On Nonparametric Methods of European Vanilla Option Pricing

By

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Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

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Abstract

In Black-Scholes and Merton (BSM) and Heston models, we implemented Empirical Esscher Transform (EET) and Empirical Characteristic Function (ECF) nonparametric methods for evaluation of European vanilla options. Further we carried out a numerical experiment where we tested the results obtained by the two nonparametric methods through closed and semi-closed pricing formulae which are available in these models respectively. Further, for testing the accuracy of EET and ECF methods in Heston model, we also implemented Monte Carlo simulation and inverse Fourier transform methods developed by Heston (1993).

In addition, we illustrate that both nonparametric methods ECF and EET, the latter relatively at a lesser degree though, provide most accurate prices for "in the money" European call options.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

Financial derivatives are financial instruments that have their returns or values derived based on the values of other financial instruments. The history of financial derivatives has begun many centuries ago. One of most widely known financial derivative is an option. Options can be traded on both OTC (Over-the-Counter)[1] markets or exchanges.[2] There are two main types of options, European and American options. This project is concerned with different pricing methods of European options.

In the late 1970s, Fisher Black, Myron Scholes and Robert Merton have developed one of the famous and widely used model for European option pricing, nowadays known as Black-Scholes Merton model (or BSM model). Black and Scholes had applied two approaches to pricing of European options, in the first approach they used "Capital Asset Pricing Theory" and in the second approach they used stochastic calculus which involes a differential equation that was solved similar to the equation from physics that describes the heat movement across an object. And at the same time, an economist from MIT, Robert Merton, also derived the same formula for European option pricing. In 1997, the Nobel prize was awarded to Myron Scholes and Robert Merton for pricing financial derivatives, while only recoginizing Black's contributions (as he died in 1995). The BSM model was built on some assumptions which might have oversimplified the financial markets yet providing a realistic mathematical model. Merton (1976) noticed about the differences between the predicted prices produced by the model and the actual prices observed in the market, pointing to the suitability of geometric Brownian motion for modelling of asset prices. According to Rubinstein (1985), the BSM model can

[1]OTC market is an unregulated market place where the option trading is conducted privately between two parties.

[2]An exchange is a regulated market where option is traded under specified terms and conditions.

create biases as it assumes the parameters of the model are constant. As Ziqun (2013) stated, the Black-Scholes Merton model was developed based on strong assumptions about the normality in the asset's returns with constant mean and constant volatility of the underlying asset over time, which cannot generate the volatility smile and skewness in the distribution of asset's return. Moodley (2005) pointed out that many empirical studies have shown the distribution of asset's log-return is non-Gaussian is to be followed by Ziqun (2013).

In Table 1.1 the Facebook European call option market prices[3] are listed as of 23 October 2018 with the time to maturity from the range from 3 to 35 days and various strike prices. In Figure 1.1 the implied volatility plot of Facebook[4] call options for varying maturity dates and strike prices is presented. The plot in Figure 1.1 illustrates the dynamics of implied volatility obtained from market prices of European call options.

Table 1.1: Facebook call option prices at 23 October 2018

Strikes	Maturities					
	3 days	7 days	14 days	21 days	28 days	35 days
145	9.55	11	11.85	13	12.1	13.86
147	7.92	10.65	10.3	11.7	10.51	10.35
149	6.25	8.85	9.9	10.35	10.47	10.4
150	5.25	8.55	9.35	9.55	9.2	10.25
152.5	3.3	7.34	7.6	8.1	8.43	8.25
155	2	5.85	6.4	6.9	7.02	7.7
157.5	0.98	4.8	5.21	5.55	6	6.32
160	0.42	3.7	4.3	4.55	5.02	5.25
162.5	0.15	2.8	3.25	3.63	3.9	4.45
165	0.05	2.13	2.51	2.85	3.1	3.32
167.5	0.03	1.55	1.96	2.28	2.46	2.59
170	0.01	1.06	1.39	1.74	1.93	2.22
172.5	0.01	0.77	1.05	1.25	1.56	1.74
175	0.01	0.53	0.79	0.96	1.15	1.42
177.5	0.01	0.4	0.58	0.7	0.9	1.07
180	0.01	0.27	0.38	0.52	0.65	0.86
185	0.01	0.13	0.18	0.3	0.34	0.5
190	0.02	0.08	0.1	0.16	0.24	0.36
195	0.01	0.03	0.05	0.11	0.27	0.18
200	0.01	0.04	0.05	0.07	0.08	0.16

[3]Source: https://www.nasdaq.com/symbol/fb/option-chain

[4]Facebook, Inc. is a social networking service company, which facilitates information sharing, photographs, websites and videos.

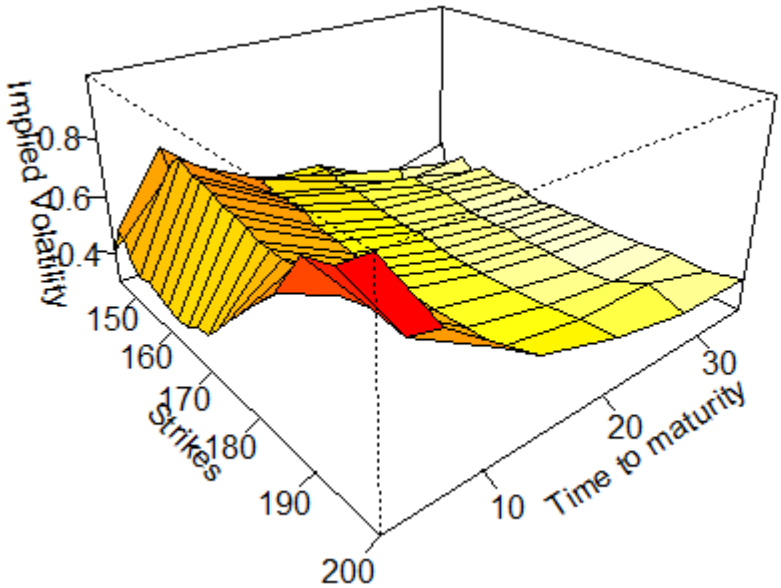

Figure 1.1: Implied volatility dynamics from Facebook European call option market prices

Theoretically, if the volatility is truly constant as assumed in BSM model, the surface of implied volatility plot should be flat. But clearly the plot of implied volatility surface above has some curvature ("a smile"), which indicates the non-constant nature of volatility parameter.

Many theoretical advancements for modelling of the asset price volatility in BSM framework were made by Hull & White (1987), E. Stein & Stein (1991) and Heston (1993). Specifically, Heston (1993) mentioned that BSM model did not allow the mean to vary and suggested a model which allowed for the randomness in the volatility of asset returns (i.e a stochastic volatility model) and provided a closed-form solution for the price of the European Call option. The stochastic volatility allows the underlying asset price to be asymetric with excess kurtosis in the assets' returns, which is a more realistic situation in the market, Crisostomo (2014), or C. O'Sullivan & O'Sullivan (2010).

Black and Scholes (1973) and Merton (1973) founded the modern option pricing theory according to which finding option's fair price is equivalent to taking is taking the expectation of the discounted cash flow under a risk-neutral probability. In a complete market, the risk-neutral probability is said to be unique, which means there exists only one risk-neutral probability measure to derive fair prices, Bingham & Kiesel (2004). However, for incomplete markets, there are an infinite number of risk-neutral probabilities for deriving the option fair prices. The nonparametric methods are often deployed for finding fair prices of options. Walker & Haley (2010) indicated that nonparametric methods can possibly avoid the risk of misspecification and biased restriction caused by parametric methods. Esscher transform is a nonparametric method which is frequently used for option pricing. Pereira & Veiga (2017) suggested an empirical version of Esscher transform method, which transforms the original physical distribution of the simulated sample data into a risk neutral distribution, by using the optimal Esscher parameter, that can be introduced into the option valuation equation to calculate the fair price. And they suggested that the martingale Esscher parameter can be obtained as the ratio of two empirical moment generating function's periods.

Another nonparametric method for pricing of European options using empirical characteristic functions was developed by Carr & Madan (1999) and Lewis (2001). Binkowski (2008) extended the empirical characteristic function method to more general Levy processes. In this project, we will apply the empirical characteristic function method to the simulated asset prices modeled from the Brownian motion process for obtaining European option prices.

The Monte Carlo simulation method will be used for checking the accuracy in BSM and Heston models. In practice, MC method is quite easy to apply but often displays a slow convergence rate (Kim (2015)).

In chapters 2 and 3, we will introduce some essential theoretical results on Brownian motion and their applications in modelling of asset price process. Besides, the Euler's discretization method will be used to simulate the paths asset price. The European option pricing problem will be discussed using the two nonparametric methods along with closed-form formulae in BSM and Heston's frameworks.

Chapter 4 will mainly present the two numerical experiments corresponding to two nonparametric and MC methods for pricing of European options.

In chapter 5, we will summarize the numerical results from the methods considered in this paper.

For this project, R is used for computation. We will mainly develop the R codes for computation ourselves. The proofs and R codes of this project are presented in Appendices A and B, respectively.

Chapter 2

Stochastic Models for Pricing of European Vanilla Options

In this chapter, we will discuss Black-Scholes Merton (BSM) and Heston Models used for pricing of European options.

2.1 Black-Scholes Merton Model

2.1.1 Preliminaries

Before giving out the Black-Scholes Merton (BSM) model formula, there are some essential theoretical backgrounds that we should be going through.

Definition 1: A stochastic process $(W_t)_{t\geq 0}$ is called a standard Brownian motion (or as standard Wiener process) if it satisfies the following conditions:

1. $W_0 = 0$ with probability 1 (or almost surely).
2. The sample paths (or trajectories) of the process W_t are continuous, almost surely.
3. (*Independent Increments*): For any partitions of time $0 = t_0 \leq t_1 \leq t_2 \leq .... \leq t_n$, the increments $W_{t_1} - W_{t_0}$, $W_{t_2} - W_{t_1}, \dots$, $W_{t_n} - W_{t_{n-1}}$ are independent from each other.
4. (*Gaussian Increments*): For any $0 \leq s \leq t$, the increments $W_t - W_s$ are normally distributed with the mean 0 and variance $t - s$, i.e. $(W_t - W_s) \sim N(0, t - s)$.

In discrete time, the standard Brownian motion represents a random walk with the time intervals of length Δt. In other words, the standard Brownian motion can be viewed as the limit of the symmetric random walk, when we narrow down to zero the step size Δt

(i.e. as $\Delta t \to 0$), Shreve (2001). The discrete analog of the standard Brownian motion (or Wiener process) can be formulated as:

$$W(t_{k+1}) = W(t_k) + Z(t_k)\sqrt{\Delta t} \tag{2.1}$$

where $W(0) = 0$ and $\Delta t = t_{k+1} - t_k$. This is achieved by using the Gaussian attribute of the Wiener process and scaling down the step size Δt (or increasing the number partitioned increments in a period of time $0 = t_0 \to t_n$) (Kordzakhia, 2017). However, for now the discrete analog of standard Brownian motion will not be used directly in pricing methodology but it will be an important part of simulation of asset price in the Heston model.

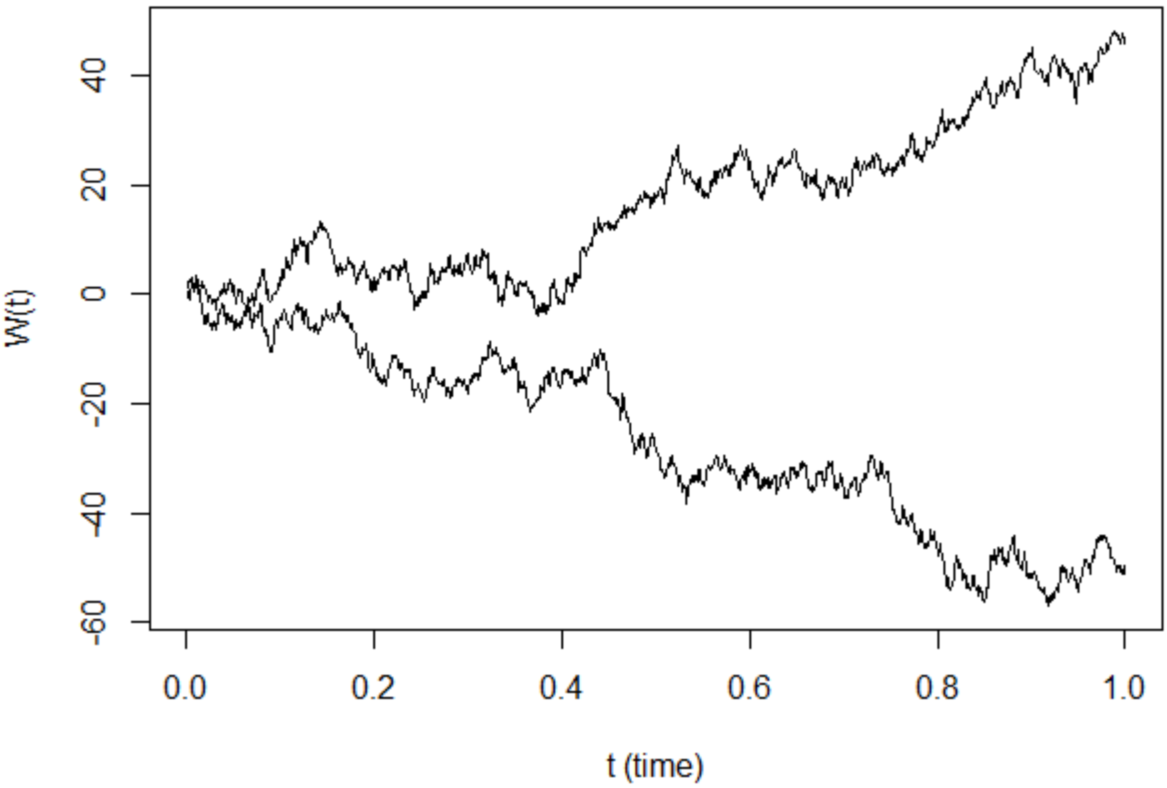

Figure 2.1: Simulated Brown Motion Paths

The BSM model provides a closed-form solution for pricing of European vanilla options. In BSM model, we make the following assumptions:

- $(C1)$ The market is assumed to be complete,
- $(C2)$ The variance of the asset price is constant, and
- $(C3)$ The price of the underlying asset at time t is assumed to follow a Geometric Brownian Motion if it satisfies the diffusion process (or stochastic differential

equation):

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{2.2}$$

where the W_t is the Brownian motion (or Wiener Process), μ is the mean return and σ is the variance of the underlying asset's price. This equation will be solved by applying the following Ito's rule, *Appendix (A.1)*.

Using Ito's lemma for standard Brownian motion, the solution to the diffusion process (2.2) can be obtained in the following form:

$$S_t = S_0 exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right) \tag{2.3}$$

and we said S_t follows a Geometric Brownian motion (The proof in *Appendix A.2*). From equation *(2.3)*, the logarithm return, i.e. $ln\left(\frac{S_t}{S_0}\right)$, will be normally distributed with the mean $(\mu - 0.5\sigma^2)t$ and the variance $\sigma^2 t$ (since $W_t \sim N(0,t)$). In other words, assuming the asset price follows the geometric Brownian motion *(2.3)*, then we will have

$$ln\left(\frac{S_t}{S_0}\right) \sim N\left(\left(\mu - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right) \tag{2.4}$$

Using the assumption $C1$, in order the risk-neutral distribution to exist, the parameter μ should be equal to the expected risk free rate of return r, Shreve (2004), or Kordzakhia (2013). Hence, the equation *(2.4)* can be rewritten as

$$ln\left(\frac{S_t}{S_0}\right) \sim N\left(\left(r - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right) \tag{2.5}$$

Definition 2: An European Call (Put) option is the right of the owner to buy (sell) an underlying asset for a specified exercise (or strike) price on the some pre-determined maturity time T.

Assuming the time maturity is T and strike price is K, its terminal payoff of European call option is defined as

$$f^c(S_T) = max(0, S_T - K) = (S_T - K, 0)^+ \tag{2.6}$$

This is because the owner of the option is willing to exercise the right to buy when the terminal asset price is greater than or equal to the strike.

Similarly, the put option owner is willing to sell the underlying asset at strike price K

when K is greater than or equal to the terminal asset price. The payoff of an European put option with maturity T and strike K is defined as

$$f^p(S_T) = max(0, K - S_T) = (K - S_T, 0)^+ . \tag{2.7}$$

An illustration of European call payoff (the upper plot) and European put payoff (the lower plot) with strike price $K = 40$ is presented below:

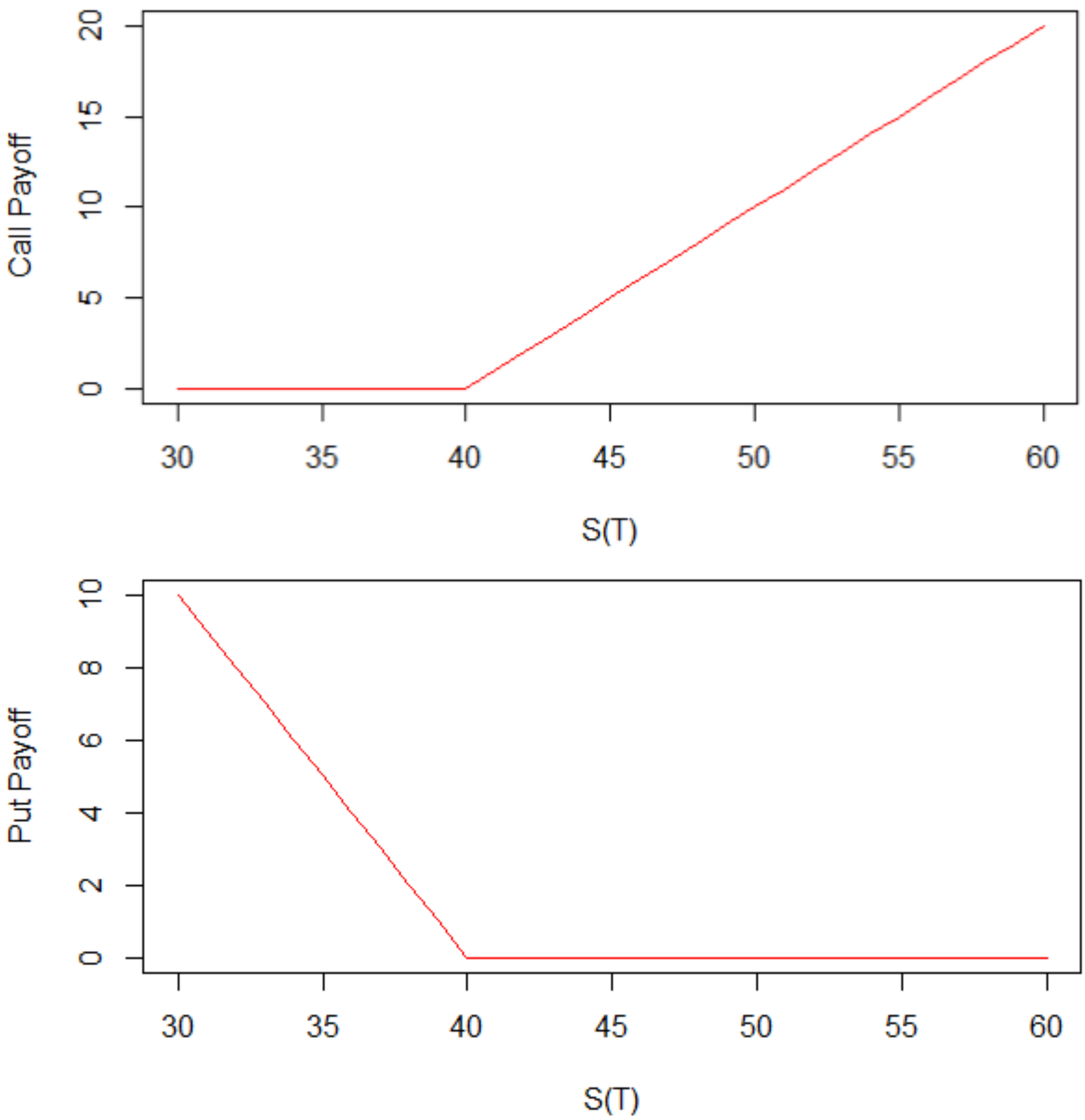

Figure 2.2: The payoff of European call option from (2.6) and put option from (2.7) with strike K = 40 at maturity T

For any asset price S_t that follows a geometric Brownian motion, Black & Scholes (1973) and Merton (1973) showed that $C(T, S_t, K)$, the European call option price at t, satisfies the partial differential equation (PDE):

$$rC(T,S_t,K) = \frac{\partial C(T,S_t,K)}{\partial t} + rS_t\frac{\partial C(T,S_t,K)}{\partial x} + 0.5\sigma^2 x^2 \frac{\partial^2 C(T,S_t,K)}{\partial x^2} \tag{2.8}$$

They derived the closed-form formula for $C(T,S_t,K)$ as the solution to (2.8).

For European Call:

$$C\left(T,S_0,K\right) = S_0 N\left(d_1\right) - Ke^{-rT}N\left(d_2\right) \tag{2.9}$$

where,

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}}\left[ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right] \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned} \tag{2.10}$$

and $N(x)$ is a standard normal distribution function.

The European put option with time maturity T and strike price K has the terminal payoff defined as $P(T,S_T,K) = (K - S_T, 0)^+$. And the corresponding value of European put option can be derived accordingly as:

Analogously for European Put Option:

$$P\left(T,S_0,K\right) = S_0 N\left(d_1 - 1\right) - Ke^{-rT}N\left(1 - d_2\right) \tag{2.11}$$

where d_1 and d_2 is defined similarly as above.

For coding, let j denotes the indicator input where $j = 1$ for call option, and $j = -1$ for put option. Then following from Kordzakhia (2013), the general formula of Black-Scholes Merton (BSM) model can be restated as:

$$V(T,S_0,K) = jS_0 N\left(j(d_2 + \sigma\sqrt{T})\right) - jKe^{-rT}N(jd_2) \tag{2.12}$$

2.1.2 Simulation Technique

We can use this Gaussian relation to simulate the values of logarithm return $ln\left(\frac{S_t}{S_0}\right)$, and given the spot asset value S_0, the asset price at time t, S_t, will be defined accordingly.

According to Graham & Talay (2013), we can alternatively use the Euler discretization scheme to simulate the stochastic terminal asset prices as follows.

1. For an interval time of length T, lets divide T into N small increments of equal size $\Delta t = t_i - t_{i-1}$.
2. Then, given the initial price S_0, at each time point $t_i = i\frac{N}{T}$, the discretization of asset price process is:

$$S_{t_i} = S_{t_{i-1}} exp\left((r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t} Z_{t_i}\right) \tag{2.13}$$

where $Z_{t_i} \sim N(0,1)$.

From experimental results, Pereira & Veiga (2017) suggested the Euler simulation scheme can facilitate understanding and has nice convergence properties. However, this method can generate bias on the terminal simulated price. Followed by Broadie & Kaya (2006), we can reduce the bias by using a large number of time steps in the simulation process. In our experiment, number of time-steps of 1000 would be used to simulate one path of the underlying asset price.

2.2 Heston Model

2.2.1 Preliminaries

Here below, we will briefly dicuss the Heston Model.

Assuming that the underlying asset's price at time t is S_t, and its corresponding stochastic volatility is v_t. Heston model is a 2-factor model, where the first factor represents the diffusion process of S_t that is similarly defined in BSM model and the second factor allows to model a stochastic volatility v_t as a stochastic process from John C. Cox & Ross (1985):

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_t^{(1)} \\ dv(t) &= \kappa[\theta - v(t)]dt + \sigma\sqrt{v(t)} dW_t^{(2)} \end{aligned} \tag{2.14}$$

where σ is the standard deviation of the volatility $v(t)$. The $dW_t^{(2)}$ and $dW_t^{(1)}$ are two Wiener processes with correlation coefficient ρ as:

$$dW_t^{(1)} dW_t^{(2)} = \rho dt \tag{2.15}$$

Here in the Heston model defined above,

- The parameter κ is interpreted as the mean reversion rate which demonstrates the speed of the convergence of the asset returns' variance $v(t)$ to its long-term mean θ.

- Given $\kappa > 0$ and θ is fixed, when $v(t) < \theta$ the drift term dt will drive the the variance $v(t)$ up to the fixed term θ in the long run, and when $v(t) > \theta$ the drift term dt would drive the variance $v(t)$ down to the fixed term θ in the long run.

Under the free-arbitrage assumptions, Heston (1993) showed that any contigent claim $f(S, v, t)$ (where S and v are values defined at that time t), fulfills the following partial differential equation:

$$\frac{1}{2}v_t S_t^2 \frac{\partial^2 f}{\partial S_t^2} + \rho\sigma v_t S_t \frac{\partial^2 f}{\partial S_t \partial v_t} + \frac{1}{2}\sigma^2 v_t \frac{\partial^2 f}{\partial v_t^2} + rS_t \frac{\partial f}{\partial S_t} + \left(\kappa\left[\theta - v_t\right] - \lambda(S, v, t)\right)\frac{\partial f}{\partial v_t} - rf + \frac{\partial f}{\partial t} = 0 \tag{2.16}$$

For vanilla European call option with strike (or exercise) price K, initial time t, and maturity time T ($t < T$), the PDE from (2.16) will be subject to the following boundary conditions:

- (1). At maturity T, and asset price is S, the payoff function is

$$f(S, v, T) = max(S - K, 0) \tag{2.17}$$

- (2). When the asset price is 0, the call option is worthless is

$$f(0, v, t) = 0 \tag{2.18}$$

- (3). As the asset price $S \to \infty$, we will have $f(\infty, v, t) = max(S - K, 0) \approx S$. Then its derivative is 1,

$$\frac{\partial f}{\partial S}(\infty, v, t) = 1 \tag{2.19}$$

- (4). When $v_t = 0$, the PDE from (2.16) will become

$$rS\frac{\partial f}{\partial S} + \kappa\theta\frac{\partial f}{\partial v}(S, 0, t) - rf(S, 0, t) + f_t(S, 0, t) = 0 \tag{2.20}$$

- (5). As the volatility $v_t \to \infty$, from the price process (2.14) we will have $\left(S_t = \int \mu S_t dt + \int \sqrt{v_t} S_t dW_t^{(1)}\right) \to \infty$. Then correspondingly, as $S \to \infty$ we have

$$f(S,\infty,t) = S \tag{2.21}$$

Heston (1993) suggested that the solution to the above PDE (2.16) should be in form of:

$$f(S,v,t) = C(S,v,t) = SP_1 - Ke^{-r(T-t)}P_2 \tag{2.22}$$

Here, P_1 and P_2 can be viewed as conditional probabilities of finishing in the money of the option at maturity, Schmelzle (2010). Let's consider $x = ln(S_t)$, Heston (1993) derived the closed-form solution for these P_1 and P_2 (or in other words, P_j where $j = \{1,2\}$), in terms of characteristic function as:

$$\begin{aligned} P_j(x,v,T;ln[K]) &= \frac{1}{2} + \frac{1}{\pi}\int_{-\infty}^{\infty} \Re\left[\frac{exp(-i\omega ln[K])f_j(x,v,t;\omega)}{i\omega}\right] \\ f_j(x,v,t;\omega) &= exp\left[C(T-t;\omega) + D(T-t;\omega)v + i\omega x\right] \end{aligned} \tag{2.23}$$

where the terms in the characteristic function f_j is defined by

$$\begin{aligned} C(T-t;\omega) &= r(T-t)i\omega + \frac{\theta\kappa}{\sigma^2}\left\{(b_j - \rho\sigma\omega i + d)(T-t) - 2ln\left[\frac{1-ge^{d(T-t)}}{1-g}\right]\right\} \\ D(T-t;\omega) &= \frac{b_j - \rho\sigma\omega i + d}{\sigma^2}\left[\frac{1-e^{d(T-t)}}{1-ge^{d(T-t)}}\right] \\ g &= \frac{b_j - \rho\sigma\omega i + d}{b_j - \rho\sigma\omega i - d} \\ d &= \sqrt{(\rho\sigma\omega i - b_j)^2 - \sigma^2(2u_j\omega i - \omega^2)} \end{aligned} \tag{2.24}$$

with the value of b_j and u_j for $j = \{1,2\}$ are:

$$\begin{aligned} b_1 &= \kappa + \lambda - \rho\sigma \\ b_2 &= \kappa + \lambda \\ u_1 &= \frac{1}{2} \\ u_2 &= -\frac{1}{2} \end{aligned} \tag{2.25}$$

However, as being reviewed by Albrecher, Mayer, Schoutens, & Tistaert (2007), the alternative solutions of Heston model derived by Gatheral (2006) can be given by changing the terms $C(T-t;\omega)$, $D(T-t;\omega)$, and g (while keeping the remaining terms unchanged) into the new forms as:

$$
\begin{aligned}
C^*(T-t;\omega) &= r(T-t)i\omega + \frac{\theta\kappa}{\sigma^2}\left\{(b_j-\rho\sigma\omega i-d)(T-t)-2ln\left[\frac{1-ge^{-d(T-t)}}{1-g}\right]\right\}\\
D^*(T-t;\omega) &= \frac{b_j-\rho\sigma\omega i-d}{\sigma^2}\left[\frac{1-e^{-d(T-t)}}{1-ge^{-d(T-t)}}\right]\\
g^* &= \frac{b_j-\rho\sigma\omega i-d}{b_j-\rho\sigma\omega i+d}
\end{aligned}
\tag{2.26}
$$

We notice here that the minus and plus signs in the original equation of Heston (1993) are flipped around and instead of using original expression of g in Heston's paper, we rewrite it as $g^* = \frac{1}{g}$ above.

The Heston characteristic function developed by using $C(T-t;\omega)$, $D(T-t;\omega)$, and g and the one derived by using $C^*(T-t;\omega)$, $D^*(T-t;\omega)$, and g^* are actually equivalent. But doing this, the new function derived by Gatheral (2006) has proved to be more stable and causes fewer numerical problems in model implementation. The latter confirmed by Mrazek & Pospisil (2017).

The parameter λ in the solution is a constant such that the price of volatility risk $\lambda(S,v,t)=\lambda v$, which is a linear function of the asset return's variance. However, this parameter λ is not easily measured from the market, and for the ease of implementation, we set $\lambda = 0$, similar to Rouah (2013).

In our experiment, by using the "trapped" expression for Heston characteristic functions, the Heston price obtained is close to the one generated by under Monte Carlo estimator. This behavior of results also being confirmed in the work of Albrecher et al. (2007).

The unstability of the orginal characteristic function derived by Heston (1993) can be visualized by plotting the integrand, for $j=\{1,2\}$,

$$
\phi_j = \Re\left[\frac{exp(-i\omega ln(K)f_j(x,v,t;\omega)}{i\omega}\right]
\tag{2.27}
$$

for characteristic function f_1 and f_2. The plots shown below being produced by using similar parameters in Albrecher et al. (2007) or Rouah (2013), with $\kappa = 1.5768$, $\rho = -0.5711$, $\sigma = 0.5751$, $\theta = 0.0398$, $S_0 = K = 100$, $T = 5$, $v_0 = 0.0175$, $r = 0.025$ and the integration range for $\omega = (0, 10]$,

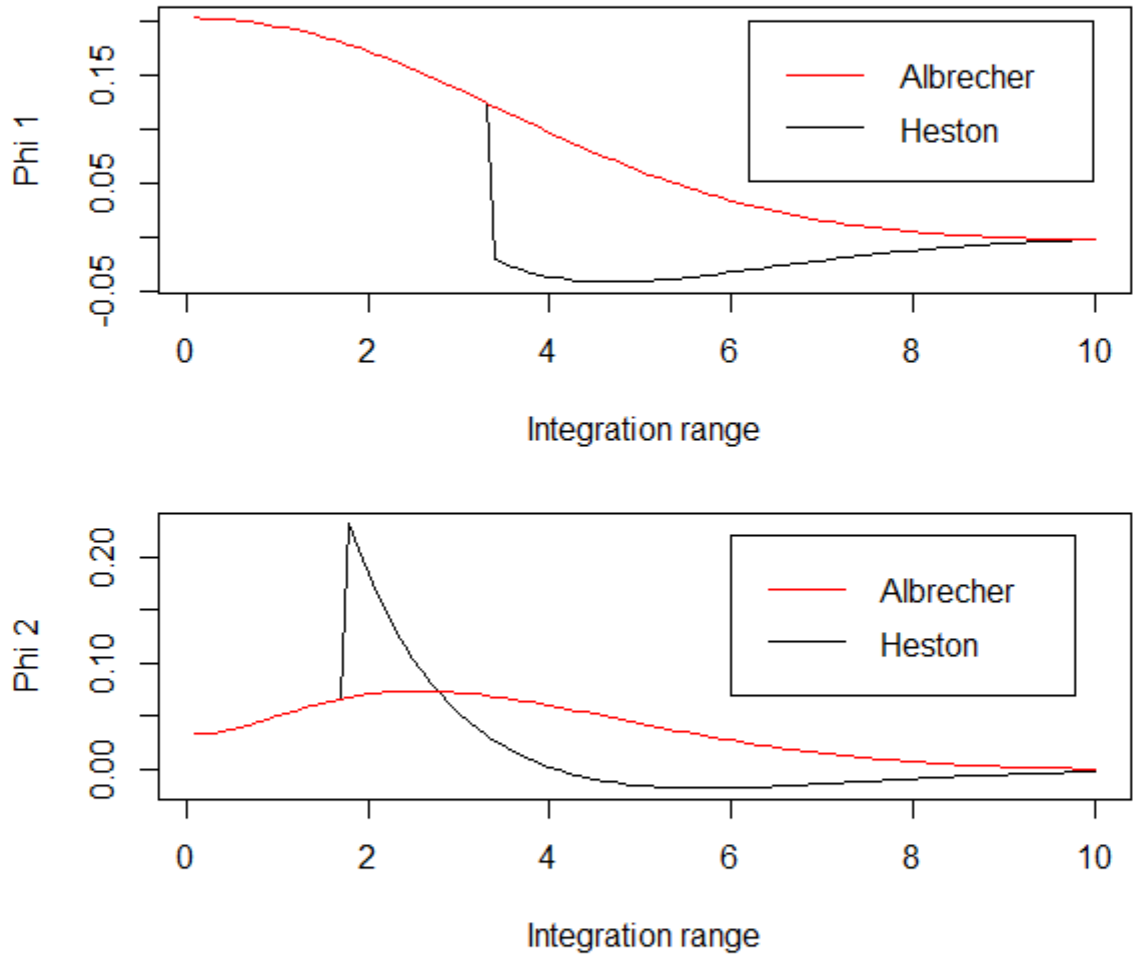

Figure 2.3: Plot of functions from (2.27)

The upper plot is visualization of (2.27) with $j = 1$ and the lower plot is with $j = 2$. From the plots, we can easily observe the discontinuities of the original Heston (1993) formula at the range of integration about $\omega = 3.7$ for the first integrand (with characteristic function f_1), and at about $\omega = 1.9$ for the second integrand (with characteristic function f_2). These discontinuities are corrected when using Albrecher et al. (2007) expression as we can observe a smooth curve in the integrand plot. Hence, we will use the expression of $C^*(T-t;\omega)$, $D^*(T-t;\omega)$, and g^* to derive the European option price under Heston model.

Using the same analogy as in the case of European call, the European put option formula can be derived as:

$$P(S,v,t) = -SP_1^C + Ke^{-r(T-t)}P_2^C \tag{2.28}$$

where,

$$P_1^C = 1 - P_1$$
$$P_2^C = 1 - P_2$$

and again, the equations for P_1 and P_2 here will contain the characteristic function f_i that would be expressed in terms of $C^*(T-t;\omega)$, $D^*(T-t;\omega)$, and g^*.

In our experiment dicussed later, for the ease of implementation, we will use the initial time $t = 0$, and hence the time period from initial point to maturity is $T - t = T$.

2.2.2 Simulation Technique

Similar to BSM model, we can use Euler discretization scheme to simulate process of asset price at the maturity time T. In Heston model, the asset price and volatility processes are correlated diffusion processes. In this model, Broadie & Kaya (2006) suggested the Euler discretization process could be performed as:

1. For an interval time of length T, divide T into N small increments of equal size Δt.
2. Then, given the initial volatility v_0, at each time point $t_i = i\frac{N}{T} = t_i - t_{i-1}$, the discretization of volatility process of asset price is:

$$\begin{aligned} v_{t_i} &= v_{t_{i-1}} + \kappa(\theta - v_{t_{i-1}})\Delta t + \sqrt{v_{t_{i-1}}}\sigma_v \Delta W_{t_i}^{(1)} \\ &= v_{t_{i-1}} + \kappa(\theta - v_{t_{i-1}})\Delta t + \sqrt{v_{t_{i-1}}}\sigma_v \sqrt{\Delta t} Z_{t_i}^{(1)} \end{aligned} \tag{2.29}$$

   where $Z_{t_i}^{(1)} \sim N(0,1)$ is an independent standard Gaussian random sequence. Note that the Wiener process $\Delta W_{t_i}^{(1)} = W_{t_i}^{(1)} - W_{t_{i-1}}^{(1)} \sim N(0, \Delta t)$. However, to avoid the negative values of volatility (when $Z < 0$), we can apply the full truncation by replacing v_t with $v_t^+ = max(0, v_t)$ as the lower bound of volality is 0. Then the corresponding simulated volatility process under Euler scheme is produced as:

$$v_{t_i} = max\left[0, \left(v_{t_{i-1}} + \kappa(\theta - v_{t_{i-1}})\Delta t + \sqrt{v_{t_{i-1}}}\sigma_v \sqrt{\Delta t} Z_{t_i}^{(1)}\right)\right] \tag{2.30}$$

3. From the given correlation between $dW_t^{(1)}$ and $dW_t^{(2)}$, and assuming we have another independent simulated Wiener process $\Delta W_t^{(3)} = \sqrt{\Delta t} Z_{t_i}^{(3)}$, we can simulate the dependent Wiener process $\Delta W_t^{(2)}$ from the two independent simulated Wiener

processes $\Delta W_t^{(3)}$ and $\Delta W_t^{(1)}$ as:

$$\begin{aligned} \Delta W_{t_i}^{(2)} &= \rho \Delta W_{t_i}^{(1)} + \sqrt{1-\rho^2}\Delta W_{t_i}^{(3)} \\ &= \sqrt{\Delta t}\left(\rho \Delta Z_{t_i}^{(1)} + \sqrt{1-\rho^2}\Delta Z_{t_i}^{(3)}\right) \end{aligned} \tag{2.31}$$

4. And given the initial price S_0, the discretization of asset price process (similar in Black-Scholes case) is then defined as:

$$\begin{aligned} S_{t_i} &= S_{t_{i-1}} exp\left((r-0.5v_t)\Delta t + \sqrt{v_{t_{i-1}}}\left(\rho \Delta W_{t_i}^{(1)} + \sqrt{1-\rho^2}\Delta W_{t_i}^{(3)}\right)\right) \\ &= S_{t_{i-1}} exp\left((r-0.5v_t)\Delta t + \sqrt{v_{t_{i-1}\Delta t}}\left(\rho \Delta Z_{t_i}^{(1)} + \sqrt{1-\rho^2}\Delta Z_{t_i}^{(3)}\right)\right) \end{aligned} \tag{2.32}$$

The procedure that we have just discussed above will be used to simulate the asset price under Heston model.

Chapter 3

Nonparametric Methods In Vanilla European Option Pricing

In this chapter, we will discuss three nonparametric methods for evaluating the European vanilla option: empirical Esscher transform (EET) method, empirical characteristic function (ECF) method, and Monte Carlo (MC) method.

3.1 The EET Method

From the work of Pereira & Veiga (2017), assuming we have a random variable X that follows a probability density function $f(x)$, then the Esscher transform (ET) density function, denoted by $h(x;\theta)$, of the orginal physical density $f(x)$ with the Esscher parameter θ is defined as follow:

$$h(x;\theta) = \frac{e^{\theta x}}{\int_{-\infty}^{\infty} f(x)dx} f(x) \tag{3.1}$$

where, the reweighting function is:

$$w(x;\theta) = \frac{e^{\theta x}}{\int_{-\infty}^{\infty} e^{\theta X} f(x)dx} \tag{3.2}$$

and the moment generating function of X is:

$$M_X(\theta) = E\left[e^{\theta X}\right] = \int_{-\infty}^{\infty} e^{\theta X} f(x)dx \tag{3.3}$$

So, the Esscher transform of $f(x)$ with the parameter θ can be restated as:

$$h(x;\theta) = w(x;\theta)f(x) = \frac{e^{\theta x}}{M(\theta)}f(x) \tag{3.4}$$

Esscher transform method can be used as a tool for transforming the distribution of log-return of the underlying asset into the risk-neutral distribution, Gerber & Shiu (1994). So, if we assume X_T be the log-return of an underlying asset for a period T, then the martingale measure of the underlying asset price can be estimated by the transformed density:

$$h(x_T;\theta) = \frac{e^{\theta x_T}}{M_X(\theta)}f(x_T). \tag{3.5}$$

From the fundamental theorem of asset pricing (Bingham & Kiesel, 2004), the risk-neutral value of a derivative that was an asset itself at the initial time would be the discounted future payoff at the maturity time T by the risk free rate of return $r_T = rT$ (where r is the constant risk free interest rate) under some martingale measure Q is

$$S_0 = e^{-r_T}E^Q\left[\frac{S_T}{S_0}\right] = e^{-r_T}E^Q\left[\frac{e^{X_T}S_0}{S_0}\right] \tag{3.6}$$

$$\longrightarrow exp(r_T) = E^Q\left[e^{X_T}\right] \tag{3.7}$$

Here, S_T is the asset price at time T, and $X_T = log\left(S_T/S_0\right)$ hence $S_T = S_0e^{X_T}$.

So, if the Esscher transformed $p.d.f$ $h(x_T,\theta)$ will be used as an equivalent risk-neutral measure Q for the underlying asset, then from the above equation, we can obtain the non-arbitrage condition for the Esscher parameter θ given as:

$$\begin{aligned} exp(r_T) &= \int_{-\infty}^{\infty} e^{x_T}h(x_T;\theta)dx_T = \int_{-\infty}^{\infty} e^{x_T}\frac{e^{\theta x_T}}{M(\theta)}f(x_T)dx_T \\ &= \frac{\int_{-\infty}^{\infty} exp\left(x_T(1+\theta)\right)f(x_T)dx_T}{M_\theta} = \frac{M_{X_T}(1+\theta)}{M_{X_T}(\theta)} \end{aligned} \tag{3.8}$$

The root of this equation will give us the value of Esscher parameter that would be essentially important to obtain an equivalent risk-neutral measure value of underlying asset at time 0 under the martingale measure Q.

Hence, the risk-neutral value Q in under Esscher transform method will be given as $h(x_T,\theta^*)$ where,

$$\theta^* = arg_\theta\left(e^{r_T} = \frac{M(1+\theta)}{M(\theta)}\right) \tag{3.9}$$

In practice, most likely the case that we would obtain a random sample of size n of the asset's price $S_{T,i}$ $(i = 1, ..., n)$, by some means of simulation technique. It is suggested to consider the empirical version of Esscher transform density defined above, Pereira & Veiga (2017):

$$g_{\theta,i} = \frac{e^{\theta X_{T,i}}}{\sum_{j=1}^{n} e^{\theta X_{T,j}}} \tag{3.10}$$

where, $X_{T,i}$ is the log-return (i.e.$ln\left(\frac{S_{T,i}}{S_0}\right)$) of the relative asset price $S_{T,i}$ at time T, and $g_{\theta,i}$ is the Empirical Esscher Transform (EET) distribution of the original probability mass function $p_i = \frac{1}{n}$. We must notice here that for the empirical simulated samples, each element in the sample will be simulated independently and identically so it is reasonble to consider the mass probability for each element in the sample with equal weight (or proportion).Hence, we can rewrite the form of EET as:

$$\begin{aligned} g_{\theta,i} &= \frac{e^{\theta X_{T,i}}}{\frac{1}{n}\sum_{j=1}^{n} e^{\theta X_{T,j}}} \frac{1}{n} = \frac{e^{\theta X_{T,i}}}{\sum_{j=1}^{n} e^{\theta X_{T,j}}} \\ &= \frac{1}{\sum_{j=1}^{n} e^{\left(\theta X_{T,j} - \theta X_{T,i}\right)}} = \frac{1}{\sum_{j=1}^{n} e^{\theta\left(X_{T,j} - X_{T,i}\right)}} \end{aligned} \tag{3.11}$$

where the equivalent empirical reweighting function distribution is

$$m\left(\theta, X_{T,i}\right) = \frac{e^{\theta X_{T,i}}}{\frac{1}{n}\sum_{j=1}^{n} e^{\theta X_{T,j}}} \tag{3.12}$$

and the empirical estimator of the moment generating function would be defined accordingly in form of

$$\hat{M}(\theta) = E\left[e^{\theta X_T}\right] = \sum_{i=1}^{n} e^{\theta X_{T,i}} p_i = \frac{1}{n}\sum_{i=1}^{n} e^{\theta X_{T,i}} \tag{3.13}$$

Under the weak law of large numbers, this estimator $\hat{M}(\theta)$ will converge in probability to the true moment generating function for density function $M(\theta)$ as the sample size n is large enough, given that the sample sequence $X_{T,i}$ is $i.d.d$ and $E\left(e^{\theta X}\right) < \infty$.

Followed from Cox & Ross (1976), the price of an European Option can be defined as the discounted of future payoff at the maturity time T, denoted by $f\left(S_0, K, T\right)$, by the risk free rate r under some equivalent martingale measure Q. Hence, the general formula of European vanilla option can be given as:

$$\tilde{V}\left(S_0, K, T\right) = exp(-rT) \int_{-\infty}^{\infty} f\left(S_0, K, T\right) h(S_T, \theta^*) dS_T \tag{3.14}$$

where $h(S_T, \theta^*)$ is the risk-neutral measure (or Esscher transformed pdf) with the value of $\theta^* = arg_\theta\left(e^{rT} = \frac{M(1+\theta)}{M(\theta)}\right)$. The terminal payoff $f\left(S_0, K, T\right) = max(0, S_T - K)$ for call option, and $f\left(S_0, K, T\right) = max(0, K - S_T)$ for put option.

Similarly, we can use the indicator value j, such that $j = 1$ for call option (or $S_{T,i} \geq K$) and $j = -1$ for put option (or $S_{T,i} \leq K$), we can define the empirical version of this European option value formula (which using the discretized version of the integral above) as:

$$\begin{aligned}\tilde{V}\left(S_0, K, T\right) &= exp(-rT) \sum_{i=1}^{n} \left[\left(S_0 e^{X_{T,i}} - K\right) j\right]^+ g(X_{T,i}, \theta^*) \\ &= exp(-rT) \sum_{i=1}^{n} \left[\left(S_0 e^{X_{T,i}} - K\right) j\right]^+ \frac{1}{\sum_{k=1}^{n} e^{\theta\left(X_{T,k} - X_{T,i}\right)}}\end{aligned} \tag{3.15}$$

where $X_{T,i} = ln\left(\frac{S_{T,i}}{S_0}\right)$ is the log-return of the underlying asset after a period of time T.

The equation (3.15) above is the empirical Esscher transform method that would be used for evaluating European vanilla options. And accordingly, the non-arbitrage condition for (3.15) is defined by the equality:

$$\begin{aligned}e^{rT} &= \frac{\hat{M}(\theta+1)}{\hat{M}(\theta)} \\ &= \frac{\frac{1}{n}\sum_{i=1}^{n} e^{(\theta+1)X_{T,i}}}{\frac{1}{n}\sum_{k=1}^{n} e^{\theta X_{T,k}}} \\ &= \frac{\sum_{i=1}^{n} e^{(\theta+1)X_{T,i}}}{\sum_{k=1}^{n} e^{\theta X_{T,k}}} \\ &= \sum_{i=1}^{n} \frac{1}{\sum_{k=1}^{n} e^{\theta\left(X_{T,k} - X_{T,i}\right) - X_{T,i}}}\end{aligned} \tag{3.16}$$

and the Esscher parameter θ^* is then defined as the root of the equation:

$$\theta^* = arg_\theta \left(e^{rT} = \sum_{i=1}^{n} \frac{1}{\sum_{k=1}^{n} e^{\theta\left(X_{T,k} - X_{T,i}\right) - X_{T,i}}}\right) \tag{3.17}$$

3.2 The ECF Method

Another alternative nonparametric method, suggested by Binkowski (2008), in European option pricing is the empirical characteristic function. But before diving into the definition of characteristic function, we will first consider the basic definition of Fourier Transform.

Definition 3: (Fourier Transform) Assuming we have an integrable function $f(x)$, and the Fourier transform $\Pi(.)$ of this integrable $f(x)$ is defined as

$$\Pi\left(f(x)\right) = \int_{-\infty}^{\infty} e^{iux} f(x) dx = \varphi_X(u) \tag{3.18}$$

and the *inverse Fourier transform* which is used to recover the function f(x) is given as

$$\Pi^{-1}\left(\varphi_X(u)\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \varphi_X(u) du \tag{3.19}$$

Definition 4: (Characteristic Function) According to Schmelzle (2010) or Carr & Madan (1999), a characteristic function $\varphi_X(u)$ of a real-valued random variable X, which is defined for an arbitrary real number u, is given in the form of

$$\varphi_X(u) = E\left[e^{iuX}\right] = \int_{-\infty}^{\infty} e^{iuX} f_X(x) dx \tag{3.20}$$

From the definition of Fourier transform, we can see easily that the characteristic function is a Fourier transform of the original density function $f_X(x)$, and by using the inversion theorem of Fourier transform, we can link the characteristic function back to the original probability distribution $f(x)$.

Definition 5: (Infinitely Divisible) A real-valued random variable X is said to be infinitely divisible (i.e. has an inifnitely divisible distribution) if for each integer $n \in N$ there exists a sequence of independent and identically distributed *(i.i.d)* random variables $X_1, X_2, ..., X_n$ such that

$$X \stackrel{d}{=} X_1 + X_2 + ... + X_n \tag{3.21}$$

i.e. X has the same distribution as the $X_1 + X_2 + ... + X_n$.

As the standard Brownian motion (or Wiener process) has independnent and stationary increments, its probability distribution is also infinitely divisible. So, for any

$t > 0$ and $n \in N$, consider the decomposition of X_t into n small increments as follow:

$$X_t = (X_t - X_{t-1}) + (X_{t-1} - X_{t-2}) + ... + (X_2 - X_1) + (X_1 - X_0) \tag{3.22}$$

Then from the independence and stationary properties of the standard Brownina motion process, we can obtain the following:

$$\begin{aligned} \varphi_{X_t}(u) &= E\left[e^{iuX_t}\right] = E\left[e^{iu\sum_{j=1}^{n}(X_j - X_{j-1})}\right] = \prod_{j=1}^{t} E\left[e^{iu(X_j - X_{j-1})}\right] \\ &= \left(E\left[e^{iu(X_1 - X_0)}\right]\right)^t = (\varphi_{X_1}(u))^t \end{aligned} \tag{3.23}$$

From the work of Lewis (2001) (Theorem 3.2), Binkowski (2008) developed an empirical characteristic function in pricing European vanilla option. Similar to what we have discussed in the case of Esscher transform, We might often obtain a simulated random sample of size n of simulated data $X_1, X_2, ..., X_n$ that are independent and identically distributed with a probability distribution $f(x)$. Also assuming there exists a characteristic function $\varphi_{X,n}(u)$ (or Fourier transform), where $\{u = x + iy : Im(u) \in (a, b) \in R\}$, for this probability distribution $f(x)$ and characteristic functions at those imaginary points $x - \frac{i}{2}$ and i, then the empirical version (or discretized version) of the characteristic function $\varphi_{X,n}(u)$ can be defined as:

$$\begin{gathered} \hat{\varphi}_{X,n}(u) = \frac{1}{n}\sum_{i=1}^{n} e^{iuX_i} \\ (u \in C) \end{gathered} \tag{3.24}$$

where C is the complex plane.

Let X_T represents the log-returns of the underlying asset by maturity T that follows a Wiener process, and assume that we divide the process of asset price from initial time $t = 0$ to the time T by m small increments with the size of $\Delta t = T/m$. Then by the infinitely divisible property of the characteristic function for applied for a Wiener process, $(X_t)_{t \geq 0}$, we have the following:

$$\varphi_{X,T}(u) = (\varphi_{X,\Delta t}(u))^m \tag{3.25}$$

Hence, equivalently, we can get the empirical version of this equation as:

$$\hat{\varphi}_{X_T,n}(u) = \left(\hat{\varphi}_{\Delta t,n}\right)^m \tag{3.26}$$

where

$$\hat{\varphi}_{\Delta t,n} = \frac{1}{n}\sum_{i=1}^{n} e^{iuX_{\Delta t,i}} \tag{3.27}$$

which is an empirical characteristic function of log-returns on the short interval time length of Δt.

So, the call option value formula derived for this empirical characteristic function with the maturity time T and strike price K can be replaced as:

$$\hat{C}\left(S_0, T, K\right) = S_0 - \frac{\sqrt{S_0 K}}{\pi} exp\left(-\frac{rT}{2} + \frac{\hat{\omega}T}{2}\right) * \\ \int_{-\infty}^{\infty} Re\left[exp\left(-iu\left(log\frac{S_0}{K} + rT + \hat{\omega}T\right)\right)\hat{\varphi}_{X_T,n}\left(-u - \frac{i}{2}\right)\right]\frac{du}{u^2 + \frac{1}{4}} \tag{3.28}$$

where $\hat{\omega}$ is defined under the martingale condition as:

$$\hat{\omega} = -\frac{ln\left(\hat{\varphi}_{\Delta t,n}(-i)\right)}{\Delta t} \tag{3.29}$$

This call function is defined and the integral inside the function is also finite, as being proved by Binkowski (2008). So, the we can define the value of Put option accordingly by the Put-Call Parity relation as:

$$\hat{P}\left(S_0, T, K\right) = \hat{C}\left(S_0, T, K\right) - S_0 + Ke^{-rT} \tag{3.30}$$

3.3 Monte Carlo Method

Monte Carlo method is a very useful technique in identifying whether the option's valuation under Black-Scholes and Heston models are appropriate. The working scheme of Monte Carlo is based on the strong law of large numbers.

Definition 6 (Strong Law of Large Numbers): Assuming we have a sequence of independent and identically distributed random variables $(X_i)_{i\geq 1}$ and further assuming that the expected value of these variables are finite, i.e.

$$E(X_i) = E(X_1) = \mu < \infty$$

Then, the strong of large numbers will hold if the empirical mean of $(X_1, X_2, ..., X_n)$ converges to the theoretical expected value of every X_i almost surely, i.e.

$$P(lim_{n\to\infty}\hat{X}_n = \mu) = 1$$

where

$$\hat{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

So, theoretically, followed by R. Korn, Korn, & Kroisandt (2010), the Monte carlo estimator, i.e.

$$\hat{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

is an unbiased estimator for $E(X_1) = \mu$, i.e. for $n \to \infty$ we will have

$$E(\hat{X}_n) = \mu Var(\hat{X}_n) = 0$$

In our experiments, let $S_{T,i}$ be the simuated asset price at maturity time T, then our goal is to estimate the $E[f(X_{T,i})]$ where the simulated random samples $(S_{T,1}, ..., S_{T,n})$ are $i.i.d$, and the Monte Carlo estimator would be defined as

$$E\left[f(S_{T,i})\right] = \frac{1}{n}\sum_{i=1}^{n} f(S_{T,i}) \tag{3.31}$$

Here, the $f(S_{T,i})$ would be the terminal payoff of the option at maturity T, i.e. for

$$f(S_{T,i}) = max\left[(S_{T,i} - K) * j, 0\right] \tag{3.32}$$

where,$j = 1$ for call option, and $j = -1$ for put option. And the corresponding option value V_n will be evaluated as the discounted value of $E\left[f(S_{T,i})\right]$ at the assumed constant risk-free rate r, i.e.

$$V_n = e^{-rT}E\left[f(S_{T,i})\right] = e^{-rT}\frac{1}{n}\sum_{i=1}^{n} f(S_{T,i}) \tag{3.33}$$

The general procedure for Monte Carlo process can be conducted as follow:

1. Generate an independent and identically distributed sample of n terminal asset price $S_{T,i}$, $i = 1, 2, ..., n$. The simulation process can be conducted under Euler scheme discussed in chapter 2 for Black-Scholes price process and Heston price process, respectively.
2. Given $j = 1$ for call price and $j = -1$ for put optiob, for each $i = 1, 2, ..., n$, we determine the discounted value of the terminal payoff as:

$$C_i = e^{-rT}[j(S_{T,i} - K)]^+ \tag{3.34}$$

3. Then the Monte Carlo price of an European call (or put) option is evaluated as:

$$C_{MC} = \frac{1}{N}\sum_{i=1}^{N} C_i \tag{3.35}$$

The variance of simulated terminal payoffs $Var[f(S_{T,i})]$ can be estimated by

$$\hat{\sigma}^2_{n-1} = \frac{1}{n-1}\sum_{i=1}^{n}[f(S_{T,i}) - V_n]^2 \tag{3.36}$$

To help us getting the feeling for the absolute error, the estimated standard deviation of the Monte Carlo estimator, which is the deviation of the difference between V_n and μ, is defined as the square root of:

$$\begin{aligned} Var(V_n) &= \frac{\frac{1}{n-1}\sum_{i=1}^{n}[f(S_{T,i}) - V_n]^2}{n} \\ &= \frac{\sum_{i=1}^{n}[f(S_{T,i}) - V_n]^2}{n(n-1)} \end{aligned} \tag{3.37}$$

Chapter 4

Numerical Results

For this chapter, numerical results from the two experiments will be reported and analysed, including experiment 1 comparing BSM model with the two nonparametric methods (EET and ECF) and experiment 2 comparing Heston model with the 2 nonparametric methods (EET and ECF).

4.1 Testing EET and ECF Methods in BSM Model

For the first experiment, we will use the spot asset price $S_0 = 41.13$, and the risk free rate is $r = 0.05$. We also assume the constant volatility of asset return in BSM model is $\sigma = 0.2$. The table 1 conducted below compares European option prices evaluated by Black-Scholes Merton model (BSM), Monte Carlo method (MC), empirical Esscher transform method (EET), and empirical characteristic function method (ECF), respectively, throughout the range of strikes $K = \{35, 41.13, 45\}$ and the range of maturities $T = \left\{\frac{1}{2}, \frac{2}{3}, 1, \frac{3}{2}\right\}$. The sample size of 300 simulated asset prices will be used to evaluate call prices in MC, EET and ECF methods. The number of time steps to simulate one terminal stochastic asset price is 1000. The figure of table is shown below:

Table 4.1: Call prices using BSM model, MC, EET and ECF methods with simulated sample size 300, and 1000 time-steps.

Time to Maturities	Strike Prices (Moneyness)	BSM	MC (Standard Deviation)	EET	ECF
T = 1/2	K = 35 (1.175)	7.2259	7.0060 (0.3005)	7.2200	7.2154
	K = 41.13 (1)	2.8333	2.5950 (0.2188)	2.7334	2.8115
	K = 45 (0.914)	1.2645	1.1001 (0.1469)	1.1782	1.2465
T = 2/3	K = 35 (1.175)	7.6239	7.8076 (0.3714)	7.6831	7.6065
	K = 41.13 (1)	3.3614	3.4356 (0.2934)	3.3481	3.3289
	K = 45 (0.914)	1.7204	1.8027 (0.2256)	1.7439	1.6909
T = 1	K = 35 (1.175)	8.3878	8.5139 (0.4331)	8.4447	8.3700
	K = 41.13 (1)	4.2983	4.4302 (0.3420)	4.3806	4.2641
	K = 45 (0.914)	2.5715	2.6629 (0.2700)	2.6276	2.5344
T = 3/2	K = 35 (1.175)	9.4520	8.2282 (0.4887)	9.3410	9.5383
	K = 41.13 (1)	5.5291	4.5561 (0.3969)	5.3975	5.6692
	K = 45 (0.914)	3.7376	2.9899 (0.3276)	3.6363	3.8908

From the table above, the call prices evaluated under the three methods and the classic BSM model are quite similar throughout all different time maturities (from $T = 1/2$ to $T = 3/2$) and strike prices (from $K = 35$ to $K = 45$). However, the performance of the Monte Carlo method seems not to be very consistent. This might be due to the effect of small simulated sample size. Lets have a closer look at the corresponding relative risk measure for these experimental prices under the three methods compared to the BSM model prices.

Table 4.2: Relative Errors (%) in call prices of MC, EET, and ECF methods compared to BSM model, by using simulated sample of size 300 and 1000 time-steps.

Time to Maturities	Strike Prices (Moneyness)	MC (Standard Deviation)	EET	ECF
T = 1/2	K = 35 (1.175)	3.04 (0.3005)	0.083	0.147
	K = 41.13 (1)	8.41 (0.2188)	3.526	0.769
	K = 45 (0.914)	13.04 (0.1469)	6.862	1.462
T = 2/3	K = 35 (1.175)	2.41 (0.3714)	0.775	0.230
	K = 41.13 (1)	2.21 (0.2934)	0.396	0.967
	K = 45 (0.914)	7.67 (0.2256)	1.7439	1.6909
T = 1	K = 35 (1.175)	1.5 (0.4331)	0.676	0.215
	K = 41.13 (1)	3.07 (0.3420)	1.915	0.796
	K = 45 (0.914)	3.57 (0.2700)	2.201	1.424
T = 3/2	K = 35 (1.175)	12.95 (0.4887)	1.174	0.913
	K = 41.13 (1)	17.6 (0.3969)	2.380	2.534
	K = 45 (0.914)	6.74 (0.3276)	2.721	4.088

From the table of relative risk errors shown above, empirical Esscher transform and empirical characteristic function methods produced the prices that are quite similar to what would be obtained under BSM model, while the Monte Carlo method with 300 simulations seems not to be effective as its relative error sizes are quite high compared to the other two, especially when $T = 3/2$. Besides, we can notice that the errors among the three methods become larger as the option is “at” or “out of the money”. And when the option is “in the money”, their prices produced will be closer to the BSM prices.

Now, lets increase the sample size of simulated terminal asset price to 5000, and repeat the above experiment. We obtained the results as below:

Table 4.3: Call prices using BSM model, MC, EET, and ECF methods with simulated sample of size 5000, and 1000 time-steps.

Time to Maturities	Strike Prices (Moneyness)	BSM	MC (Standard Deviation)	EET	ECF
T = 1/2	K = 35 (1.175)	7.2259	7.1873 (0.0767)	7.2029	7.2326
	K = 41.13 (1)	2.8333	2.8031 (0.0561)	2.8132	2.8504
	K = 45 (0.914)	1.2645	1.2441 (0.0383)	1.2499	1.2808
T = 2/3	K = 35 (1.175)	7.6239	7.5765 (0.0877)	7.6372	7.6479
	K = 41.13 (1)	3.3614	3.3236 (0.0663)	3.3647	3.4130
	K = 45 (0.914)	1.7204	1.6901 (0.0488)	1.7164	1.7725
T = 1	K = 35 (1.175)	8.3878	8.4331 (0.1057)	8.3842	8.4009
	K = 41.13 (1)	4.2983	4.3019 (0.0845)	4.2664	4.3222
	K = 45 (0.914)	2.5715	2.5523 (0.0677)	2.5268	2.5966
T = 3/2	K = 35 (1.175)	9.4520	9.6818 (0.1340)	9.4468	9.4647
	K = 41.13 (1)	5.5291	5.7243 (0.1134)	5.5411	5.5489
	K = 45 (0.914)	3.7376	3.9074 (0.0975)	3.7619	3.7588

and the corresponding table of relative errors:

Table 4.4: Relative Errors (%) in call prices of MC, EET, and ECF methods compared to BSM model, by using simulated sample of size 5000 and 1000 time-steps.

Time to Maturities	Strike Prices (Moneyness)	MC (Standard Deviation)	EET	ECF
T = 1/2	K = 35 (1.175)	0.5356 (0.0767)	0.3197	0.0913
	K = 41.13 (1)	1.0659 (0.0561)	0.7094	0.6035
	K = 45 (0.914)	1.6522 (0.0383)	1.1937	1.2490
T = 2/3	K = 35 (1.175)	0.6230 (0.0877)	0.1731	0.3135
	K = 41.13 (1)	1.1245 (0.0663)	0.0982	1.5351
	K = 45 (0.914)	1.7612 (0.0488)	0.2325	3.0284
T = 1	K = 35 (1.175)	0.5377 (0.1057)	0.0453	0.1538
	K = 41.13 (1)	0.0838 (0.0845)	0.7422	0.5560
	K = 45 (0.914)	0.7466 (0.0677)	1.7383	0.9761
T = 3/2	K = 35 (1.175)	2.4312 (0.1340)	0.0550	0.1344
	K = 41.13 (1)	3.5304 (0.1134)	0.2170	0.3581
	K = 45 (0.914)	4.5318 (0.0975)	0.6394	0.5564

From the results obtained, as increasing the sample zise to 5000, the Monte Carlo method tends to produce the call prices that is quite close to the BSM prices. Besides, the relative errors in prices obtained by EET and ECF methods compared to BSM model have been reduced, especially when option is out of the money (i.e. when $K > S_0 = 41.13$). So, as the sample size increases, the MC, EET and ECF methods tend to reproduce the BSM prices. The behavior of EET prices is consistent to what was expected from the work of Gerber & Shiu (1994) or Pereira & Veiga (2017) as the sample size increases.

4.2 Testing EET and ECF Methods in Heston Model

For this second experiment, we will fix the parameters of Heston model based on the values suggested from the work of Pereira M. and Veiga A.(2017) with $\kappa = 3$, $\theta = 0.04$, $v_0 = 0.04$, $\sigma = 0.4$, $\rho = -0.5$, and $r = 0.05$. Note that σ here is the standard deviation of the stochastic volatility process.

The implied volatility surface plot of option call prices from Heston model with these fixed parameters is shown as below:

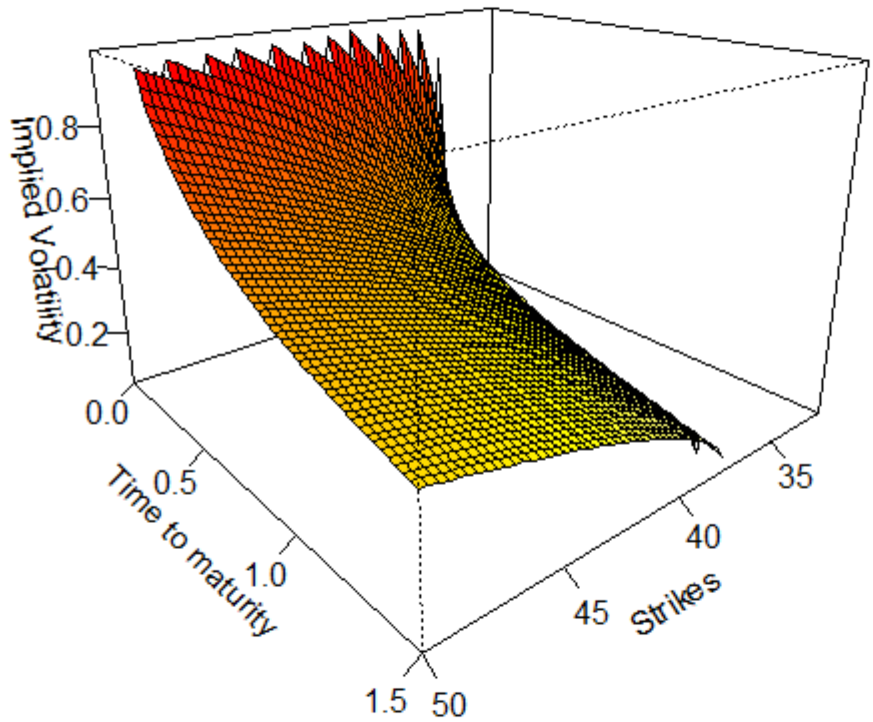

Figure 4.1: Implied Volatility of Heston model

Similar to what we observed in BSM prices, the implied volatility surface plot of call prices from Heston model also has the smile effect. And this is indeed true as the volatility in real market is not constant, and so the surface could not be flat. This would indicate the modelling of volatility process in Heston model would be more realistic.

Lets compare the prices of call options under Heston model with those obtained under Black-Scholes model. The values of parameters are fixed at $S_0 = 41.13$, $\kappa = 3$, $\theta = 0.04$, $v_0 = 0.04$, $\sigma = 0.4$, $\rho = -0.5$, and $r = 0.05$. We note here that the Heston price equals BSM price when the correlation $\rho = 0$ and standard deviation of variance process $\sigma = 0$, this woud lead to the volatility in asset price process equals the initial volaility v_0. So, in Black-Scholes world the volatility of asset price process is $v_0 = 0.04$:

Table 4.5: Call prices using Heston model and BSM model

Time to Maturities	Strike Prices (Moneyness)	Heston	BSM
T = 1/2	K = 35 (1.175)	7.3431	6.9942
	K = 41.13 (1)	2.7911	1.1343
	K = 45 (0.914)	1.0804	0.0045
T = 2/3	K = 35 (1.175)	7.7568	7.2774
	K = 41.13 (1)	3.3162	1.4542
	K = 45 (0.914)	1.5199	0.0233
T = 1	K = 35 (1.175)	8.5336	7.8370
	K = 41.13 (1)	4.2569	2.0871
	K = 45 (0.914)	2.3681	0.1403
T = 3/2	K = 35 (1.175)	9.6022	8.6590
	K = 41.13 (1)	5.5031	3.0249
	K = 45 (0.914)	3.5577	0.5379

The results show that by taking the process of volatility into account, the call prices obtained under Heston model is higher the those obtained under BSM model for any specific combination of time maturity and strike price. Note here that those Heston prices are obtained by assuming the price of volatility risk is 0 (i.e. $\lambda = 0$). For any estimation of $\lambda > 0$ in the market, the theoretical Heston call price would be higher than those obtained in the table above.

Lets assuming further that we will use a sample size of 300 simulated asset prices, obtained from the Euler discretization process with 1000 time steps, to evaluate call prices under MC, EET and ECF methods. In ECF method, the Δt we used is still 1/252 (i.e. each trading day). The table comparing the Heston prices with 3 nonparametric methods, MC, EET and ECF, from the given fixed parameters is shown below:

Table 4.6: Call prices using Heston model, MC, EET and ECF methods with simulated sample of size 300, and 1000 time-steps.

Time to Maturities	Strike Prices (Moneyness)	Heston	MC (Standard Deviation)	EET	ECF
T = 1/2	K = 35 (1.175)	7.3431	7.7684 (0.3035)	7.4602	7.2406
	K = 41.13 (1)	2.7911	3.1114 (0.2234)	2.9187	2.8705
	K = 45 (0.914)	1.0804	1.2632 (0.1541)	1.1576	1.3001
T = 2/3	K = 35 (1.175)	7.7568	7.6281 (0.3334)	7.8574	7.5908
	K = 41.13 (1)	3.3162	3.3366 (0.2371)	3.4792	3.2831
	K = 45 (0.914)	1.5199	1.4904 (0.1657)	1.5724	1.6388
T = 1	K = 35 (1.175)	8.5336	8.7628 (0.4092)	8.5259	8.3279
	K = 41.13 (1)	4.2569	4.5574 (0.3158)	4.3927	4.1877
	K = 45 (0.914)	2.3681	2.6381 (0.2420)	2.5259	2.4552
T = 3/2	K = 35 (1.175)	9.6022	9.9769 (0.4975)	9.5915	9.4612
	K = 41.13 (1)	5.5031	5.7038 (0.4212)	5.4098	5.5491
	K = 45 (0.914)	3.5577	3.7251 (0.3573)	3.499	3.7625

From the table, it seems likely to be the case that the prices obtained under EET and ECF methods are quite similar and close to the values of prices obtained under Heston model while the Monte Carlo method produced the prices that are quite different from the Heston prices as its relative errors are quite high. Lets take a closer look at the corresponding relative errors table below:

Table 4.7: Relative Errors (%) in call prices of MC, EET, and ECF methods compared to Heston model, by using simulated sample of size 300 and 1000 time-steps.

Time to Maturities	Strike Prices (Moneyness)	MC (Standard Deviation)	EET	ECF
T = 1/2	K = 35 (1.175)	5.792 (0.3035)	1.5947	1.396
	K = 41.13 (1)	11.476 (0.2234)	4.5717	2.845
	K = 45 (0.914)	16.920 (0.1541)	7.1455	20.335
T = 2/3	K = 35 (1.175)	1.659 (0.3334)	1.2969	2.140
	K = 41.13 (1)	0.615 (0.2371)	4.9153	0.998
	K = 45 (0.914)	1.941 (0.1657)	3.4542	7.823
T = 1	K = 35 (1.175)	2.686 (0.4092)	0.0902	2.410
	K = 41.13 (1)	7.059 (0.3158)	3.1901	1.626
	K = 45 (0.914)	11.402 (0.2420)	6.6636	3.678
T = 3/2	K = 35 (1.175)	3.902 (0.4975)	0.1114	1.468
	K = 41.13 (1)	3.647 (0.4212)	1.6954	0.836
	K = 45 (0.914)	4.705 (0.3573)	1.6499	5.757

In overall, from the errors reported in the table, the longer the maturity or the lower the strike price, the better the approximations of Heston call prices obtained under the 3 discussed methods as the error sizes decrease. Among the three methods, EET seems to be the most effective method that can generate the price with low error values for all combinations of strike price and time maturity considered. Monte Carlo and empirical characteristic function methods performed poorly, especially when $K = 45$ and $T = 1/2$.

We now repeat the experiment with the simulated sample of size 5000. Theoretically, by increasing the sample size the error values of MC and EET should be reduced. The results are reported below:

Table 4.8: Call prices using Heston model, MC, EET and ECF methods with simulated sample of size 5000, and 1000 time-steps.

Time to Maturities	Strike Prices	Heston	MC (Standard Deviation)	EET	ECF
T = 1/2	K = 35	7.3431	7.2959 (0.0705)	7.3418	7.2271
	K = 41.13	2.7911	2.7582 (0.0493)	2.7859	2.8350
	K = 45	1.0804	1.0611 (0.0312)	1.0752	1.2655
T = 2/3	K = 35	7.7568	7.7721 (0.0807)	7.7757	7.6406
	K = 41.13	3.3162	3.3191 (0.0596)	3.3214	3.3927
	K = 45	1.5199	1.5223 (0.0422)	1.5237	1.7494
T = 1	K = 35	8.5336	8.3416 (0.0942)	8.5546	8.3963
	K = 41.13	4.2569	4.1057 (0.0719)	4.2502	4.3100
	K = 45	2.3681	2.2403 (0.0543)	2.3359	2.5816
T = 3/2	K = 35	9.6022	9.7406 (0.1191)	9.5658	9.4505
	K = 41.13	5.5031	5.6100 (0.0981)	5.4792	5.5240
	K = 45	3.5577	3.6508 (0.0817)	3.5516	3.7304

and we also have the corresponding table of relative errors reported as:

Table 4.9: Relative Errors (%) in call prices of MC, EET, and ECF methods compared to Heston model, by using simulated sample of size 5000 and 1000 time-steps.

Time to Maturities	Strike Prices (Moneyness)	MC (Standard Deviation)	EET	ECF
T = 1/2	K = 35 (1.175)	0.6428 (0.3035)	0.0177	1.58
	K = 41.13 (1)	1.1787 (0.2234)	0.1863	1.57
	K = 45 (0.914)	1.7864 (0.1541)	0.4813	17.13
T = 2/3	K = 35 (1.175)	0.1972 (0.3334)	0.2437	1.5
	K = 41.13 (1)	0.0874 (0.2371)	0.1568	2.31
	K = 45 (0.914)	0.1579 (0.1657)	0.2500	15.1
T = 1	K = 35 (1.175)	2.2499 (0.4092)	0.2461	1.61
	K = 41.13 (1)	3.5519 (0.3158)	0.1574	1.25
	K = 45 (0.914)	5.3967 (0.2420)	1.3597	9.02
T = 3/2	K = 35 (1.175)	1.4413 (0.4975)	0.3791	1.58
	K = 41.13 (1)	1.9425 (0.4212)	0.4343	0.38
	K = 45 (0.914)	2.6169 (0.3573)	0.1715	4.85

By increasing the sample size to 5000, the errors produced by MC method in overall have been reduced significantly, and ranges from about 0.0874% to about 5.4%. The values of errors produced by EET method are also reduced compared to those obtained by using the sample of size 300. In other words, the both the call values obtained under MC and EET methods are quite similar to the one obtained under Heston model. This is also expected from the work of Pereira & Veiga (2017). The performance of empirical characteristic function also improves slightly than before, but there are still records of large relative errors observed at the combination of $\left(T = \frac{1}{2}, K = 45\right)$ (with error of 17.13%), $\left(T = \frac{2}{3}, K = 45\right)$ (with error of 15.10%), and $(T = 1, K = 45)$ (with error of 9.02%).

As shown above, when increasing the sample size to 5000 (with 1000 time steps of simulation process), the prices obtained under empirical characteristic function are still different from those Heston prices when the option is “out of the money”. Notice here, for $K = 45$, the error size produced by ECF method decreases when the time maturity T increases. So, it is suggested to extend the length of time maturity and observe the behavior of the price produced by ECF compared to the corresponding Heston price.

Fixing $K = 45$ and the set of time maturities $T = \left\{\frac{1}{2}, \frac{2}{3}, 1, \frac{3}{2}, 2, 3, 4, 5\right\}$, by using simulated sample of size 5000 to evaluate the ECF prices, we obtain:

Table 4.10: Call prices using Heston model and ECF method with simulated sample of size 5000, and 1000 time-steps over time maturities 1/2 to 5.

Strike Price (Moneyness)	**Time to Maturities**	**Heston Call Prices**	**ECF Call Prices**
K = 45 (0.914)	T = 1/2	1.0804	1.2794
	T = 2/3	1.5199	1.7389
	T = 1	2.3681	2.5956
	T = 3/2	3.5577	3.7680
	T = 2	4.6623	4.8465
	T = 3	6.6788	6.8129
	T = 4	8.5045	8.5985
	T = 5	10.1859	10.2486

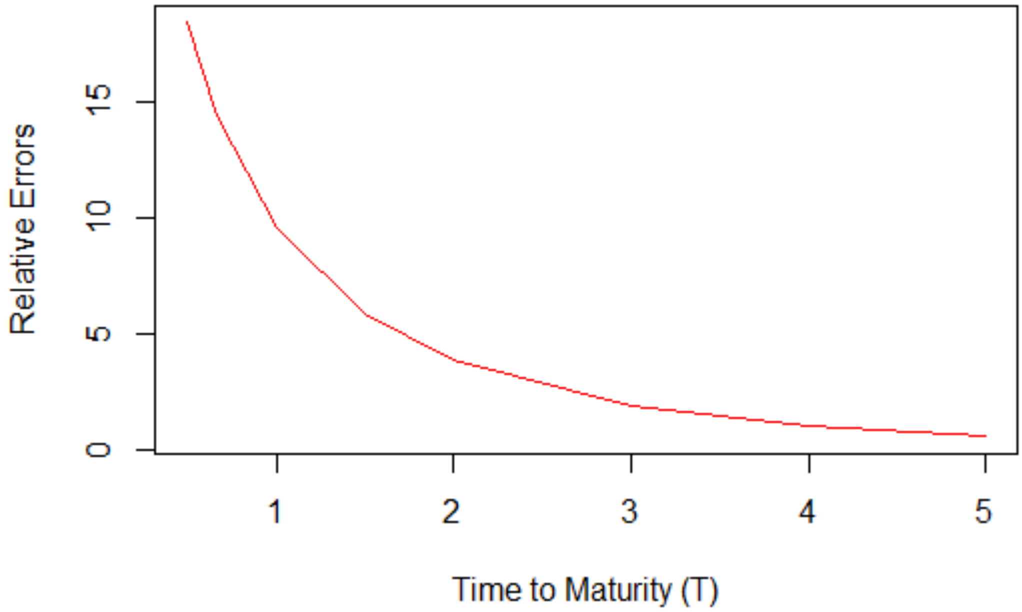

Figure 4.2: Plot of relative errors of ECF call prices vs Heston call prices from table 4.10

In summary, the prices produced by ECF method are higher than the Heston prices. However, the size of relative error reduces as the time to maturity T increases. In addition we found that the ECF method is likely to reproduce the Heston call prices for “out of the money” European type options when the time to maturity is sufficiently large.

Chapter 5

Conclusion

In this project, two numerical experiments have been conducted to compare the empirical Esscher transform (EET) and empirical characteristic function (ECF) nonparametric methods for pricing European vanilla options in BSM and Heston models for which analytical pricing formulae are available in closed and semi-closed form respectively. In each model, the Monte Carlo simulation procedure has been established using Euler's discretization algorithm.

In each model we established the separate numerical experiment where we tested the results obtained based on EET and ECF methods. In BSM model, EET outperformed ECF method in terms of accuracy of prices of European Call options computed using BSM formula which was also used for debugging the code for MC simulations. The more the call is out of the money, the less efficient EET accuracy improvement becomes.

In Heston model, EET, ECF and MC methods tend to reproduce the prices which were obtained by implementation of semi-closed analytical formulae developed by Heston (1993) as the sample size increases. In Heston's model, surprisingly, the EET method outperformed ECF method when tested on prices of European call prices. However, MC method remained the best out of all three methods even though it was rather time consuming to use. We also looked at the problem occuring with Heston's semi-closed analytical formula for far "out of money" European call options.

Further testing is required to explore the advantage of using EET method versus ECF and MC methods for pricing of European vanilla options as well as evaluation of implied volatility surfaces in Heston's model.

Appendix A

Proof of Equations

A.1 Ito's lemma

Given any function $f(t, x)$ that is continuously differentiable at t and twice continuously differentiable at x, then the for any diffusion process X_t that is defined as equation *(2.2)*, we will have the following result:

$$df\left(t, X_t\right) = \left[\frac{\partial f\left(t, X_t\right)}{\partial t} + \frac{1}{2}\frac{\partial^2 f\left(t, X_t\right)}{\partial x^2}\right] dt + \frac{\partial f\left(t, X_t\right)}{\partial x} dW_t$$

A.2 Equation (2.3): Solutions of Geometric Brownian Motion

To prove the equation of form (2.3) is a solution of the diffusion process:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where $dW_t \sim N(0, \Delta t)$, we will apply the Ito's rule on the standard Brownian motion factor. Lets consider the twice differentiable function

$$f(t, x) = S_0 * exp\{(\mu - 0.5\sigma^2)t + \sigma x\}$$

Then, we have the corresponding results:

$$\frac{\partial f(t,x)}{\partial x} = \sigma * S_0 * exp\{(\mu - 0.5\sigma^2)t + \sigma x\} = \sigma * f(t,x)$$
$$\frac{\partial^2 f(t,x)}{\partial x^2} = \sigma^2 * S_0 * exp\{(\mu - 0.5\sigma^2)t + \sigma x\} = \sigma^2 * f(t,x)$$
$$\frac{\partial f(t,x)}{\partial t} = (\mu - 0.5\sigma^2) * S_0 * exp\{(\mu - 0.5\sigma^2)t + \sigma x\} = (\mu - 0.5\sigma^2) * f(t,x)$$

and thus by the Ito's lemma, we have:

$$\begin{aligned} df(t,x) &= \left\{\frac{\partial f(t,x)}{\partial t} + 0.5 * \frac{\partial^2 f(t,x)}{\partial x^2}\right\} dt + \frac{\partial f(t,x)}{\partial x} dW_t \\ &= \{(\mu - 0.5\sigma^2) * f(t,x) + 0.5 * \sigma^2 * f(t,x)\}dt + \sigma * f(t,x) * dW_t \\ &= \mu * f(t,x) * dt + \sigma * f(t,x) * dW_t \end{aligned}$$

Thus, we will have:

$$S_t = f(t,x) = S_0 * exp\{(\mu - 0.5\sigma^2)t + \sigma x\}$$

Appendix B

R Codes

B.1 R code for equation (2.7): Simulation of Terminal Asset Price in BSM

```
P.simp = function(S0,mu,sig,t,n,m) {
  # S0: initial asset value
  # mu: drift of the diffussion process
  # sig: volatility of asset price (assumed constant)
  # t: the time period of interest
  # n: number of simulated final asset prices
  # m: number of time steps of the brownian process
  ST=matrix(0,nrow=n,ncol=1)
  for (i in 1:n){
    St=matrix(0,nrow=m,ncol=1)
    for (j in 1:m){
      if (j==1){
        St[j]=S0*exp((mu-0.5*sig^2)*t/m+sig*sqrt(t/m)*rnorm(1,0,1))
      } else{
        St[j]=St[j-1]*exp((mu-0.5*sig^2)*t/m+sig*sqrt(t/m)*rnorm(1,0,1))
      }
    }
    ST[i]=St[m]
  }
```

```
  return(ST)   # return vector of simulated final asset value
}
```

B.2 R codes for equation (2.12): Black-Scholes Merton Model

```
BSM=function(S0,K,r,t,sigma,j){
  # S0: initial asset price
  # K: Strike Price
  # r: risk-free rate
  # t: time to maturity
  # sigma: volatility of asset price (assumed constant)
  # j: j=1 for "Call Option" , and j=-1 for "Put Option"

  if (abs(j)==1){
    d2=(log(S0/K)+(r-0.5*sigma^2)*t)/(sigma*sqrt(t))
    d1=d2+sigma*sqrt(t)
    Price=j*S0*pnorm(j*d1)-j*K*exp(-r*t)*pnorm(j*d2)
    return(Price)
  } else{
    return(NA)
  }
}
```

B.3 R codes for equation (2.17 & 2.18): Simulation of Terminal Asset Price in Heston case

```
P.Hes = function(m,S0,r,v0,t,n,rho,sigma,k,theta) {
  # m: the sample size of simulation
  # S0: intial asset price
  # r: risk-free rate (assumed constant)
  # v0: intial volatility of asset's return
```

```
# t: time to maturity
# n: number of time steps in Wiener process
# rho: correlation coefficients
# sigma: square root of variance of asset return's volatility
# k: speed of mean reversion
# theta: long run mean of volatility

# (a) Create Euler discretization process to generate 1 terminal asset price
Price=function(S0,r,v0,t,n,rho,sigma,k,theta){
  W=rnorm(n,0,1)  # an independent Wiener process
  Z=rnorm(n,0,1)  # an independent Wiener process in volatility process
  W.hat=rho*Z+sqrt(1-rho^2)*W # Wiener process in asset price process

  # Simulate the volatility process
  v=matrix(0,nrow=n,ncol=1)
  for (i in 1:n){
    if (i==1){
      v[1]=max(0,v0+k*(theta-v0)*t/n+sigma*sqrt(v0)*sqrt(t/n)*Z[1])
    } else{
      v[i]=max(0,v[i-1]+k*(theta-v[i-1])*t/n+
                 sigma*sqrt(v[i-1]*t/n)*sqrt(t/n)*Z[i])
    }
  }

  # Simulate the asset price process
  St=matrix(0,nrow=n,ncol=1)
  for (i in 1:n){
    if (i==1){
      St[i]=S0*exp((r-0.5*v0)*t/n+sqrt(v0)*sqrt(t/n)*W.hat[i])
    } else {
      St[i]=St[i-1]*exp((r-0.5*v[i-1])*t/n+sqrt(v[i-1])*sqrt(t/n)*W.hat[i])
    }
  }
```

```
    ST=St[n]
    return(ST)
  }
  # (b) replicate the Euler discretization function above m times of simulation
  St.sample=replicate(m,Price(S0,r,v0,t,n,rho,sigma,k,theta))
  return(St.sample)
}
```

B.4 R code for equation (2.20): Heston model

```
HesOV=function(S0,K,t,r,v0,theta,rho,kappa,sig,j){
  # S0: initial asset price
  # K: Strike price
  # t: time to maturity
  # r: risk free rate
  # v0: initial volatility
  # theta: long run mean of volatility
  # rho: correlation coefficient
  # kappa: speed of mean reversion
  # sig: square root of variance of volatility
  # j: j=1 for "Call Option" , and j=-1 for "Put Option"

  if (abs(j)==1){
  P1=function(ohm,S0,K,t,r,v0,theta,rho,kappa,sig){
    i=1i
    p=Re(exp(-i*log(K)*ohm)*cfHeston(ohm-i,S0,t,r,v0,
                                   theta,rho,kappa,sig)/(i*ohm*S0*exp(r*t)))
    return(p)
  }
  P2=function(ohm,S0,K,t,r,v0,theta,rho,kappa,sig){
    i=1i
    p=Re(exp(-i*log(K)*ohm)*cfHeston(ohm,S0,t,r,v0,theta,rho,kappa,sig)/(i*ohm))
    return(p)
```

```
  }
  cfHeston=function(ohm,S0,t,r,v0,theta,rho,kappa,sig){
    d=sqrt((rho*sig*1i*ohm-kappa)^2+sig^2*(1i*ohm+ohm^2))
    g2=(kappa-rho*sig*1i*ohm-d)/(kappa-rho*sig*1i*ohm+d)
    cf1=1i*ohm*(log(S0)+r*t)
    cf2=theta*kappa/(sig^2)*((kappa-rho*sig*1i*ohm-d)*t-
                                  2*log((1-g2*exp(-d*t))/(1-g2)))
    cf3=v0/sig^2*(kappa-rho*sig*1i*ohm-d)*(1-exp(-d*t))/(1-g2*exp(-d*t))
    cf=exp(cf1+cf2+cf3)
    return(cf)
  }
  vP1=0.5+j*1/pi*integrate(P1,lower = 0,upper=200,S0,K,t,r,v0,theta,
                             rho,kappa,sig)$value
  vP2=0.5+j*1/pi*integrate(P2,lower = 0,upper=200,S0,K,t,r,v0,theta,
                             rho,kappa,sig)$value
  price=j*S0*vP1-j*exp(-r*t)*K*vP2
  return(price)
  } else{
    return(NA)
  }
}
```

B.5 R codes for equation (3.16): Risk-Neutral Esscher Parameter

```
Esscher.par=function(r,t,S0,St.sample,a,b){
  # r: risk-free rate (assuemd constant)
  # t: time to maturity (in fraction of years)
  # S0: initial asset price
  # St.sample: simulated terminal asset price sample
  # a: lower bound for interval search
  # b: upper bound for interval search
```

```
  n=length(St.sample)
  Xt=log(St.sample/S0)  # sample of log-returns

  # Use 'uniroot' to find optimal solution of Esscher Parameter
  Esscher=function (theta){
    f1=matrix(0,ncol=1,nrow=length(Xt))
    for (i in 1:length(Xt)){
      f=matrix(0,ncol=1,nrow=length(Xt))
      for (j in 1:length(Xt)){
        f[j]=exp(theta*(Xt[j]-Xt[i])-Xt[i])
      }
      f1[i]=1/sum(f)
    }
    f2=exp(r*t)-sum(f1)
    return(f2)
  }
  if ((Esscher(a)*Esscher(b))>0){
    return(NA)
  } else {
    theta=uniroot(Esscher,interval=c(a,b))$root
    return(theta)
  }
}
```

B.6 R codes for equation (3.14): Esscher Function for European Prices

```
EET.P= function(K,S0,St.sample,t,r,j,a,b) {
  # K: Strike price
  # S0: initial asset price
  # St.sample: simulated terminal asset price sample
```

```
  # t: time to maturity (in fraction of years)
  # r: risk-free rate (assumed constant)
  # j: 1 (for call option), or -1 (for put option)

  n=length(St.sample) # n: size of simulated sample
  theta=Esscher.par(r,t,S0,St.sample,a,b) # find the risk-neutral
                                          #     Esscher parameter
  Xt=log(St.sample/S0) # log-return
  q=matrix(0,nrow=n,ncol=1)
  for (i in 1:n){
    f=matrix(0,ncol=1,nrow=n)
    for (k in 1:n){
      f[k]=exp(theta*(Xt[k]-Xt[i]))
    }
    q[i]=1/sum(f)  # Esscher transformed risk-neutral measure
  }
  if (abs(j)==1){
    Pay.off=matrix(0,nrow=n,ncol=1)
    for (l in 1:n){
      Pay.off[l]=max(j*(exp(Xt[l])*S0-K),0)
    }
    Price=sum(Pay.off*q)*exp(-r*t) # European Option Price
    return(Price)
  } else {
    return(NA)
  }
}
```

B.7 R codes for equation (3.27 & 3.28 & 3.29): Empirical Characteristic Function

```
ECF=function(S0,K,St.sample,r,t,delta,j){
  # S0: initial asset price
  # K: Strike price
  # St.sample: sample of the simulated asset price over 'delta t' time period
  # r: annual risk-free rate
  # t: time to maturity (in fraction of years)
  # delta: size of time increment (in fraction of years) between observations
  # j : 1 for "call option", or -1 for "put option"

  if (abs(j)==1){
  X=log(St.sample/S0)
  p=t/delta  # number of time increments
  n=length(X)
  i=1i
  CF.hat=matrix(0,nrow=n,ncol=1)
  for (s in 1:n){
    CF.hat[s]=(1/n)*exp(X[s])  # ECF of log returns
  }
  ohm.h=-log(sum(CF.hat))/delta # martingale condition (equation 3.28)

  # The integrand inside equation (3.27)
  Integrand=function(u){
    CF.n=matrix(0,nrow=n, ncol=1)
    for(s in 1:n){
      CF.n[s]=(1/n)*exp((-i*u+1/2)*X[s])
    }
    CFT=(sum(CF.n))^p
    Int=Re(exp(-i*u*(log(S0/K)+r*t+ohm.h*t))*CFT)/(u^2+1/4)
    return(Int)
  }
```

```
  # Option price under Empirical Characteristic Function:
  Price=(S0-(sqrt(S0*K)/pi)*exp(-r*t/2+ohm.h*t/2)*integrate(Integrand,
                              lower = 0,upper = Inf)$value + min(0,j)*(S0-K*exp(-r*t)))
  return(Price)
  } else{
      return(NA)
    }
}
```

B.8 R code for equation (3.33 & 3.34): Monte Carlo Method

```
MC=function(St,t,K,r,j){
  # St: Simulated terminal asset price sample
  # t: the time interval (or time to maturity)
  # K: Strike price
  # r: the risk-free rate (assumed constant)
  # j: -1 for "Put option", and 1 for "Call option"

  if (abs(j)==1){
    n=length(St)
    payoff=matrix(0,nrow=n,ncol=1)
    for (i in 1:n){
      payoff[i]=max(0,j*(St[i]-K))*exp(-r*t)
    }
    # Price obtained by Monte Carlo Method
    Price=sum(payoff)*(1/n)
    return(Price)
  } else {
    return(NA)
  }
}
```

B.9 R code for equation (3.37): Standard Deviation of Monte Carlo Estimator

```
MC.sig=function(St,t,K,r,j){
  # St:  Simulated terminal asset price sample
  # t: time to maturity
  # K: Strike Price
  # r: the risk-free rate (assumed constant)
  # j: -1 for "Put option", and 1 for "Call option"

  if (abs(j)==1){
    n=length(St)
    payoff=matrix(0,nrow=n,ncol=1)
    for (i in 1:n){
      payoff[i]=max(0,j*(St[i]-K))*exp(-r*t)
    }
    # Price obtained by Monte Carlo Method
    Price=sum(payoff)*(1/n)
    stdev=sqrt((sum((payoff-Price)^2)/(n-1))/n)
    return(stdev)
  } else {
    return(NA)
  }
}
```

B.10 R code for Implied Volatility from BSM

```
ImpliedVol<-function(S0,K,r,t,Price,j,a,b) {
  # S0: initial asset price
  # K: Strike price
  # r: risk free rate
  # t: time to maturity
  # Price: actual price
```

```
  # j: -1 (for put option), or 1 (for call option)
  # a: lower bound of search interval
  # b: upper bound of search interval

  f=function(x){
    d2=(log(S0/K)+(r-0.5*x^2)*t)/(x*sqrt(t))
    d1=d2+x*sqrt(t)
    BSM_Price=j*S0*pnorm(j*d1)-j*K*exp(-r*t)*pnorm(j*d2)
    g= BSM_Price-Price
    return(g)
  }
  if (f(a)*f(b)>0){
    return(NA)
  } else{
    Sigma=uniroot(f,c(a,b))$root
    print(Sigma)}
}
```

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