Change of Measure II

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Outline

- 1.Review of the change of measure formula
- 2. Martingale as the Radon Nikodym Theorem
- 3.Change of measure for Stochastic Process

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Review

- Change of the measure for the expectation

$$\mathbb{E}^{\mathbb{Q}}[Y] = \mathbb{E}^{\mathbb{P}}[Y \frac{d\mathbb{Q}}{d\mathbb{P}}].$$

- Change of the measure for the conditional expectation

$$\mathbb{E}^{\mathbb{Q}}[Y|\mathcal{G}]\mathbb{E}^{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{G}] = \mathbb{E}^{\mathbb{P}}[Y \frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{G}].$$

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Radon Nikodym Derivative Process

- We construct such process with the exponential martingale

Proposition 1.10. *Assume that* $\mathbb{E}^{\mathbb{P}}[\mathcal{E}_\sigma(T)] = 1$*, where* $\mathcal{E}_\sigma(t)$ *is defined on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and* M_t *is an adapted process w.r.t. a filtration* $\mathcal{F}_t$*. A probability measure* $\mathbb{Q}$ *is defined by*

$$\mathbb{Q}(A) = \int_A \mathcal{E}_\sigma(T)\, d\mathbb{P}, \quad or \quad \frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}_\sigma(T).$$

Then M_t *is a* $\mathbb{Q}$*-martingale if and only if* $M_t\mathcal{E}_\sigma(t)$ *is a* $\mathbb{P}$*-martingale.*

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Girsanov Theorem

Theorem 2.3 (Girsanov Theorem). *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* B_t $(0 \leq t \leq T)$ *be a standard Brownian motion w.r.t.* $\mathbb{P}$. *Then*

$$B_t^{\mathbb{Q}} = B_t - \int_0^t \sigma_s \, ds,$$

is a standard Brownian motion w.r.t. $\mathbb{Q}$, *where* $\mathbb{Q}(A) = \int_A \mathcal{E}_\sigma(T) \, d\mathbb{P}$, $A \in \mathcal{F}$ *and* $\mathcal{E}_\sigma(t)$ *is defined in* (1.1) *and* $\mathbb{E}^{\mathbb{P}}[\mathcal{E}_\sigma(T)] = 1$.