Fitting dynamic models using integrated nested Laplace approximations - INLA

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Abstract

Inference in state-space models usually relies on recursive forms for filtering and smoothing of the state vectors regarding the temporal structure of the observations, an assumption that is, from our view point, unnecessary if the data set is fixed, that is, completely available before analysis. In this paper we propose a computational framework to perform approximate full Bayesian inference in linear and generalized dynamic linear models based on the Integrated Nested Laplace Approximation (INLA) approach. The proposed framework directly approximates the posterior marginals of interest disregarding the assumption of recursive updating/estimation of the states and hyperparameters in the case of fixed data sets and, therefore, enable us to do fully Bayesian analysis of complex state-space models more easily and in a short computational time. The proposed framework overcomes some limitations of current tools in the dynamic modeling literature and is vastly illustrated with a series of simulated as well as worked real-life examples, including realistically complex models with correlated error structures and models with more than one state vector, being mutually dependent on each other.

Keywords: Approximate Bayesian inference, state-space models, Laplace approximation, augmented model, spatio-temporal dynamic models

1 Introduction

State space models, also known as *dynamic models* in the Bayesian literature, are a broad class of parametric models with time varying parameters where both, the parameter variation and the available data information are described in a probabilistic way. They find application in the modeling and forecasting of time series data and regression (for a comprehensive treatment see for example West and Harrison, 1997; Durbin and Koopman, 2001). As their static analogues, they can also be generalized to deal with responses belonging to the exponential family of distributions (e.g., West et al., 1985). Inference in these models usually relies on recursive forms for filtering and smoothing of the state vectors regarding the temporal structure of the observations, an assumption that is, from our view point, unnecessary if the data set is fixed, that is, completely available before analysis. In this paper we propose, and illustrate through a series of examples, a computational framework to perform approximate full Bayesian inference

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in linear and generalized dynamic linear models based on the Integrated Nested Laplace Approximation (INLA) approach. The proposed framework directly approximates the posterior marginals of interest disregarding the assumption of recursive updating/estimation of the states and hyperparameters in the case of fixed data sets and, therefore, enable us to do fully Bayesian analysis of complex state-space models more easily.

INLA is a recent approach proposed by Rue and Martino (2007) and Rue et al. (2009) to perform fast full Bayesian inference through the accurate approximation of the marginal posterior densities of hyperparameters and latent variables in latent Gaussian models. This class of statistical models embraces a wide range of models commonly used in applications, including generalized linear models, generalized additive models, smoothing spline models, semi-parametric regression, spatial and spatio-temporal models, log-Gaussian Cox processes, geostatistical and geoadditive models, besides state-space models. Most of these latent models have been successfully fitted using the INLA library, a user friendly interface for using INLA with the `R` programming language (R Development Core Team, 2010).

Currently the INLA library provides tools to fit univariate dynamic models with a simple random walk evolution form for the states, such as first order and dynamic regression models, and assuming in the specification of the latent model that the errors are independent. However, for more complex cases, such as growth models and spatio-temporal dynamic models, where there can be more than one state vector, being mutually dependent on each other, and where the error terms can be structured as a matrix with correlated values, the INLA approach does not provide direct ways to perform inference. In this paper we show how the INLA approach can be extended to allow for the formulation of specific latent models in a state-space form in order to perform approximate Bayesian inference on them using INLA, even in the above complex cases

A computational framework to inference is proposed to achieve this goal and illustrated with simulated as well as real-life examples of linear and generalized dynamic linear models. In a first approach existing model options in the INLA library are used to directly model first order random walk evolution and seasonal behavior of simpler state-space models. Furthermore, we build on some functionalities present in the INLA library to manipulate several likelihoods and correlated error structures to develop a framework that enable the formulation and fitting of dynamic models in a more general setting using INLA. This generic approach becomes useful in the case of more complex models, as those mentioned above, and consists in merging the actual observations from the observational equation with "pseudo" observations coming from the evolution (system) equations of the dynamic model in a unique structure and fit this augmented latent model in INLA considering different likelihoods for the observations and states. The combination of the two approaches is also possible. We show how this inference framework enables the fitting of several kinds of dynamic models, including realistically complex spatio-temporal models, in a short computational time and in a user friendly way.

The rest of the paper is organised as follows. In Section 2 we briefly introduce dynamic models and the main computational approaches in the literature to perform inference on this class of models. Section 3 describes the basics of the INLA computational approach. The proposed framework to fit state-space models using INLA is illustrated in Section 4 through a series of simulated examples, ranging from simple univariate models to realistically complex spatio-temporal dynamic models. In Section 5 some well known worked examples from the literature are considered and their fitting using the INLA library is compared with current computational tools. Concluding remarks and future work are stated in Section 6.

2 Dynamic models

According to Migon et al. (2005), dynamic models can be seen as a generalization of regression models, allowing changes in parameter values throughout time by the introduction of an equation governing the temporal evolution of regression coefficients. In the linear Gaussian case they consist of the couple of equations

$$y_t = F_t' x_t + \nu_t, \qquad \nu_t \sim N(0, V_t) \tag{1}$$

$$x_t = G_t x_{t-1} + \omega_t, \qquad \omega_t \sim N(0, W_t), \tag{2}$$

where y_t is a time sequence of scalar observations and x_t is a sequence of state (latent) parameters describing locally the system. It is assumed that y_t is conditionally independent given x_t. F_t is a vector of explanatory variables, while G_t represents a matrix describing the states evolution. Both, F_t and G_t are usually defined by the modeler according to model design principles (see West and Harrison, 1997). The disturbances ν_t and ω_t are assumed to be both, serially independent and also independent of each other. Therefore, the model is completely specified by the quadruple $\{F_t; G_t; V_t; W_t\}$. When these quantities are known, inference on the states x_t can be performed analytically through an iterative procedure using the Kalman filter algorithm (for details see for example West and Harrison, 1997).

On the other hand, if either of the variances V_t and W_t is unknown, inference in dynamic linear models is not available analytically. In order to circumvent this problem, several proposals to perform approximate inference in DLMs have appeared in the literature, including approaches based on the extended Kalman filter (Anderson and Moore, 1979), Gaussian quadratures (Pole and West, 1990), data augmentation (Frühwirth-Schnatter, 1994), Laplace approximations (Ehlers and Gamerman, 1996) and assumed density approaches (Zoeter and Heskes, 2006). In recent years, attention has been mostly concentrated in simulation-based approaches, such as sequential Monte Carlo also known as particle filters (e.g., Gordon et al., 1993; Storvik, 2002; Doucet and Tadić, 2003) and Markov chain Monte Carlo (MCMC) methods (e.g., Carter and Kohn, 1994, 1996; Gamerman, 1998; Reis et al., 2006) or still in a combination of these two methods (e.g., Andrieu et al., 2010; Whiteley et al., 2010). MCMC is currently the most common approach to inference in dynamic models due to its generality and capability to obtain samples from the posterior distribution of all unknown model parameters in an efficient way. However, MCMC implementation is more involved and it suffers from a series of well known problems that have hindered its wider utilization in applied settings. For example, convergence can be quite difficult to diagnose and the computational cost may become prohibitively high for complex models, as is the case of spatio-temporal dynamic models. The Monte Carlo errors are also intrinsically large and strong correlation among parameters is common, leading the algorithms slow.

A sort of computational tools to fit state-space models using some of the inference methods mentioned above have also appeared in the literature to help end users to benefit from methodological developments. The first of them was the Bats software (West et al., 1988; Pole et al., 1994), a package for time series analysis and forecasting using Bayesian dynamic modeling, developed in the late 80's by the "Bayesian Forecasting Group" of Warwick University. It deals with univariate time series and dynamic regression models. It performs sequential estimation and uses a discount factor approach to model the unknown variances.

The SsfPack library (Koopman et al.,1999), a module for the programming language Ox, provides functions for likelihood evaluation and signal extraction of linear Gaussian state-space models, with support for estimating some non-Gaussian and nonlinear models using importance sampling and MCMC methods.

More recently some R packages and functions to fit linear and generalized linear dynamic models have been developed. The function StructTS written by Bryan Ripley (see Ripley, 2002) fits linear Gaussian state-space models (also called structural models) for univariate time series by maximum likelihood, by decomposing the series in trend and/or seasonal components. The dlm package (Petris, 2010), performs maximum likelihood estimation, Kalman filtering and smoothing, and Bayesian analysis (through a Gibbs sampler) of Gaussian linear dynamic models. The algorithms used for Kalman filtering, likelihood evaluation, and sampling from the state vectors are based on the singular value decomposition of the relevant variance matrices.

The sspir package (Dethlefsen and Lundbye-Christensen, 2006) includes functions for Kalman filtering and smoothing of linear and generalized dynamic linear models. Estimation of variance matrices can be performed using the EM algorithm in the Gaussian case, but it requires that the variance matrices to be estimated are constant. Non-Gaussian state space models are approximated to a Gaussian state-space model through an iterated extended Kalman filtering approach. The KFAS package (Helske, 2010), also provides functions for Kalman filtering and smoothing of univariate exponential family state space models. Yet another implementation is given in the FKF package (Luethi et al., 2010), which implements a fast Kalman filter for fitting high-dimensional linear state-space models to large datasets.

These computational tools have contributed to a wider use of dynamic models in applied contexts. However, support remains incomplete in some particular modeling aspects. For example, as we leave the Gaussian univariate case, estimation of hyperparameters and its uncertainty is not straightforward to obtain. Estimation of more complex dynamic models as is the case of the spatio-temporal ones is also not possible with these tools. All these gaps can however be filled using the INLA approach.

Another concern is related to the fact that parameter estimation with the above software tools is performed, in general, using a "dynamic" (recursive) algorithm, for example, following Kalman's ideas, where filtering and smoothing steps are used in order to estimate the hyperparameters and states, considering the temporal order of the observations. Their inference procedure is based on the (not always needed) assumption that sequential updating/estimation is required. This seems to be also built in stone in reading books on the subject. This thinking is justified in problems where on-line (i.e., real time) estimation and prediction is required every time that a new observation arrives, both from the point of view of storage costs as well as for rapid adaptation to changing signal characteristics. Examples of applied settings where this situation is common include computer vision, economics and financial data analysis, feed-back control systems, mobile communications, radar surveillance systems, etc. In that cases, a recursive filter is a convenient solution and several proposals have appeared in the literature to efficiently solve the problem based, for example, on sequential Monte Carlo algorithms (for a review see Andrieu et al., 2004; Cappé et al., 2007, and references therein). However, when the data set is fixed, in the sense that all observations were already measured, and the interest is in the estimation of parameters and states using just this information, there is no reason why the procedure for inference should also be "dynamic". This is the idea pursued in the INLA approach, where the posteriors of interest are directly approximated avoiding look (at least computationally) at the temporal structure of the data. From our view point it seems more natural and makes full Bayesian analysis (that is, assessment of uncertainty for the hyperparameters, predictive marginals, etc.) possible in an easy way, even for complex state-space models.

Yet another advantage of the INLA approach is its suitability to perform model comparison. Marginal likelihoods, for example, which can be used as a basis to compare competing models through the Bayes factor, can be easily computed with INLA. Additionally, the deviance information criterion (Spiegelhalter et al., 2002) and two predictive measures, the conditional

predictive ordinate and the probability integral transform, used to validate and compare models and as a tool to detect atypical observations, are available from the INLA output (see Martino and Rue, 2010, for implementation details).

In the next two sections we show, through a series of examples, how to take advantage of the INLA features to obtain an improved inference in linear and generalized dynamic linear models in a simple, yet flexible way.

3 The Integrated Nested Laplace Approximation (INLA) approach

INLA is a computational approach, recently introduced by Rue and Martino (2007) and Rue et al. (2009), to perform fast Bayesian inference in the broad class of latent Gaussian models, that is, models of an outcome variable y_i that assume independence conditional on some underlying (unknown) latent field $\boldsymbol{\xi}$ and a vector of hyperparameters $\boldsymbol{\theta}$. It was proposed as an alternative to the usually time consuming MCMC methods. Unlike MCMC where posterior inference is sample-based, the INLA computational approach directly approximates the posteriors of interest with a closed form expression. Therefore, problems of convergence and mixing, inherent to MCMC runs, are not an issue. The main aim of the INLA approach is to approximate the marginal posteriors for the latent variables as well as for the hyperparameters of the Gaussian latent model, given by

$$\pi(\xi_i \mid \boldsymbol{y}) = \int \pi(\xi_i \mid \boldsymbol{\theta}, \boldsymbol{y}) \pi(\boldsymbol{\theta} \mid \boldsymbol{y}) d\boldsymbol{\theta} \tag{3}$$

$$\pi(\theta_j \mid \boldsymbol{y}) = \int \pi(\boldsymbol{\theta} \mid \boldsymbol{y}) d\theta_{-j}. \tag{4}$$

This approximation is based on an efficient combination of (analytical Gaussian) Laplace approximations to the full conditionals $\pi(\boldsymbol{\theta} \mid \boldsymbol{y})$ and $\pi(\xi_i \mid \boldsymbol{\theta}, \boldsymbol{y})$, $i = 1, \cdots, n$, and numerical integration routines to integrate out the hyperparameters $\boldsymbol{\theta}$.

The INLA approach as proposed in Rue et al. (2009) includes three main approximation steps to obtain the marginal posteriors in (3) and (4). The first step consists in approximate the full posterior $\pi(\boldsymbol{\theta} \mid \boldsymbol{y})$. To achieve this, firstly and approximation to the full conditional distribution of $\boldsymbol{\xi}$, $\pi(\boldsymbol{\xi} \mid \boldsymbol{y}, \boldsymbol{\theta})$, is obtained using a multivariate Gaussian density $\tilde{\pi}_G(\boldsymbol{\xi} \mid \boldsymbol{y}, \boldsymbol{\theta})$ (for details see Rue and Held, 2005) and evaluated at its mode. Then the posterior density of $\boldsymbol{\theta}$ is approximated by using the Laplace approximation

$$\tilde{\pi}(\boldsymbol{\theta} \mid \boldsymbol{y}) \propto \left. \frac{\pi(\boldsymbol{\xi}, \boldsymbol{\theta}, \boldsymbol{y})}{\tilde{\pi}_G(\boldsymbol{\xi} \mid \boldsymbol{\theta}, \boldsymbol{y})} \right|_{\boldsymbol{\xi} = \boldsymbol{\xi}^*(\boldsymbol{\theta})},$$

where $\boldsymbol{\xi}^*(\boldsymbol{\theta})$ is the mode of the full conditional of $\boldsymbol{\xi}$ for a given $\boldsymbol{\theta}$. Since no exact closed form is available for $\boldsymbol{\xi}^*(\boldsymbol{\theta})$, an optimization scheme is necessary. Rue et al. (2009) computes this mode using a Newton-Raphson algorithm. The posterior $\tilde{\pi}(\boldsymbol{\theta} \mid \boldsymbol{y})$ will be used later to integrate out the uncertainty with respect to $\boldsymbol{\theta}$ when approximating the posterior marginal of ξ_i.

The second step computes the Laplace approximation of the full conditionals $\pi(\xi_i \mid \boldsymbol{y}, \boldsymbol{\theta})$ for selected values of $\boldsymbol{\theta}$. These values of $\boldsymbol{\theta}$ must be carefully chosen, as explained below, as they will also be used as evaluation points in the numerical integration applied to obtain the posterior marginals of ξ_i in (3). The density $\pi(\xi_i \mid \boldsymbol{\theta}, \boldsymbol{y})$ is approximated using the Laplace approximation

defined by:

$$\tilde{\pi}_{LA}(\xi_i \mid \boldsymbol{\theta}, \boldsymbol{y}) \propto \left. \frac{\pi(\boldsymbol{\xi}, \boldsymbol{\theta}, \boldsymbol{y})}{\tilde{\pi}_G(\boldsymbol{\xi}_{-i} \mid \xi_i, \boldsymbol{\theta}, \boldsymbol{y})} \right|_{\boldsymbol{\xi}_{-i} = \boldsymbol{\xi}^*_{-i}(\xi_i, \boldsymbol{\theta})}, \tag{5}$$

where $\boldsymbol{\xi}_{-i}$ denotes the vector $\boldsymbol{\xi}$ with the ith component omitted, $\tilde{\pi}_G(\boldsymbol{\xi}_{-i} \mid \xi_i, \boldsymbol{\theta}, \boldsymbol{y})$ is the Gaussian approximation of $\pi(\boldsymbol{\xi}_{-i} \mid \xi_i, \boldsymbol{\theta}, \boldsymbol{y})$, treating ξ_i as fixed (observed) and $\boldsymbol{\xi}^*_{-i}(\xi_i, \boldsymbol{\theta})$ is the mode of $\pi(\boldsymbol{\xi}_{-i} \mid \xi_i, \boldsymbol{\theta}, \boldsymbol{y})$.

The approximation of $\pi(\xi_i \mid \boldsymbol{\theta}, \boldsymbol{y})$ using (5) can be quite expensive, since $\tilde{\pi}_G(\boldsymbol{\xi}_{-i} \mid \xi_i, \boldsymbol{\theta}, \boldsymbol{y})$ must be recomputed for each value of ξ_i and $\boldsymbol{\theta}$. Two alternatives are proposed in Rue et al. (2009) to obtain these full conditionals in a cheapest way. The first one is just the Gaussian approximation $\tilde{\pi}_G(\xi_i \mid \boldsymbol{\theta}, \boldsymbol{y})$, which provides reasonable results in short computational time. However, according to Rue and Martino (2007), its accuracy can be affected by errors in the location and/or errors due to the lack of skewness. These weaknesses can be corrected at a moderate extra cost, using a simplified version of the Laplace approximation, defined as the series expansion of $\tilde{\pi}_{LA}(\xi_i \mid \boldsymbol{\theta}, \boldsymbol{y})$ around $\xi_i = \mu_i(\boldsymbol{\theta})$, the mean of $\tilde{\pi}_G(\xi_i \mid \boldsymbol{y}, \boldsymbol{\theta})$ (for details see Rue et al., 2009).

Finally, in the third step the full posteriors obtained in the previous two approximation steps are combined and the marginal posterior densities of ξ_i and θ_j are obtained by integrating out the irrelevant terms. The approximation for the marginal of the latent variables can be obtained by the expression

$$\pi(\xi_i \mid \boldsymbol{y}) = \int \pi(\xi_i \mid \boldsymbol{y}, \boldsymbol{\theta}) \pi(\boldsymbol{\theta} \mid \boldsymbol{y}) d\boldsymbol{\theta} \approx \sum_k \tilde{\pi}(\xi_i \mid \boldsymbol{\theta}_k, \boldsymbol{y}) \tilde{\pi}(\boldsymbol{\theta}_k \mid \boldsymbol{y}) \triangle_k, \tag{6}$$

which is evaluated using numerical integration on a set of grid points for $\boldsymbol{\theta}$, with area weights $\triangle_k$ for $k = 1, 2, \cdots, K$. According to Rue et al. (2009), since these integration points are selected in a regular grid, it is feasible to take all the area weights $\triangle_k$ to be equal. A similar numerical integration procedure is used for the evaluation of the marginals $\pi(\theta_j \mid \boldsymbol{y})$. Since the dimension of $\boldsymbol{\theta}$ is assumed small (i.e., ≤ 7), these numerical routines are efficient in returning a discretized representation of the marginal posteriors.

A good choice of the set of evaluation points is crucial to the accuracy of the above numerical integration steps. In order to do that, Rue et al. (2009) suggest to compute the negative Hessian matrix S at the mode, $\boldsymbol{\theta}^*$, of $\tilde{\pi}(\boldsymbol{\theta} \mid \boldsymbol{y})$ and to consider its spectral value decomposition, $S^{-1} = Q \Lambda Q^T$. Then $\boldsymbol{\theta}$ is defined via a standardized variable, z, as

$$\boldsymbol{\theta}(z) = \boldsymbol{\theta}^* + Q \Lambda^{1/2} z \qquad \text{or} \qquad z = Q^T \Lambda^{-1/2} (\boldsymbol{\theta} - \boldsymbol{\theta}^*)$$

and a collection, Z, of z values is found, such that the corresponding $\boldsymbol{\theta}(z)$ points are located around the mode $\boldsymbol{\theta}^*$. Starting from $z = 0$ ($\boldsymbol{\theta} = \boldsymbol{\theta}^*$), each component entry of z is searched in the positive and negative directions in step sizes of η_z. All z-points satisfying

$$\log \tilde{\pi}(\boldsymbol{\theta}(0) \mid \boldsymbol{y}) - \log \tilde{\pi}(\boldsymbol{\theta}(z) \mid \boldsymbol{y}) < \eta_\pi$$

are taken to be in Z. The set of evaluation points is finally based on the values in Z. An appropriate tuning of the η_z and η_π values should be performed in order to produce accurate approximations.

An efficient computational implementation of the procedures needed by the INLA approach was built on the open source library GMRFLib (Rue and Follestad, 2002), a C-library for fast and exact simulation of Gaussian Markov random fields. It particularly benefits from sparse

matrix algorithms available in the GMRFLib library to quickly model the Gaussian latent field. Its user friendly `R` interface is available from the web page `http://www.r-inla.org/`, which also includes many examples of fitting for most of the latent models mentioned in the introduction.

The following two sections show, through a series of examples, how the INLA approach can be extended to deal with inference in dynamic models in an easy way using the INLA library.

4 INLA for state-space models

In this section we illustrate, through a series of simulated data sets, the proposed computational framework to formulate and fit the most common types of dynamic models using the INLA approach. Firstly, the steps of the proposed framework to perform fast Bayesian inference on these models using the INLA library are described in detail using a simple univariate dynamic linear model. Next, we consider examples of state-space models that can be directly fitted using model options that already exist in the INLA library, that is, models with a simple random walk evolution form and with independently distributed error terms. Finally, the proposed framework is applied on models that could not be fitted using INLA's standard tools, as is the case of models with several state vectors dependent among them as well as models whose error terms are correlated. The results evidence the capability of INLA to fit realistically complex state-space models.

All the examples in this section were fitted using different log-gamma priors for the log-precisions of observations and states specified as follows: an informative prior, with mean equal to the real value and coefficient of variation equal to 0.5, a vague prior also centered on the true simulated value but with coefficient of variation equal to 10 and the default INLA log-gamma prior. As little sensitivity to the prior specification was, in general, observed in all the cases and due to space constraints, marginal posterior densities for the hyperparameters with these priors are only detailed in the first illustrative example. Full results for all the examples can be found at the technical report by Ruiz-Cárdenas et al (2010).

Main parts of the `R` code, considered as relevant to a better understanding of the proposed approach are included in this section. The full `R` code to simulate and fit all the examples in this section is available from the INLA web page (see the supplementary material section). Further coding details can also be found in Ruiz-Cárdenas et al (2010).

4.1 A toy example

We begin with a very simple simulated example of a first order univariate dynamic linear model in order to gain insight into the specification of dynamic models for use within INLA. The model has the following observational and system equations:

$$y_t = x_t + \nu_t, \qquad \nu_t \sim N(0, V), \qquad t = 1, \cdots, n \tag{7}$$

$$x_t = x_{t-1} + \omega_t, \qquad \omega_t \sim N(0, W), \qquad t = 2, \cdots, n \tag{8}$$

That is, we are assuming that $F_t = G_t = 1$, $Vt = V$ and $W_t = W$, for all t. Therefore, the model has $n + 2$ unknown parameters. Following the notation in section 3, the vector of hyperparameters is given by $\boldsymbol{\theta} = \{V, W\}$, while the latent field corresponds to $\boldsymbol{\xi} = \{x_1, \cdots, x_n\}$. Since the evolution of states in this simple model follows a first order random walk process, it could just be fitted with the INLA library using existing model options according to the following code:

```
i <- 1:n          # indices for x_t
formula <- y ~ f(i, model="rw1", constr=F) -1
r <- inla(formula, data = data.frame(i,y))
```

However, we will use this simple example to propose and illustrate a generic approach to fit dynamic models with INLA based on an augmented model structure, which enables the fitting of complex state-space models as will be shown later on in this section. The key feature of this generic approach consists in equating to zero the system equation, that is, we re-write (8) as

$$0 = x_t - x_{t-1} + \omega_t, \qquad \omega_t \sim N(0, W), \qquad t = 2, \cdots, n \tag{9}$$

and then we build an augmented model with dimension $n + (n-1)$ merging these "faked zero observations" from system equation (9) with the actual observations from Eq. (7) in a unique structure, as shown in Diagram 1. The first column of this structure is associated to the actual observations, $\boldsymbol{y}_t = \{y_1, \cdots, y_n\}$, which occupy the first n elements. The second column is associated to the system equation, whose last $n-1$ elements, corresponding to the number of state parameters in Eq. (9), are forced to be zero. All the other entries in the structure are filled in with NA's. This augmented structure will have additional columns when the state-space model contains more than one evolution equation.

$$\left[\begin{array}{c:c} y_1 & \text{NA} \\ \vdots & \vdots \\ y_n & \text{NA} \\ \hdashline \text{NA} & 0 \\ \vdots & \vdots \\ \text{NA} & 0 \end{array}\right] \left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array}\right\} \text{n-1 elements}$$

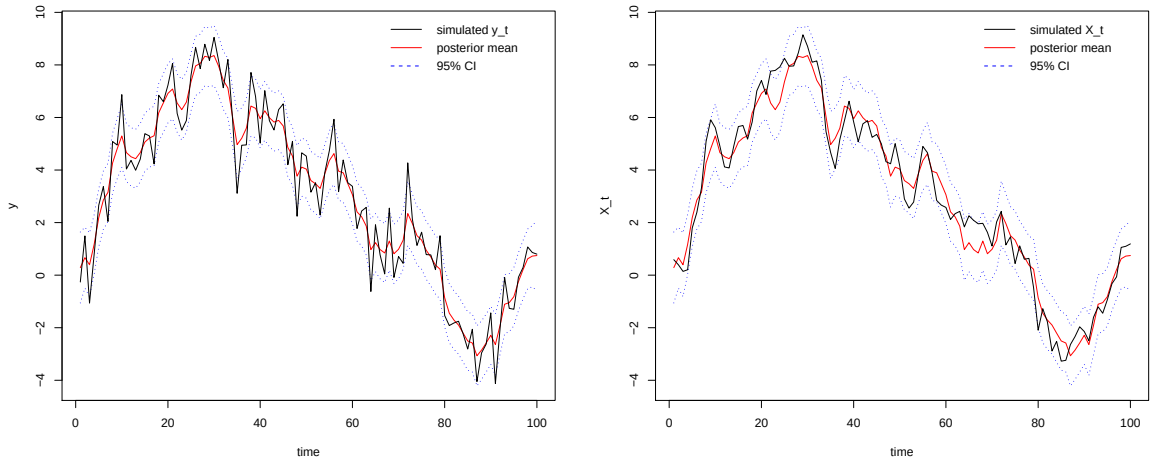
Diagram 1. Schematic representation of the data structure for the augmented model.

Inference in this augmented model using the INLA approach is performed considering two different likelihoods, one for each column of the augmented structure. Hence, given the states, x_t, the actual data points in the first column of the augmented structure are assumed to follow a Gaussian distribution with unknown precision V^{-1}. The artificial observations in the second column are deterministically known (i.e., with zero variance) conditioned on the values of x_t, x_{t-1} and ω_t in (9). This condition is represented in INLA by assuming that these "faked" observations follow a Gaussian distribution with a high and fixed precision.

In order to estimate the states, x_t in this model, it is necessary just to know the perturbations ω_t, $t = \{2, \cdots, T\}$, which are the only stochastic terms in the system equation (8). Thus, we are considering that there is no information about x_t beyond its temporal evolution form. In the formulation of the model this information is induced through vectors of indices, say i, j and l, associated to x_t, x_{t-1} and ω_t terms, respectively. The x_{t-1} terms are modeled as a copy of the x_t terms, making the value at position $k+1$ in index vector j equal to the value at position k in index vector i; this lag induces the temporal evolution of x_t. Since we are not considering stochasticity in the states, for the formulation of the model we must left the terms x_t free to assume values in any region of the parametric space. This is achieved defining $\boldsymbol{x}$ as a vector of independent and Gaussian distributed random variables with a fixed and low precision (high variance). Let $\boldsymbol{y}$ be a $n \times 1$ vector of observations simulated from the dynamic linear model described above. The following `R` code implements the augmented approach to this model using the INLA library:

```
## building the augmented model
m <- n-1
Y <- matrix(NA, n+m, 2)
Y[1:n,     1] <- y                # actual observations
Y[1:m + n, 2] <- 0                # faked observations
## indices for the INLA library
i  <- c(1:n, 2:n)                 # indices for x_t
j  <- c(rep(NA,n), 2:n -1)        # indices for x_{t-1}
w1 <- c(rep(NA,n), rep(-1,m))     # weights for x_{t-1}
l  <- c(rep(NA,n), 1:m)           # indices for w_t
## formulating the model
formula <- Y ~ f(i, model="iid", initial=-10, fixed=T) +
               f(j, w1, copy="i") + f(l, model ="iid") -1
## call to fit the model
require(INLA)
r <- inla(formula, data = data.frame(i,j,w1,l),
          family = c("gaussian", "gaussian"),
          control.data = list(list(), list(initial=10, fixed=T)))
```

The series of simulated and estimated values for the vectors of observations and states, respectively, are illustrated in Figure 1. Precisions of the perturbation terms were well estimated in all cases, as can be seen in Figure 2, with credibility intervals always including the true simulated values.



(a) (b)

Figure 1: Simulated and predicted values (posterior mean and 95% credibility interval) for the observations (a) and states (b) in the toy example.

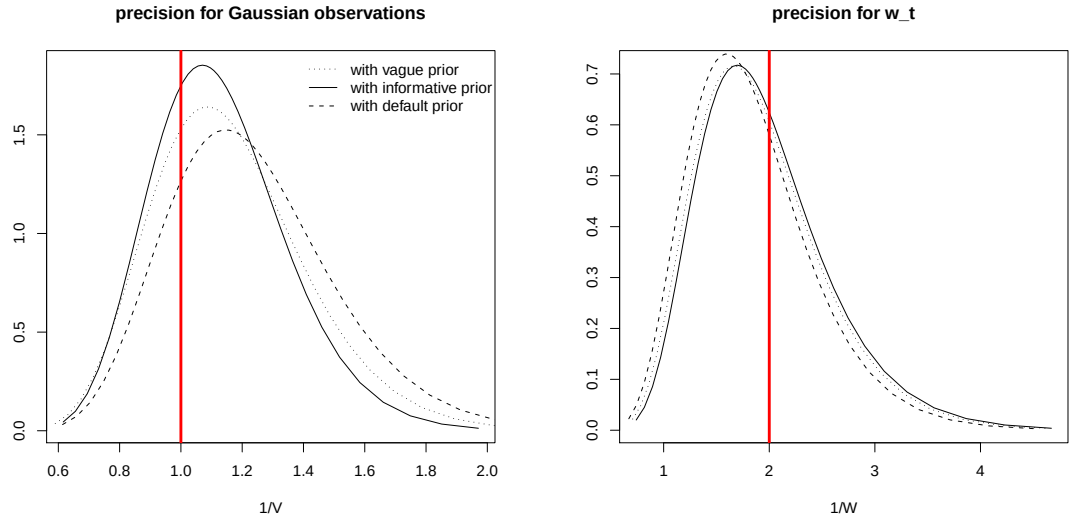


Figure 2: Posterior densities for the hyperparameters in the toy example. Red lines indicate true simulated values.

4.2 Applying current INLA modeling options

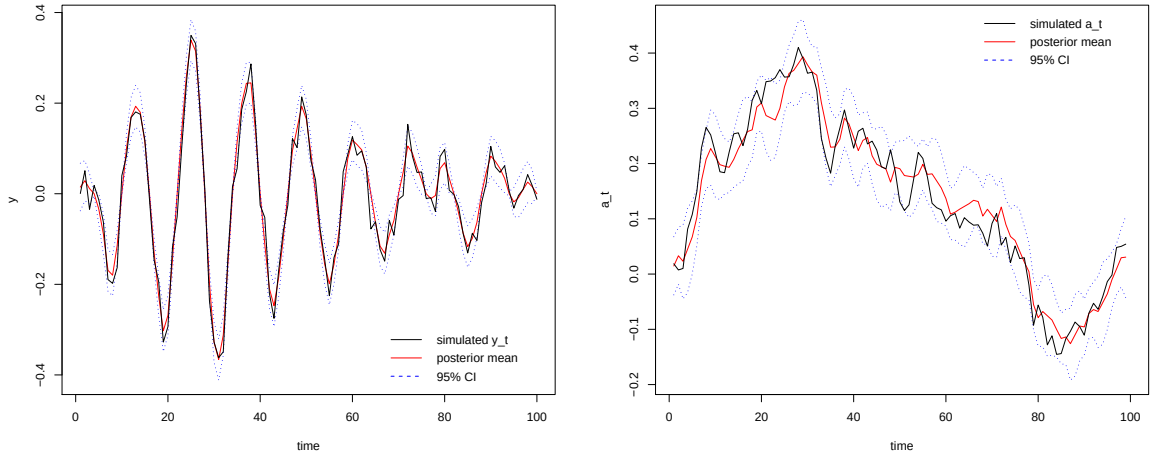
The INLA approach currently offers some functionalities to deal with first and second order random walk and with seasonal models that can be directly used to fit simpler state-space models. This is illustrated in this subsection through two examples where a generalized dynamic regression model and a dynamic seasonal model are fitted using existing model options from the INLA library. What both of these examples have in common is the independence among the error terms and the simple evolution form for the states. Examples of more complex models, where these assumptions are not valid and that are unable to be fitted using INLA's standard tools, are considered in the next subsection.

Example 1: A dynamic seasonal model

In this example we simulate a monthly time series with an annual seasonal pattern by using a cosine form. The response y_t is assumed to be normal. The DLM formulation is then defined as

$$\begin{aligned} y_t &= a_t \cos\left(\frac{\pi(t-1)}{6}\right) + \nu_t, & \nu_t &\sim N(0, V), & t &= 1, \cdots, n \\ a_t &= a_{t-1} + \omega_{1t}, & \omega_{1t} &\sim N(0, W_1), & t &= 2, \cdots, n \end{aligned}$$

This model can easily be fitted with INLA using a first order random walk option model. Results for predicted values and states are summarized in Figure 3:



(a) (b)

Figure 3: Simulated and predicted values (posterior mean and 95% credibility interval) for the observations (a) and for a_t state vector (b) in the dynamic seasonal model.

Example 2: A generalized dynamic regression model

Here we simulate data from a multiple Poisson regression model with two regressors, Z_{1t} and Z_{2t}. Thus, the linear predictor is given by $\lambda_t = \boldsymbol{F}_t\boldsymbol{x}_t$, where $\boldsymbol{F}_t = (1, Z_{1t}, Z_{2t})$ and the regression coefficients $\boldsymbol{x}_t = (\beta_{0t}, \beta_{1t}, \beta_{2t})$ follow a simple random walk evolution. The model has the following observational and system equations:

$$
\begin{aligned}
(y_t \mid \mu_t) &\sim \text{Poisson}(\mu_t) \\
\log(\mu_t) = \lambda_t &= \beta_{0_t} + \beta_{1_t} Z_1 + \beta_{2_t} Z_2 && && t = 1, \cdots, n \\
\beta_{0t} &= \beta_{0,t-1} + \omega_{0t}, && \omega_{0t} \sim N(0, W_0), && t = 2, \cdots, n \\
\beta_{1t} &= \beta_{1,t-1} + \omega_{1t}, && \omega_{1t} \sim N(0, W_1), && t = 2, \cdots, n \\
\beta_{2t} &= \beta_{2,t-1} + \omega_{2t}, && \omega_{2t} \sim N(0, W_2), && t = 2, \cdots, n
\end{aligned}
$$

Results using the first order random walk model from the INLA library for modeling the evolution of the regression coefficients are shown in Figure 4:

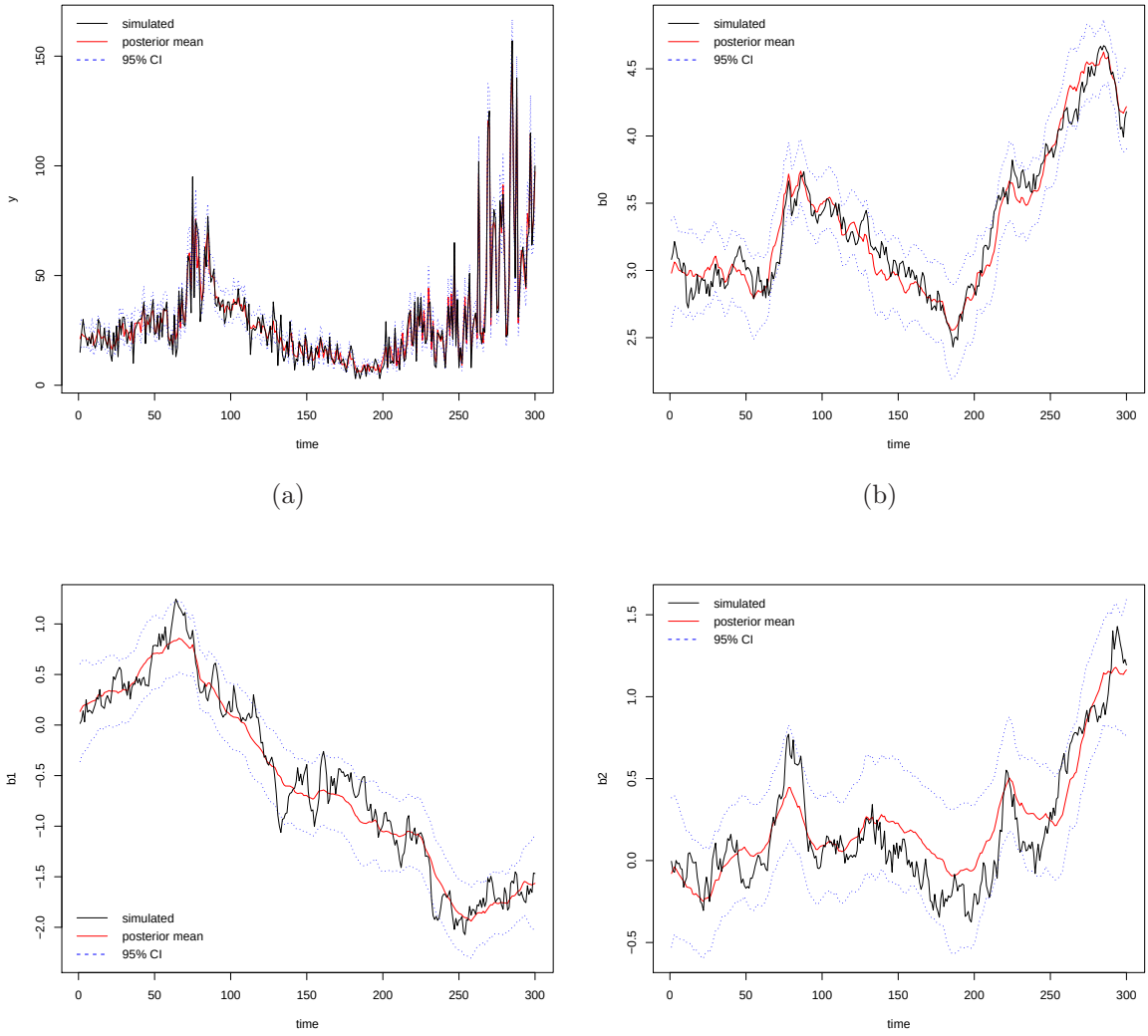


Figure 4: Simulated and predicted values (posterior mean and 95% credibility interval) for the observations (a) and regression coefficients, β_0, β_1 and β_2 (b-d) in the generalized dynamic regression example.

4.3 Extending the INLA modeling capabilities

As seen in the above subsection, the class of state-space models that can be fitted under the INLA approach using current tools to fit latent Gaussian models, is restricted to models with a simple temporal evolution form for the states and with an independence assumption for the error terms. This is not the case, for example, of growth models and spatio-temporal dynamic models, where there can be more than one state vector, being mutually dependent on each other, and where the error terms can be structured as a matrix with correlated values. The proposed framework for inference described in subsection 4.1 extends the INLA approach in order to perform approximate Bayesian inference in these more complex cases. The examples in this subsection illustrate how to apply the proposed framework in the above mentioned situations.

4.3.1 Models with different evolution structure for the states

Here a second order polynomial dynamic model and a dynamic seasonal model with two harmonics are considered to exemplify how to deal, under the INLA approach, with state-space models when the temporal evolution structure for the states involve more than one state vector.

Example 3: A second order polynomial dynamic model

This is a growth model with a state vector that comprises two elements, $x_t = (x_{1t}, x_{2t})$, the first representing the current level and the second representing the current rate of change in the level. The observational and system equations for this model are simulated as

$$y_t = x_{1t} + \nu_t, \qquad \nu_t \sim N(0, V), \qquad t = 1, \cdots, n \tag{10}$$

$$x_{1t} = x_{1,t-1} + x_{2,t-1} + \omega_{1t}, \qquad \omega_{1t} \sim N(0, W_1), \qquad t = 2, \cdots, n \tag{11}$$

$$x_{2t} = x_{2,t-1} + \omega_{2t}, \qquad \omega_{2t} \sim N(0, W_2), \qquad t = 2, \cdots, n \tag{12}$$

Note that the temporal evolution for x_{1t} in (11) depends on both, $x_{1,t-1}$ and $x_{2,t-1}$. This evolution structure can not be directly accounted for the INLA library, therefore, we made use here of the augmented model structure described in subsection 4.1, equating (11) and (12) to zero and then merging these "pseudo" observations with the actual observations from Eq. (10). The augmented structure has dimension $n + 2(n-1)$ and three different likelihoods, being the elements in the first column Gaussian distributed with unknown precision V^{-1}, while the artificial observations in the other two columns are modeled as Gaussian with a high and fixed precision (see the toy example for details). Results of simulated and predicted values for observations and states are presented in Figure 5.

Example 4: A seasonal dynamic model with harmonics

In this example we simulate a monthly time series with an annual seasonal pattern by using a sum of sine and cosine terms with the same frequency. This model has a two parameter state $x_t = (a_t, b_t)$, where a_t models an arbitrary amplitude (seasonal peak to trough variation) and b_t allows the cycle maximum and minimum to be phase shifted. The state-space formulation for this model is defined as follows:

$$x_t = \begin{pmatrix} a_t \\ b_t \end{pmatrix}; \quad F_t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad G_t = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}; \quad \text{where } \phi = \frac{\pi}{6},$$

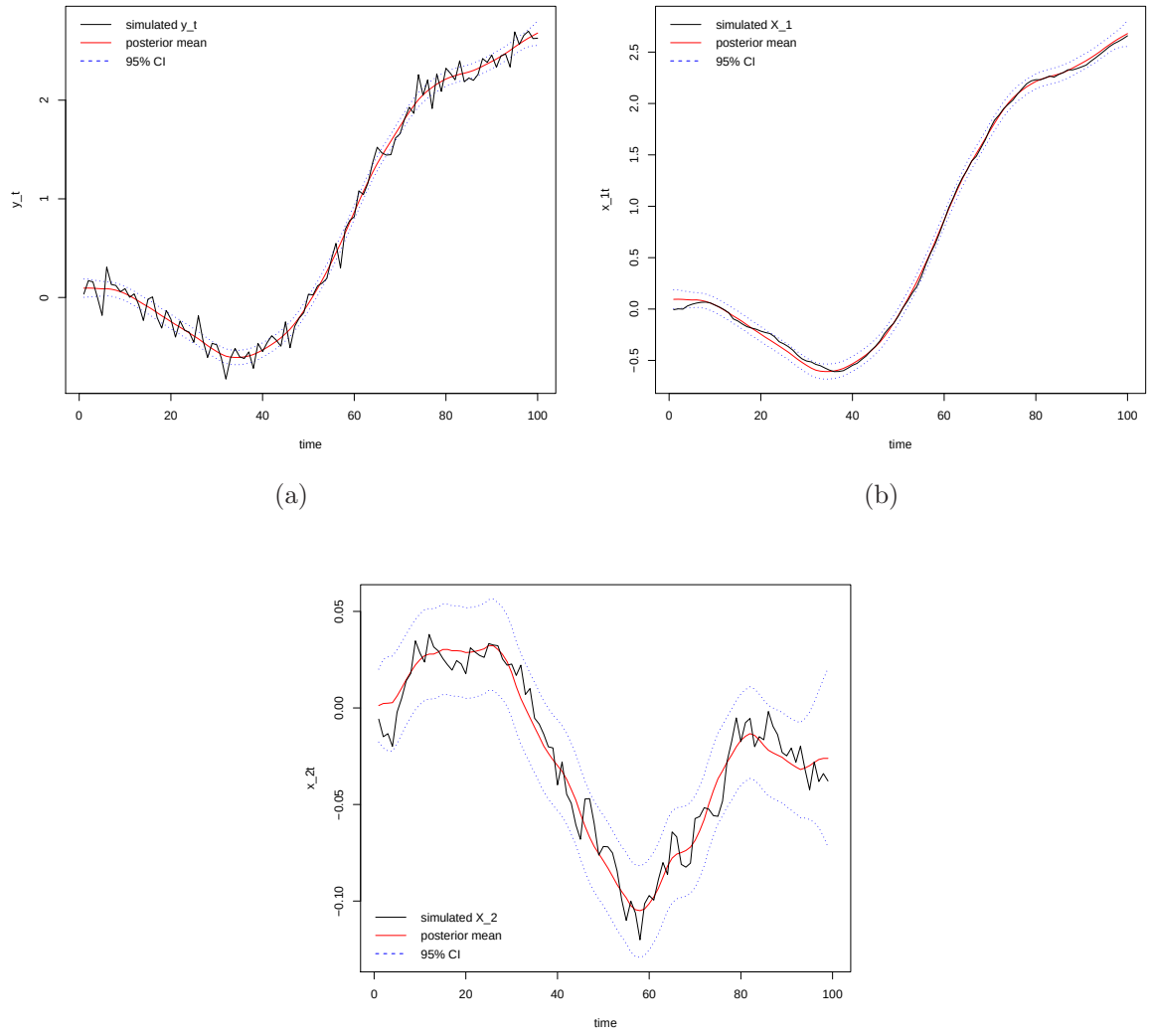
with observational and system equations given by

$$y_t = a_t + \nu_t, \qquad \nu_t \sim N(0, V), \qquad t = 1, \cdots, n$$

$$a_t = \cos(\phi)a_{t-1} + \sin(\phi)b_{t-1} + \omega_{1t}, \qquad \omega_{1t} \sim N(0, W_1), \qquad t = 2, \cdots, n \tag{13}$$

$$b_t = -\sin(\phi)a_{t-1} + \cos(\phi)b_{t-1} + \omega_{2t}, \qquad \omega_{2t} \sim N(0, W_2), \qquad t = 2, \cdots, n \tag{14}$$

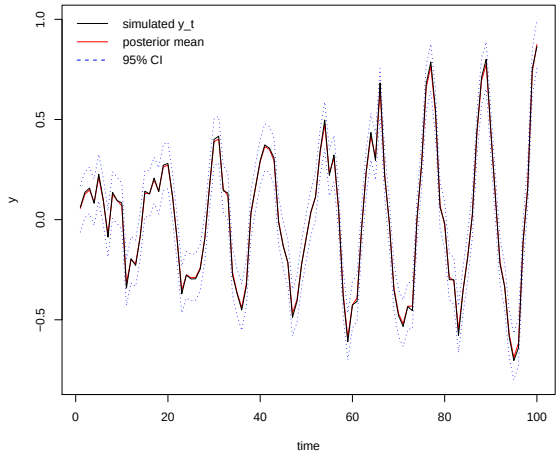
The augmented structure approach applied to this case yields the results shown in Figures 6.



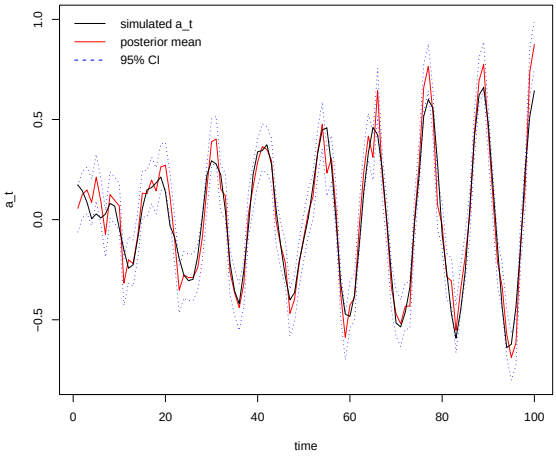
(a) (b)

(c)

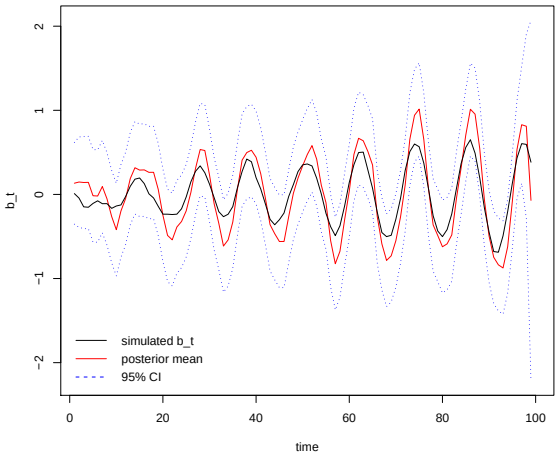
Figure 5: Simulated and predicted values (posterior mean and 95% credibility interval) for the observations (a) and for x_{1t} (b) and x_{2t} (c) state vectors in the second order polynomial dynamic model.



(a)



(b)



(c)

Figure 6: Simulated and predicted values (posterior mean and 95% credibility interval) for the observations (a) and for a_t (b) and b_t (c) state vectors in the harmonic seasonal model.

4.3.2 Models with correlated error structure

In the following examples we simulate data from two different versions of a Gaussian spatio-temporal dynamic model for aerial data (Vivar and Ferreira, 2009) in order to demonstrate how the INLA library also can easily deal with correlated error structures in complex spatio-temporal dynamic models.

Example 5: A first order spatio-temporal dynamic model with covariates

We begin with a non-stationary first-order Gaussian model with one spatially structured covariate, where for each time t and area s, $t = \{1, \cdots, T\}$; $s = \{1, \cdots, S\}$, the response y_{ts} is specified as:

$$\boldsymbol{y}_t = \boldsymbol{F}_t' \boldsymbol{x}_t + \boldsymbol{\Upsilon}_t \boldsymbol{z}_t + \boldsymbol{\omega}_{1t}, \qquad \boldsymbol{\omega}_{1t} \sim PGMRF\left(\boldsymbol{0s}, \boldsymbol{W}_1^{-1}\right) \tag{15}$$

$$\boldsymbol{x}_t = \boldsymbol{G}_t \boldsymbol{x}_{t-1} + \boldsymbol{\omega}_{2t}, \qquad \boldsymbol{\omega}_{2t} \sim PGMRF\left(\boldsymbol{0s}, \boldsymbol{W}_2^{-1}\right) \tag{16}$$

$$\boldsymbol{z}_t = \boldsymbol{z}_{t-1} + \boldsymbol{\omega}_{3t}, \qquad \boldsymbol{\omega}_{3t} \sim PGMRF\left(\boldsymbol{0s}, \boldsymbol{W}_3^{-1}\right) \tag{17}$$

with $\boldsymbol{y}_t = (y_{t1}, \cdots, y_{tS})'$ and $\boldsymbol{\Upsilon}_t = (\Upsilon_{t1}, \cdots, \Upsilon_{tS})'$ denoting the observed field and the vector of covariates at time t, respectively. $\boldsymbol{F}_t = \boldsymbol{Is}$, $\boldsymbol{G}_t = \rho \boldsymbol{Is}$, $\boldsymbol{0s}$ is an $S \times S$ null matrix, and $\boldsymbol{Is}$ is the $S \times S$ identity matrix. The vectors of errors $\boldsymbol{\omega}_{i1}, \cdots, \boldsymbol{\omega}_{iT}$, $i = \{1, 2, 3\}$, are assumed to be independent and modeled as proper Gaussian Markov random fields (PGMRF). Matrices $\boldsymbol{W}_i$ describe the spatial covariance structure of $\boldsymbol{\omega}_{it} = (\omega_{it1}, \cdots, \omega_{itS})'$. Precision matrices $\boldsymbol{W}_i^{-1}$ are modeled as $\boldsymbol{W}_i^{-1} = \tau_i \left(\boldsymbol{Is} - \dfrac{\phi_i}{\lambda_{max}} \boldsymbol{C} \right)$, with $\boldsymbol{C}$ being a structure matrix defined as

$$C_{k,l} = \begin{cases} c_k & \text{if} \quad k = l, \\ -h_{k,l} & \text{if} \quad k \in d_l, \\ 0 & \text{otherwise}, \end{cases}$$

d_l is the set of neighbors of area l, $h_{k,l} > 0$ is a measure of similarity between areas k and l (here we assume that $h_{k,l} = 1$) and $c_k = \sum_{l \in d_k} h_{k,l}$. λ_{max} is the maximum eigenvalue of matrix $\boldsymbol{C}$; τ_i are scale parameters and $0 \leq \phi_i < 1$ control the degree of spatial correlation.

We simulated a time series of 100 times for each of the 26 counties in the map of Eire (that is, $S = 26$ and $T = 100$). This map is available in `R` from `spdep` package (Bivand, 2010). Inference is performed for the state vectors $\boldsymbol{x}_t$ and $\boldsymbol{z}_t$, as well as for the scale and correlation parameters, τ_i, ϕ_i, $i = \{1, 2, 3\}$, but not for ρ, whose value was fixed in one before analysis, leading to a non-stationary process.

Predicted values closely followed the simulated series of observations and states in all cases, as illustrated in Figures 7 and 8 for the 18th area and its neighbors. Precision and correlation parameters were also well estimated even when default INLA values for hyperprior parameters and initial values were specified (results not shown here).

For a comparison between simulated and predicted observations and states for all areas at some instant times see the maps in Figures 15 to 17 at the Supplementary material section.

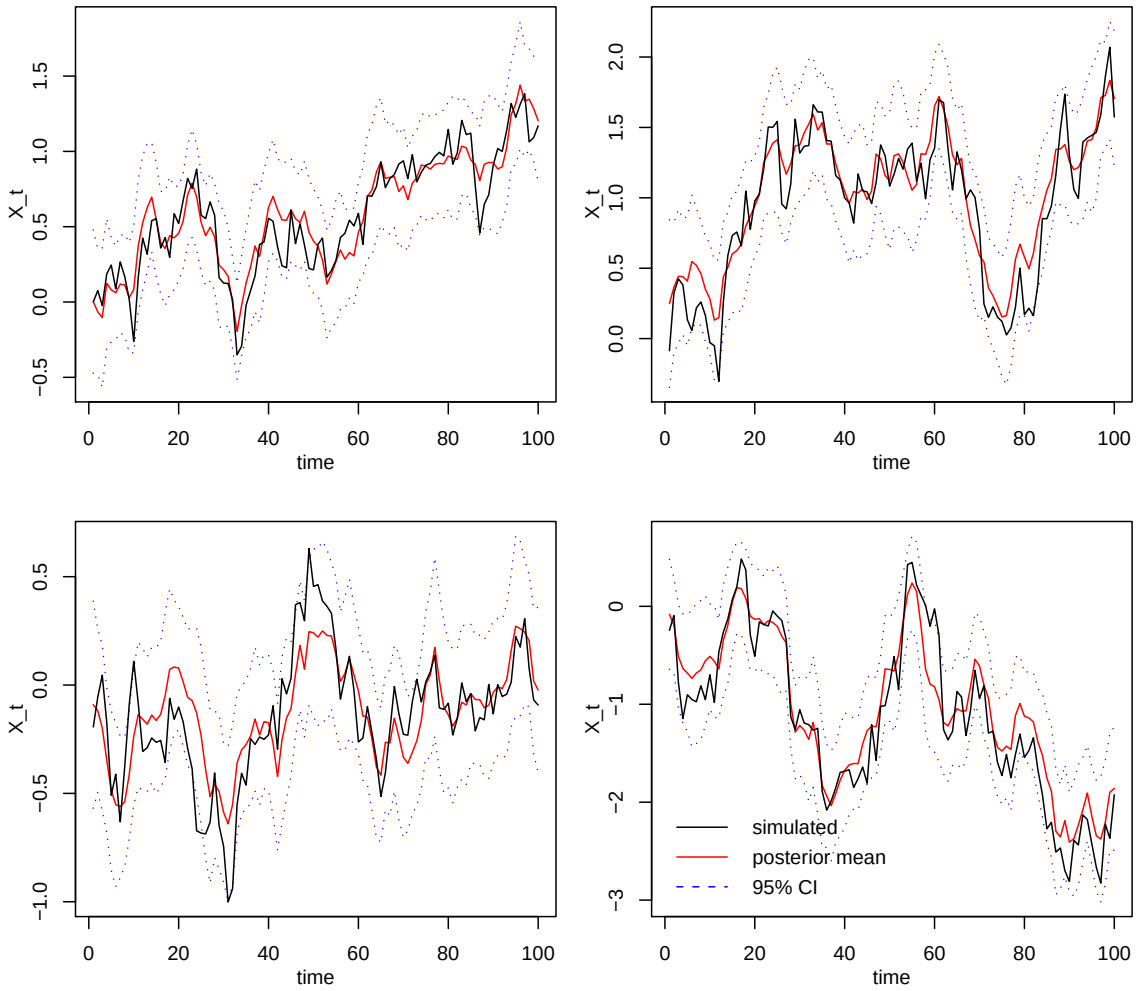


Figure 7: Simulated and predicted values (posterior mean and 95% credibility interval) for the states $\boldsymbol{x}_t$ in the 18th area (a) and in its neighbors (b-d) in the first order spatio-temporal dynamic model.

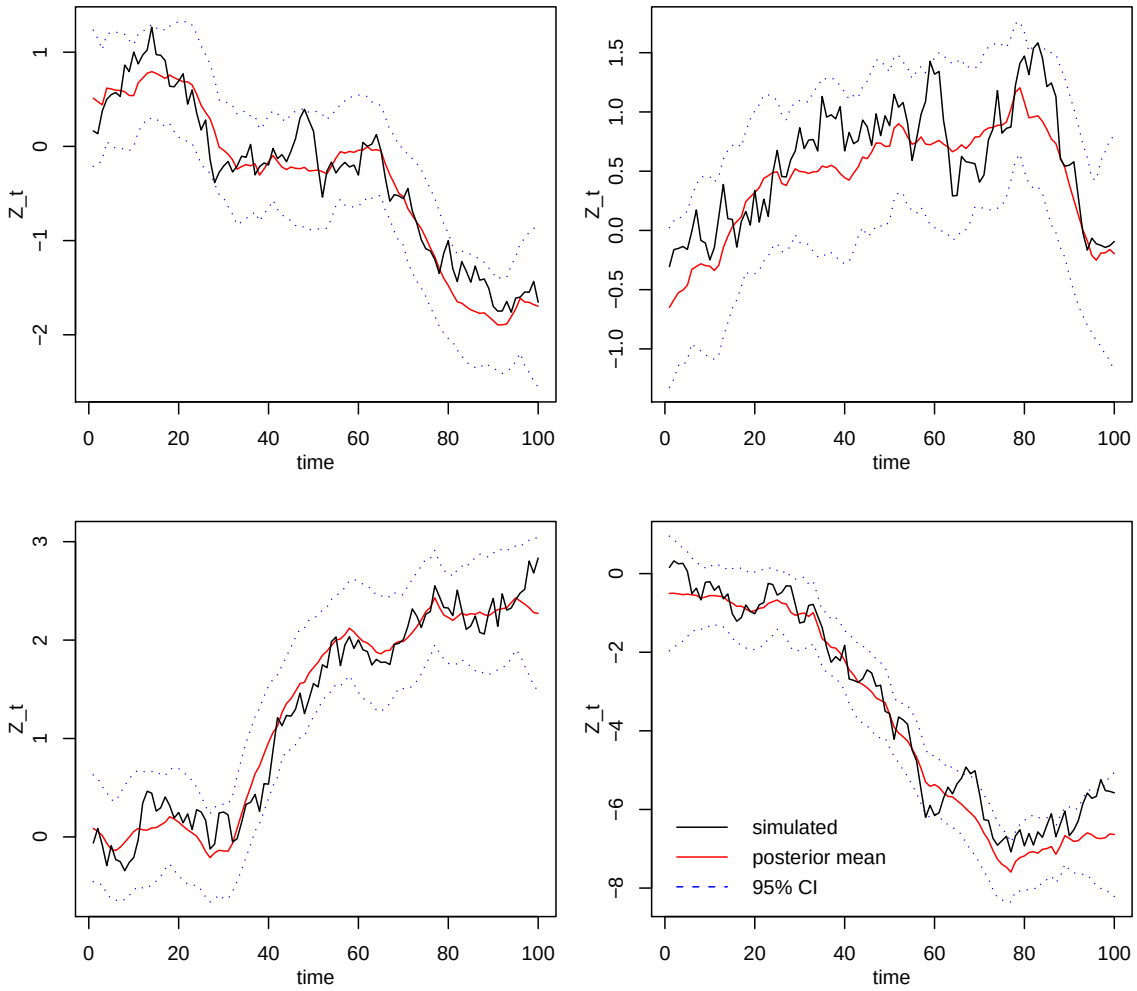


Figure 8: Simulated and predicted values (posterior mean and 95% credibility interval) for the states $\boldsymbol{z}_t$ in the 18th area (a) and in its neighbors (b-d) in the first order spatio-temporal dynamic model.

Example 6: A second order spatio-temporal dynamic model

Now we will simulate data from a non-stationary second-order Gaussian model without covariates (Vivar and Ferreira, 2009). In this example both, the correlated error structure and the growth structure for the evolution of the states are present. The model is specified as:

$$\boldsymbol{y}_t = \boldsymbol{F}_t'\boldsymbol{x}_t + \boldsymbol{\omega}_{1t}, \qquad \boldsymbol{\omega}_{1t} \sim PGMRF\left(\boldsymbol{0s}, \boldsymbol{W}_1^{-1}\right) \tag{18}$$

$$\boldsymbol{x}_t = \boldsymbol{G}_t\boldsymbol{x}_{t-1} + \boldsymbol{\omega}_{23t}, \qquad \boldsymbol{\omega}_{23t} \sim PGMRF\left(\boldsymbol{0s}, \boldsymbol{W}_{23}^{-1}\right) \tag{19}$$

where,

$$\boldsymbol{x}_t = \begin{pmatrix} \boldsymbol{x}_{1t} \\ \boldsymbol{x}_{2t} \end{pmatrix}, \quad \boldsymbol{F}_t = \begin{pmatrix} \boldsymbol{Is} \\ \boldsymbol{0s} \end{pmatrix}, \quad \boldsymbol{G}_t = \begin{pmatrix} \rho_1\boldsymbol{Is} & \rho_1\boldsymbol{Is} \\ \boldsymbol{0s} & \rho_2\boldsymbol{Is} \end{pmatrix}, \quad \boldsymbol{\omega}_{23t} = \begin{pmatrix} \boldsymbol{\omega}_{2t} \\ \boldsymbol{\omega}_{3t} \end{pmatrix} \text{ and } \boldsymbol{W}_{23}^{-1} = \begin{pmatrix} \boldsymbol{W}_2^{-1} & \boldsymbol{0s} \\ \boldsymbol{0s} & \boldsymbol{W}_3^{-1} \end{pmatrix},$$

and the rest of notation follows example 5.

The simulated time series here are for the 100 counties in the map of North Carolina at 30 instant times. Inference is performed for the state vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ and for the scale and correlation parameters, τ_j, ϕ_j, $j = \{1, 2, 3\}$. A non-stationary process is defined assuming that $\rho_1 = \rho_2 = 1$. Relevant parts of the code to fit this model using the INLA library were included at the supplementary material section. Further details can also be found in the full `R` script (available online).

Predicted values here also closely agree with the simulated series of observations and states in all counties. Figures 9 and 10 exemplify this for the 20th county and its neighbors. Maps comparing simulated and predicted observations and states for all areas at some instant times are also shown in Figures 15 and 16 as supplementary material.

In order to check how this model responds to changes in initial values of the hyperparameters as well as to different prior hyperparameter specification, we perform a small sensitivity analysis. Firstly, the model was fitted with informative, vague and default log-gamma priors for τ_i, as specified in the toy example (section 4.1). In this case, priors for the correlation parameters, ϕ_i, and initial values for all the hyperparameters were specified at their default settings. Furthermore, we consider default prior specification for all the hyperparameters and the model was fitted under two situations: (i) with different initial values for log-precision parameters, τ_i, and default initial values for correlation parameters, and (ii) with different initial values for correlation parameters, ϕ_i, and default initial values for precision parameters. The model was not very sensitive to prior specification of precision parameters (results not shown here, for details see Ruiz-Cárdenas et al, 2010). In a similar way, there was a low impact of the choice of initial hyperparameter values on the approximation of the marginal posterior density for these hyperparameters.

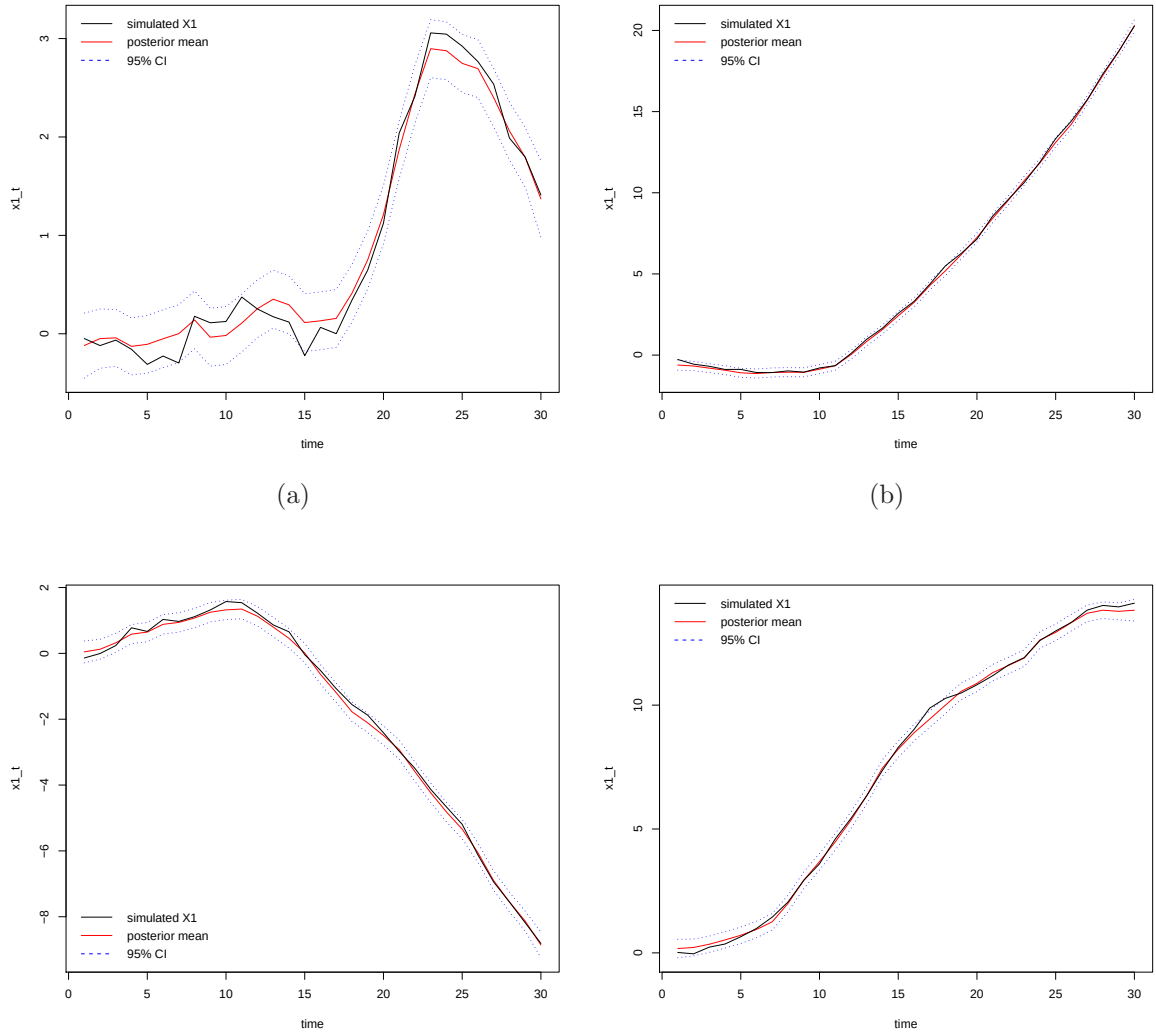


Figure 9: Simulated and predicted values (posterior mean and 95% credibility interval) for the states $x_{1,t}$ in the 20th area (a) and in its neighbors (b-d) in the second order spatio-temporal dynamic model.

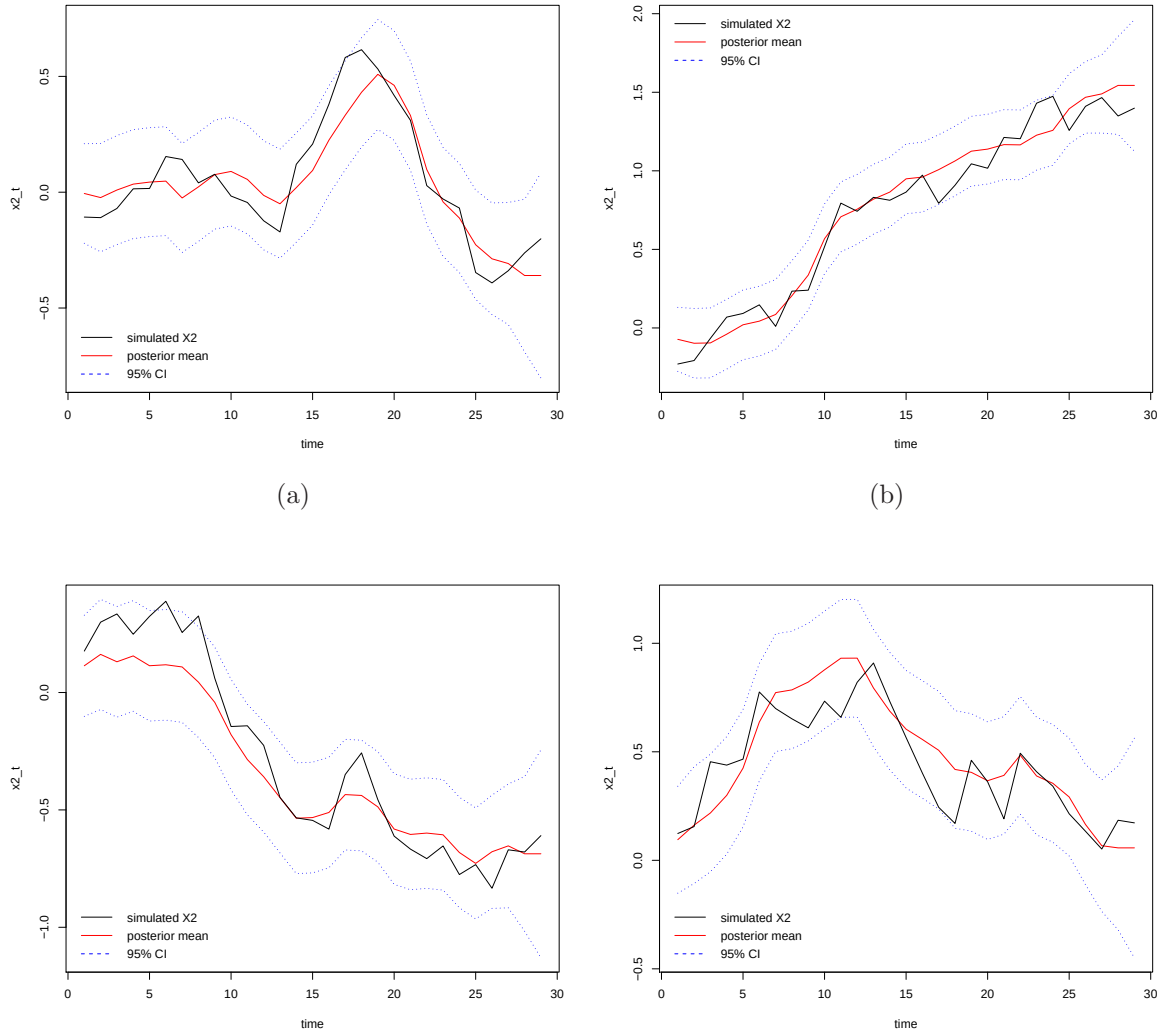


Figure 10: Simulated and predicted values (posterior mean and 95% credibility interval) for the states $x_{2,t}$ in the 20th area (a) and in its neighbors (b-d) in the second order spatio-temporal dynamic model.

5 Case studies

In this section we use some well known examples from the literature to illustrate how a relevant data analysis with DLMs can be performed using the INLA approach. The examples include dynamic models with Gaussian and Poisson observation densities, temporal trend and seasonality components as well as external covariates. When possible, comparison with results from the literature using other inference methods is provided. In order to facilitate understanding, relevant parts of the `R` code used in model fitting, for some representative examples, were included in an appendix at the end of this paper. The full code and data sets needed to fit all the examples in this section is also provided from the INLA web page (see the Supplementary material).

Example 7: UK Gas consumption

The first worked example to be analyzed corresponds to the quarterly UK gas consumption from 1960 to 1986, in millions of therms. Details on this dataset can be found in Durbin and Koopman (2001, p. 233). Following Dethlefsen and Lundbye-Christensen (2006) here we use the (base 10) logarithm of the UK gas consumption as response, which is assumed to be normally distributed and we fit a model with a first order polynomial trend (T_t) with time-varying coefficients and an unstructured seasonal component (S_t), also varying over time. Therefore, the observational and system equations are given by

$$y_t = \log_{10}(UKgas)_t = T_t + S_t + \nu_t, \qquad \nu_t \sim N(0, V), \qquad t = 1, \cdots, n \quad (20)$$

$$T_t = T_{t-1} + \beta_{t-1} + \omega_{1t}, \qquad \omega_{1t} \sim N(0, W_1), \qquad t = 2, \cdots, n \quad (21)$$

$$\beta_t = \beta_{t-1} + \omega_{2t}, \qquad \omega_{2t} \sim N(0, W_2), \qquad t = 2, \cdots, n \quad (22)$$

$$S_t = -(S_{t-1} + S_{t-2} + S_{t-3}) + \omega_{3t}, \qquad \omega_{3t} \sim N(0, W_3), \qquad t = 4, \cdots, n \quad (23)$$

Approximate Bayesian inference in this case was performed using a mixed approach. Firstly, we utilized an augmented structure merging the observational equation (20) with the polynomial trend (21). The slope and seasonal terms in equations (22) and (23), were modeled with standard model options from the INLA library. Specifically, for β_t we used a first order random walk model, while the INLA's seasonal model was used for S_t term.

The decomposition of the time series in trend, slope and seasonal components and its comparison with results obtained by Dethlefsen and Lundbye-Christensen (2006) using the `sspir` package, which uses an extended Kalman filtering approach to inference, are shown in Figure 11. Results with the two approaches were very similar. The amplitude of the seasonal term remains virtually constant from 1960–1971, then it increases during the period 1971-1979 and finally it stabilizes again.

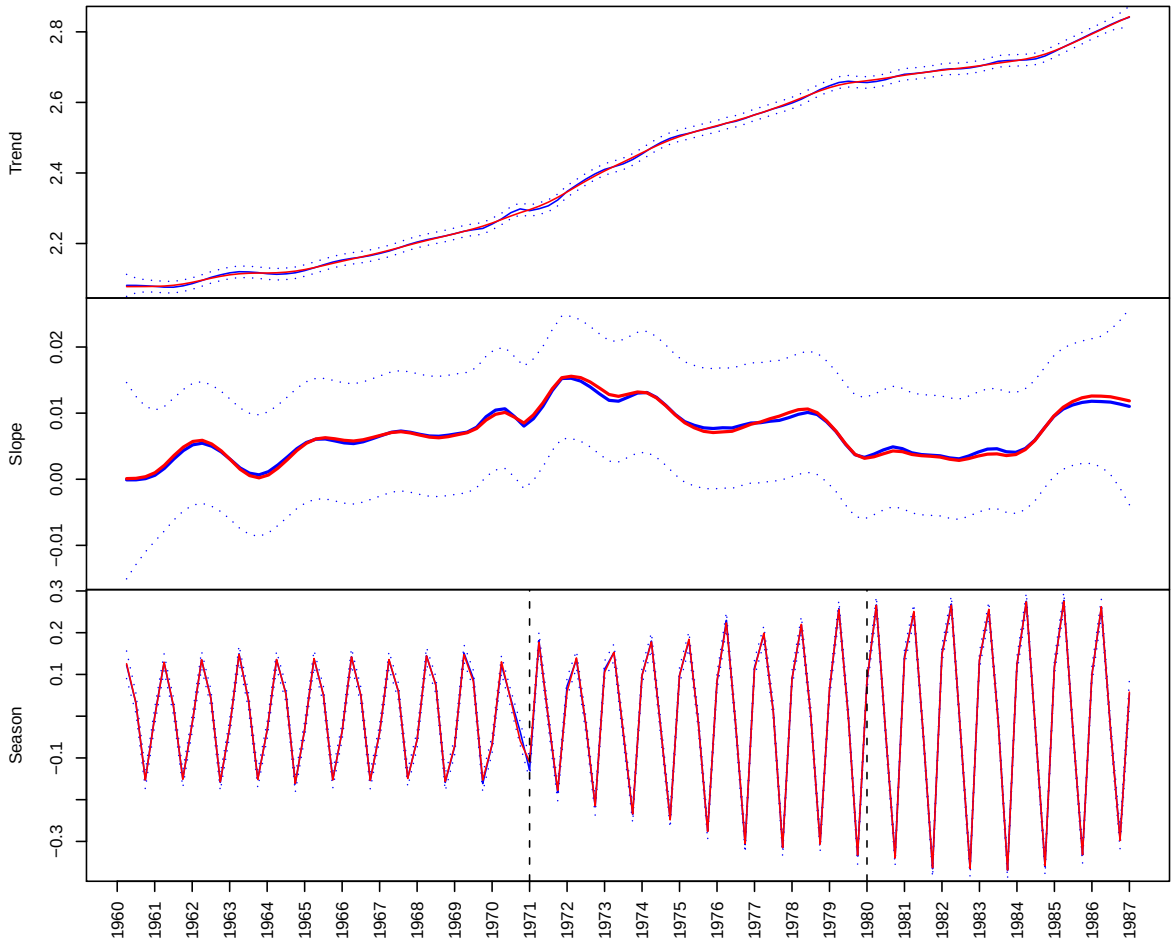


Figure 11: Time-varying trend, slope, and seasonal components in the UK gas consumption series obtained with INLA (blue lines) and with the `sspir` package (red lines). Dotted lines represent 95% credibility intervals for INLA estimates.

Example 8: Van drivers

This is a classical example of a generalized dynamic linear model. Here the response y_t corresponds to the monthly numbers of light goods van drivers killed in road accidents in Great Britain, from January 1969 to December 1984 (192 observations). A seat belt law (intervention) was introduced on January 31st, 1983. The interest is in quantifying the effect of the seat belt legislation law on the number of deaths. For further information about the data set see Harvey and Durbin (1986) and Durbin and Koopman (2000).

This dataset has been previously analised with INLA in a time series setting, assuming that the squared root of the counts, y_t, follows a Gaussian distribution. For details of this implementation see Martino and Rue (2010). Following Dethlefsen and Lundbye-Christensen (2006), here we use a generalized dynamic linear model for Poisson data with a 13-dimensional latent process, consisting of an intervention parameter, seat belt, changing value from zero to one in February 1983, a constant monthly seasonal term (S_t), and a temporal trend (T_t),

modeled as a random walk. The observational and system equations for this model are as follows

$$
\begin{aligned}
& y_t \sim Poisson(\mu_t) \\
\log(\mu_t) = \lambda_t &= T_t + \alpha * \text{seatbelt} + S_t, && t = 1, \cdots, n && (24)\\
T_t &= T_{t-1} + \omega_{1t}, && \omega_{1t} \sim N(0, W), \quad t = 2, \cdots, n && (25)\\
S_t &= -(S_{t-1} + \cdots + S_{t-11}), && t = 12, \cdots, n && (26)
\end{aligned}
$$

The trend and seasonal terms in linear predictor (24) can be directly modeled using existing first order random walk and seasonal model options from the INLA library, as shown in the following code:

```
i <- j <- 1:n             # indices for T_t and S_t
formula <- y ~ belt + f(i, model="rw1", param=c(1,0.0005), constr=F) +
               f(j, model="seasonal", season.length=12) -1
r <- inla(formula, data = data.frame(belt, i, j, y), family = "poisson")
```

The estimated trend and the effect of the seat belt intervention, as well as its comparison with results obtained with the `sspir` package (Dethlefsen and Lundbye-Christensen, 2006) can be displayed in Figure 12.

The posterior mean for α parameter in Eq. (24), which represents the effect of the seat belt law on the number of deaths, was -0.283; this corresponds to a reduction in the number of deaths of 24.63%. This result agrees with the corresponding values reported by Durbin and Koopman (2000) and Dethlefsen and Lundbye-Christensen (2006) for this parameter, which were -0.280 and -0.285, respectively.

Example 9: Mumps

The response y_t in this example is the monthly number of registered cases of mumps in New York City from January 1928 to June 1972. This data set was previously studied by Hipel and McLeod (1994). According to Dethlefsen and Lundbye-Christensen (2006), the incidence of mumps is known to show seasonal behavior and a variation in trend during the study period. For this data set we used a generalized dynamic linear model for Poisson observations, where the mumps incidence was modeled with a first order polynomial trend (T_t) with time-varying coefficients and a time-varying harmonic seasonal component (H_t) as suggested in Dethlefsen and Lundbye-Christensen (2006). The observational and system equations for this model are as follows

$$
\begin{aligned}
& y_t \sim Poisson(\mu_t) \\
\log(\mu_t) = \lambda_t &= T_t + H_t, && t = 1, \cdots, n && \\
T_t &= T_{t-1} + \beta_{t-1} + \omega_{1t}, && \omega_{1t} \sim N(0, W_1), \quad t = 2, \cdots, n && (27)\\
\beta_t &= \beta_{t-1} + \omega_{2t}, && \omega_{2t} \sim N(0, W_2), \quad t = 2, \cdots, n && (28)\\
H_t &= a_t \cos\left(\frac{2\pi}{12} t\right) + b_t \sin\left(\frac{2\pi}{12} t\right), && t = 1, \cdots, n && (29)\\
a_t &= a_{t-1} + \omega_{3t}, && \omega_{3t} \sim N(0, W_3), \quad t = 2, \cdots, n && (30)\\
b_t &= b_{t-1} + \omega_{4t}, && \omega_{4t} \sim N(0, W_4), \quad t = 2, \cdots, n && (31)
\end{aligned}
$$

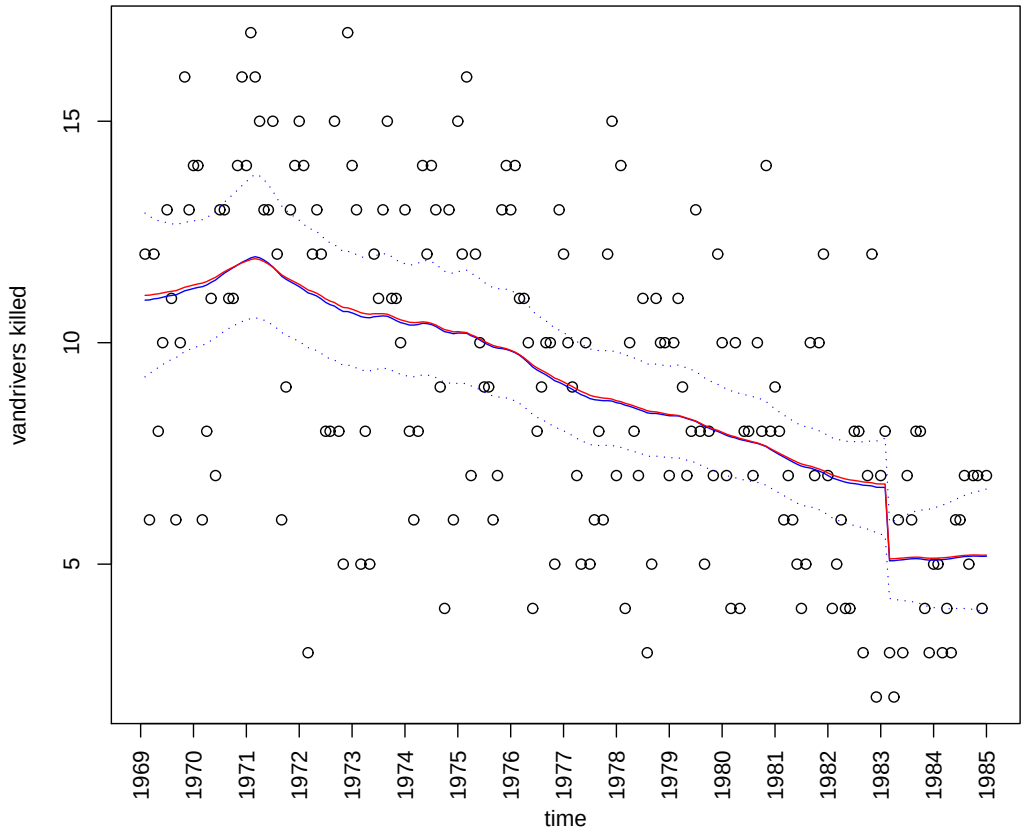


Figure 12: Number of vandrivers killed and estimated trend + intervention. Solid and dotted lines in blue correspond to the posterior mean and 95% credibility intervals, respectively, obtained with the INLA library. The line in red corresponds to the estimated trend + intervention with the `sspir` package.

As in example 7, a mixed approach is suitable here, where the polynomial trend in system equation (27) is merged with the observational equation, yielding an augmented model with two different likelihoods (Poisson for the n actual observations and Gaussian for the $n-1$ "pseudo" observations). The slope and seasonal terms in equations (28) to (31) follow a random walk evolution and were just modeled with a first order random walk model option from the INLA library. The relevant parts of the code to formulate and fit this model with INLA are presented in the supplementary material.

The comparison of results obtained with the INLA library and with the `sspir` package (Dethlefsen and Lundbye-Christensen, 2006) for the variation of mumps incidence are shown in Figure 13. They were very similar for the two approaches. According to Figure 13, seasonal pattern of incidence changes slowly, as can be seen in the decreasing behavior of the peak-to-trough ratio and peak location series. The location of the incidence's peak also changes from middle/late April in the beginning of the study period to late May in the last four years.

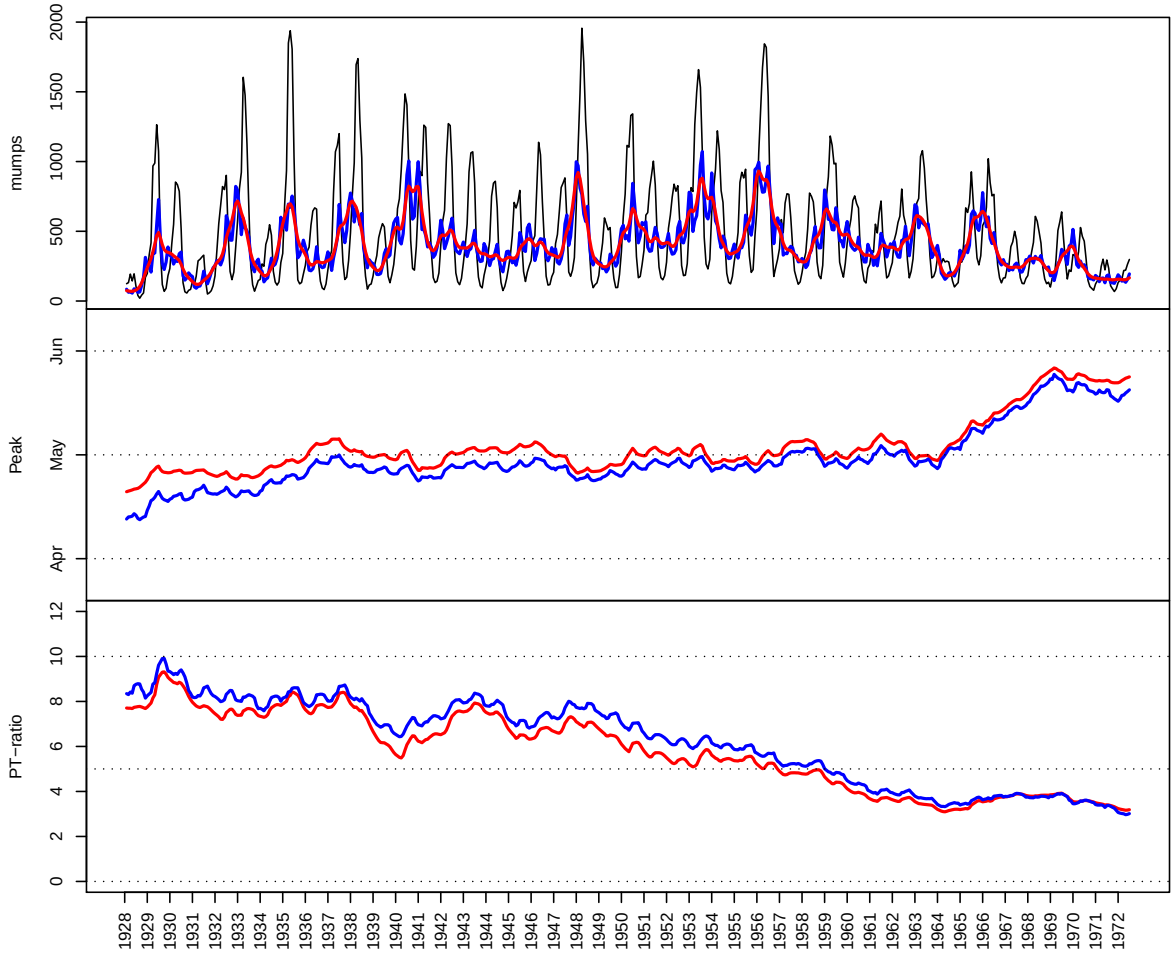


Figure 13: Comparison between INLA (blue lines) and `sspir` (red lines) results for the variation in the incidence of mumps in New York city from 1927 to 1972. The upper frame shows the observed number of cases jointly with the de-seasonalized trend. The location of the peak of the seasonal pattern is shown in the middle frame, while the lower frame is for the variation in the peak-to-trough ratio over the study period.

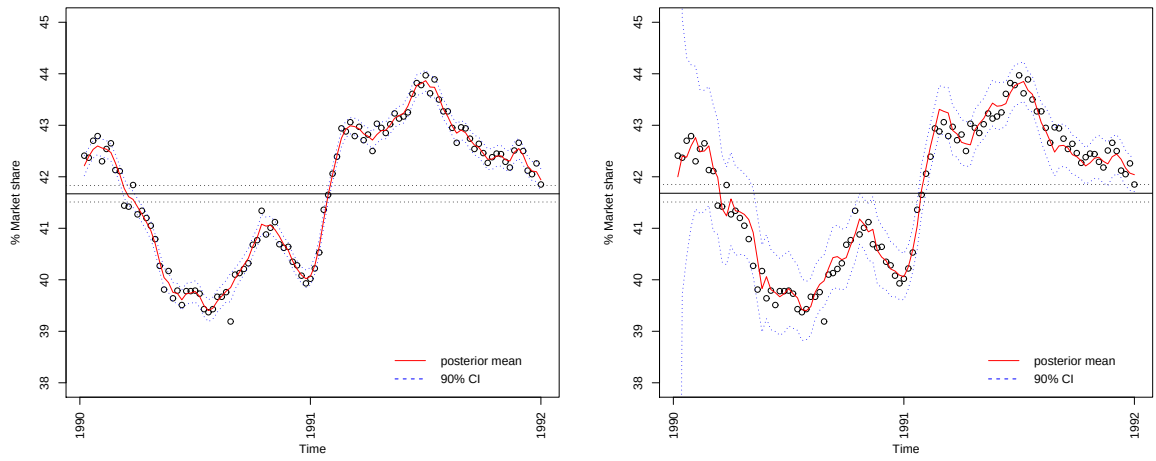
Example 10: Market share

In our last worked example we analize percent market share for a consumer product. This example was fully analized in Pole et al. (1994, Chapter 5). The model for market share utilize weekly available information for 1990 and 1991 on product price and measures of promotional activity. The objectives are to determine a model with good predictive power and to assess the importance of suggested explanatory variables. The response y_t is assumed to be Gaussian distributed. For this data set we use a dynamic regression model with three covariates, `price`, `prom` and `cprom`, where `price` is the measured price relative to a number competitor's average prices; `prom` and `cprom` are producer and competitor promotion indices. Following Pole et al. (1994), the level is modeled as fixed and the regression coefficients have a random walk evolution. One point identified as outlier on week 34 of 1990 and excluded from the analysis in Pole et al. (1994), was also excluded in our analysis for comparison purposes. The observational and

system equations for this model are as follows

$$
\begin{aligned}
y_t &= \alpha_t + \beta_{1t}\texttt{price}_t + \beta_{2t}\texttt{prom}_t + \beta_{3t}\texttt{cprom}_t + \nu_t, && \nu_t \sim N(0, V), && t = 1, \cdots, n \\
\beta_{1t} &= \beta_{1,t-1} + \omega_{1t}, && \omega_{1t} \sim N(0, W_1), && t = 2, \cdots, n \quad (32) \\
\beta_{2t} &= \beta_{2,t-1} + \omega_{2t}, && \omega_{2t} \sim N(0, W_2), && t = 2, \cdots, n \quad (33) \\
\beta_{3t} &= \beta_{3,t-1} + \omega_{3t}, && \omega_{3t} \sim N(0, W_3), && t = 2, \cdots, n \quad (34)
\end{aligned}
$$

The model was formulated in INLA considering a simple random walk evolution form for each regression coefficient. Figure 14 shows the predicted market share values and the estimated level obtained with the INLA library and that reported in Pole et al. (1994) using the Bats software.



(a) (b)

Figure 14: Observed and predicted values (posterior mean and 90% credibility interval) for the market share example using the INLA library (a) and the Bats software (b). Horizontal black lines in both plots indicate the estimated level with its 90% credibility interval.

Results of the week-by-week forecasts for the first five weeks of 1992 under four different scenarios, as considered in Pole et al. (1994), are shown in Table 1, using the INLA library and the Bats software. The four scenarios were:

1. `prom` and `cprom` indices set to 0,
2. `prom` set to its first five values of 1990 and `cprom` set to zero,
3. `prom` set to zero and `cprom` set to its first five values of 1990,
4. `prom` and `cprom` set to their first five values of 1990.

Relative price was maintained fixed in all cases at 0.206, the final value for 1991.

Table 1: Week-by-week forecasts for percent market share for the first five weeks of 1992 with INLA and with Bats software for four different promotion scenarios (see text for details). 0.05 and 0.95 quantiles are indicated by $0.05q$ and $0.95q$, respectively.

Week	INLA			Bats		
	mean	0.05q	0.95q	mean	0.05q	0.95q
	Scenario 1			Scenario 1		
1992/1	41.40	41.19	41.61	41.40	41.04	41.76
1992/2	41.40	41.18	41.61	41.40	41.04	41.76
1992/3	41.40	41.18	41.61	41.40	41.03	41.77
1992/4	41.40	41.18	41.62	41.40	41.03	41.77
1992/5	41.40	41.18	41.62	41.40	41.03	41.77
	Scenario 2			Scenario 2		
1992/1	41.36	41.14	41.59	41.36	41.00	41.73
1992/2	41.32	41.07	41.56	41.31	40.93	41.69
1992/3	41.27	41.00	41.54	41.25	40.86	41.65
1992/4	41.25	40.97	41.53	41.23	40.83	41.63
1992/5	41.22	40.92	41.52	41.19	40.77	41.61
	Scenario 3			Scenario 3		
1992/1	41.28	40.91	41.64	41.23	40.84	41.62
1992/2	41.27	40.90	41.66	41.22	40.82	41.62
1992/3	41.27	40.88	41.67	41.22	40.82	41.62
1992/4	41.27	40.87	41.68	41.22	40.81	41.63
1992/5	41.26	40.83	41.71	41.20	40.79	41.62
	Scenario 4			Scenario 4		
1992/1	41.24	40.86	41.63	41.19	40.79	41.59
1992/2	41.19	40.77	41.62	41.13	40.71	41.55
1992/3	41.14	40.69	41.62	41.07	40.63	41.51
1992/4	41.12	40.64	41.62	41.05	40.60	41.50
1992/5	41.08	40.55	41.63	41.00	40.53	41.47

Forecasting trend was similar under the two approaches for all scenarios considered.

6 Concluding remarks

In this paper we propose a computational framework to perform full approximate Bayesian inference in linear and generalized dynamic linear models based on the INLA approach. We illustrate our proposal through a series of simulated and worked examples ranging from simple univariate models to realistically complex spatio-temporal dynamic models. Our approach allows an easy specification of complex dynamic models in `R` using a formula language, as is routinely done with the most common linear and generalized linear models. The proposed

framework outperforms computational tools currently available in the literature of dynamic models in some important respects:

- Unknown precision parameters and its credibility intervals are straightforwardly estimated with INLA, jointly with the state parameters, unlike other approaches in the literature which do not estimate the unknown variance parameters automatically. The `sspir` package (Dethlefsen and Lundbye-Christensen, 2006), for example, requires the combination of numerical maximization algorithms with the output of the iterated extended Kalman smoother, while the Bats software (Pole et al., 1994) uses a discount factor approach to model unknown variances. The SsfPack package (Koopman et al., 1999) provides punctual estimates of the hyperparameters of state space models, but it requires further Monte Carlo simulation in order to get the confidence intervals through some bootstrap procedure as proposed, for example, in Franco et al. (2008).

- Our approach is able to deal with complex spatio-temporal observations in an easy way, as shown in examples 5 and 6. To the best of our knowledge there are no other computational tools currently available in the literature to deal with this kind of data in a framework of dynamic models.

The direct approximation of the posteriors for the states and hyperparameters performed by the INLA approach, following the framework proposed in this paper, allows a fast yet easier way to inference, even for complex state-space models. Our approach improves current recursive inference methods where estimation is performed with filtering and smoothing steps, based on the temporal structure of the observations, which become difficult to apply as the complexity of the models increases. From our view point the procedure for inference should not be confused with the dynamic nature of the model. In all the examples considered in this paper the observations were fixed, that is, all them have been already measured. Hence, disregarding (computationally) the temporal structure of the data, like INLA does, possibilitates their full Bayesian analysis, even for realistically complex cases.

Reasonable results were found for most of the examples considered in this paper using the default INLA values for the hyperprior parameters and for the initial values of these parameters. However, for some models this choice can greatly impact the final results. Therefore, a sensitivity analysis to address the choice of those values is highly recommended.

The extension of the proposed framework to consider multivariate observations is straightforward. The approach has also potential to be applied/extended to other classes of models such as models with errors in covariates. This is subject of current research.

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Supplementary Material:

The full R code to simulate and fit all the examples in this paper can be found at
<http://www.math.ntnu.no/inla/r-inla.org/papers/DLM_examples_11112010.R>.
For further details see also the technical report by Ruiz-Cárdenas et al. (2010).

A R script for fitting the second order dynamic spatio-temporal model in example 6

```
## simulating the data set

## Loading North Carolina's map (it has 100 areas)
require(spdep)
ncfile <- system.file("etc/shapes/sids.shp", package="spdep")[1]
nc <- readShapePoly(ncfile)

# building the structure matrix (C)
nc.nb <- poly2nb(nc)
d <- sapply(nc.nb, length)                    # vector with number of neighbors
C <- diag(d) - nb2mat(nc.nb, style="B")       # structure matrix
n <- length(d)

## simulated values for tau_i and phi_i (i=1,2,3)
tau <- c(30, 50, 50)
phi <- c(0.8, 0.9, 0.9)

# building the precision matrix
lamb.max <- max(eigen(C, only.values=TRUE)$values) # maximum eigenvalue of C matrix
Q1 <- (diag(n)-phi[1]/lamb.max*C)
Q2 <- (diag(n)-phi[2]/lamb.max*C)
Q3 <- (diag(n)-phi[3]/lamb.max*C)

myrmvnorm <- function(n, mu, S)
  sweep(matrix(rnorm(n*nrow(S)), n)%*%chol(S), 2, mu)

## defining the length of time series (number of years)
k <- 30

set.seed(1)

## simulating obsevational and innovation errors
w1 <- t(myrmvnorm(k, rep(0,n), solve(tau[1]*Q1)))
w2 <- t(myrmvnorm(k, rep(0,n), solve(tau[2]*Q2)))
w3 <- t(myrmvnorm(k, rep(0,n), solve(tau[3]*Q3)))

## generating the time series for observations and states
yy <- x1 <- x2 <- matrix(0, n, k)
x1[,1] <- w2[,1]
x2[,1] <- w3[,1]
for (i in 2:k) {
  x2[,i] <- x2[,i-1] + w3[,i]
  x1[,i] <- x1[,i-1] + x2[,i-1] + w2[,i]
}
yy <- x1 + w1

## defining the Cmatrix to use with model='generic1' for w1, w2 and w3
st.cmat <- kronecker(C, diag(k))
```

```
c.mat <- list(i=unlist(apply(st.cmat!=0, 1, which)),
              j=rep(1:nrow(st.cmat), rowSums(st.cmat!=0)),
              values=st.cmat[st.cmat!=0])

## building the augmented model
## ----------------------------
nd <- n*k
Y <- matrix(NA, nd*3-2*n, 3)
Y[1:nd             , 1] <- as.vector(t(yy))
Y[1:(nd-n) + nd    , 2] <- 0
Y[1:(nd-n) + 2*nd-n, 3] <- 0

## indices for the f() function
## ----------------------------
id1  <- (1:nd)[-((1:n)*k)]
id2  <- (1:nd)[-c(1,((1:(n-1))*k)+1)]
ix1  <- c(1:nd, id2, rep(NA,nd-n))                  ## indices for x1_t
ix1b <- c(rep(NA,nd), id1, rep(NA,nd-n))            ## indices for x1_{t-1}
wx1b <- c(rep(NA,nd), rep(-1,nd-n), rep(NA,nd-n))   ## weights for x1_{t-1}
ix2  <- c(rep(NA,nd),rep(NA,nd-n), id2)             ## indices for x2_t
ix2b <- c(rep(NA,nd), rep(id1, 2))                  ## indices for x2_{t-1}
wx2b <- c(rep(NA,nd), rep(-1,2*(nd-n)))             ## weights for x2_{t-1}
iw1  <- c(1:nd, rep(NA,2*(nd-n)))                   ## indices for w1_t
iw2  <- c(rep(NA,nd), id2, rep(NA,nd-n))            ## indices for w2_t
iw3  <- c(rep(NA,nd),rep(NA,nd-n), id2)             ## indices for w3_t

## formulating the model
## with default prior for precision parameters and initial \phi=0.5
formula1 <- Y ~ f(iw1, model="generic1", Cmatrix=c.mat, initial=c(log(50),0.5)) +
                f(ix1, model="iid", initial=-10, fixed=T) +
                f(ix1b, wx1b, copy="ix1") +
                f(ix2, model="iid", initial=-10, fixed=T) +
                f(ix2b, wx2b, copy="ix2") +
                f(iw2, model="generic1", Cmatrix=c.mat, initial=c(log(100),0.5)) +
                f(iw3, model="generic1", Cmatrix=c.mat, initial=c(log(100),0.5)) -1

## call to fit the model
## ---------------------
r1 <- inla(formula1, data = data.frame(ix1,ix1b,wx1b,ix2,ix2b,wx2b,iw1,iw2,iw3),
           family = rep("gaussian",3),
           control.data = list(list(initial=10, fixed=T),
           list(initial=10, fixed=T), list(initial=10, fixed=T)))
```

Model option "generic1", used in the formula for the specification of precision matrices, requires that the structure matrix $\boldsymbol{C}$ be passed as a file containing only the non-zero entries of the matrix. The file must contain three columns, where the first two ones contain the row and column indices of the non-zero entries of matrix $\boldsymbol{C}$, and the third column contain the corresponding non-zero values of structure matrix $\boldsymbol{C}$.

B R script for fitting the Mumps case study (example 9)

```
# building the augmented model
# ----------------------------
m <- n-1
Y <- matrix(NA, n+m, 2)
Y[1:n,             1] <- mumps
Y[1:m + n,         2] <- 0

## indices for the INLA library
# ---------------------------
i       <- c(1:n, 2:n)               # indices for T_t
j       <- c(rep(NA,n), 1:m)         # indices for T_{t-1}
weight1 <- c(rep(NA,n), rep(-1,m))   # weights for T_{t-1}
l       <- c(rep(NA,n), 1:m)         # indices for \beta_{t-1}
weight2 <- c(rep(NA,n), rep(-1,m))   # weights for \beta_{t-1}
w1      <- c(rep(NA,n), 2:n)         # indices for w_{1,t}
q       <- c(1:n, rep(NA,m))         # indices for a_t
cosine  <- c(cosw,rep(NA,m))         # weights for a_t
rr      <- c(1:n, rep(NA,m))         # indices for b_t
sine    <- c(sinw,rep(NA,m))         # weights for b_t

# formulating the model
# ---------------------
formula = Y ~ f(q, cosine, model="rw1",param=c(1,0.01),initial=4, constr=F) +
              f(rr, sine, model="rw1",param=c(1,0.01),initial=4, constr=F) +
              f(l, weight2, model="rw1",param=c(1,0.2),initial=4, constr=F) +
              f(i, model="iid", initial=-10, fixed=TRUE) +
              f(j, weight1, copy="i") + f(w1, model ="iid") -1

# call to fit the model
# ---------------------
r <- inla(formula, data = data.frame(cosine,sine,i,j,weight1,l,weight2,q,rr,w1),
          family = c("poisson","gaussian"),
          control.data = list(list(),list(initial=10, fixed=TRUE)))
```

It is important to note here that in the formulation of this model, the seasonal terms, following a first order random walk process, must be declared first in the formula to be passed to the `inla` function, followed by the terms in the equations that forms the augmented model. Otherwise the INLA library can make a wrong interpretation of the indices for these terms, which can lead to misleading results.

C Maps of simulated and predicted values for observations and states at some instant times in examples 5 and 6

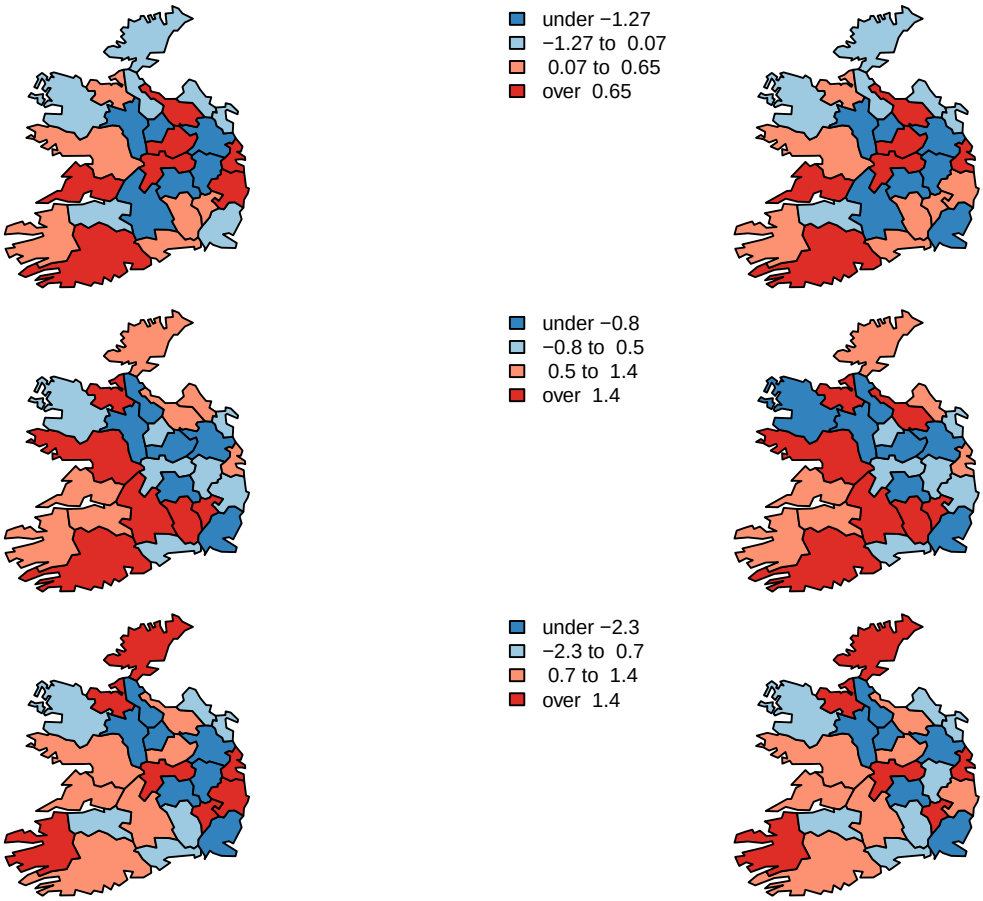

Figure 15: Maps of simulated (left) and predicted values (right) for observations, $\boldsymbol{y}_t$, at times 34, 64 and 91 in example 5.

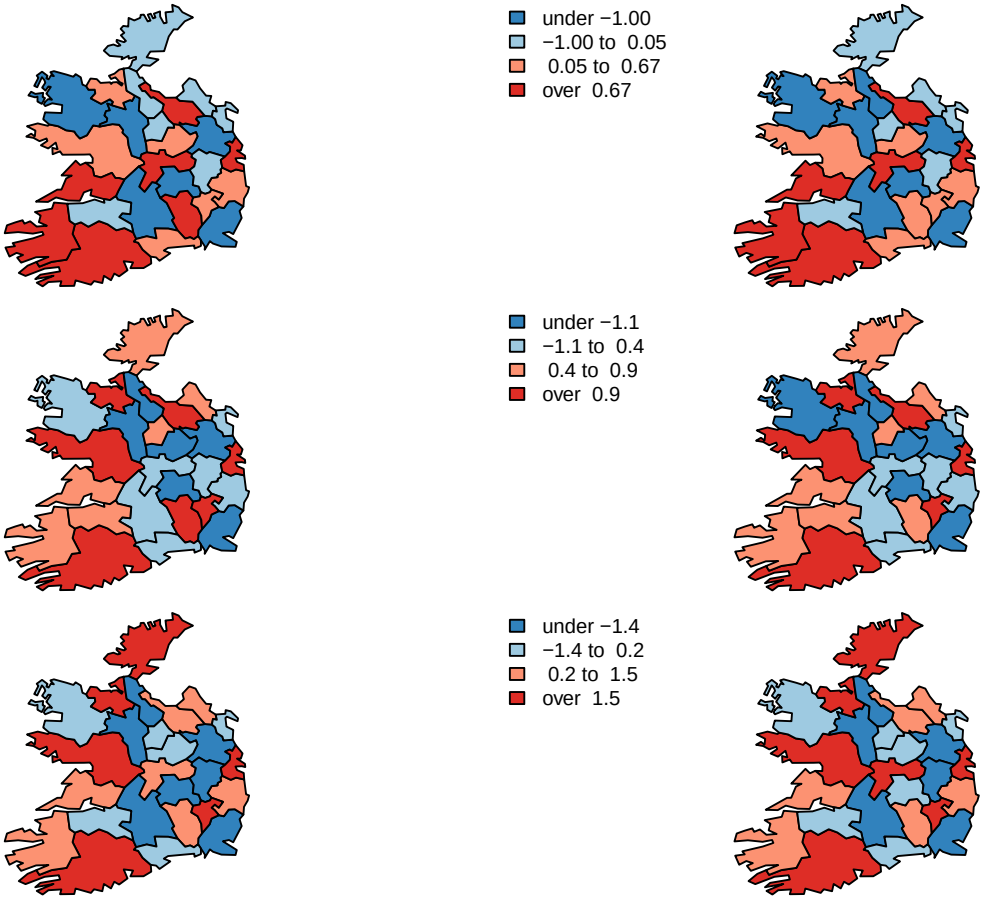

Figure 16: Maps of simulated (left) and predicted values (right) for state vector, $\boldsymbol{x}_t$, at times 34, 64 and 91 in example 5.

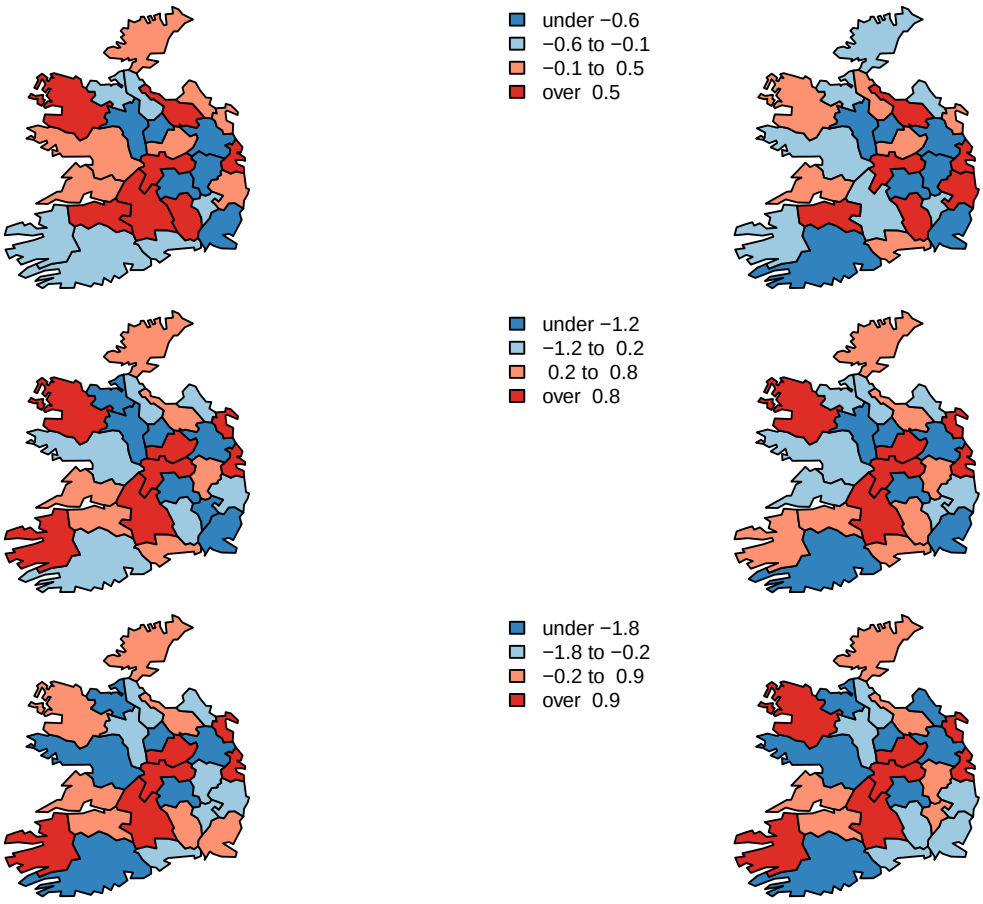

Figure 17: Maps of simulated (left) and predicted values (right) for state vector, $\boldsymbol{z}_t$, at times 34, 64 and 91 in example 5.

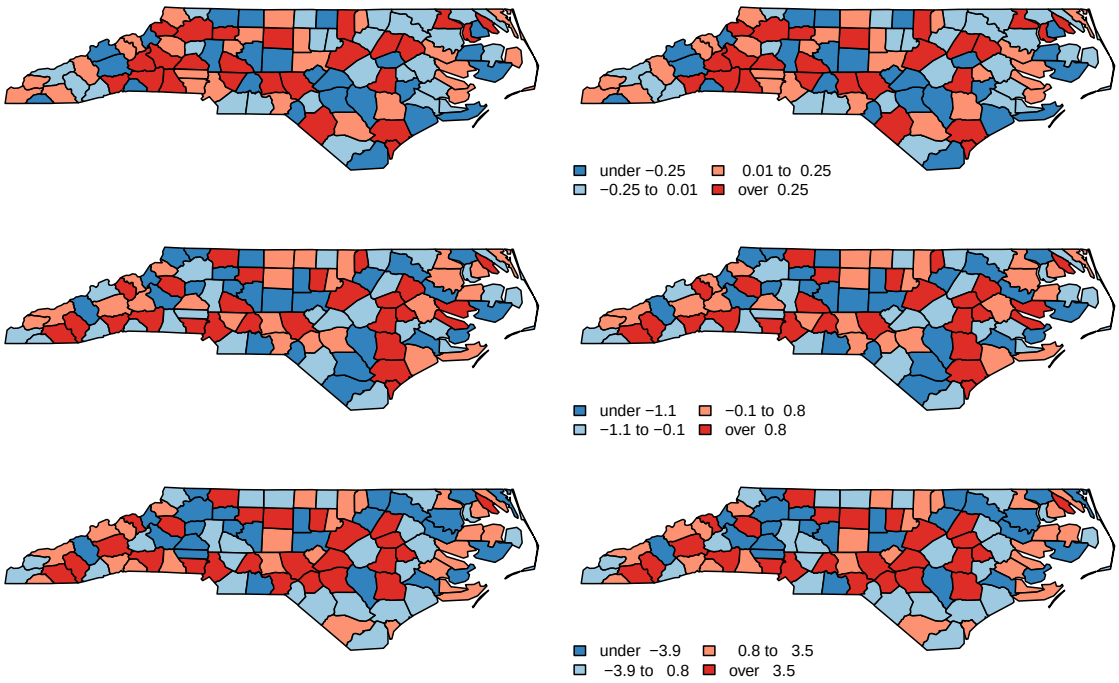

Figure 18: Maps of simulated (left) and predicted values (right) for observations, $\boldsymbol{y}_t$, at times 2, 7 and 15 in example 6.

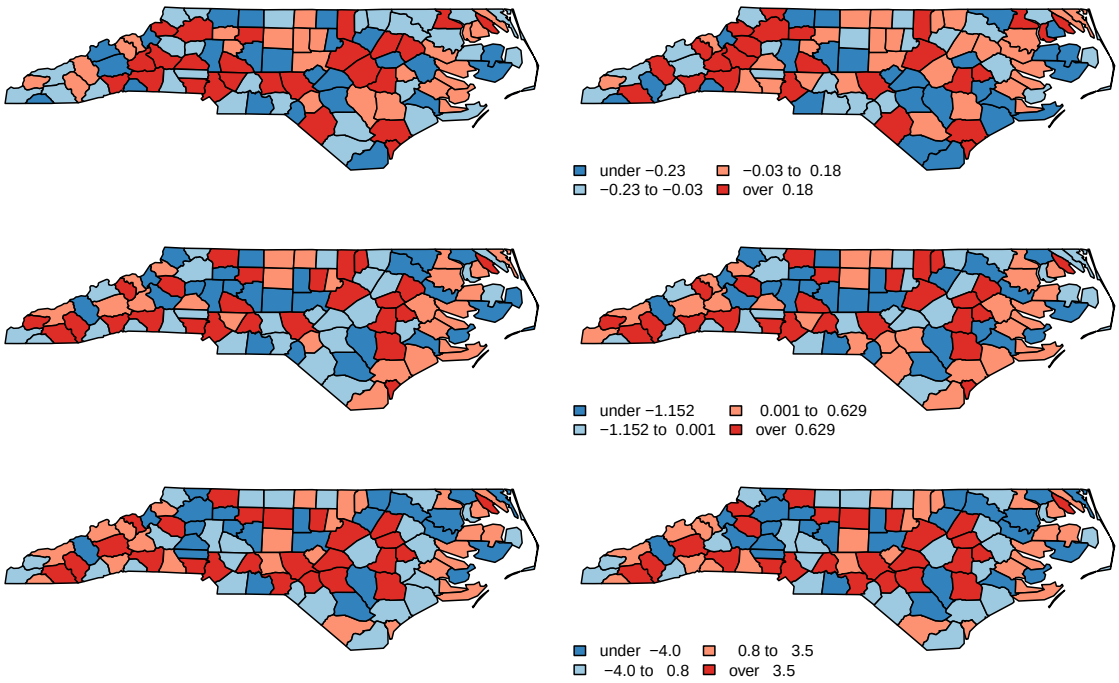

Figure 19: Maps of simulated (left) and predicted values (right) for X_1 state vector at times 2, 7 and 15 in example 6.

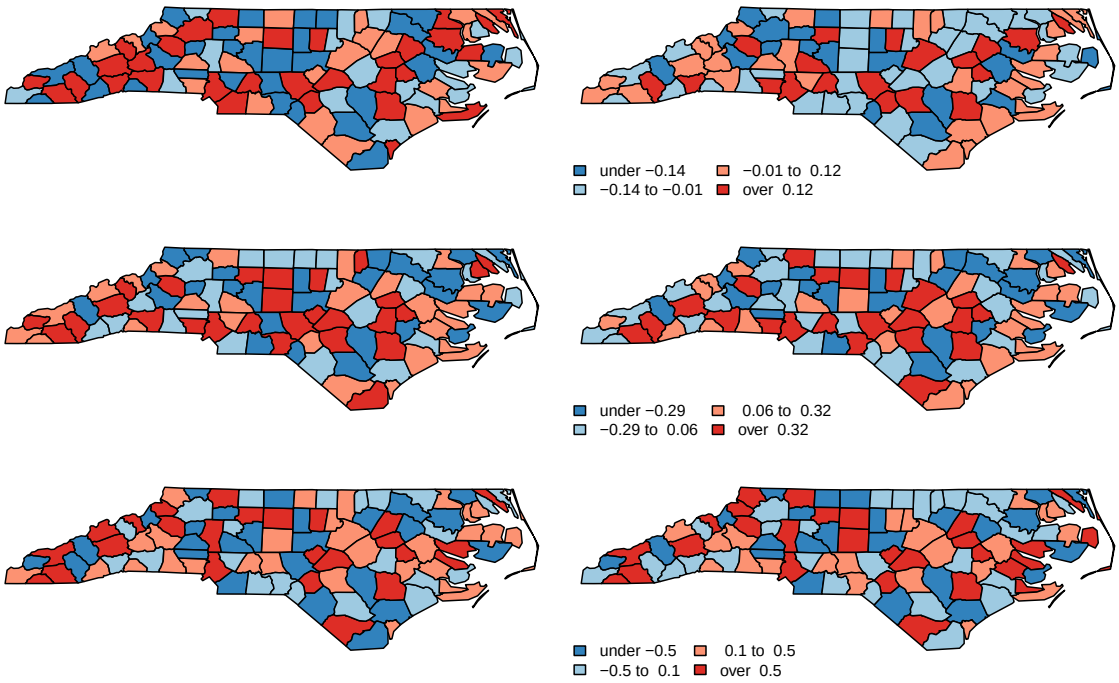

Figure 20: Maps of simulated (left) and predicted values (right) for X_2 state vector at times 2, 7 and 15.