

Zero-inflated models: Poisson, Binomial, negative Binomial and BetaBinomial

Parametrisation

There is support two types of zero-inflated models, which we name type 0 and type 1. These are defined for both the Binomial, the Poisson, the negative Binomial and BetaBinomial likelihood. For simplicity we will describe only the Poisson as the other cases are similar.

Type 0

The (type 0) likelihood is defined as

$$\text{Prob}(y \mid \dots) = p \times 1_{[y=0]} + (1 - p) \times \text{Poisson}(y \mid y > 0)$$

where p is a hyperparameter where

$$p = \frac{\exp(\theta)}{1 + \exp(\theta)}$$

and θ is the internal representation of p ; meaning that the initial value and prior is given for θ . This is model is called `zeroinflatedpoisson0` (and `zeroinflatedbinomial0`).

Type 1

The (type 1) likelihood is defined as

$$\text{Prob}(y \mid \dots) = p \times 1_{[y=0]} + (1 - p) \times \text{Poisson}(y)$$

where p is a hyperparameter where

$$p = \frac{\exp(\theta)}{1 + \exp(\theta)}$$

and θ is the internal representation of p ; meaning that the initial value and prior is given for θ . This is model is called `zeroinflatedpoisson1` (and `zeroinflatedbinomial1`).

Link-function

As for the Poisson, the Binomial the negative Binomial and the BetaBinomial.

Hyperparameters

For Poisson and the Binomial, there is one hyperparameter; where

$$p = \frac{\exp(\theta)}{1 + \exp(\theta)}$$

and the prior and initial value is is given for θ .

For the negative Binomial and BetaBinomial, there are two hyperparameters. The overdispersion parameter n for the negative Binomial is represented as

$$\theta_1 = \log(n)$$

and the prior is defined on θ_1 . The zero-inflation parameter p , is represented as

$$p = \frac{\exp(\theta_2)}{1 + \exp(\theta_2)}$$

and the prior and initial value is is given for θ_2 . For the BetaBinomial it is similar.

Specification

- family = zeroinflatedbinomial0
- family = zeroinflatedbinomial1
- family = zeroinflatednbinomial0
- family = zeroinflatednbinomial1
- family = zeroinflatedpoisson0
- family = zeroinflatedpoisson1
- family = zeroinflatedbetabinomial0
- family = zeroinflatedbetabinomial1
- Required arguments: As for the Binomial, the negative Binomial, BetaBinomial and Poisson likelihood.

Hyperparameter specification and default values

Zeroinflated Binomial Type 0

doc Zero-inflated Binomial, type 0

hyper

theta

hyperid 90001

name logit probability

short.name prob

initial -1

fixed FALSE

prior gaussian

param -1 0.2

to.theta function(x) log(x/(1-x))

from.theta function(x) exp(x)/(1+exp(x))

survival FALSE

discrete FALSE

link default logit cauchit probit cloglog loglog

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Zeroinflated Binomial Type 1

doc Zero-inflated Binomial, type 1

hyper

theta

hyperid 91001

name logit probability
short.name prob
initial -1
fixed FALSE
prior gaussian
param -1 0.2
to.theta function(x) log(x/(1-x))
from.theta function(x) exp(x)/(1+exp(x))

survival FALSE

discrete FALSE

link default logit cauchit probit cloglog loglog

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Zeroinflated NegBinomial Type 0

doc Zero inflated negBinomial, type 0

hyper

theta1

hyperid 95001
name log size
short.name size
initial 2.30258509299405
fixed FALSE
prior pc.mgamma
param 7
to.theta function(x) log(x)
from.theta function(x) exp(x)

theta2

hyperid 95002
name logit probability
short.name prob
initial -1
fixed FALSE
prior gaussian
param -1 0.2
to.theta function(x) log(x/(1-x))
from.theta function(x) exp(x)/(1+exp(x))

survival FALSE

discrete FALSE

link default log

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Zeroinflated NegBinomial Type 1

doc Zero inflated negBinomial, type 1

hyper

theta1

hyperid 96001
name log size
short.name size
initial 2.30258509299405
fixed FALSE
prior pc.mgamma
param 7
to.theta function(x) log(x)
from.theta function(x) exp(x)

theta2

hyperid 96002
name logit probability
short.name prob
initial -1
fixed FALSE
prior gaussian
param -1 0.2
to.theta function(x) log(x/(1-x))
from.theta function(x) exp(x)/(1+exp(x))

survival FALSE

discrete FALSE

link default log

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Zeroinflated BetaBinomial Type 0

doc Zero-inflated Beta-Binomial, type 0

hyper

theta1

hyperid 88001
name overdispersion
short.name rho
initial 0
fixed FALSE
prior gaussian
param 0 0.4
to.theta function(x) log(x/(1-x))

```

    from.theta function(x) exp(x)/(1+exp(x))
theta2
  hyperid 88002
  name logit probability
  short.name prob
  initial -1
  fixed FALSE
  prior gaussian
  param -1 0.2
  to.theta function(x) log(x/(1-x))
  from.theta function(x) exp(x)/(1+exp(x))

survival FALSE

discrete TRUE

link default logit cauchit probit cloglog loglog

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```

Zeroinflated BetaBinomial Type 1

doc Zero-inflated Beta-Binomial, type 1

hyper

```

  theta1
    hyperid 89001
    name overdispersion
    short.name rho
    initial 0
    fixed FALSE
    prior gaussian
    param 0 0.4
    to.theta function(x) log(x/(1-x))
    from.theta function(x) exp(x)/(1+exp(x))
  theta2
    hyperid 89002
    name logit probability
    short.name prob
    initial -1
    fixed FALSE
    prior gaussian
    param -1 0.2
    to.theta function(x) log(x/(1-x))
    from.theta function(x) exp(x)/(1+exp(x))

```

survival FALSE

discrete TRUE

link default logit cauchit probit cloglog loglog

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Zeroinflated Poisson Type 0

doc Zero-inflated Poisson, type 0

hyper

theta

hyperid 85001
name logit probability
short.name prob
initial -1
fixed FALSE
prior gaussian
param -1 0.2
to.theta function(x) log(x/(1-x))
from.theta function(x) exp(x)/(1+exp(x))

survival FALSE

discrete FALSE

link default log

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Zeroinflated Poisson Type 1

doc Zero-inflated Poisson, type 1

hyper

theta

hyperid 86001
name logit probability
short.name prob
initial -1
fixed FALSE
prior gaussian
param -1 0.2
to.theta function(x) log(x/(1-x))
from.theta function(x) exp(x)/(1+exp(x))

survival FALSE

discrete FALSE

link default log

pdf zeroinflated

Example

In the following example we estimate the parameters in a simulated example for both type 0 and type 1.

Poisson

```
## type 0
n=100
a = 1
b = 1
z = rnorm(n)
eta = a + b*z
p = 0.2
E = sample(c(1,5,10,15), size=n, replace=TRUE)
lambda = E*exp(eta)

## first sample y|y>0
y = rpois(n, lambda = lambda)
is.zero = (y == 0)
while(sum(is.zero) > 0)
{
  y[is.zero] = rpois(sum(is.zero), lambda[is.zero])
  is.zero = (y == 0)
}
## then set some of these to zero
y[ rbinom(n, size=1, prob=p) == 1 ] = 0

data = list(y=y,z=z)
formula = y ~ 1+z
result0 = inla(formula, family = "zeroinflatedpoisson0", data = data, E=E)
summary(result0)

## type 1
y = rpois(n, lambda = lambda)
y[ rbinom(n, size=1, prob=p) == 1 ] = 0
data = list(y=y,z=z)
formula = y ~ 1+z
result1 = inla(formula, family = "zeroinflatedpoisson1", data = data, E=E)
summary(result1)
```

Binomial

```
## type 0
n=100
a = 1
b = 1
z = rnorm(n)
eta = a + b*z
p = 0.2
Ntrials = sample(c(1,5,10,15), size=n, replace=TRUE)
prob = exp(eta)/(1 + exp(eta))

y = rbinom(n, size = Ntrials, prob = prob)
is.zero = (y == 0)
while(sum(is.zero) > 0)
{
```

```

    y[is.zero] = rbinom(sum(is.zero), size = Ntrials[is.zero], prob = prob[is.zero])
    is.zero = (y == 0)
}
y[ rbinom(n, size=1, prob=p) == 1 ] = 0
data = list(y=y,z=z)
formula = y ~ 1+z
result0 = inla(formula, family = "zeroinflatedbinomial0", data = data, Ntrials = Ntrials)
summary(result0)

## type 1
y = rbinom(n, size = Ntrials, prob = prob)
y[ rbinom(n, size=1, prob=p) == 1 ] = 0
data = list(y=y,z=z)
formula = y ~ 1+z
result1 = inla(formula, family = "zeroinflatedbinomial1", data = data, Ntrials=Ntrials)
summary(result1)

```


Advanced example

In the following example we estimate the parameters in a simulated example for a type0 likelihood, where one linear predictor enters the zero-probability and one other linear predictor enters the non-zero Poisson for example. The same trick can be used for other models of type0. The trick is that the likelihood

$$p^* 1_{[y=0]} + (1 - p^*) P(y|y > 0)$$

can be reformulated as a Bernoulli likelihood for the “class”-variable

$$z = \begin{cases} 1, & \text{if } y = 0 \\ 0, & \text{if } y > 0. \end{cases}$$

where p^* is the probability for success, and zero-inflated type0 likelihood (with fixed $p = 0$) for those $y > 0$. Since p^* and the linear predictor in P is separated into two likelihoods, we can apply one linear predictor to each one, hence extend the basic model to cases where p^* also depends on a linear predictor. Here is a small simulated example doing this.

```
require(INLA)

n = 100
a = 0.5
b = 1.5
x1 = rnorm(n, sd = 0.5)

eta.z = -a - b*x1
z = rbinom(n, 1, inla.link.logit(eta.z, inverse=TRUE))
n.y = sum(z)

x2 = rnorm(n.y, sd = 0.5)
eta.y = a + b*x2
lambda = exp(eta.y)
y = rpois(n.y, lambda)

is.zero = (y == 0)
while(sum(is.zero) > 0)
{
  y[is.zero] = rpois(sum(is.zero), lambda[is.zero])
  is.zero = (y == 0)
}

Y = matrix(NA, n + n.y, 2)
Y[1:n, 1] = z
Y[n + 1:n.y, 2] = y

form = Y ~ 0 + mu.z + mu.y + cov.z + cov.y
ldat = list(
  Y=Y,
  mu.z=rep(1:0, c(n, n.y)),
  mu.y=rep(0:1, c(n, n.y)),
  cov.z=c(x1, rep(NA,n.y)),
  cov.y=c(rep(NA, n), x2))
```

```

res <- inla(form, data=ldat,
            family=c('binomial', 'zeroinflatedpoisson0'),
            control.family=list(
                list(),
                list(hyper = list(
                    prob = list(
                        initial = -20,
                        fixed = TRUE))))))
round(res$summary.fix, 4)

```

Notes

None.

Extentions

There are some extentions available which currently is only implemented for the cases where its needed/requested.

Type 2 Is like Type 1 but where (for the Poisson)

$$p = 1 - \left(\frac{E \exp(x)}{1 + E \exp(x)} \right)^\alpha$$

where $\alpha > 0$ is the hyperparameter instead of p (and $E \exp(x)$ is the mean). Available for Poisson as **zeroinflatedpoisson2**, for binomial as **zeroinflatedbinomial2** and for the negative binomial as **zeroinflatednbinomial2**.

The internal representation is $\theta = \log(\alpha)$ and prior is defined on $\log(\alpha)$.

Zeroinflated Poisson Type 2

doc Zero-inflated Poisson, type 2

hyper

theta

```

hyperid 87001
name log alpha
short.name a
initial 0.693147180559945
fixed FALSE
prior gaussian
param 0.693147180559945 1
to.theta function(x) log(x)
from.theta function(x) exp(x)

```

survival FALSE

discrete FALSE

link default log

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Zeroinflated Binomial Type 2

doc Zero-inflated Binomial, type 2

hyper

theta

hyperid 92001
name alpha
short.name alpha
initial -1
fixed FALSE
prior gaussian
param -1 0.2
to.theta function(x) log(x)
from.theta function(x) exp(x)

survival FALSE

discrete FALSE

link default logit cauchit probit cloglog loglog

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Zeroinflated Negative Binomial Type 2

doc Zero inflated negBinomial, type 2

hyper

theta1

hyperid 99001
name log size
short.name size
initial 2.30258509299405
fixed FALSE
prior pc.mgamma
param 7
to.theta function(x) log(x)
from.theta function(x) exp(x)

theta2

hyperid 99002
name log alpha
short.name a
initial 0.693147180559945
fixed FALSE
prior gaussian
param 2 1
to.theta function(x) log(x)

```

    from.theta function(x) exp(x)
survival FALSE
discrete FALSE
link default log
pdf zeroinflated

```

Zeroinflated Negative Binomial Type 1 Strata 2

doc Zero inflated negBinomial, type 1, strata 2

hyper

theta1

```

    hyperid 97001
    name log size
    short.name size
    initial 2.30258509299405
    fixed FALSE
    prior pc.mgamma
    param 7
    to.theta function(x) log(x)
    from.theta function(x) exp(x)

```

theta2

```

    hyperid 97002
    name logit probability 1
    short.name prob1
    initial -1
    fixed FALSE
    prior gaussian
    param -1 0.2
    to.theta function(x) log(x/(1-x))
    from.theta function(x) exp(x)/(1+exp(x))

```

theta3

```

    hyperid 97003
    name logit probability 2
    short.name prob2
    initial -1
    fixed FALSE
    prior gaussian
    param -1 0.2
    to.theta function(x) log(x/(1-x))
    from.theta function(x) exp(x)/(1+exp(x))

```

theta4

```

    hyperid 97004

```

```

name logit probability 3
short.name prob3
initial -1
fixed TRUE
prior gaussian
param -1 0.2
to.theta function(x) log(x/(1-x))
from.theta function(x) exp(x)/(1+exp(x))
theta5
hyperid 97005
name logit probability 4
short.name prob4
initial -1
fixed TRUE
prior gaussian
param -1 0.2
to.theta function(x) log(x/(1-x))
from.theta function(x) exp(x)/(1+exp(x))
theta6
hyperid 97006
name logit probability 5
short.name prob5
initial -1
fixed TRUE
prior gaussian
param -1 0.2
to.theta function(x) log(x/(1-x))
from.theta function(x) exp(x)/(1+exp(x))
theta7
hyperid 97007
name logit probability 6
short.name prob6
initial -1
fixed TRUE
prior gaussian
param -1 0.2
to.theta function(x) log(x/(1-x))
from.theta function(x) exp(x)/(1+exp(x))
theta8
hyperid 97008
name logit probability 7
short.name prob7
initial -1
fixed TRUE

```

```

    prior gaussian
    param -1 0.2
    to.theta function(x) log(x/(1-x))
    from.theta function(x) exp(x)/(1+exp(x))
theta9
    hyperid 97009
    name logit probability 8
    short.name prob8
    initial -1
    fixed TRUE
    prior gaussian
    param -1 0.2
    to.theta function(x) log(x/(1-x))
    from.theta function(x) exp(x)/(1+exp(x))
theta10
    hyperid 97010
    name logit probability 9
    short.name prob9
    initial -1
    fixed TRUE
    prior gaussian
    param -1 0.2
    to.theta function(x) log(x/(1-x))
    from.theta function(x) exp(x)/(1+exp(x))
theta11
    hyperid 97011
    name logit probability 10
    short.name prob10
    initial -1
    fixed TRUE
    prior gaussian
    param -1 0.2
    to.theta function(x) log(x/(1-x))
    from.theta function(x) exp(x)/(1+exp(x))

status experimental

survival FALSE

discrete FALSE

link default log

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```

Zeroinflated Negative Binomial Type 1 Strata 3

doc Zero inflated negBinomial, type 1, strata 3

hyper

theta1

hyperid 98001
name logit probability
short.name prob
initial -1
fixed FALSE
prior gaussian
param -1 0.2
to.theta function(x) log(x/(1-x))
from.theta function(x) exp(x)/(1+exp(x))

theta2

hyperid 98002
name log size 1
short.name size1
initial 2.30258509299405
fixed FALSE
prior pc.mgamma
param 7
to.theta function(x) log(x)
from.theta function(x) exp(x)

theta3

hyperid 98003
name log size 2
short.name size2
initial 2.30258509299405
fixed FALSE
prior pc.mgamma
param 7
to.theta function(x) log(x)
from.theta function(x) exp(x)

theta4

hyperid 98004
name log size 3
short.name size3
initial 2.30258509299405
fixed TRUE
prior pc.mgamma
param 7
to.theta function(x) log(x)
from.theta function(x) exp(x)

theta5

hyperid 98005
name log size 4
short.name size4
initial 2.30258509299405
fixed TRUE
prior pc.mgamma
param 7
to.theta function(x) log(x)
from.theta function(x) exp(x)

theta6

hyperid 98006
name log size 5
short.name size5
initial 2.30258509299405
fixed TRUE
prior pc.mgamma
param 7
to.theta function(x) log(x)
from.theta function(x) exp(x)

theta7

hyperid 98007
name log size 6
short.name size6
initial 2.30258509299405
fixed TRUE
prior pc.mgamma
param 7
to.theta function(x) log(x)
from.theta function(x) exp(x)

theta8

hyperid 98008
name log size 7
short.name size7
initial 2.30258509299405
fixed TRUE
prior pc.mgamma
param 7
to.theta function(x) log(x)
from.theta function(x) exp(x)

theta9

hyperid 98009
name log size 8
short.name size8


```

    initial 2.30258509299405
    fixed TRUE
    prior pc.mgamma
    param 7
    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta10
    hyperid 98010
    name log size 9
    short.name size9
    initial 2.30258509299405
    fixed TRUE
    prior pc.mgamma
    param 7
    to.theta function(x) log(x)
    from.theta function(x) exp(x)
theta11
    hyperid 98011
    name log size 10
    short.name size10
    initial 2.30258509299405
    fixed TRUE
    prior pc.mgamma
    param 7
    to.theta function(x) log(x)
    from.theta function(x) exp(x)

status experimental
survival FALSE
discrete FALSE
link default log
pdf zeroinflated

```

0.0.1 Zero and N -inflated Binomial likelihood: type 3

This is the case where

$$\begin{aligned}
 \text{Prob}(y|\dots) &= p_0 \times 1_{[y=0]} + \\
 &\quad p_N \times 1_{[y=N]} + \\
 &\quad (1 - p_0 - p_N) \times \text{binomial}(y, N, p)
 \end{aligned}$$

where:

$$p = \frac{\exp(\eta)}{1 + \exp(\eta)} \quad p_0 = \frac{p^{\alpha_0}}{1 + p^{\alpha_0} + (1 - p)^{\alpha_N}} \quad p_N = \frac{(1 - p)^{\alpha_N}}{1 + p^{\alpha_0} + (1 - p)^{\alpha_N}}$$

There are 2 hyperparameters, α_0 and α_N , governing zero-inflation where: The zero-inflation parameters α_0 and α_N are represented as $\theta_0 = \log(\alpha_0)$; $\theta_N = \log(\alpha_N)$ and the prior and initial value is given for θ_0 and θ_N respectively.

Here is an example

```

nsim<-10000
x<-rnorm(nsim)
alpha0<-1.5
alphaN<-2.0
p = exp(x)/(1+exp(x))
p0 = p^alpha0 / (1 + p^alpha0 + (1-p)^alphaN)
pN = (1-p)^alphaN / (1 + p^alpha0 + (1-p)^alphaN)
P<-cbind(p0, pN, (1-p0 -pN))
N<-rpois(nsim,20)
y<-rep(0,nsim)
for(i in 1:nsim)
  y[i]<-sum(rmultinom(1,size = 1,P[i,])*c(0,N[i],rbinom(1,N[i],p[i])))
formula = y ~1 + x
r = inla(formula, family = "zeroninflatedbinomial3", Ntrials = N, verbose = TRUE,
  data = data.frame(y, x))

```

and the default settings

doc Zero and N inflated binomial, type 3

hyper

theta1

```

hyperid 93101
name alpha0
short.name alpha0
initial 1
fixed FALSE
prior loggamma
param 1 1
to.theta function(x) log(x)
from.theta function(x) exp(x)

```

theta2

```

hyperid 93102
name alphaN
short.name alphaN
initial 1
fixed FALSE
prior loggamma
param 1 1
to.theta function(x) log(x)
from.theta function(x) exp(x)

```

status experimental

survival FALSE

discrete FALSE

link default logit cauchit probit cloglog loglog

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