1. (De Mere problems) Calculate the probabilities that

1) by throwing dice 4 times, one gets at least one 6 (ans: 0.518)

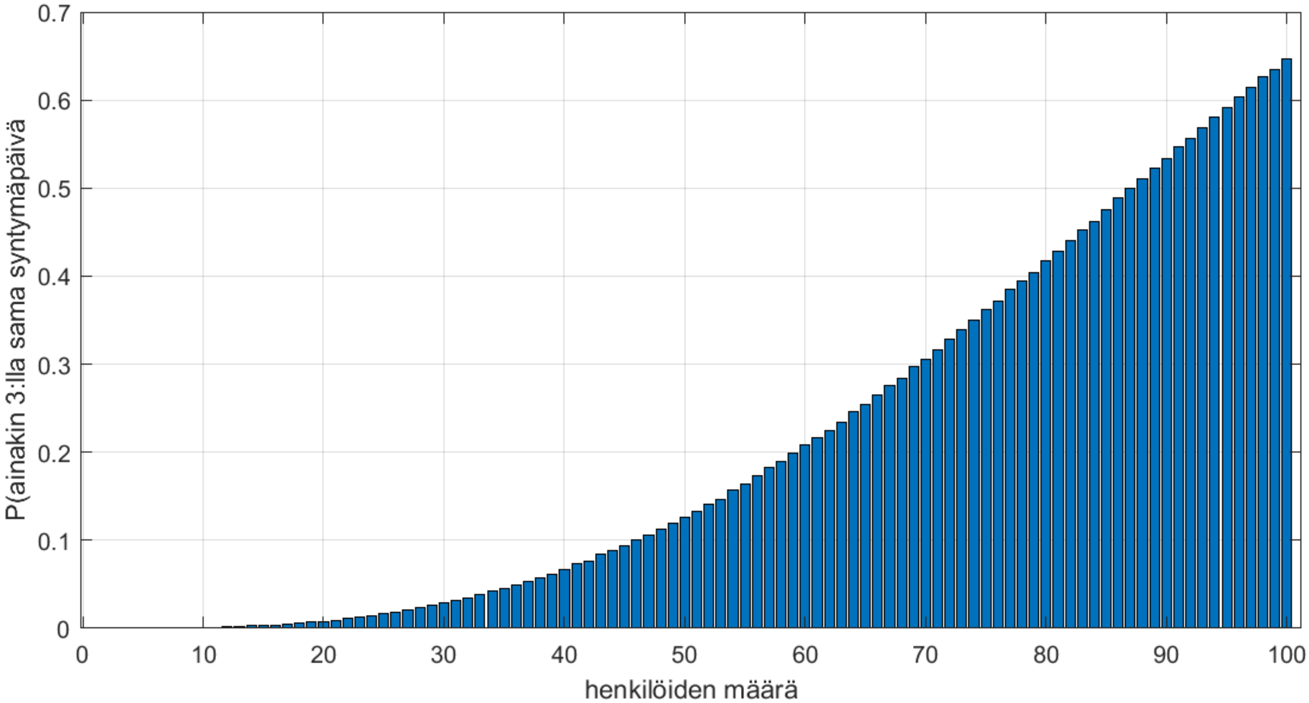
2) by throwing two dices 24 times, one gets at least one pair of 6 (ans: 0.491)

hint: $P(\text{at least one}) = 1 - P(\text{none})$

Test by simulation

result1=randint(1,7,4) and result2=randint(1,7,(2,24))

2. Calculate by simulation the probability that among n persons at least 3 have same birthday

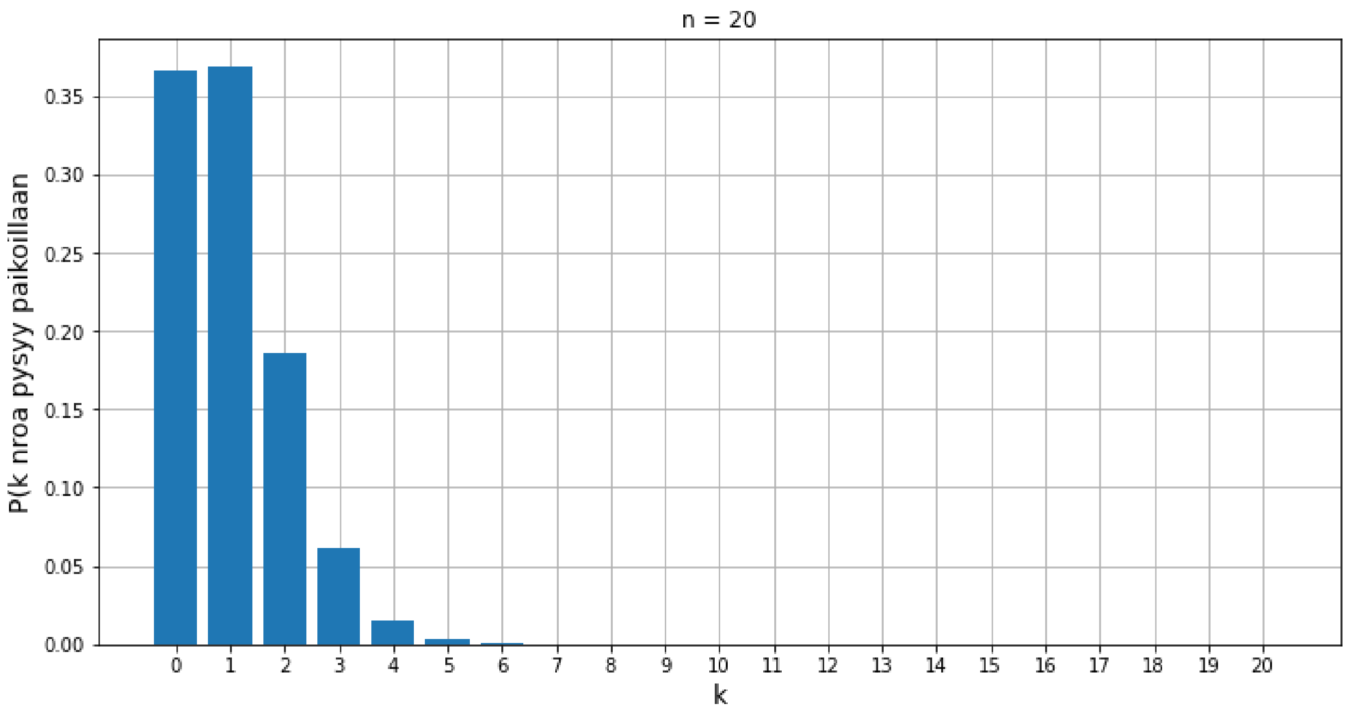


hint: u,c=np.unique(birthdays,return_counts=True), calculate for one n

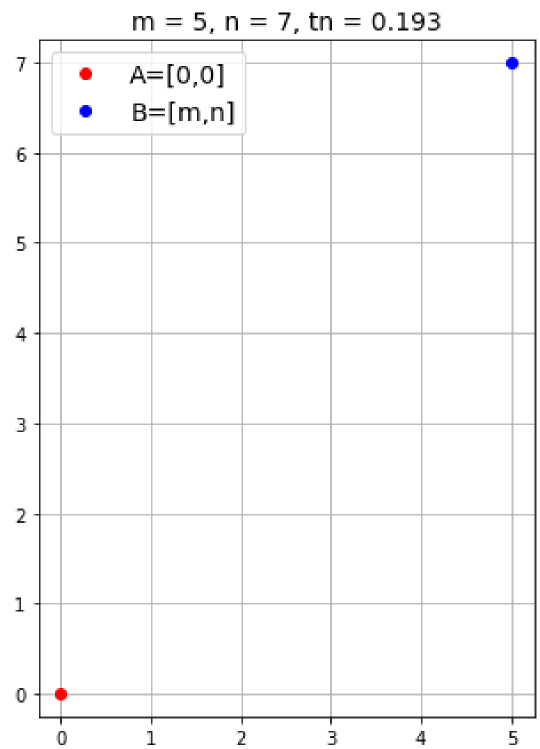
3. (Coupon collector's problem) In probability theory, the coupon collector's problem describes "collect all coupons and win" contests. It asks the following question: If each box of a brand of cereals contains a coupon, and there are n different types of coupons, calculate by simulation the probability that by buying t boxes, one finds all n different coupons. (ans: $n = 10, t = 20/30/40/50 \to 0.215/0.63/0.86/0.95$)

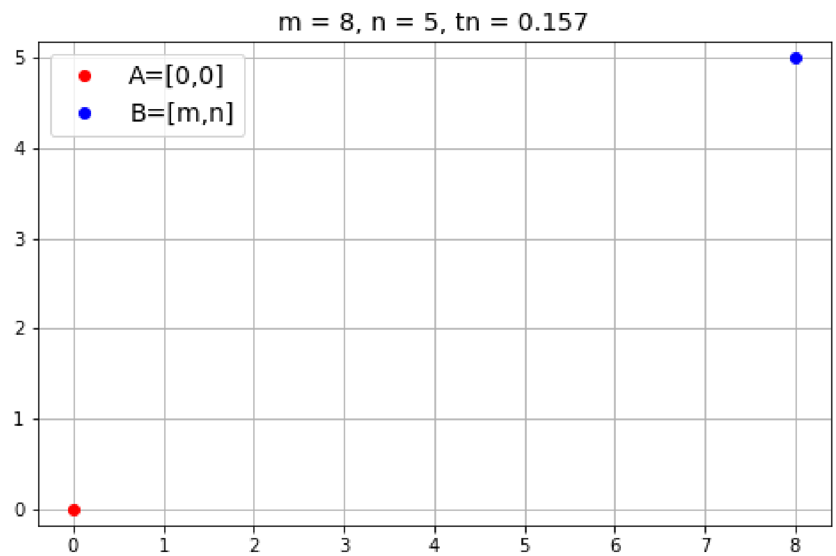
hint: coupons=randint$(1, n+1, t)$

4. Arrange numbers $1, 2, \ldots, n$ to a random order. Calculate by simulation the probability that $k = 0, 1, \ldots, n$ numbers stay at their original places and draw a picture like below.



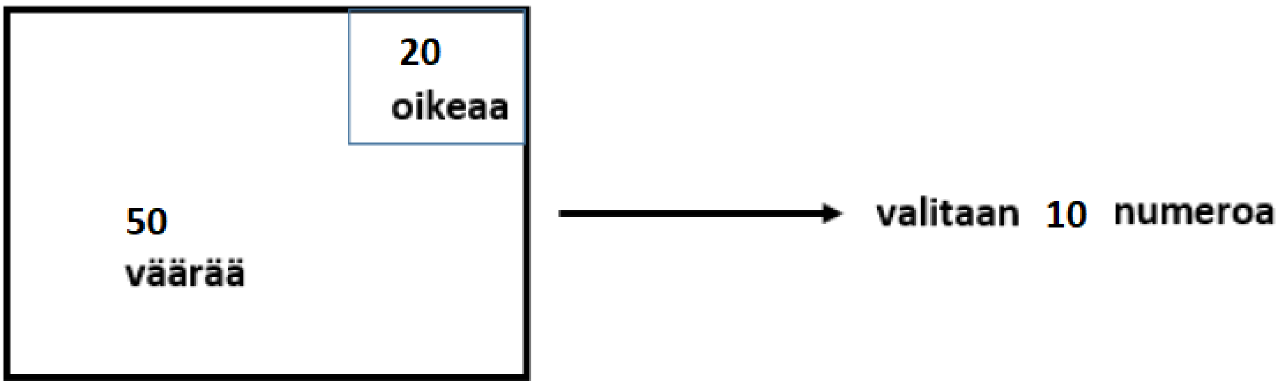
5. In the grid below the traveller can move either one step right or up. Traveller starts at $A = [0, 0]$ and chooses the moves randomly. Given m and n, calculate the probability that traveller goes through point $B = [m, n]$. Test by simulation.

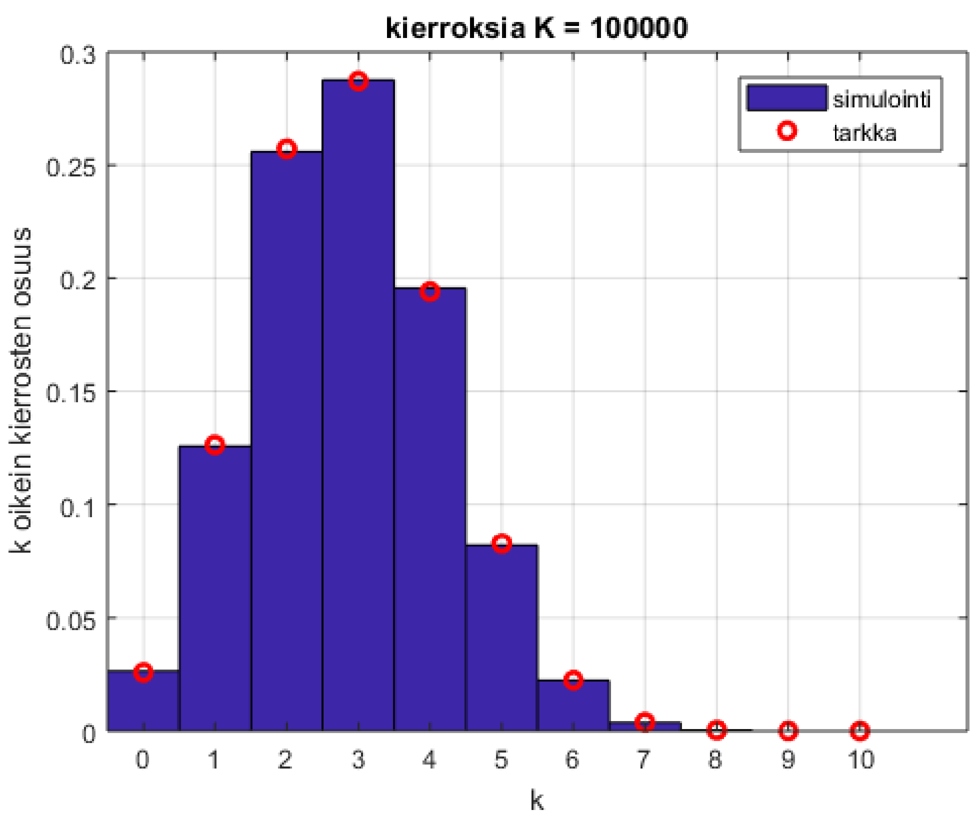




hint: route $=$ randint$(0, 2, m + n)$

6. Keno: player chooses 10 numbers from 1,2,...,70. There are 20 winning and 50 non-winning numbers. Calculate the probabilities that player chooses 0,1,...,10 winning numbers and test by simulation.





7. In lotto, 7 numbers from 1,2,...,39 are chosen. Calculate by simulation the probability there are no consecutive numbers (ans. 0.28)

hint: sort the numbers, np.sort(numbers)

8. Deal 52 cards to 4 persons, 13 to each.
Calculate by simulation the probabilities

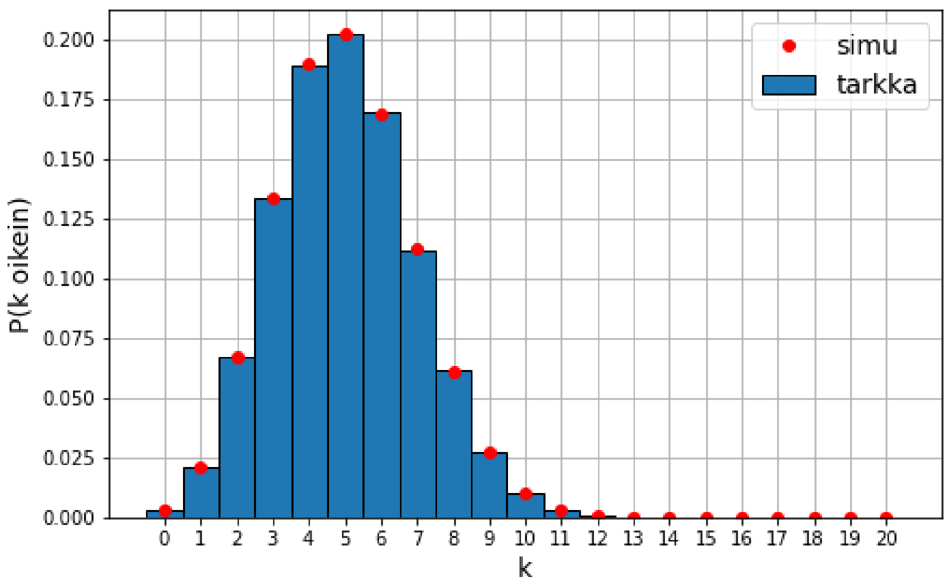
P(each person gets one ace (= number 1))

P(each person gets cards from each 4 suites)

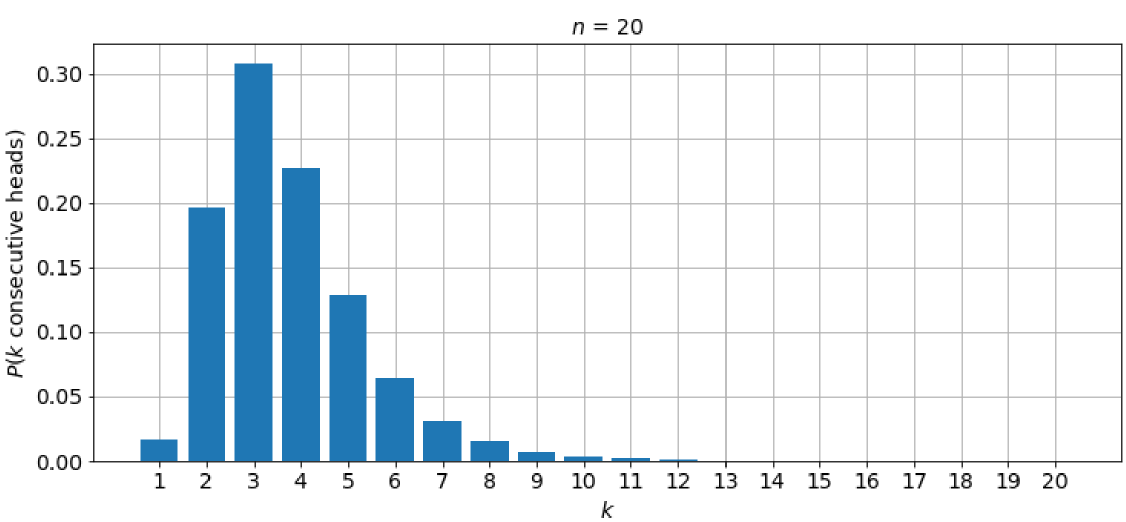
(ans: 0.0106/0.1055)

hint: deck as in poker-simulation, shuffle
s=permutation(52)
sdeck=deck[s , :]
cards 1-13 to person 1, 14-26 to person 2 etc i.e
person1=sdeck[:13 , :] etc

9. Exam contains 20 questions, each with 4 answer alternatives (of which one is right). Calculate probabilities that one gets $k = 0, 1, 2, \ldots, 20$ questions right by guessing and test by simulation.



10. Calculate by simulation the probability that when flipping a coin n times, the maximum number of consecutive heads is $k = 1, 2, \ldots, n$ and draw a picture like below

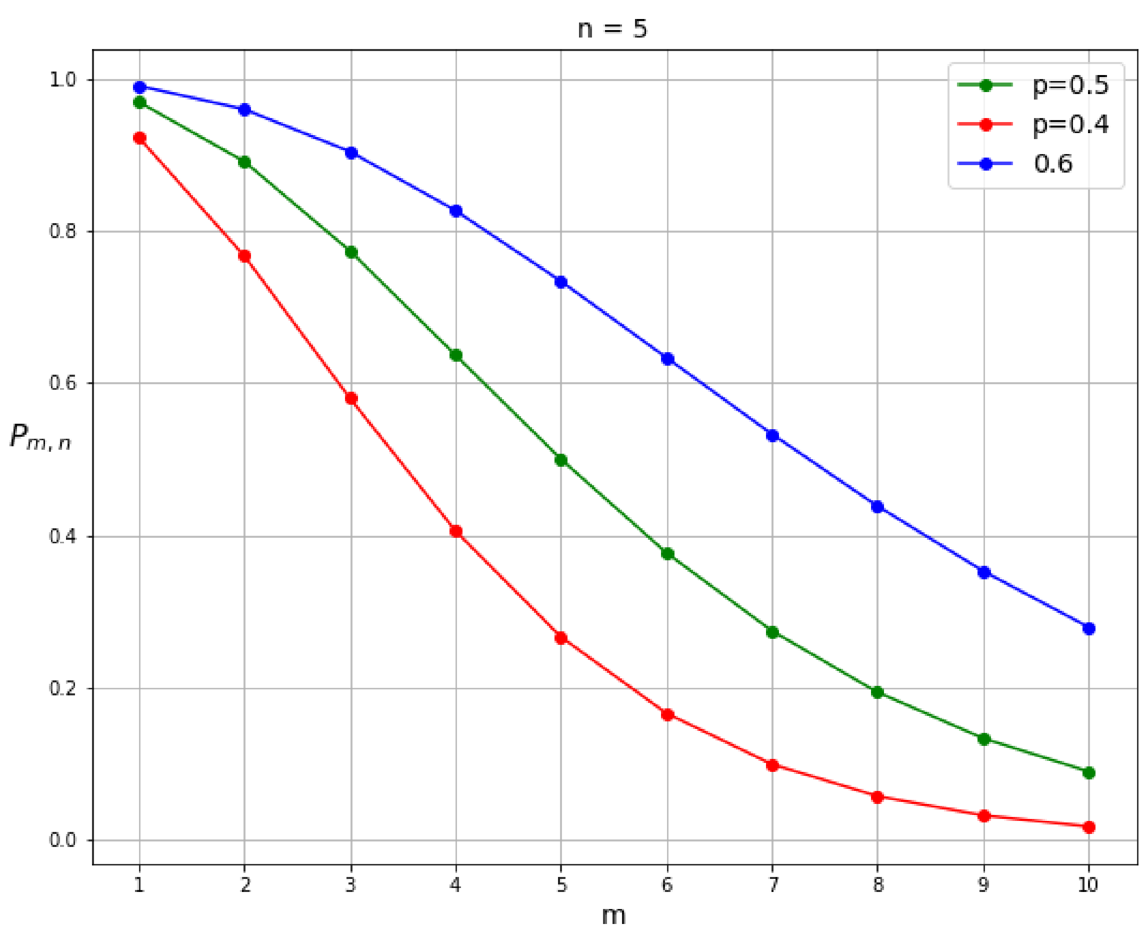


11. Problem of points

A and B play a game which A wins with probability p and B with probability $1-p$. They continue until one of them has won N games. If A has won $N-m$ games and B has won $N-n$ games, then A wins the series (i.e wins m games before B wins n games) with probability

$$P_{m,n} = \sum_{r=m}^{m+n-1} \binom{m+n-1}{r} p^r (1-p)^{m+n-1-r}$$

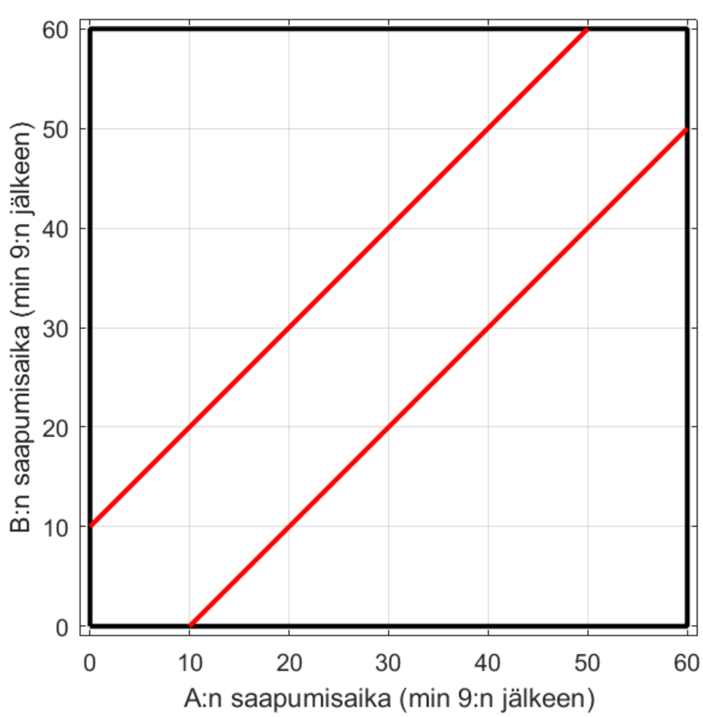
Given m, n and p, calculate $P_{m,n}$ and check by simulation.



12. Craps

Player twrows two dice. If the sum is 7 or 11, player wins, and if the sum is 2,3 or 12, player loses. If sum is 4,5,6,8,9 or 10, player continues throwing the two dice until sum is the same as in the first throw and player wins, or the sum is 7 and player loses.
Calculate by simulation the probability that player wins (ans. 0.493)

13. A and B arrive to a cafe at random times between 9-10, and stay for 10 minutes. Calculate the probability that are in the cafe simultaneously using the picture below (ans: 0.305).
Test by simulation (create arrival times $(0 \ldots 60$ minutes after 9) from uniform distribution).



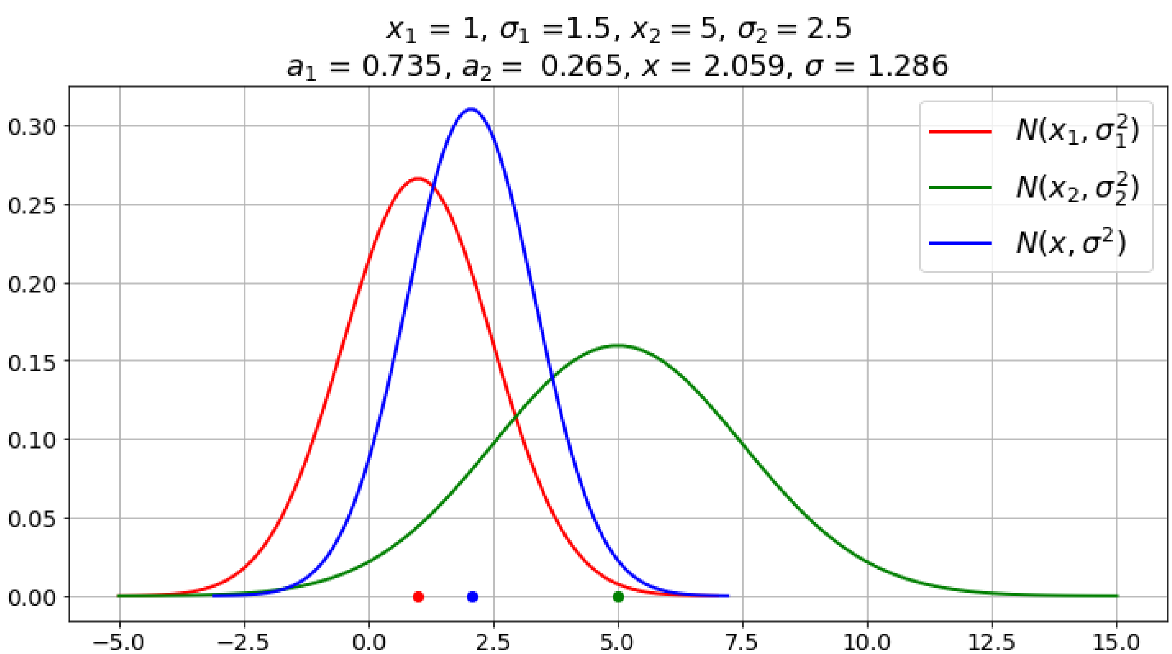
14. Given measurements x_1 and x_2 and their standard deviations σ_1 and σ_2, calculate weights

$$a_1 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \quad \text{and} \quad a_2 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$$

the weighted average $x = a_1 x_1 + a_2 x_2$ and it's standard deviation

$$\sigma = \sqrt{\frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}}$$

and draw the Gaussians corresponding to the normal distributions $N(x_1, \sigma_1^2)$, $N(x_2, \sigma_2^2)$ and $N(x, \sigma^2)$.



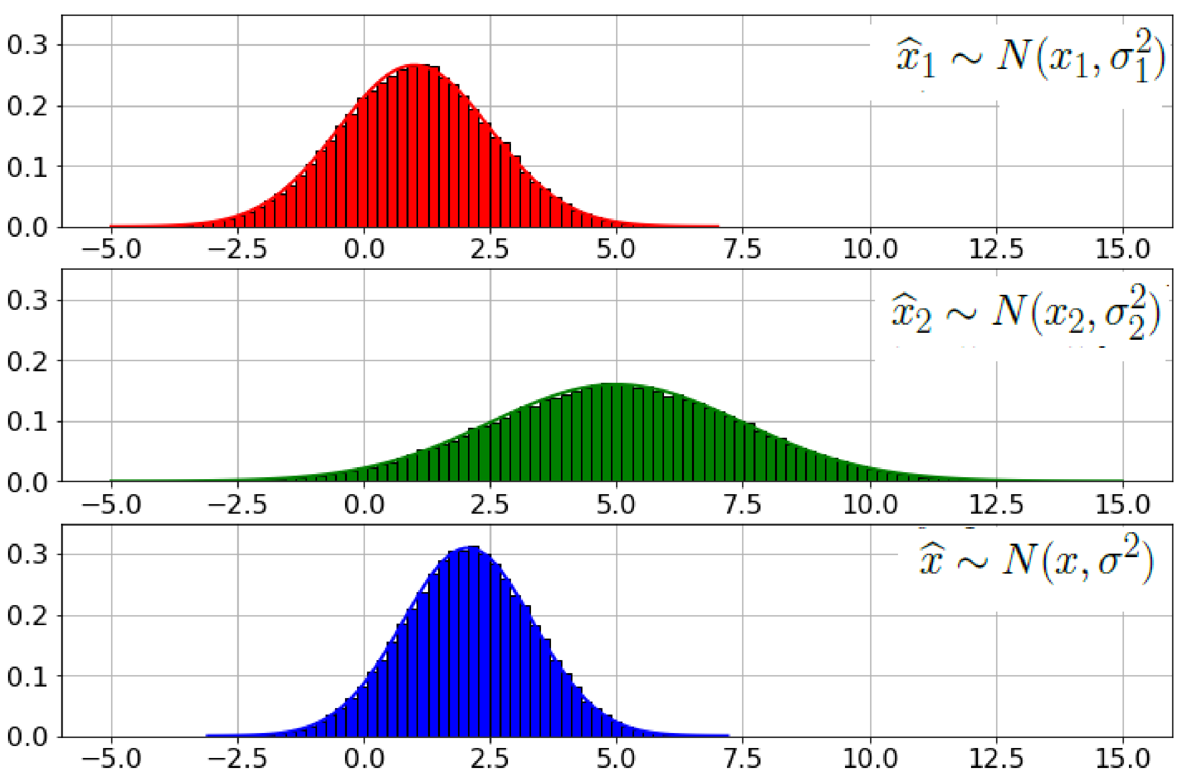
Test by simulation the following fact:
if

$$\widehat{x}_1 \sim N(x_1, \sigma_1^2) \text{ and } \widehat{x}_2 \sim N(x_2, \sigma_2^2)$$

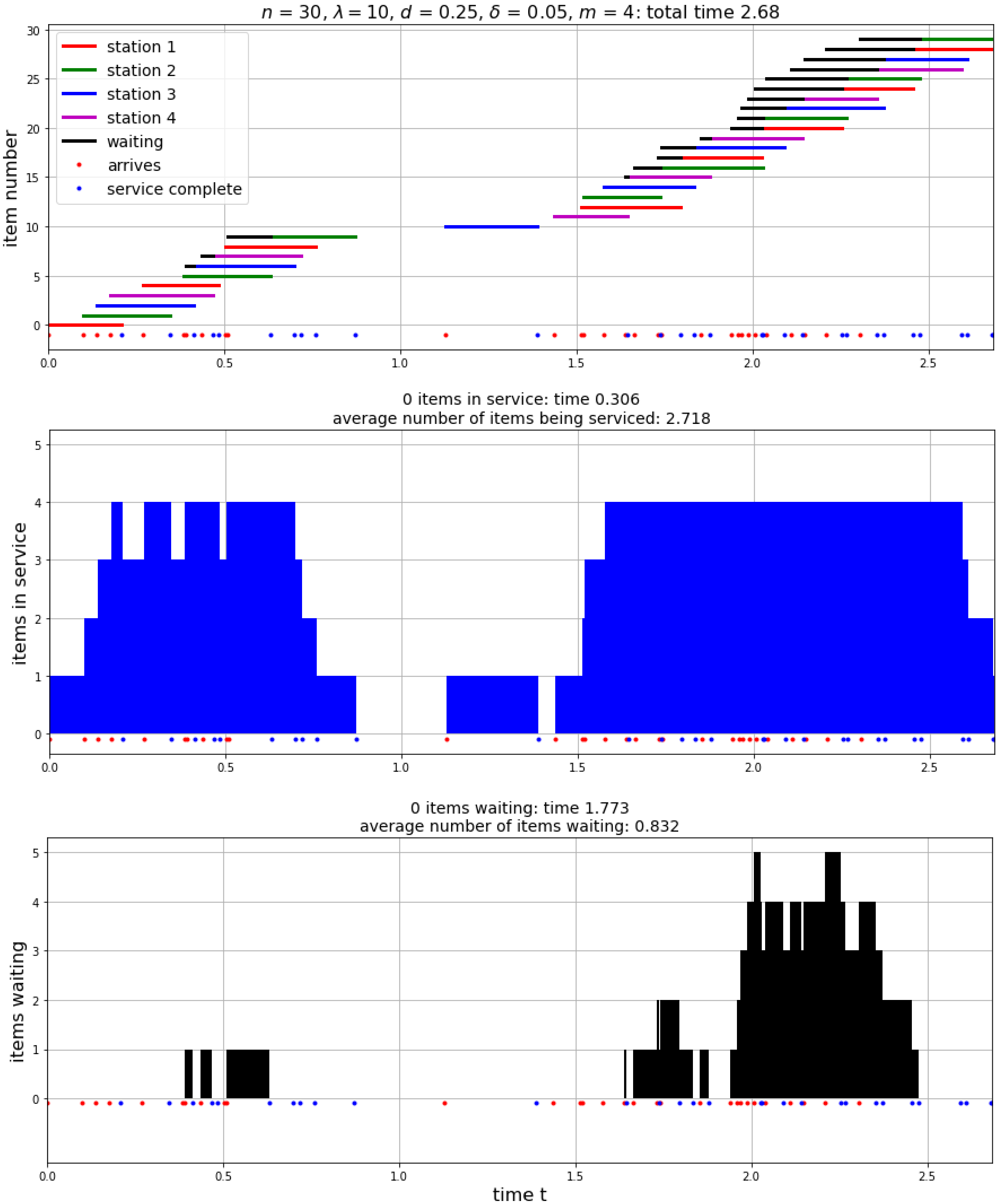
then

$$\widehat{x} = a_1\widehat{x}_1 + a_2\widehat{x}_2 \sim N(x, \sigma^2)$$

i.e create vectors $\widehat{x}_1 \sim N(x_1, \sigma_1^2)$, $\widehat{x}_2 \sim N(x_2, \sigma_2^2)$ and $\widehat{x} = a_1\widehat{x}_1 + a_2\widehat{x}_2$ of length (for example) 100000, and draw their distributions.



15. n items arrive to service such that the differences of arrival times are $\mathrm{Exp}(\lambda)$-distributed and service times are uniformly distributed between $d \pm \delta$.
There are m service stations and an item goes to the station next available (i.e the service of the previous item is completed).
Given n, λ, d, δ ja m, simulate the arrivals and services and calculate the total service time, time when there is 0 items in service / waiting to be serviced and the averages of items being serviced / waiting to be serviced, and draw pictures like below.

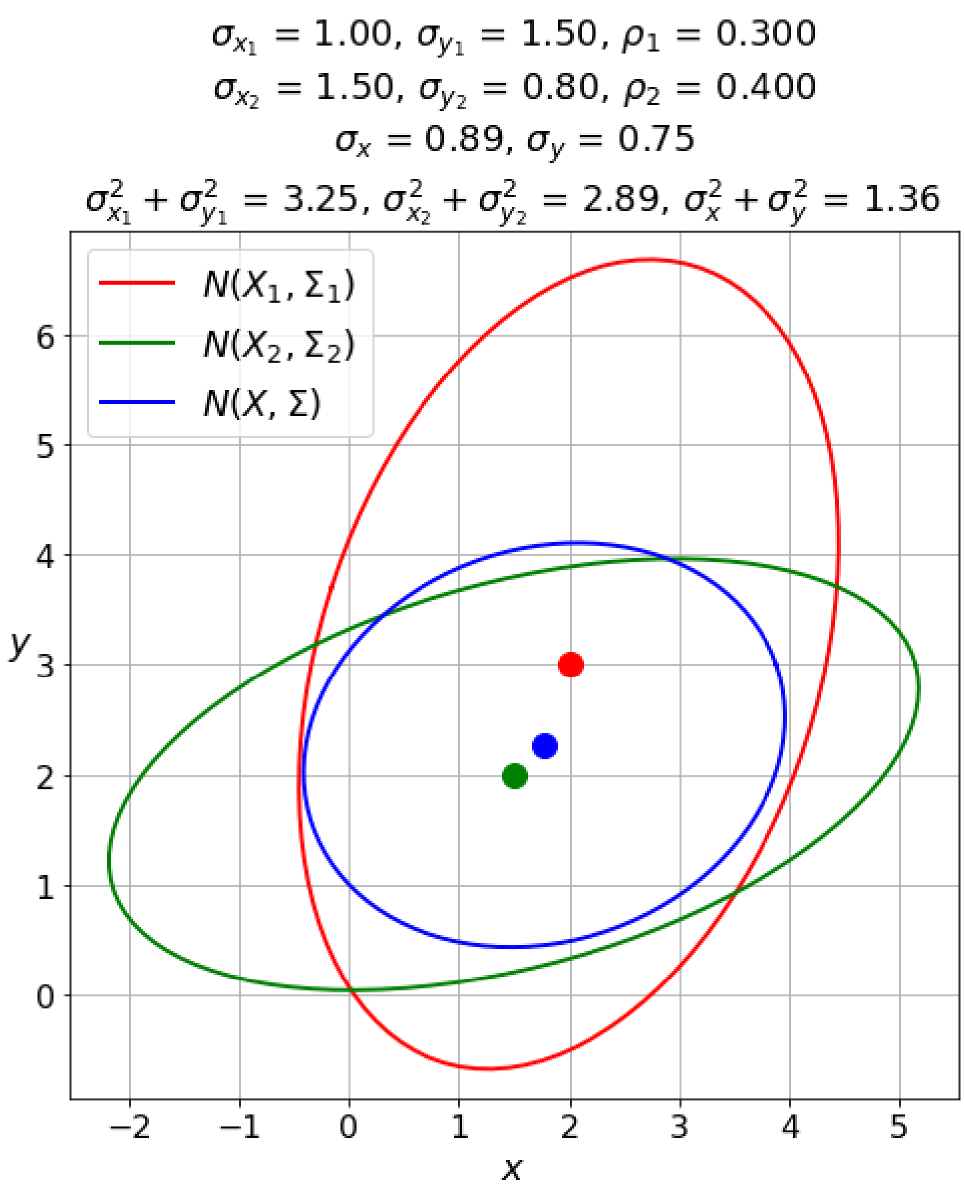


n = 30, λ = 10, d = 0.25, δ = 0.05, m = 4: total time 2.68
station 1
station 2
station 3
station 4
waiting
arrives
service complete
item number
0 items in service: time 0.306
average number of items being serviced: 2.718
items in service
0 items waiting: time 1.773
average number of items waiting: 0.832
items waiting
time t

16. Given 2D-normally distributed measurements/ estimates X_1 and X_2 and their covariance matrices Σ_1 and Σ_2, calculate coefficient matrices

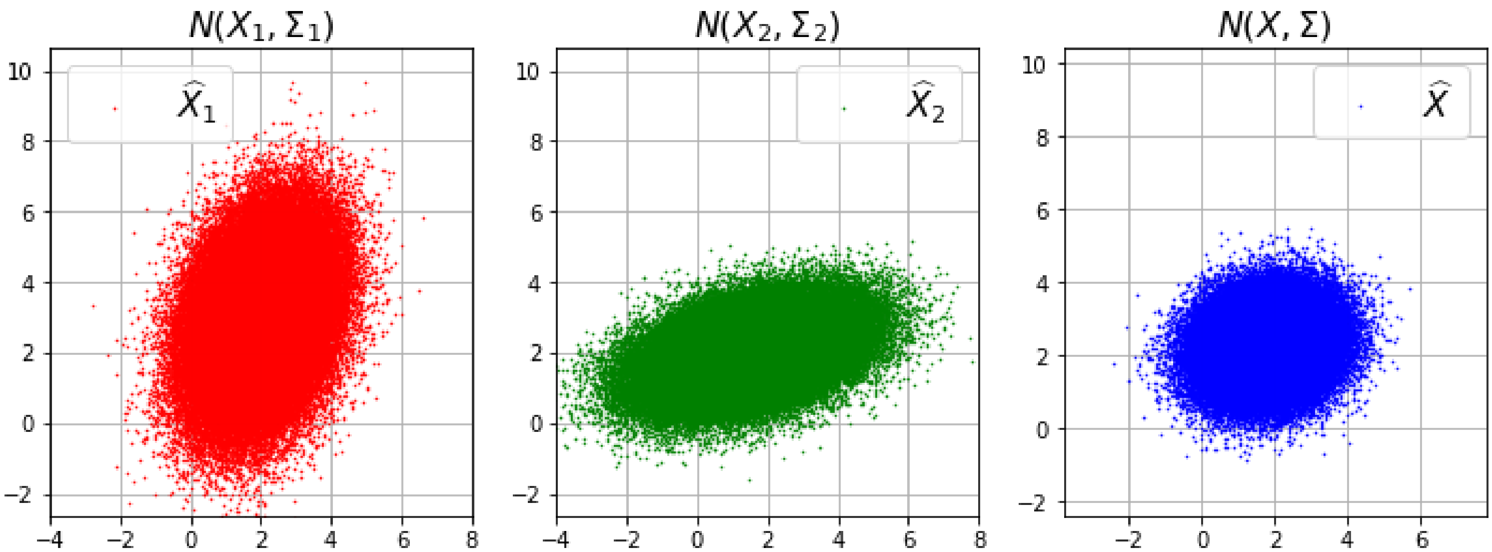
$$A_1 = \Sigma_2(\Sigma_1 + \Sigma_2)^{-1}, \quad A_2 = \Sigma_1(\Sigma_1 + \Sigma_2)^{-1}$$

weighted average $X = A_1 X_1 + A_2 X_2$
and it's covariance matrix $\Sigma = \Sigma_2(\Sigma_1 + \Sigma_2)^{-1}\Sigma_1$
and draw the 95 % ellipses of the distributions $N(X_1, \Sigma_1)$, and $N(X, \Sigma)$



Test by simulation the fact:
if $\widehat{X}_1 \sim N(X_1, \Sigma_1)$ and $\widehat{X}_2 \sim N(X_2, \Sigma_2)$,
then $\widehat{X} = A_1\widehat{X}_1 + A_2\widehat{X}_2 \sim N(X, \Sigma)$

i.e create (for example) matrices $\widehat{X}_1 \sim N(X_1, \Sigma_1)$, $\widehat{X}_2 \sim N(X_2, \Sigma_2)$ and $\widehat{X} = A_1\widehat{X}_1 + A_2\widehat{X}_2$ containing (for example) 100000 points (as columns) , and draw a picture



Hint: X_1, X_2 and X are (2,1) column vectors

$\widehat{X}_1, \widehat{X}_2$ and $\widehat{X}$ are $(2, n)$ matrices
matrix multiplication @, matrix inverse np.linalg.inv