微分方程的初值问题

一阶
$$\begin{cases} \dfrac{dy}{dt} = f(y,t) \\ y(t=0) = y_0 \end{cases}$$

$$\begin{cases} \dfrac{y_{n+1} - y_n}{\Delta t} = f(y_n, t_n) \\ y(t_0) = y_0 \end{cases}$$

$$\begin{cases} y_{n+1} = y_n + \Delta t f(y_n, t_n) \\ y_0 = t_0 \end{cases}$$

欧拉法

改进的欧拉法

$$\begin{cases} \dfrac{y_{n+1}-y_n}{\Delta t} = \dfrac{1}{2}(f(y_n,t_n)+f(y_{n+1},t_{n+1})) \\ y_0 = t_0 \end{cases}$$

$$\begin{cases} y_{n+1} = y_n + \dfrac{\Delta t}{2}(f(y_n,t_n)+\underline{f(y_{n+1},t_{n+1})}) \\ y_0 = t_0 \end{cases}$$

欧拉法

二阶Runge-Kutta法

$$\begin{cases} \dfrac{y_{n+1}-y_n}{\Delta t} = f(y_{n+1/2}, t_{n+1/2}) \\ y_0 = y_0 \end{cases}$$

$$\begin{cases} y_{n+1} = y_n + \Delta t f(\underline{y_{n+1/2}}, t_{n+1/2}) \\ y_0 = t_0 \end{cases}$$

$$y_{n+1/2} = y_n + \frac{\Delta t}{2} f(y_n, t_n)$$

用欧拉法预测

微分方程的初值问题

一阶
$$\begin{cases} \dfrac{dy}{dt} = f(y,t) \\ y(t=0) = y_0 \end{cases}$$

二阶
$$\begin{cases} y''(t) = f(y, y', t) \\ y(0) = y_0 \\ y'(0) = y_0^{'} \end{cases} \Rightarrow \begin{cases} y' = z \\ z' = f(y, z, t) \\ y(0) = y_0 \\ z(0) = y_0^{'} \end{cases}$$

微分方程的边值问题

$$\begin{cases} y''(t) = f(y, y', t) \\ y(0) = y_0 \\ y(L) = y_L \end{cases}$$

习惯写为：

$$\begin{cases} y''(x) = f(y, y', x) \\ y(0) = y_0 \\ y(L) = y_L \end{cases}$$

$$\begin{cases} y''(x) = f(y, y', x) \\ y(0) = y_0 \\ y(L) = y_L \end{cases}$$

边值问题的差分解法

$$\begin{cases} u(x)y'' + v(x)y' + w(x)y = f(x) \\ y(a) = \alpha, \qquad y(b) = \beta \end{cases}$$ 第一类边值条件

$$y''(x_i) = (y_{i+1} - 2y_i + y_{i-1}) / h^2$$

$$y'(x_i) = (y_{i+1} - y_{i-1}) / 2h$$

代入上式，得到

$$\begin{cases} a_i y_{i-1} + b_i y_i + c_i y_{i+1} = d_i \quad i = 2,3,\ldots,n-1 \\ y_1 = \alpha, \quad y_n = \beta \end{cases}$$

where $a_i = u(x_i) - \dfrac{h}{2} v(x_i) \qquad b_i = h^2 w(x_i) - 2u(x_i)$

$$c_i = u(x_i) + \frac{h}{2} v(x_i) \qquad d_i = h^2 f(x_i)$$

矩阵形式

$$\begin{bmatrix} 1 & 0 & & & \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} \alpha \\ d_2 \\ \vdots \\ d_{n-1} \\ \beta \end{bmatrix}$$

边值问题的差分解法

$$\begin{cases} u(x)y''+v(x)y'+w(x)y = f(x) \\ y'(a)=\alpha, \qquad y'(b)=\beta \end{cases}$$

第二类边值条件

$$y''(x_i)=(y_{i+1}-2y_i+y_{i-1})/h^2$$

$$y'(x_i)=(y_{i+1}-y_{i-1})/2h$$

$$(y_2-y_1)/h=\alpha$$

$$(y_n-y_{n-1})/h=\beta$$

步骤类似。。。

$$\begin{cases} u(x)y''+v(x)y'+w(x)y = f(x) \\ y(a)=\alpha, \qquad y(b)=\beta \end{cases}$$

$$\begin{cases} u(x)y''+v(x)yy'+w(x)y = f(x) \\ y(a)=\alpha, \qquad y(b)=\beta \end{cases}$$

边值问题的打靶法

更一般的方程

$$\begin{cases} y''(x) = f(y, y', x) \\ y(a) = \alpha \\ y(b) = \beta \end{cases}$$

第一类边值条件

步骤：

1) 猜测一个**$y'(a)$**的值，如**$y'(a)=m_1$**
2) 从**$y(a)$**和**$y'(a)$**可解初值问题
3) 计算**$y(b)$**，并和β比较，如在误差范围内相等则停止，否则回到**1)**
4) 猜测的过程可以用搜索法、二分法、牛顿法等。

例：为了使烟花从地面发射5s后在距离地面40m的空中爆炸，初始的发射速度应为多大？

$$\begin{cases} \dfrac{d^2 y}{dt^2} = -10 - \gamma \dfrac{dy}{dt} \\ y(0) = 0 \\ y(5) = 40 \end{cases} \Rightarrow \begin{cases} \dfrac{dv}{dt} = -10 - \gamma v \\ \dfrac{dy}{dt} = v \end{cases}$$

$$y(0) = 0$$

$$v(0) = ?$$

$$\Downarrow$$

$$y(5) = 40$$

$$\begin{cases} v_{i+1} = v_i + \underline{(-10 - \gamma v_i)} \cdot h \\ y_{i+1} = y_i + \underline{v_i} \cdot h \end{cases}$$

欧拉法

$$\begin{cases} v_{i+1} = v_i + \underline{(-10 - \gamma v_{i+1/2})} \cdot h \\ y_{i+1} = y_i + \underline{v_{i+1/2}} \cdot h \end{cases}$$

二阶龙格库塔法

$$v_{i+1/2} = v_i + (-10 - \gamma v_i) \cdot h / 2$$

The analytic solution

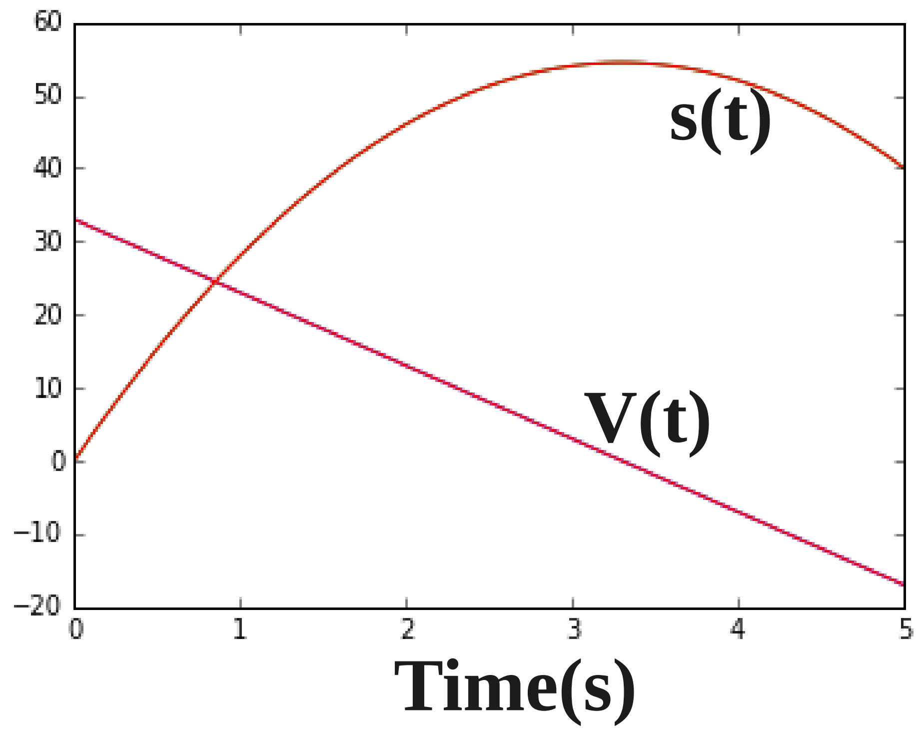
当空气阻力为零时

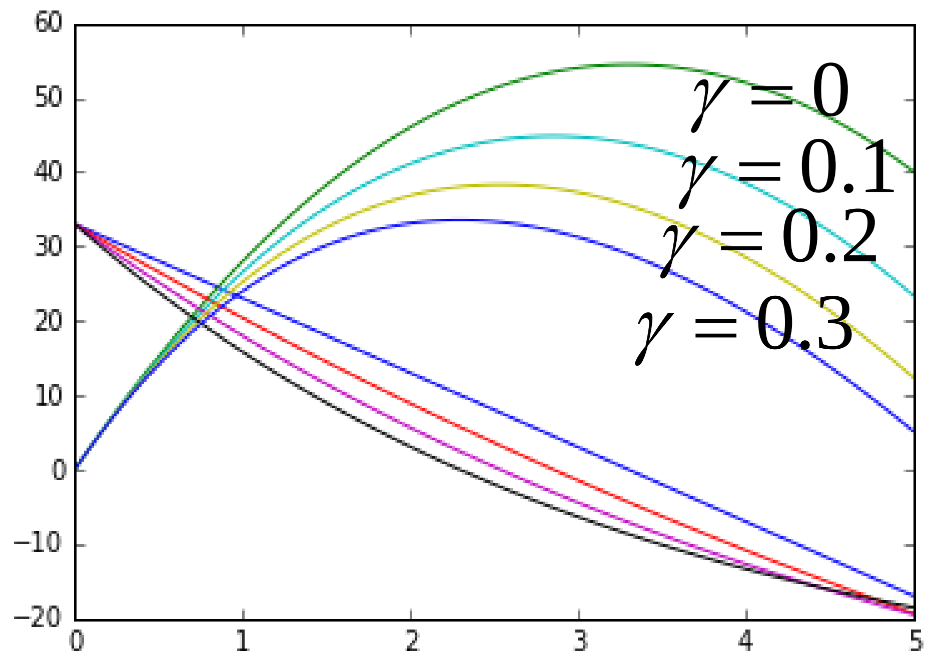
$$v = v_0 - gt$$

$$s = v_0 t - \frac{1}{2} g t^2$$

$$40 = v_0 \cdot 5 - \frac{1}{2} 10 \cdot 5^2$$

$$v_0 = (40 + 125) / 5 = 33$$





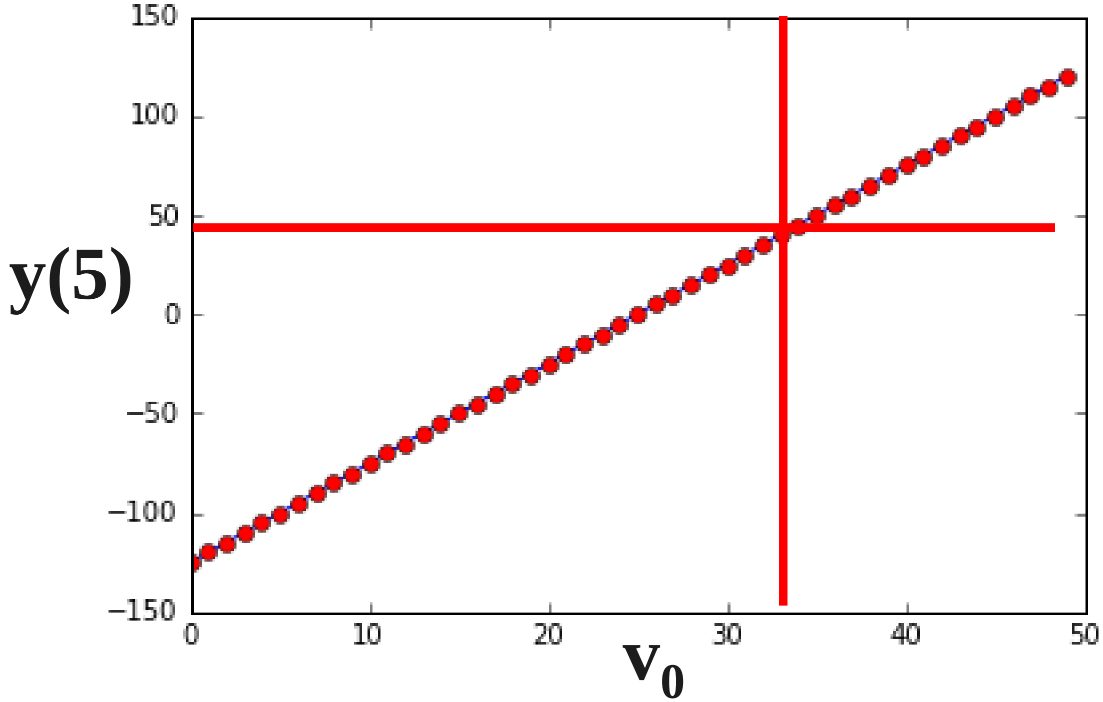
打靶法

$$\begin{cases} \dfrac{d^2y}{dt^2} = -10 - \gamma \dfrac{dy}{dt} \\ y(0) = 0 \\ y(5) = 40 \end{cases}$$

$$y(0) = 0$$

$$v(0) = ?$$

$$y(5) = 40$$

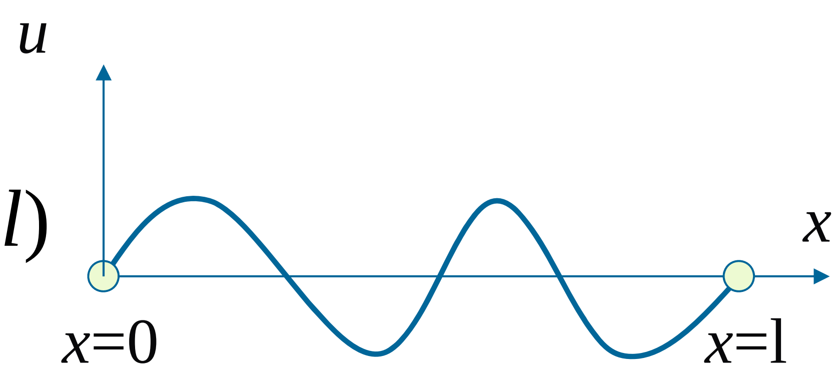


v_0 = 33.000001311302185 when friction is zero.

打靶法求解本征值问题

物理问题：两端固定的均匀弦自由振动

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0 & (0<x<l) \\ u\big|_{x=0} = 0 \quad u\big|_{x=l} = 0 \\ u\big|_{t=0} = \varphi(x) \quad u_t\big|_{t=0} = \psi(x) \end{cases}$$



设 $u(x,t) = \phi(x)T(t)$ 代入上述波动方程和边界条件得

$$\begin{cases} \phi T'' - a^2 \phi'' T = 0 \\ \phi(0)T(t) = 0 \\ \phi(l)T(t) = 0 \end{cases}$$

用 $a^2\phi T$ 遍除各项即得 $\phi T''-a^2\phi''T=0$

$$\frac{T''}{a^2T}=\frac{\phi''}{\phi}$$

上式成立要求：

$$\frac{T''}{a^2T}=\frac{\phi''}{\phi}=-k^2$$

即有：

$$\begin{cases}\dfrac{d^2\phi}{dx^2}=-k^2\phi\\ \phi(x=0)=\phi(x=1)=0.\end{cases}$$

和

$$\frac{d^2T}{dt^2}=-k^2a^2T$$

本征值问题

$$\begin{cases} \dfrac{d^2\phi}{dx^2} = -k^2\phi \quad (0 < x < l) \\ \phi(x=0) = \phi(x=1) = 0. \end{cases}$$

特点：1　只对一些特定的k值才能找到满足边界条件的解

2　方程是线性的和齐次的

本征值：k_n　　　解析解：$k_n = n\pi, \quad n = 1,2\cdots$

本征函数: $\phi_n(x)$　　　$\phi_n(x) \propto \sin n\pi x$

数值求解：打靶法

$$\begin{cases} \dfrac{d^2\phi}{dx^2} = -k^2\phi \quad (0 < x < l) \\ \phi(x=0) = \phi(x=1) = 0. \end{cases}$$

1. 先猜一个试验本征值
2. 对微分方程作为初值问题求解
3. 检验所得解是否满足边界条件
4. 若满足，则该试验本征值为真实本征值，对应的解为本征函数，否则重复1, 2, 3步

对弦振动问题

$$\begin{cases} \dfrac{d^2\phi}{dx^2} = -k^2\phi \quad (0 < x < l) \\ \phi(x=0) = \phi(x=1) = 0. \end{cases}$$

1. 选取*k*值
2. 从x=0开始求微分方程，初始条件为

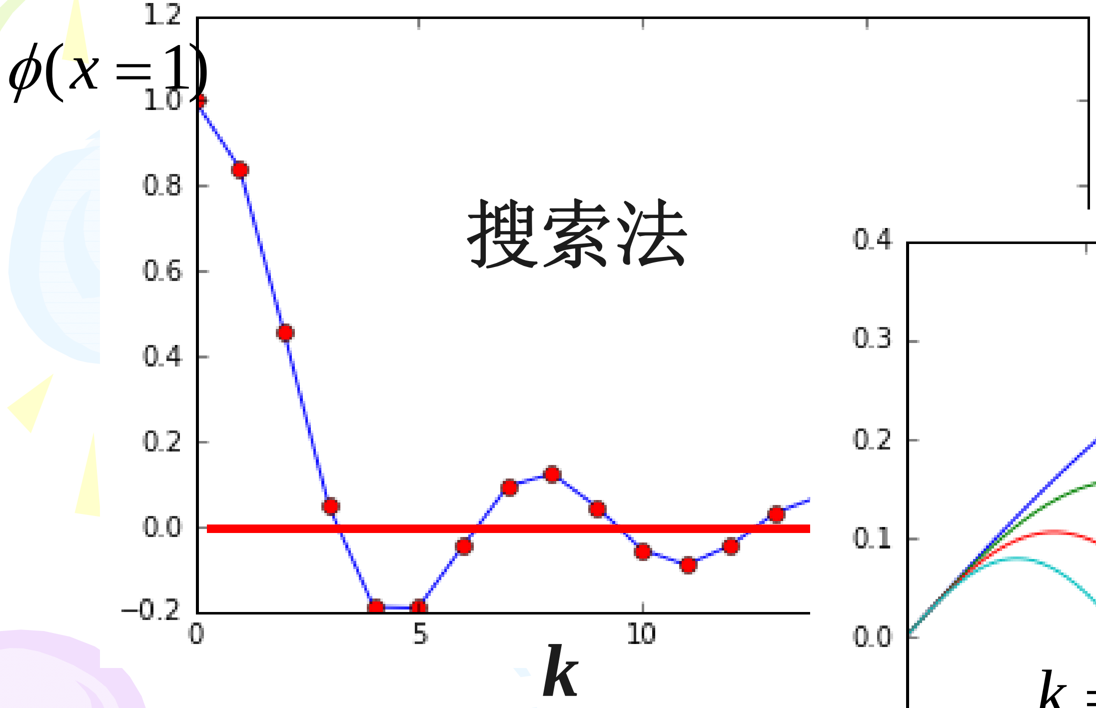
$$\phi(x=0) = 0, \phi'(x=0) = \delta \qquad (\delta\text{取值任意})$$

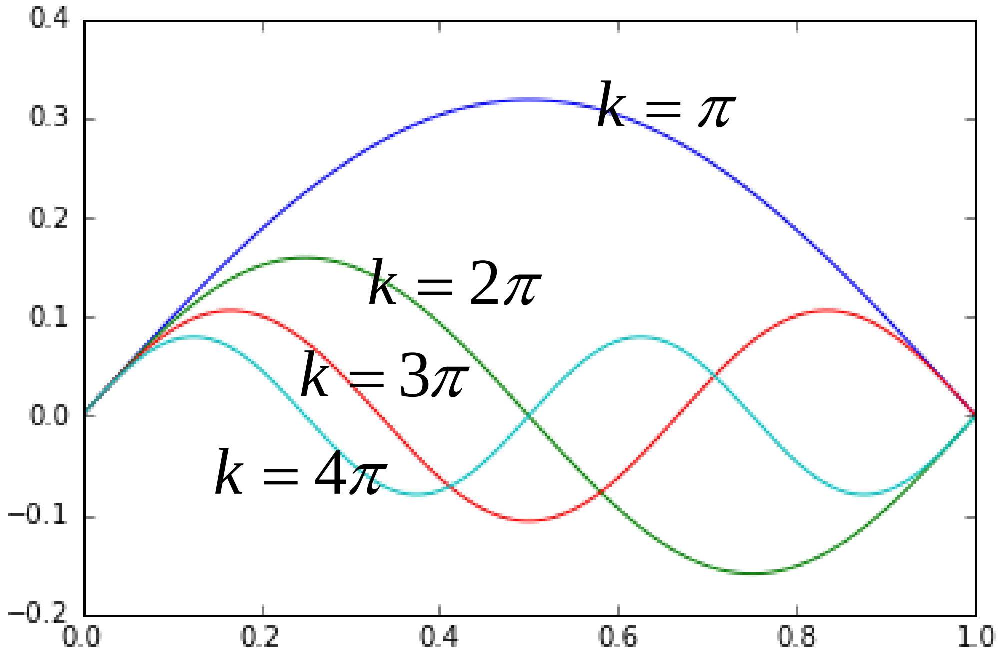
3. 求出x=1时的 ϕ 值，并判断是否为0
4. 重新调整k，并再度求解微分方程，直到x=l时 ϕ 的值为0，这时就找到了本征值以及对应的本征函数

$$
\begin{cases}
\dfrac{d^2\phi}{dx^2} = -k^2\phi \quad (0 < x < l) \\
\phi(x=0) = \phi(x=1) = 0
\end{cases}
\Rightarrow
\begin{cases}
\dfrac{d\psi}{dx} = -k^2\phi \\
\dfrac{d\phi}{dx} = \psi
\end{cases}
$$

$$\phi(x=0)$$

$$\phi'(x=0) = \psi = \delta$$





物理问题2：一维薛定谔方程的定态解

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V(x)\psi=E\psi}$$

$$\frac{d^2\psi}{dx^2}+k(x)\psi=0$$

$$k(x)=\frac{2m}{\hbar^2}\left[E-V(x)\right]$$

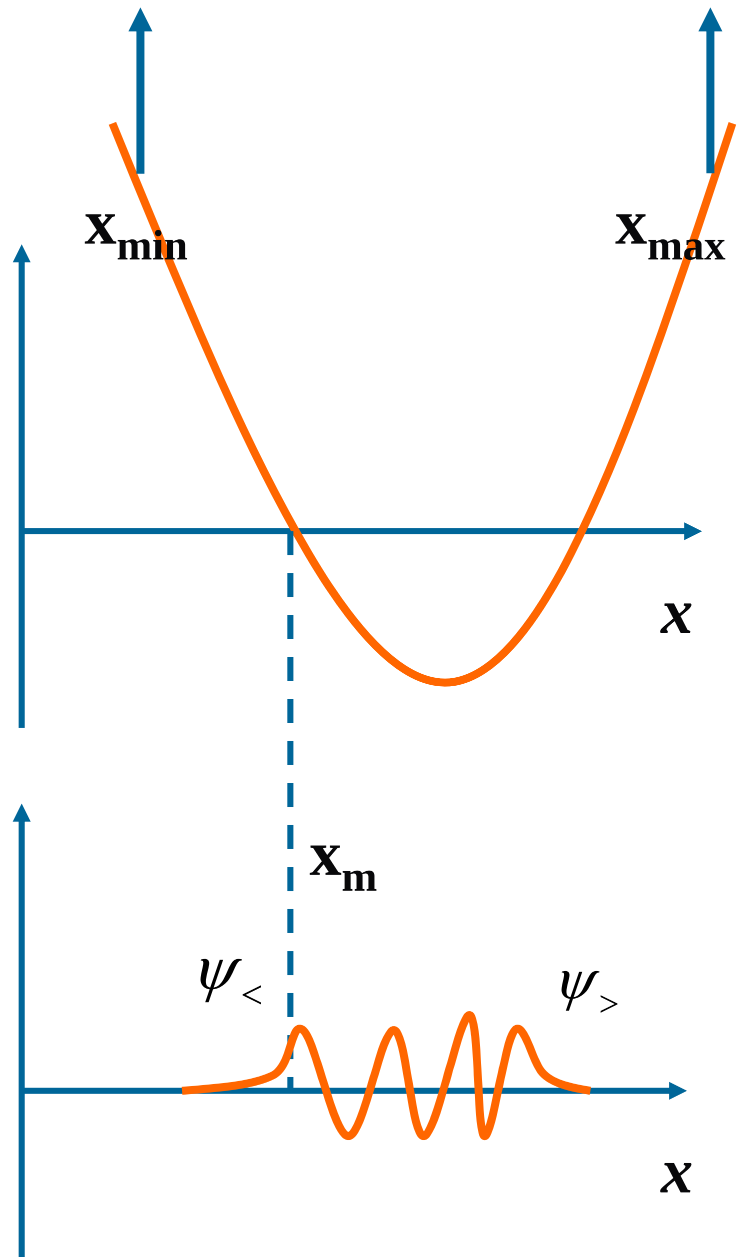
E只能取特定的离散值

波函数在任何一点光滑连接：

$$\begin{cases} \psi_<(x_m) = \psi_>(x_m) \\ \dfrac{d\psi_<}{dx}\Big|_{x_m} = \dfrac{d\psi_>}{dx}\Big|_{x_m} \end{cases}$$

$$\psi_<(x_{\min}) = \psi_>(x_{\max}) = 0$$

对特定的E



求解步骤：打靶法

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V(x)=E\psi$$

设置能量本征值的试验值 E

在x_{min}和x_{max}两点，波函数为0，即：

$$\psi_<(x_{\min})=\psi_>(x_{\max})=0$$

$$\psi_<(x_{\min}+\Delta x)=\psi_>(x_{\max}-\Delta x)=\delta$$

用Runge-Kutta 法解微分方程，计算x_m处的波函数值及其导数，x_m原则上可任取。

令$\psi_<(x_m)=\psi_>(x_m)$定出系数

判断$\frac{d\psi_<}{dx}\Big|_{x_m}-\frac{d\psi_>}{dx}\Big|_{x_m}=0$是否成立

课 堂 练 习

例：为了使烟花从地面发射5s后在距离地面40m的空中爆炸，初始的发射速度应为多大？

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$$y(0) = 0$$

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$$\Downarrow$$

$$y(5) = 40$$

$$\begin{cases} v_{i+1} = v_i + \underline{(-10 - \gamma v_i)} \cdot h \\ y_{i+1} = y_i + \underline{v_i} \cdot h \end{cases}$$

欧拉法

$$\begin{cases} v_{i+1} = v_i + \underline{(-10 - \gamma v_{i+1/2})} \cdot h \\ y_{i+1} = y_i + \underline{v_{i+1/2}} \cdot h \end{cases}$$

二阶龙格库塔法

$$v_{i+1/2} = v_i + (-10 - \gamma v_i) \cdot h / 2$$

```
gg = 10.0 #acceleration

def derv_f(y1,y2,x):
    gamma = 0.1
    s = - gg - gamma*y1
    return s

def derv_g(y1,y2,x):
    s = y1
    return s

def Runge_Kutta(x, y1, y2, h):
    y1half = y1+derv_f(y1,y2,x)*h*0.5
    y2half = y2+derv_g(y1,y2,x)*h*0.5
    y1next = y1 +
      derv_f(y1half,y2half,x+0.5*h)*h
    y2next = y2 +
       derv_g(y1half,y2half,x+0.5*h)*h
    return (y1next,y2next)
```

```
def Integeral(xx,y1,y2,N,h):
    for i in range(N):
        y1[i+1], y2[i+1] =
            Runge_Kutta(xx[i], y1[i], y2[i], h)
    return

xa = 0.0
xb = 5.0
N = 100
h = (xb-xa)/N
xx = np.linspace
            (xa,xb,N+1,dtype=np.float64)
y1 = np.zeros(N+1,dtype=np.float64)
y2 = np.zeros(N+1,dtype=np.float64)

y1[0] = 30.0 #initial velocity
y2[0] = 0.0  #initial position

Integeral(xx,y1,y2,N,h)
```

作　业

用打靶法求解弦振动问题，给出算法公式和计算程序，求出至少前三个本征值，并画出相应的本征振动的图形。

$$\begin{cases} \dfrac{d^2\phi}{dx^2} = -k^2\phi \quad (0 < x < l) \\ \phi(x=0) = \phi(x=1) = 0. \end{cases}$$

下周上课前，发送至： **njuphyhw@126.com**

1) 规范邮件标题：作业**5**-姓名-学号
2) **python**程序作为附件发送。
3) 把**word**文件转化为**pdf**发送。

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