual dot notation to denote the average taken over the dotted subscripts, that is, $\overline{Y}_{i..} = \sum_{j,k} Y_{ijk}/JK$, $\overline{Y}_{ij.} = \sum_k Y_{ijk}/K$, and so on. Also we use the usual formulas for the SS's and the MS's in the ANOVA table; in particular, $MS_{AB} = SS_{AB}/(I-1)(J-1)$, where $SS_{AB} = K\sum_{i,j}(\overline{Y}_{ij.} - \overline{Y}_{i..} - \overline{Y}_{.j.} + \overline{Y}_{...})^2$. Finally, let $Q_{p,\nu}$ denote a Studentized range variable, which is the ratio of the range of p iid $N(0,1)$ variables and an independently distributed $(\chi_\nu^2/\nu)^{1/2}$ variable, and let $Q_{p,\nu}^{(\alpha)}$ denote the upper α point of its distribution.

Theorem. Under model (2.2) the random variable

$$
\max_{1 \leq i < i' \leq I} \frac{|\overline{Y}_{i..} - \overline{Y}_{i'..} - (\mu_i - \mu_{i'})|}{\sqrt{MS_{AB}/JK}} \qquad (3.1)
$$

is distributed as a Studentized range variable $Q_{I,(I-1)(J-1)}$, and hence the simultaneous $100(1-\alpha)\%$ confidence intervals for $\mu_i - \mu_{i'}$ $(1 \leq i < i' \leq I)$ are given by the probability statement

$$
P\left\{\mu_i - \mu_{i'} \in \left[\overline{Y}_{i..} - \overline{Y}_{i'..} \pm Q_{I,(I-1)(J-1)}^{(\alpha)} \sqrt{\frac{MS_{AB}}{JK}}\right],\ \forall i < i'\right\} = 1 - \alpha. \qquad (3.2)
$$

Proof. First make the transformation

$$
X_{ijk} = Y_{ijk} - b\overline{Y}_{.j.}, \qquad (3.3)
$$

where the real constant b is chosen so that the X_{ijk} for the same j and different i are independent. (Of course, X_{ijk} and $X_{i'j'k'}$ for $j \neq j'$ are independent for any choice of b.) That is, b solves the quadratic equation

$$
\begin{aligned}
&\operatorname{cov}(X_{ijk}, X_{i'jk'}) \quad \text{for } i \neq i' \\
&= \left\{\rho_2 - \frac{2b}{IK}\left[1 + \rho_1(K-1) + \rho_2 K(I-1)\right]\right. \\
&\quad \left. + \frac{b^2}{IK}\left[1 + \rho_1(K-1) + \rho_2 K(I-1)\right]\right\}\sigma_Y^2 \\
&= 0,
\end{aligned}
$$

and the solution can be verified to be real by using the fact that $\rho_1 \geq \rho_2$. Next make the transformation,

$$
\begin{aligned}
Z_{ijk} &= X_{ijk} - c\overline{X}_{ij.} \\
&= Y_{ijk} - b(1-c)\overline{Y}_{.j.} - c\overline{Y}_{ij.}, \qquad (3.4)
\end{aligned}
$$

where now the real constant c is chosen so that the Z_{ijk} for the same j and same i are independent. (Of course, Z_{ijk} and $Z_{i'j'k'}$ for $i \neq i'$ or $j \neq j'$ are independent for any choice of c.) That is, c solves the quadratic equation

$$
\begin{aligned}
&\operatorname{cov}(Z_{ijk}, Z_{ijk'}) \quad \text{for } k \neq k' \\
&= \left\{(\rho_1 - \rho_2) - \frac{2c}{K}\left[\frac{\sigma_X^2}{\sigma_Y^2} + (K-1)(\rho_1 - \rho_2)\right]\right. \\
&\quad \left. + \frac{c^2}{K}\left[\frac{\sigma_X^2}{\sigma_Y^2} + (K-1)(\rho_1 - \rho_2)\right]\right\}\sigma_Y^2 \\
&= 0,
\end{aligned}
$$

where $\sigma_X^2 = \operatorname{var}(X_{ijk})$, and the solution to this quadratic equation can also be verified to be real.

The transformations (3.3) and (3.4) make the Z_{ijk} mutually independent normal variables with $E(Z_{ijk}) = (1-c)(\mu_i - b\overline{\mu}_.)$ (from (3.4)) and $\operatorname{var}(Z_{ijk}) = \sigma_Z^2$ (say), which is the same for all i, j, k. The Z_{ijk} then follow the usual fixed-effects balanced one-way layout model. This fact along with (3.4) gives the following:

$$
\begin{aligned}
&\overline{Z}_{i..} - (1-c)(\mu_i - b\overline{\mu}_.) \\
&\quad = (1-c)(\overline{Y}_{i..} - \mu_i - b(\overline{Y}_{...} - \overline{\mu}_.)) \\
&\quad \sim N\left(0, \frac{\sigma_Z^2}{JK}\right) (1 \leq i \leq I), \qquad (3.5)
\end{aligned}
$$

$$
\begin{aligned}
SS_B^Z &= IK\sum_j (\overline{Z}_{.j.} - \overline{Z}_{...})^2 \\
&= (1-b)^2(1-c)^2\, SS_B \sim \sigma_Z^2\chi_{J-1}^2, \qquad (3.6) \\
SS_{AB}^Z &= K\sum_{i,j}(\overline{Z}_{ij.} - \overline{Z}_{i..} - \overline{Z}_{.j.} + \overline{Z}_{...})^2 \\
&= (1-c)^2\, SS_{AB} \sim \sigma_Z^2\chi_{(I-1)(J-1)}^2, \qquad (3.7) \\
SS_E^Z &= \sum_{i,j,k}(Z_{ijk} - \overline{Z}_{ij.})^2 = SS_E \sim \sigma_Z^2\chi_{IJ(K-1)}^2, \qquad (3.8)
\end{aligned}
$$

where the SS's without superscripts are the SS's in terms of the Y's. Note that (3.5)–(3.8) are independently distributed of each other. Since the Z's are unobservable while the Y's are observable, and since b and c are constants that depend on unknown parameters, it is clear after an examination of (3.5)–(3.8) that, although σ_Z^2 can also be estimated from SS_B^Z and SS_E^Z, only SS_{AB}^Z can be used to form the Studentized range statistic

$$\max_{1\le i<i'\le I} \frac{|\bar{Z}_{i..} - \bar{Z}_{i'..} - (1-c)(\mu_i - \mu_{i'})|}{\left\{SS_{AB}^Z/JK(I-1)(J-1)\right\}^{1/2}}$$

$$= \max_{1\le i<i'\le I} \frac{(1-c)|\bar{Y}_{i..} - \bar{Y}_{i'..} - (\mu_i - \mu_{i'})|}{(1-c)\sqrt{MS_{AB}/JK}}$$

$$\sim Q_{I,(I-1)(J-1)},$$

which proves (3.1). The result (3.2) follows immediately.

Corollary. Simultaneous confidence intervals for all contrasts $\sum_i c_i\mu_i$, where $\sum_i c_i = 0$, are given by the probability statement

$$P\left\{\sum_i c_i\mu_i \in \left[\sum_i c_i\bar{Y}_{i..} \pm Q_{I,(I-1)(J-1)}^{(\alpha)} \sqrt{\frac{MS_{AB}}{JK}} \times \sum_i \frac{|c_i|}{2}\right] \text{ for all contrasts}\right\} = 1-\alpha.$$

Proof. Use Lemma 1 of Miller (1966, p. 44).

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The Geometry of Rank-Order Tests

WADE D. COOK and LAWRENCE M. SEIFORD*

This article examines the geometry of rank-order tests. We show that the set of rankings of n objects can be represented as the extreme points of a polyhedron determined by a set of linear constraints. Various rank-order statistics are interpreted via this geometric model. The model allows a unified presentation and illustrates the mechanics of rank-order tests.

1. INTRODUCTION

Although the development of distribution-free statistical tests can be traced as far back as 1710, the basis for many of the best-known distribution-free tests is Fisher's (1951) method of randomization. If applied directly to the original observations it produces efficient but impractical tests. However, the sample space for the test statistics can be standardized with the replacement of the original observations by their ranks. The resulting rank-order tests maintain much of the high efficiency while becoming vastly superior in practicality and ease of application. The statistical efficiency, ease, speed, and scope of application, however, only partly account for the success of distribution-free rank-order tests. If the data available relate solely to order or deal with a qualitative characteristic that can be ranked but not measured, the use of rank-order tests is inescapable.

In this article, we investigate rankings and statistics based on rankings. We show that the space of rankings can be characterized algebraically by a set of linear constraints. The resulting polyhedron is a geometric model in which the interpretation of various rank-order statistics becomes exceedingly transparent. To the authors' knowledge, despite the fact that rank-order tests are widely employed, this lucid interpretation of rank-order statistics has (with one exception (Schulman 1979)) apparently been ignored. The geometric model presented in Section 2 has many advantages not present in other characterizations. It captures the discrete set of rankings as the extreme points of a polyhedron determined by a set of linear constraints. The spacial visualization afforded by this model should aid in the development of improved rank-order models. In addition it has proven, in the authors' experience, to be a most effective pedagogical tool, in that it allows for a unified representation of many familiar distribution-free tests.

*Wade D. Cook is Professor, Faculty of Administrative Studies, York University, Toronto, Canada M3J 2R6. Lawrence M. Seiford is Associate Professor, Department of General Business, University of Texas, Austin, TX 78712. The research was supported in part under NSERC Grant A8966.