Physics 106 Lecture 9

Newton's Law of Gravitation

SJ 7th Ed.: Chap 13.1 to 2, 13.4 to 5

- Historical overview
- Newton's inverse-square law of gravitation
  - Force
  - Gravitational acceleration "g"
- Superposition
- Gravitation near the Earth's surface
- Gravitation inside the Earth (concentric shells)
- Gravitational potential energy
  - Related to the force by integration
  - A conservative force means it is path independent
  - Escape velocity

Gravitation – Introduction

Why do things fall?
Why doesn't everything fall to the center of the Earth?
What holds the Earth (and the rest of the Universe) together?
Why are there stars, planets and galaxies, not just dilute gas?

- **Aristotle** – Earthly physics is different from celestial physics
- **Kepler** – 3 laws of planetary motion, Sun at the center. Numerical fit / no theory
- **Newton** – English, 1665 (age 23)
  - Physical Laws are the same everywhere in the universe (same laws for legendary falling apple and planets in solar orbit, etc).
  - Invented differential and integral calculus (so did Liebnitz)
  - Proposed the law of "universal gravitation"
  - *Deduced* Kepler's laws of planetary motion
  - Revolutionized "Enlightenment" thought for 250 years
    - Reason ←→ prediction and control, versus faith and speculation
    - Revolutionary view of clockwork, deterministic universe (now dated)
- **Einstein** - Newton + 250 years (1915, age 35)
  - General Relativity – mass is a form of concentrated energy ($E=mc^2$), gravitation is a distortion of space-time that bends light and permits black holes (gravitational collapse).
- **Planck, Bohr, Heisenberg, et al** – Quantum mechanics (1900–27)
  - Energy & angular momentum come in fixed bundles (quanta): atomic orbits, spin, photons, etc.
  - Particle-wave duality: determinism breaks down.
  - There should be a "graviton" (quantum gravity particle). No progress yet.
- **Current Issues**
  - Dark Matter – Luminous mass of galaxies is too small to explain stars' orbits.
  - Dark Energy and inflation – Possible anti-gravity at long range fuels accelerating expansion of the universe, and also early Big Bang.

Gravitation – Basic Concepts

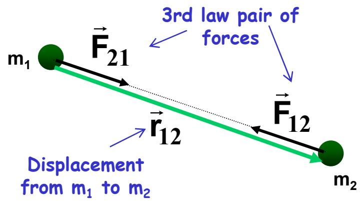
Every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them

For a pair of masses: $$|F_{12}| = \frac{G\, m_1 m_2}{r_{12}^2}$$

- **Inertial mass:**
  - Measures resistance to acceleration, e.g.: F = ma.
  - Measures **response** to gravitational acceleration - *a field.*
- **Gravitational mass:**
  - Mass is the **source** of the gravitational acceleration field - always
  - Gravitational mass measures **strength** of a gravitational field produced.
- **Duality/Equivalence:**
  - Every bit of mass acts as both inertial and gravitational mass with the **same value of m in each role.**
- **Gravitational Force**
  - No contact needed: "*action at a distance*".
  - Cannot be screened out, unlike electrical forces.
  - Always attractive unlike electrical forces (except for "dark energy", maybe).
  - Very weak compared to electrical forces. Too small to notice between most human-scale objects and smaller (e.g., p^+ and e^-).

Gravitation is long range, has cosmological effects over long times. But it is a weak force on the human scale.

Newton's Law of Universal Gravitation



Force on m₂ due to m₁ — Gravitational Constant $G = 6.67\times10^{-11}\ \mathrm{Nm^2/kg^2}$

$$\vec{F}_{12} = -\frac{G\, m_1 m_2}{r_{12}^2}\,\hat{r}_{12}$$

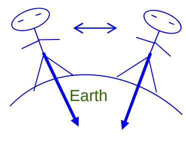
Unit vector along r_{12}

$$\hat{r}_{12} \equiv \frac{\vec{r}_{12}}{|\vec{r}_{12}|} = -\hat{r}_{21}$$

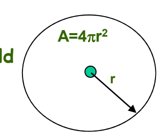
- Force is between pairs of point masses
- Symmetric in m_1 & m_2 so $F_{12} = -F_{21}$
- Not screened or affected by other bodies
- Easy to miss between masses near a third large mass (e.g. on Earth surface)

IF m_2 IS REMOVED, IS ANYTHING AT POINT 2 DIFFERENT BECAUSE m_1 IS STILL AT POINT 1?
FIELD transmits the force (no contact, action at a distance)

Acceleration field $$\vec{F}_{12} \equiv m_2 \vec{g}_{12} \Rightarrow \vec{g}_{12} = -\frac{G m_1}{r_{12}^2}\hat{r}_{12}$$



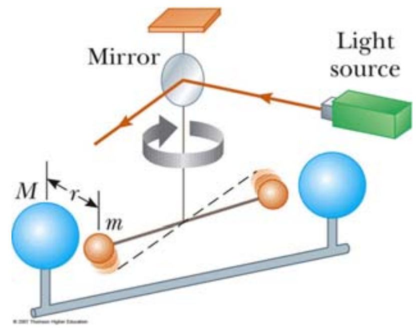
Inverse square law: as sphere grows field (or force) x area is constant



field at location of m_2 due to m_1

Finding the Value of G

- Henry Cavendish first measured G directly (1798)
- Two masses *m* are fixed at the ends of a light horizontal rod (torsion pendulum)
- Two large masses M were placed near the small ones
- The angle of rotation was measured
- Results were fitted into Newton's Law



$G=6.67\times10^{-11}\ N.m^2/kg^2$

G versus g:

- G is the universal gravitational constant, the same everywhere
- $g = a_g$ is the acceleration due to gravity. It varies by location.
- $g = 9.80\ m/s^2$ at the surface of the Earth

Why was the Law of Gravitation not obvious (except to Newton).
How big are gravitational forces between ordinary objects?

$$|F_{12}| = \frac{G\, m_1 m_2}{r_{12}^2}$$

1 Newton is about the force needed to support 100 grams of mass on the Earth

m_1	m_2	r_{12}	F_{12}
1 kg a liter of soda	1 kg sandwich	1 meter	6.67×10^{-11} N.
100 kg a person	100 kg another person	1 meter	6.67×10^{-7} N.
10^6 kg a ship	10^6 kg another ship	100 meters	0.67 N. still hard to detect

Conclusion:

- G is very small, so...need huge masses to get perceptible forces

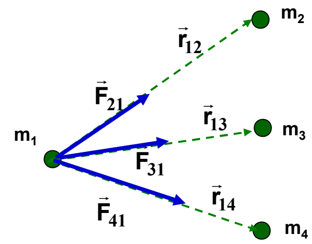
Does gravitation play a role in atomic physics & chemistry?

9.1×10^{-31} kg electron	1.7×10^{-27} kg proton	5×10^{-11} meter orbit radius	$\boxed{4\times10^{-47}\ N.}$

Superposition:

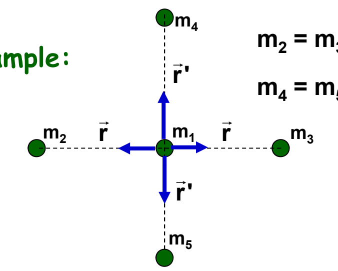
The net force on a point mass when there are many others nearby is the vector sum of the forces taken one pair at a time

$$\boxed{\vec{F}_{on\,1} = \sum_{i \neq 1} \vec{F}_{i,1} = \vec{F}_{2,1} + \vec{F}_{3,1} + \vec{F}_{4,1}}$$



All gravitational effects are between pairs of masses. No known effects depend directly on 3 or more masses.

Example:



$m_2 = m_3$

$m_4 = m_5$

$\vec{F}_{on\ m_1} = m_1 \vec{g}_{at\ m_1} = 0$ by symmetry

$$g_{i,1} \equiv a_{gi,1} = -\frac{Gm_i}{r_{1,i}^2}$$

For continuous mass distributions, integrate

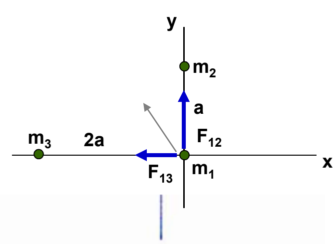
$$\boxed{\vec{F}_{on\,1} = \int_{\text{mass dist}} d\vec{F}_1}$$

Numerical Example:

Find net force on m_1 due to m_2 and m_3

- Use superposition
- Basic forces at right angles

$$\vec{F}_{net,j} = \sum_{i \neq j} \vec{F}_{i,j}$$



$F_{12} = G\frac{m_1 m_2}{a^2}$ $\quad F_{13} = G\frac{m_1 m_3}{(2a)^2}$

$m_1 = 6$ kg
$m_2 = 4$. kg
$m_3 = m_2$
$a = 2$ cm $= .02$ M.

$(m_3 = m_2)$ $\quad F_{13} = G\frac{m_1 m_2}{4a^2} = \frac{F_{12}}{4}$.

$F_{12} = \frac{6.67 \times 10^{-11} \times 6 \times 4}{(.02)^2} = 4.0 \times 10^{-6}$ N.

$F_{13} = 1.0 \times 10^{-6}$ N.

DIRECTIONS $\quad F_{12} \perp F_{13}$.

$|F_{NET}| = (F_{12}^2 + F_{13}^2)^{\frac{1}{2}} = 10^{-6} \times (1^2 + 4^2)^{\frac{1}{2}} = \sqrt{17} \times 10^{-6} =$

$\boxed{|F_{NET}| = 4.12 \times 10^{-6}\ N}$

ANGLE θ WITH +X AXIS ; $\quad \theta = \tan^{-1}\left(\frac{F_y}{F_x}\right) = \tan^{-1}\left(\frac{+F_{12}}{-F_{13}}\right) = \tan^{-1}(-4)$

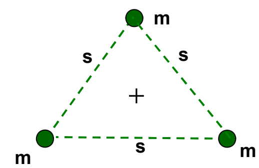
QUADRANT II

$\boxed{\theta = \begin{cases} -75.9° & ✗ \\ +104° & ✓ \end{cases}}$

Superposition for a triangle

- 9.1. In the sketch, equal masses are placed at the vertices of an equilateral triangle, each of whose sides equals “s”. In which direction would the top-most chunk of mass try to accelerate (ignore the Earth’s gravity) with the bottom two held in place?

A) ↖ B) ↗ C) ↓ D) ↑ E) $a = 0$

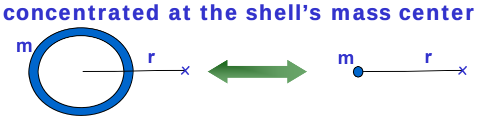


- 9.2. Another chunk of mass is placed at the exact center of the triangle in the sketch. In which direction does it tend to accelerate?

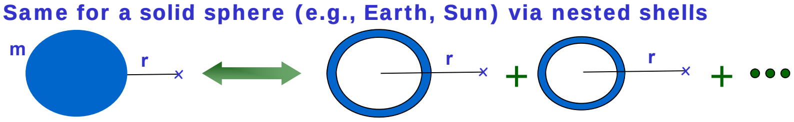
$$\boxed{\vec{F}_{net,j} = \sum_{i \neq j} \vec{F}_{i,j}}$$

Shell Theorem: superposition for masses with spherical symmetry

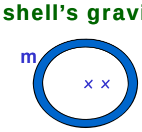
1. For a test mass **OUTSIDE** of a **uniform spherical shell** of mass, the shell’s gravitational force (field) is the same as that of a **point mass** concentrated at the shell’s mass center



Same for a solid sphere (e.g., Earth, Sun) via nested shells

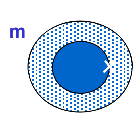


2. For a test mass **INSIDE** of a **uniform spherical shell** of mass, the shell’s gravitational force (field) is zero



- Obvious by symmetry for center
- Elsewhere, need to integrate over sphere

3. For a solid sphere, the force on a test mass INSIDE includes only the mass closer to the CM than the test mass.

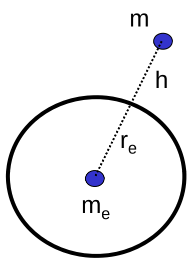


- Example: On surface, measure acceleration g a distance r from center
- Example: Halfway to center, $a_g = g/2$

$$V_{sphere} = \frac{4}{3}\pi r^3$$

Gravitation near the surface of the Earth:

What do "g" and "weight = mg" mean?



- Earth's mass acts as like a point mass m_e at the center (by the Shell Theorem)
- Radius of Earth = r_e
- Object with mass m is at altitude h… …above the surface, so $r = r_e + h$
- Weight $W = ma_g$ with acceleration given by Newtons Law of Gravitation (any altitude)

$$\vec{a}_g = -\frac{G\,m_e}{(r_e + h)^2}\hat{r} \quad \text{at any altitude}$$

When m is "on or near the surface:"

$$h << r_e \quad \text{or, in other words} \quad r_e + h \approx r_e$$

$$g \equiv \left|\vec{a}_g\right| \cong \frac{G\,m_e}{r_e^2} \quad \text{where} \quad g \cong 9.8\ \text{m/s}^2$$

Example: Use the above to find the mass of the Earth, given:

- $g = 9.8\ \text{m/s}^2$ (measure in lab)
- $G = 6.67\text{x}10^{-11}\ \text{m}^3/\text{kg.s}^2$ (lab)
- $r_e = 6370$ km (average - measure)

$$\Rightarrow \quad m_e = \frac{g\,r_e^2}{G} = \frac{9.8 \times 6370 \times 10^3}{6.67 \times 10^{-11}}$$

$$m_e = 5.98 \times 10^{24}\ \text{kg}$$

Altitude dependence of *g*

- Weight decreases with altitude h
- The work needed to increase Δh declines, since weight decreases

TABLE 13.1

Free-Fall Acceleration *g* at Various Altitudes Above the Earth's Surface

Altitude *h* (km)	*g* (m/s^2)
1 000	7.33
2 000	5.68
3 000	4.53
4 000	3.70
5 000	3.08
6 000	2.60
7 000	2.23
8 000	1.93
9 000	1.69
10 000	1.49
50 000	0.13
∞	0

Free fall acceleration

9.3 What is the magnitude of the free-fall acceleration at a point that is a distance $2r_e$ above the surface of the Earth, where r_e is the radius of the Earth?

a) 4.8 m / s²
b) 1.1 m / s²
c) 3.3 m / s²
d) 2.5 m / s²
e) 6.5 m / s²

$$a_g = \frac{G\, m_e}{(r_e + h)^2} \quad \text{at any altitude}$$

$$g = 9.8\ \mathrm{m/s^2}$$

Gravitational "field" transmits the force

▪ A piece of mass m_1 placed somewhere creates a "gravitational field" that has values described by some function $g_1(r)$ everywhere in space.
▪ Another piece of mass m_2 feels a force proportional to $g_1(r)$ and in the same direction, also proportional to m_2.

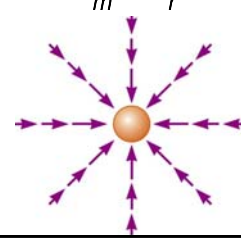
$$\boxed{\vec{F}_{12} = m_2\vec{g}_1(r) = -\frac{G\, m_1 m_2}{r_{12}^2}\hat{r}_{12}}$$

Concepts for g-fields:

- No contact needed: "*action at a distance*".
- *Acceleration Field* created by gravitational mass transmits the force as a distortion of space that another (inertial) mass responds to.
- Gravitational field is "Conservative" (i.e. can have a potential energy function).
- g cannot be screened out, unlike electrical fields.
- g is always attractive (except cosmologically, maybe), unlike electrical fields.

▪ The field $g_1(r)$ is the gravitational force per unit mass created by mass m_1, present at all points whether or not there is a test mass m_2 located there
▪ The gravitational field vectors point in the direction of the acceleration a particle would experience if placed in the field at each point.

$$\vec{g} = \frac{\vec{F}_g}{m} = -\frac{GM}{r^2}\hat{r}$$



Field lines help to visualize strength and direction.

- close together → strong field,
- direction → force on test mass

Gravitational Potential Energy ΔU

$\Delta U = mg\Delta h$ fails unless a_g is constant – force depends on r

Definition: Work done by gravity on a test mass m moved through dr $\rightarrow$ $dW \equiv \vec{F}_g \bullet d\vec{r} = -dU_g$ $\leftarrow$ potential energy change

force varies along path

displacement

Choose: gravitational potential = zero at R = ∞ where the force = zero

i.e.: U(R) = → 0 as R → ∞.

Mass M creates the g-field. Integrate along a radial path from R to infinity

$$\Delta U \equiv -\int_R^\infty dW = -\int_R^\infty \vec{F}_g \bullet d\vec{r} = -\int_R^\infty \frac{GmM}{r^2}dr$$

$$= -GmM\int_R^\infty \frac{dr}{r^2} = \frac{GmM}{r}\Big|_R^\infty$$

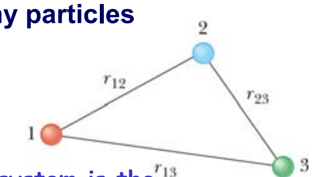
$$\boxed{U_g = -\frac{GmM}{R}}$$

Note: NOT R^2

- The gravitational potential energy between any two particles varies as 1/R. The force varies as $1/R^2$
- The potential energy is negative because the force is attractive and we chose the potential energy to be zero at infinite separation.
- Another form of energy (external work or kinetic energy) is converted when the potential energy and separation between masses increase.

Gravitational Potential Energy

Mutual potential energy of a system of many particles



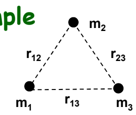
$$U_{total} = \sum_{\text{all pairs}} U_{ij}(r_{ij})$$

shared, sum over all possible pairings

The total gravitational potential energy of the system is the sum over all pairs of particles.

Gravitational potential energy obeys the superposition principle

Example



3 possible pairs

$$U_{total} = U_{12} + U_{13} + U_{23} = -\left\{\frac{Gm_1m_2}{r_{12}} + \frac{Gm_1m_3}{r_{13}} + \frac{Gm_2m_3}{r_{23}}\right\}$$

The slope of the potential energy curve is related to the force.

Recall: $dW \equiv \vec{F}_g \bullet d\vec{r} = -dU_g$

force due to gravitation $\rightarrow$ $\vec{F}_g = -\frac{dU_g}{dr}$ $\leftarrow$ Slope of potential energy function (derivative, gradient)

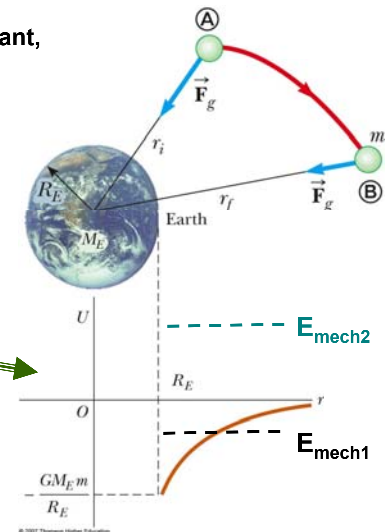
minus

Gravitational Potential Energy, cont

- As a particle moves from A to B, its gravitational potential energy changes by ΔU
- But the mechanical energy remains constant, independent of path, so long as no other force is acting

$$E_{mech} = K + U(r)$$

- Graph of the gravitational potential energy *U* versus *r* for an object above the Earth's surface
- The potential energy goes to zero as *r* approaches infinity
- The mechanical energy may be positive, negative, or zero



Conservation of mechanical energy with gravitation

- E_{mech} determines whether motion is bound, free, or at escape threshold

E_{mech} is constant

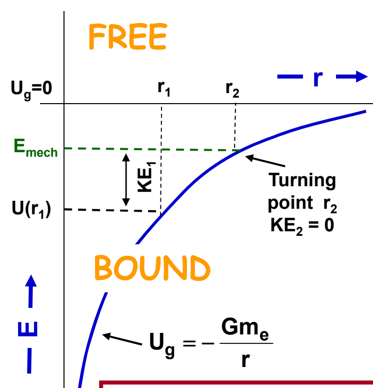
$$E_{mech} = K + U_g(r)$$

$$U_g(r) = -\frac{Gm_e m}{r} \quad \text{always negative}$$

For $E_{mech} < 0$, particle is bound and cannot escape. It cannot move beyond a turning point (e.g., r_2)

For $E_{mech} > 0$, particle is free. It can reach r = infinity and still have some KE left

$E_{mech} = 0$ is the escape condition. A particle at any location r would need at least KE = $-U_g(r)$ to move off to the right and never return.



$$\vec{F}_g = -\frac{dU_g}{dr}$$

The tangent to the potential energy graph measures the gravitational force

How much energy does it cost per kilogram to escape completely from the surface of the Earth?

$$U_g = -\frac{Gm_e x1}{r_e} = 17.4 \text{ KWH/kg}$$

Escape speed formula – derivation and example

Escape condition for object of mass m from the surface:

$$E_{mech} \equiv K + U_g = 0 = \frac{1}{2}mv_{esc}^2 - \frac{Gm_e m}{r_e}$$

The mass m cancels:

$$0 = \frac{1}{2}v_{esc}^2 - \frac{Gm_e}{r_e} \Rightarrow \boxed{v_{esc} = \sqrt{\frac{2Gm_e}{r_e}} = \sqrt{2gr_e}}$$

Example: Find the escape speed from the Earth's surface

$g = 9.8\ m/s^2$ $r_e = 6370$ km

$$v_{esc} = \sqrt{2x9.8x6.370x10^6} = 11{,}100\ m/s \approx 7\ mi/s$$

Example: Jupiter has 300 times the Earth's mass and 10 times the Earth's diameter. How does the escape velocity for Jupiter compare to that for the Earth?

$$v_{jup} = \sqrt{\frac{2.G.m_{jup}}{r_{jup}}} = \sqrt{\frac{2.G.m_e.300}{r_e.10}} = \sqrt{30}\ v_e \approx 5.5\ v_e$$

How much does g vary across objects near the earth's surface?

Fractional change in g over range dr = dg / g

$$g = G\frac{m_e}{r^2} \qquad r \approx r_e = 6.37x10^6\ m.$$

dr ~ size of object

Optional Topic

$$\frac{dg}{dr} = -\frac{2Gm_e}{r^3} \qquad dg = -2g\frac{dr}{r_e}$$

$$\therefore \left|\frac{dg}{g}\right| = \frac{2dr}{r_e}$$

depends on ratio of object size to Earths radius

EXAMPLES

OBJECT	SIZE	%VARIATION
PERSON	2 m	6.3×10^{-5} %
OFFICE TOWER	800 m	9×10^{-3} %
EVEREST (30,000 ft)	9000 m	2.9×10^{-1} %

conclusion: can treat g as constant

Spring 2005

When is it valid to approximate U_g by $mg\Delta h$?

FOR $\Delta h << r_e$, SHOW $\Delta U_g \simeq mg\Delta h$.

Optional Topic

PROOF $\Delta U_g = U(r_e+\Delta h) - U(r_e)$

$$= -\frac{GM_em}{r_e+\Delta h} + \frac{GM_em}{r_e}$$

$$= \frac{GM_em}{r_e}\left\{1 - \frac{1}{1+\Delta h/r_e}\right\}$$

APPROXIMATION

FOR $|x|<1$ $\quad \frac{1}{1+x} \simeq 1 - x + \cancel{x^2/2!} - \cdots$ neglect

$$\Delta U_g \approx \frac{GM_em}{r_e}\left(1-1+\Delta h/r_e\right)$$

$$\simeq \frac{GM_e}{r_e^2} m \Delta h = mg\Delta h\,!$$

Answer: when $\Delta h << r_e$