

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

PSI STUDY TEXT

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# Statistical Physics

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## Abstract

This text provides a concise introduction to *critical phenomena* in statistical physics. Phase transitions and critical exponents will be discussed, as well as their explanation through mean field theory and renormalization group techniques. A number of sources stated below have been used.

- Thermodynamics:
  - C. J. Adkins, *Equilibrium thermodynamics* [1]
- Standard books on statistical mechanics:
  - L.D. Landau and E.M. Lifshitz: *Statistical physics, part I* [2]
  - R.K. Pathria, *Statistical mechanics* [3]
- Critical phenomena:
  - A. Burkov, PSI lectures 2013/2014 and 2014/2015.
  - N. Goldenfeld, *Lectures on phase transitions and the renormalization group* [4]
  - J. Cardy, *Scaling and renormalization in statistical physics* [5]
  - M. Kardar, *Statistical Physics of Fields* [6]
  - E. Brezin, *Introduction to statistical field theory* [7]
  - J. M. Yeomans, *Statistical mechanics of phase transitions* [8]
  - M.E.J. Newman and G. T. Barkema, *Monte Carlo Methods in Statistical Physics* [9]
  - S. Sachdev, *Quantum Phase Transitions* [10]

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# Chapter 1: Preliminaries

## 1.1 Review of Thermodynamics

As apparent from the following quote by Sir Arthur Stanley Eddington [11], one can hardly overestimate the meaning of thermodynamics:

*"If someone points out to you that your pet theory of the universe is in disagreement with Maxwell's equations-then so much the worse for Maxwell's equations. If it is found to be contradicted by observation-well these experimentalists do bungle things sometimes. But if your theory is found to be against the second law of thermodynamics I can give you no hope; there is nothing for it but to collapse in deepest humiliation."*

### Basic concepts

Let us briefly overview the main concepts of equilibrium thermodynamics, following an excellent book [1] (which I encourage you to read if you are interested to know more).

- Thermodynamics enables one to interrelate the directly observable *macroscopic quantities* (such as pressure and volume) without making any microscopic assumptions. This is done with the help of new (abstract) quantities that enable us to characterize processes and conditions in a simple way. Of these new concepts, three are fundamental, each following from one law of thermodynamics: the zeroth law gives *temperature*  $T$ , the first law defines *internal energy*  $U$ , and the second law *entropy*  $S$ .
- Thermodynamic variables are either *intensive*: ‘local in character’ (e.g. pressure, electric field, density) or *extensive*: ‘measure the system as a whole’ (mass, volume, internal energy). *Specific* variables are extensive quantities per unit mass (volume) of the system. Many of the direct observables form *conjugate pairs* such that their product has the dimensions of energy, e.g. pressure/volume, temperature/entropy, surface tension/area.
- Thermodynamic equilibrium is a state of the system when no further changes take place. This in particular includes a thermal equilibrium (after being in thermal contact for a long time). A process is reversible if its direction can be reversed by an infinitesimal change in the conditions. This happens when i) the process is *quasi-static*: every state through which the system passes may be considered as an equilibrium state (slow enough) and ii) no *hysteresis* occurs: e.g. magnetization of iron (corresponding to long-lived metastable state due to inhomogeneities).

## Thermodynamic laws

### The zeroth law

**The zeroth law:** *If two systems are separately in thermal equilibrium with a third, then they must be also in thermal equilibrium with each other.*

This allows one to define empirical temperature as follows. It is an experimental fact that the state of a gas system is fully characterized by specifying  $P$  and  $V$ . However, if we demand that such a system is in thermal equilibrium with another one, only one of these quantities is independent. Let us now use the zeroth law, considering three gas systems, taking the system 3 as a reference frame (thermometer).

Since systems 1 and 2 are in thermal equilibrium with 3, we must have

$$F_1(P_1, V_1, P_3, V_3) = 0, \quad F_2(P_2, V_2, P_3, V_3) = 0. \quad (1.1)$$

Solving both equations for  $P_3$  we have

$$P_3 = P_3(P_1, V_1, V_3), \quad P_3 = P_3(P_2, V_2, V_3). \quad (1.2)$$

Hence, equating the two and solving for  $P_1$  we have

$$P_3(P_1, V_1, V_3) = P_3(P_2, V_2, V_3) \Leftrightarrow P_1 = P_1(V_1, V_2, V_3, P_2). \quad (1.3)$$

By the zeroth law, systems 1 and 2 must also be in thermal equilibrium,  $F_3(P_1, P_2, V_1, V_2) = 0$ , giving

$$P_1 = P_1(V_1, V_2, P_2). \quad (1.4)$$

Comparing this with (1.3), we conclude that in (1.3)  $V_3$  has to drop out, in both first and second equations, and so we can define a new function  $f$  so that

$$f(P_1, V_1) = f(P_2, V_2) \Rightarrow f(P, V) = \Theta = \text{const.} \quad (1.5)$$

So  $\Theta$ , called the empirical temperature is the same for all such systems in equilibrium. The latter equation is called the equation of state. Note that once established for the gas system, the empirical temperature can be extended to other substances, using gas system as a thermometer.

### The First law

**The First law:** historically marks the recognition of *heat* as a form of energy. It represents a generalized law of conservation of energy. We proceed in two steps:

- *Step 1:* For a thermally isolated system, there exists a function of state called the internal energy  $U$ , such that we have

$$dU = dW, \quad (1.6)$$

where  $W$  is the work done on the system. Since this is independent of path, it is a total differential (the amount of work needed depends solely on the change effected and not on the means by which the work is performed nor on the intermediate stages through which the system passes between its initial and final states).

- *Step 2:* If the system is not thermally isolated, one can also change its state without doing work on it. The heat  $\delta Q$  measures to which extent the change is not adiabatic (breaks thermal isolation) and the first law modifies to

$$dU = \delta Q + \delta W. \quad (1.7)$$

We fix the signs as follows.  $dU = U_f - U_i$  is a change of internal energy of the system going from its initial to final state, it is a total differential (independent of path).  $\delta W$  represents the work done on the system and  $\delta Q$  is the heat transferred to the system, both not being total differentials. (Note that only their sum is a function of state; work and heat cannot be defined separately unless we know exactly the route by which the system passes from its initial to its final state.) This convention allows one to define *hotter* and *colder*: when two systems are placed in thermal contact, the heat flows from hotter to colder.

For example, the work done by hydrostatic pressure on the fluid is given by

$$\delta W = -PdV. \quad (1.8)$$

Work against surface tension  $\tau$  to increase the surface area  $A$  is  $\delta W = \tau\delta A$ , see tutorial.

Having defined the heat, we can also introduce the generalized heat capacity

$$C_{\beta,\gamma,\dots}^{(\alpha)} = \frac{\delta Q_{\beta,\gamma,\dots}}{d\alpha}, \quad (1.9)$$

expresses the rate at which heat is absorbed when the variable  $\alpha$  is changed. For  $n$ -parameter system, we need to specify  $(n-1)$  constraints  $(\beta, \gamma, \dots)$  to specify the path. For example, we have

$$C_V \equiv C_V^{(T)} = \left( \frac{\delta Q}{dT} \right)_V, \quad C_P \equiv C_P^{(T)} = \left( \frac{\delta Q}{dT} \right)_P. \quad (1.10)$$

### The second law

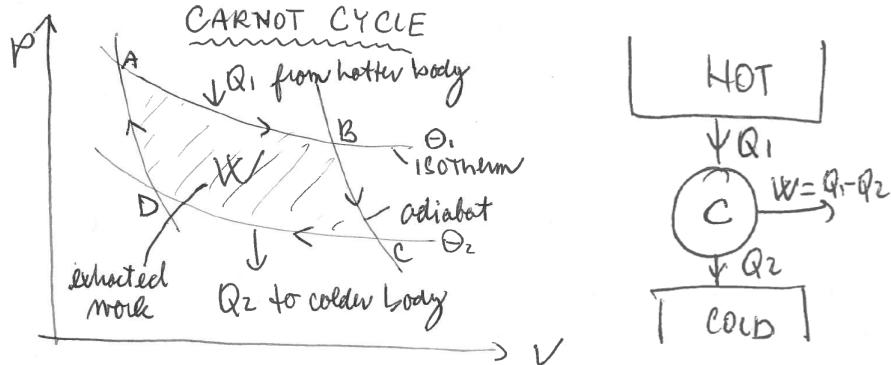
- The second law. Not all processes that are energetically possible (consistent with the first law) actually occur in nature—there is an essential *irreversibility*. At the same time there is a limit to the *efficiency* with which heat can be converted to work. To convert heat to work we shall use the heat engine (works in cycles, returning after each cycle to its initial state). It works with a *thermal efficiency*  $\eta$ ,

$$\eta = \frac{\text{work out}}{\text{heat in}} = \frac{W}{Q_1}, \quad (1.11)$$

per cycle. Denoting by  $Q_2$  rejected heat, we have  $W = Q_1 - Q_2$  and hence

$$\eta = 1 - \frac{Q_2}{Q_1}. \quad (1.12)$$

The Carnot cycle is a cyclic process that consists of two adiabatics and two isotherms, see figure:

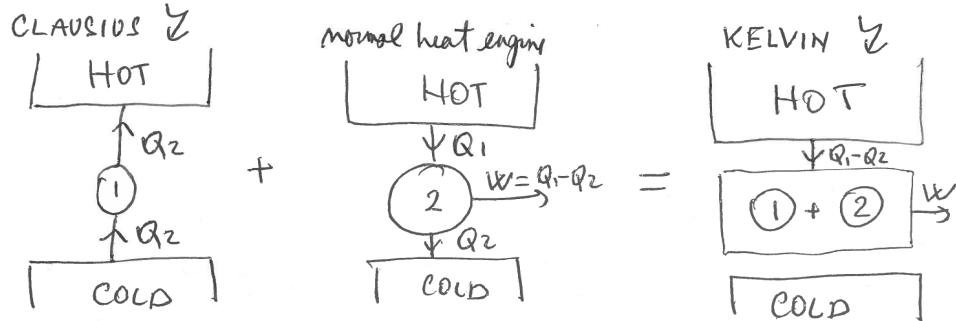


Note that the process is *reversible*: one can heat hotter body and cool colder one at the expense of mechanical work  $W$ .

**The second law (Kelvin).** *No process is possible whose sole result is the complete conversion of heat into work.*

**The second law (Clausius).** *No process is possible whose sole result is the transfer of heat from a colder to a hotter body.*

The proof of equivalence of the two formulations is sketched in the following figure:



Note that the first law forbids *perpetual motion of the first kind*: a machine cannot operate continuously by creating its own energy. The second law forbids *perpetual motion of the second kind*: a machine cannot be made that runs continuously by using the internal energy of a single heat reservoir.

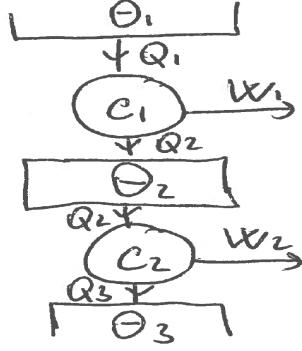
**Carnot's (self-centered) theorem:** *No engine operating between two given reservoirs can be more efficient than the Carnot engine.*

**Corollary:** *All reversible engines operating between the same reservoirs are equally efficient.*

We shall prove both these statements in today's tutorial. The latter means that for any reversible engine

$$\frac{Q_1}{Q_2} = f(\Theta_1, \Theta_2), \quad (1.13)$$

where  $f$  is a universal function. Composing two Carnot engines as in the picture,



we thus have

$$\frac{Q_1}{Q_2} = f(\Theta_1, \Theta_2), \quad \frac{Q_2}{Q_3} = f(\Theta_2, \Theta_3), \quad \frac{Q_1}{Q_3} = f(\Theta_1, \Theta_3). \quad (1.14)$$

This means that

$$f(\Theta_1, \Theta_3) = f(\Theta_1, \Theta_2)f(\Theta_2, \Theta_3) \Rightarrow f(\Theta_1, \Theta_2) = \frac{T(\Theta_1)}{T(\Theta_2)}. \quad (1.15)$$

So

$$\boxed{\frac{Q_1}{Q_2} = \frac{T_1}{T_2}, \quad \eta_r = 1 - \frac{T_2}{T_1}}, \quad (1.16)$$

where  $T$  is the *thermodynamic temperature*.

- Clausius theorem and entropy. Due to Carnot's theorem we have  $\eta \leq \eta_r$ , giving

$$1 - \frac{Q_2}{Q_1} \leq 1 - \frac{T_2}{T_1} \Rightarrow \frac{Q_1}{T_1} - \frac{Q_2}{T_2} \leq 0 \quad \text{or} \quad \sum \frac{Q_i}{T_i} \leq 0, \quad (1.17)$$

adopting a convention that heat entering a system is positive. Slightly more generally, this implies:

**Clausius theorem:** *for any closed cycle*

$$\boxed{\oint \frac{\delta Q}{T} \leq 0}, \quad (1.18)$$

*with equality for a reversible cycle.*

**Definition.** Entropy  $S$  is defined as<sup>1</sup>

$$\boxed{dS = \frac{\delta Q_r}{T}}. \quad (1.19)$$

---

<sup>1</sup>A more abstract but direct definition of entropy follows from the Caratheodory formulation of 2nd law, see e.g. Chapter 6 in [1].

Note that this requires reversible change! Using the Clausius theorem,  $S$  is a function of state, that is  $dS$  is a total differential. So the temperature  $1/T$  is an integrating factor for  $\delta Q_r$ . Considering further a cycle described in the picture and Clausius theorem again:

$$\int_A^B \frac{\delta Q}{T} + \int_B^A \frac{\delta Q_r}{T} \leq 0$$

$$\int_A^B \frac{\delta Q}{T} \leq \int_A^B \frac{\delta Q_r}{T} = S_B - S_A$$

we conclude that

$$dS \geq \frac{\delta Q}{T}, \quad (1.20)$$

with the equality iff the change is reversible.

In particular, for a thermally isolated system,  $\delta Q = 0$ , and hence

$$dS \geq 0, \quad (1.21)$$

that is, the entropy of an isolated system cannot decrease. (For final equilibrium configuration, the entropy is as large as possible.) The law of increase of entropy provides ‘arrow of time’, a direction to the sequence of natural events. “The arrow of time results from there not being thermodynamic equilibrium throughout the universe.”

Let us finally note that the first law can now be written as

$$dU = TdS - PdV, \quad (1.22)$$

irrespective of whether or not the changes are reversible, as long as there is some reversible path connecting the initial and final configurations. (For irreversible  $\delta W \geq -PdV$ , e.g. friction.)

Note also that using Clausius, we have

$$C = \frac{\delta Q}{dT} = \frac{TdS}{dT} \Rightarrow C_V = T \left( \frac{\partial S}{\partial T} \right)_V. \quad (1.23)$$

### The third law

The third law is concerned with the limiting behavior of systems in equilibrium as the temperature approaches absolute zero. In particular, it provides the ‘integration constant’ for the entropy:

**The third law.** *The entropy of an equilibrium system at absolute zero is exactly equal to zero.*

Mathematically this can be formulated in steps as follows. Integrating the Clausius relation, we have

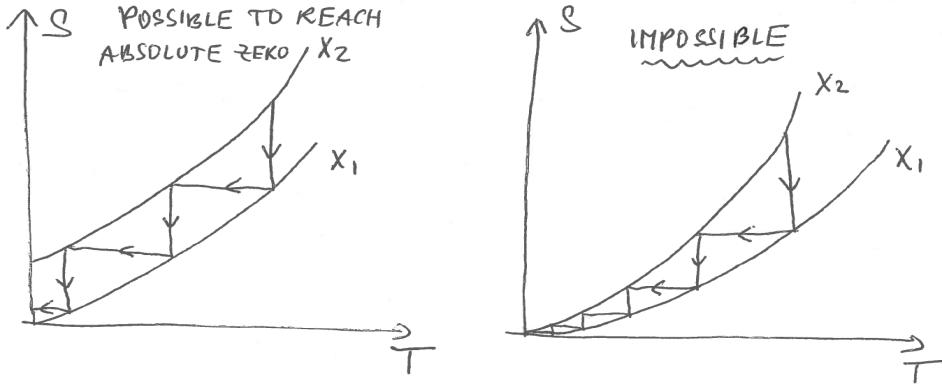
$$S(T, X) = S(0, X) + \int_0^T \frac{C(T', X)}{T'} dT', \quad (1.24)$$

where  $X$  stands for all other thermodynamic parameters. We require that the value  $S(0, X)$  is independent of  $X$ ,

$$S(0, X) = S(0) \Leftrightarrow \lim_{T \rightarrow 0} \left( \frac{\partial S(T, X)}{\partial X} \right)_T = 0. \quad (1.25)$$

Classically, one can choose  $S(0) = 0$ .

It is a direct consequence of the third law that: *It is impossible to cool the system to absolute zero in a finite number of steps.* This is illustrated in the following figure:



## Thermodynamic potentials

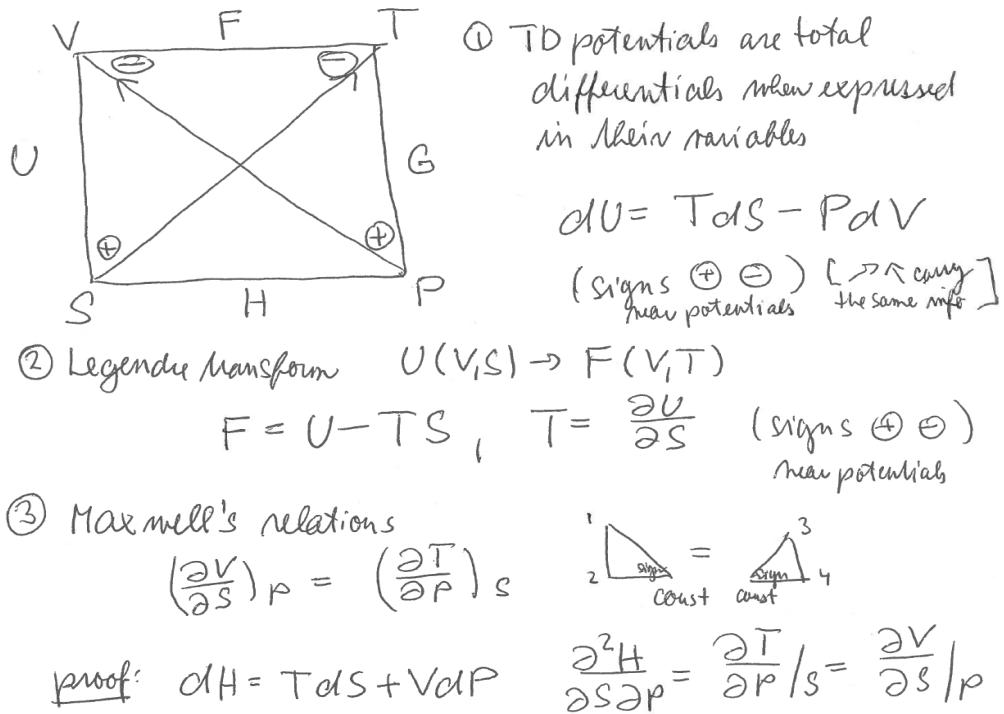
Thermodynamic potentials “determine the equilibrium state of a system under various constraints.” Each potential is a function of state and has proper independent variables. When its proper variables are held constant, stable equilibrium corresponds to the minimum of the potential.

We have the following table of basic thermodynamic potentials:

Potential	Expression	Variables	Differential form
Internal energy	$U$	$S, V$	$dU = TdS - PdV$
Enthalpy	$H = U + PV$	$S, P$	$dH = TdS + VdP$
Free Energy	$F = U - TS$	$T, V$	$dF = -SdT - PdV$
Gibbs Energy	$G = U - TS + PV$	$T, P$	$dG = -SdT + VdP$

(1.26)

I also remind you the magic square.



## 1.2 Basics of statistical physics

### Quantum vs. classical

Let us briefly compare the description of quantum and classical systems, providing a dictionary that allows one to “translate” one to another.

- In quantum systems a *microstate* is described by a pure state in Hilbert space,  $|\psi\rangle \in \mathcal{H}$ . A macrostate corresponds to a density matrix  $\rho$  where a pure state  $|\psi_\lambda\rangle$  appears with a probability  $p_\lambda$ :

$$\rho = \sum_{\lambda} p_{\lambda} |\psi_{\lambda}\rangle \langle \psi_{\lambda}|. \quad (1.27)$$

In this sense, the density matrix describes a “statistical ensemble of several quantum states”. It is normalized,  $\text{Tr}\rho = 1$ , and its time evolution is described by von Neumann equation:

$$\boxed{\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar} [H, \rho].} \quad (1.28)$$

[**Hint:** If you don't believe this try to prove it for a pure state  $\rho = |\psi\rangle \langle \psi|$  employing Schrodinger's equation.] Having a quantum mechanical observable described

by an operator  $A$ , the density matrix allows us to calculate its *expectation value* according to

$$a = \langle A \rangle = \text{Tr}(\rho A) . \quad (1.29)$$

- Let us get a very crude estimate for validity of the classical physics description using the *Heisenberg's uncertainty principle*, estimating the thermal fluctuations and the distance in between the molecules:

$$\frac{(\Delta p)^2}{2m} \approx k_B T, \quad \Delta x \approx \left( \frac{V}{N} \right)^{1/3}, \quad (1.30)$$

gives

$$\Delta x \Delta p >> \hbar \quad \Rightarrow \quad \sqrt{2mk_B T} \left( \frac{V}{N} \right)^{1/3} >> \hbar \quad \Rightarrow \quad k_B T >> n^{2/3} \frac{\hbar^2}{2m} . \quad (1.31)$$

This requires large temperature and/or small particle densities,  $n = N/V$ , and is usually satisfied, apart from some weirdo's as for example  ${}^4\text{He}$ .

- In classical physics a *microstate* describing  $N$  particles corresponds to a vector in  $6N$ -dimensional phase space  $\Sigma_N = \mathbb{R}^{6N}$ :  $(\vec{q}_1, \vec{p}_1, \dots, \vec{q}_N, \vec{p}_N) \in \epsilon_N$  whose evolution is given by Hamilton's canonical equations. A *macrostate* corresponds to a probability distribution  $w_N$  in the phase space (parameterizing our ignorance about the system) with the measure

$$\mu = w(\vec{q}_1, \vec{p}_1, \dots, \vec{q}_N, \vec{p}_N, t) d\tau_N, \quad d\tau_N = \frac{1}{N!(2\pi\hbar)^{3N}} d^3\vec{p}_1 d^3\vec{q}_1 \dots d^3\vec{p}_N d^3\vec{q}_N , \quad (1.32)$$

where the weird prefactor guarantees the right normalization,  $\int w d\tau_N = 1$ .<sup>2</sup>

One can think of a statistical ensemble as having a large number of boxes with the same macrostate, so that if I choose one randomly, I find the corresponding microstate with a probability  $\mu$ .

Observable is a function on microstates  $A = A(p, q)$  whole average value is calculated as

$$\langle A \rangle = \int A(p, q) w(p, q, t) d\tau_N . \quad (1.34)$$

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<sup>2</sup>One can justify the appearance of  $2\pi\hbar$  from Heisenberg's principle which implies that the phase space should not be divided into smaller cells than  $2\pi\hbar$ . Note also that  $w$  describes the  $N$ -particle probability distribution. 1-particle distribution  $f$  is given by

$$f(\vec{p}, \vec{q}, t) = \int d\tau_{N-1}(2, 3, \dots, N) w(\vec{p}, \vec{q}, \vec{p}_2, \vec{q}_2, \dots), \quad (1.33)$$

giving  $f(\vec{p}, \vec{q}) d\tau_1 = dN$ , a number of particles in a given volume of the phase space.

In summary, we have the following dictionary:

Description	Quantum	Classical	
stage	Hilbert space $\mathcal{H}$	Phase space $\Sigma_N$	
math	$\text{Tr}(\cdot)$	$\int \cdot d\tau_N$	
statistical ensemble	density matrix $\rho$	probability distribution $w_N$	
normalization	$\text{Tr}\rho = 1$	$\int w d\tau_N = 1$	
observable	operator $A$	phase space function $A(p, q)$	
average	$\langle A \rangle = \text{Tr}(A\rho)$	$\langle A \rangle = \int A(p, q)w(p, q, t)d\tau_N$	(1.35)

## Liouville's theorem

In theoretical mechanics one can understand the Liouville's theorem as a consequence of the fact that any Hamiltonian vector field (in particular the time evolution) preserves the symplectic 2-form and hence also the volume of the phase space (our cat no matter how deformed preserved its volume). Now we show that this is a ‘consequence’ of the underlying quantum mechanical description of our world and the fact that quantum mechanics is unitary.

The von Neumann's equation (1.28) reflects the fact that quantum mechanics is unitary. Let us write down its classical counterpart. The correspondence between classical and quantum mechanics states that

$$\{q, p\} = 1 \quad \leftrightarrow \quad \frac{1}{i\hbar}[\hat{q}, \hat{p}] = 1 \quad \leftrightarrow \quad [\hat{f}, \hat{g}] \quad \leftrightarrow \quad i\hbar\{f, g\}, \quad (1.36)$$

for any two observables  $f$  and  $g$ , where we replaced the commutators  $[ , ]$  with the Poisson brackets  $\{ , \}$ . In particular, the density matrix  $\rho$  becomes a distribution function on a phase space,  $w$ , and the von Neumann equation has the following classical limit:  $i\hbar\frac{\partial w}{\partial t} = i\hbar\{H, w\}$ . Hence we recovered (using the antisymmetry of Poisson's brackets) the classical *Liouville's equation*:

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial t} + \{w, H\} = 0.} \quad (1.37)$$

This equation has an immediate consequence of crucial interest to statistical physics. The ‘simplest way’ to satisfy the Liouville's equation is to set  $w = \text{const}$ . This corresponds to the so called *microcanonical ensemble*.<sup>3</sup> More generally, it means that  $w$  is an integral of motion. A generic system has the following 7 integrals of motion:  $E, \vec{P}, \vec{M}$ .<sup>4</sup> So we must have

$$w = w(E, \vec{P}, \vec{M}). \quad (1.38)$$

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<sup>3</sup> $w = \text{constant}$  means that the system can be found with equal probability in any microstate consistent with the known energy and composition. Although conceptually simple, the microcanonical ensemble is not very practical for calculations. One has to deal with distributions coming from the fact that the motion in the phase space occurs on a fixed energy hypersurface which causes many troubles.

<sup>4</sup>Extra symmetries of integrable systems are a luxury that disappears in the thermodynamic limit. For example, the Runge–Lenz vector exists only for a potential  $1/r$ .

However, one can often remove  $\vec{P}$  and  $\vec{M}$  by going to a ‘comoving corotating frame’ (unless you want to study statistical physics of rotating droplets), and we are left with energy  $E$ . This is an additive quantity. Taking two subsystems  $w_{12} = w_1 w_2$ , we have  $\log w_{12} = \log w_1 + \log w_2$ . This precisely corresponds to the fact that their energies add, and suggest we set

$$\log w = \alpha + \beta E \quad \Rightarrow \quad w(q_i, p_i, t) = e^{\alpha + \beta E(p_i, q_i, t)}. \quad (1.39)$$

This is a tremendous simplification. We just described the Gibbs distribution for the *canonical ensemble*. We shall re-derive this distribution in the next section, starting from a different perspective.

Let us make two more observations. First, since the probability distribution is normalized,  $\int w d\tau_N = 1$ , by differentiating it along a trajectory we find

$$0 = \frac{d}{dt} \int w d\tau_N = \int \frac{dw}{dt} d\tau_N + \int w \frac{d\tau_N}{dt} = \int w \frac{d\tau_N}{dt}. \quad (1.40)$$

This then implies that

$$\boxed{\frac{d\tau_N}{dt} = 0}, \quad (1.41)$$

that is, the volume of the phase space is preserved along the time evolution. This is the Liouville’s theorem we have seen in Theoretical Mechanics.

Second, by employing the Hamilton’s canonical equations,

$$\frac{\partial H}{\partial q^i} = -\dot{p}_i, \quad \frac{\partial H}{\partial p_i} = \dot{q}^i, \quad (1.42)$$

Eq. (1.37) rewrites as follows

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \{\rho, H\} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q^i} \frac{\partial w}{\partial p_i} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial q^i} \dot{q}^i + \dot{p}_i \frac{\partial w}{\partial p_i} = \frac{\partial w}{\partial t} + \dot{X}^A \nabla_A w, \quad (1.43)$$

where we have introduced a notation  $X^A = (q^i, p_i)$  and  $\nabla_A = (\frac{\partial}{\partial q^i}, \frac{\partial}{\partial p_i})$ . Using Hamilton’s equation once more, we find that  $\nabla \cdot \dot{X} = \nabla_A \dot{X}^A = 0$ . Hence we can write the Liouville’s equation as a *continuity equation in the phase space*:

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial t} + \nabla \cdot (w \dot{X}) = 0.} \quad (1.44)$$

## Gibbs distribution

Standardly, the Gibbs distribution is derived for specific ensembles, starting from a microcanonical one, canonical one, and so on, using nice physical arguments e.g. [2]. Let us instead, use a ‘quantum mechanical’ derivation of the most general Gibbs distribution.

We start with the notion of statistical (von Neumann) entropy. This is defined by

$$\boxed{S = -k_B \text{Tr}(\rho \log \rho),} \quad (1.45)$$

where  $\rho$  stands for the quantum mechanical density matrix  $\rho$ . From now on as per usual we shall set the Boltzmann constant  $k_B = 1$ . Let us further assume that we measure quantum mechanical observables described by operators  $A_i$  ( $i = 1, \dots, n$ ) to have values  $a_i$ , i.e., we have

$$\text{Tr}(\rho A_i) = a_i, \quad \text{Tr}\rho = 1. \quad (1.46)$$

To find the explicit form of  $\rho$  we shall use the following:

**Principle of maximal mess:** *The density matrix  $\rho$  representing the equilibrium state of the microsystem is such that it maximizes the statistical entropy, subject to the (macroscopical) constraints (1.46).*

To exploit the principle of maximal mess we shall use the method of *Lagrange multipliers*. We want to maximize the corresponding functional:

$$I = S - \sum_i \lambda_i [\text{Tr}(\rho A_i) - a_i] - \lambda_0 [\text{Tr}\rho - 1]. \quad (1.47)$$

Here  $\lambda_i$  and  $\lambda_0$  are the Lagrange multipliers. Varying this functional with respect to  $\rho$ , we find

$$\begin{aligned} \delta I &= \delta S - \sum_i \lambda_i \text{Tr}(A_i \delta \rho) - \lambda_0 \text{Tr}(\delta \rho) \\ &= -\text{Tr}(\log \rho \delta \rho + \delta \rho) - \sum_i \lambda_i \text{Tr}(A_i \delta \rho) - \lambda_0 \text{Tr}(\delta \rho) \\ &= \text{Tr}[\delta \rho B], \quad B \equiv -\log \rho - \sum_i \lambda_i A_i - (\lambda_0 + 1)\mathbf{1}. \end{aligned} \quad (1.48)$$

We require the action to be maximized, that is  $\delta I = \text{Tr}[\delta \rho B] = 0$ . Since this is valid for any hermitian matrix  $\delta \rho$  we must have  $B = 0$ ,<sup>5</sup> i.e.,  $\log \rho = -\sum_i \lambda_i A_i - (1 + \lambda_0)\mathbf{1}$ . In other words, we found that

$$\rho = e^{-\sum_i \lambda_i A_i - (1 + \lambda_0)\mathbf{1}} = e^{-(1 + \lambda_0)} e^{-\sum_i \lambda_i A_i}. \quad (1.49)$$

By imposing  $\text{Tr}\rho = 1$ , we eliminate  $\lambda_0$ , and recover the *Gibbs distribution*

$$\boxed{\rho = \frac{1}{Z} e^{-\sum_i \lambda_i A_i}, \quad Z \equiv \text{Tr}(e^{-\sum_i \lambda_i A_i})}. \quad (1.50)$$

---

<sup>5</sup>Provided that  $A$  is any matrix, the statement  $\text{Tr}[AB] = 0 \Rightarrow B = 0$  is evident. This can then be ‘decomposed’ into  $A$  hermitian and  $A$  anti-hermitian, showing the statement needed in the derivation.

Quantity  $Z$  is called the partition function. Of course, this can be translated to a classical result using the above correspondence.

Let us next show the following important results:

$$\boxed{a_i = -\frac{\partial}{\partial \lambda_i} \log Z, \quad S = \sum_i \lambda_i a_i + \log Z, \quad \lambda_i = \frac{\partial S}{\partial a_i}.} \quad (1.51)$$

Indeed, to prove the first one we calculate

$$-\frac{\partial}{\partial \lambda_i} \log Z = -\frac{1}{Z} \frac{\partial Z}{\partial \lambda_i} = \frac{1}{Z} \text{Tr}[A_i e^{-\sum_i \lambda_i A_i}] = \text{Tr}[\rho A_i] = a_i. \quad (1.52)$$

We also have

$$S = -\text{Tr}(\rho \log \rho) = \text{Tr}(\rho \sum_i \lambda_i A_i + \rho \log Z) = \sum_i \lambda_i a_i + \log Z, \quad (1.53)$$

which immediately implies that  $\lambda_i = \frac{\partial S}{\partial a_i}$ , since  $Z$  is a function of  $\lambda_i$  but not  $a_i$ .

## Canonical ensemble

Let us now consider a special case of a system where only the energy is allowed to fluctuate:

$$U = \langle H \rangle = \text{Tr}(\rho H). \quad (1.54)$$

Such ensemble is called the canonical ensemble and the corresponding Gibbs distribution simplifies to (denoting  $\lambda_H = \beta$  as per usual)

$$\boxed{\rho = \frac{1}{Z} e^{-\beta H}, \quad Z = \text{Tr}[e^{-\beta H}].} \quad (1.55)$$

Let us now connect to thermodynamics. We make the following definitions for the infinitesimal *work*  $\delta W$  and the infinitesimal *heat*  $\delta Q$ :

$$\boxed{\delta W \equiv \text{Tr}(\rho \delta H), \quad \delta Q \equiv \text{Tr}(H \delta \rho).} \quad (1.56)$$

These definitions are quite intuitive. For example, taking  $\rho = \sum_m p_m |m\rangle \langle m|$  and  $H = \sum_m e_m |m\rangle \langle m|$ , the *heat transfer* corresponds to the redistribution of probabilities of states,  $\delta Q = \sum_m e_m \delta p_m$ , whereas *work* corresponds to ‘shifting’ the energy levels,  $\delta W = \sum_m (\delta e_m) p_m$ . The latter can easily be achieved by applying conservative forces (e.g. electric field), while the former is much more ‘dirty’.

In any case, with these definitions we recover the first law

$$dU = d\text{Tr}(\rho H) = \text{Tr}(H \delta \rho) + \text{Tr}(\rho \delta H) = \delta Q + \delta W. \quad (1.57)$$

Using the definition (1.45) and eq. (1.55), we have

$$\begin{aligned}\delta S &= -\delta \text{Tr}[\rho \log \rho] = -\text{Tr}[\log \rho \delta \rho] - \text{Tr} \delta \rho = -\text{Tr}[\delta \rho(-\beta H - \log Z)] \\ &= \beta \text{Tr}(H \delta \rho) = \beta \delta Q,\end{aligned}\quad (1.58)$$

where we have used the fact  $\text{Tr} \delta \rho = 0$  from the normalization of the density matrix. Making further a contact with thermodynamics by imposing the *Clausius relation*  $\delta Q = T \delta S$ , we must have

$$\boxed{\beta = \frac{1}{T}}. \quad (1.59)$$

That is, the Lagrange multiplier  $\beta$  gives the thermodynamic temperature. The partition function then becomes

$$\boxed{Z = \text{Tr}(e^{-\beta H}), \quad \beta = \frac{1}{T}}. \quad (1.60)$$

Let us finally define the *free energy*  $F$  by

$$\boxed{F = -T \log Z}. \quad (1.61)$$

Using expression (1.53), we find

$$S = \beta U + \log Z \Leftrightarrow F = U - TS. \quad (1.62)$$

The latter is the standard relation between free energy  $F$  and internal energy  $U$ , known from classical thermodynamics.

Let us remark that one can use other (more complicated) ensembles, for example the *grand-canonical ensemble* where both the energy and number of particles are allowed to fluctuate,

$$Z_G = \text{Tr}(e^{-\beta H + \beta \mu N}), \quad (1.63)$$

where  $N$  is the operator for number of particles and the Lagrange multiplier  $\mu$  stands for the chemical potential. However, it turns out the in the thermodynamic limit ( $N \rightarrow \infty$ ) it does not really matter which ensemble is used. For this reason the most used is the simplest canonical ensemble. See, however, the following table of ensembles:

Ensemble	Variables	Partition function	Potential	Differential
canonical	$T, N, V$	$Z_C = \text{Tr}_N(e^{-\beta H_N})$	$F = -T \log Z_C$	$dF = -SdT + \mu dN - PdV$
grand-can.	$T, \mu, V$	$Z_G = \text{Tr}(e^{-\beta H + \beta \mu N})$	$A = -T \log Z_G$	$dA = -SdT - Nd\mu - PdV$
isobaric	$T, N, P$	$Z_i = \text{Tr}(e^{-\beta H + \beta PV})$	$G = -T \log Z_i$	$dG = -SdT + \mu dN + VdP$

(1.64)

## 1.3 Thermodynamic approach to phase transitions

### Statistical physics machinery at work: ideal gas

- The statistical physics machinery goes as follows. Given the Hamiltonian  $H$ , we calculate the (canonical ensemble) partition function (2.40) and derive from it the

free energy (1.61). The physical observables are then derived using the well known thermodynamic relations. For example, since the free energy  $F = F(T, V, N)$  satisfies

$$dF = -SdT - PdV + \mu dN, \quad (1.65)$$

we can immediately calculate

$$P = -\frac{\partial F}{\partial V}\Big|_{T,N}, \quad S = -\frac{\partial F}{\partial T}\Big|_{V,N}, \quad \mu = \frac{\partial F}{\partial N}\Big|_{V,T}. \quad (1.66)$$

Let us demonstrate this machinery on an example of ideal gas.

- Ideal gas. Let us consider an ideal gas in  $d$  number of spacetime dimensions (*sorry for that joke*). The partition function of  $N$  non-interacting particles with positions and momenta  $\vec{r}_i, \vec{p}_i, i = 1, \dots, N$ , is given by

$$Z = K_N \int d\vec{r}_1 \dots d\vec{r}_N d\vec{p}_1 \dots d\vec{p}_N e^{-\beta H}, \quad H = \sum_i \frac{\vec{p}_i^2}{2m}, \quad (1.67)$$

where  $K_N$  stands for the normalization factor (we set it equal to one for simplicity). Integrating over momenta, using  $\int_{-\infty}^{\infty} dp \exp(-\beta \frac{p^2}{2m}) = \sqrt{\frac{2\pi m}{\beta}}$ , and denoting by  $V$  the volume of the gas, we get

$$Z = \left(\frac{2\pi m}{\beta}\right)^{\frac{N(d-1)}{2}} V^N. \quad (1.68)$$

The free energy then reads

$$F = -T \lg Z = -\frac{d-1}{2} TN \lg(2\pi mT) - TN \lg V. \quad (1.69)$$

Employing (1.66) we then find

$$P = -\frac{\partial F}{\partial V}\Big|_{T,N} = \frac{TN}{V}, \quad (1.70)$$

which is the equation of state of *ideal gas*

$$\boxed{Pv = T, \quad v = \frac{V}{N}}. \quad (1.71)$$

The entropy  $S$  reads

$$S(T, V, N) = -\frac{\partial F}{\partial T}\Big|_{V,N} = \frac{d-1}{2} N \lg(2\pi mT) + N \lg V + \frac{d-1}{2} N, \quad (1.72)$$

Let us note the following interesting fact. Calculating the internal energy,  $U = F + TS$ , we find

$$U = F + TS = \frac{d-1}{2} NT. \quad (1.73)$$

Obviously  $U = U(T)$ . (The independence of  $U$  on  $V$  can be considered as a characteristic of ideal gas.) This is in such a way that there is  $\frac{1}{2}NT$  contribution to  $U$  per dof, which is the famous *partition theorem*. We also note that when we calculate the pressure, and since  $U = U(T)$ , we have  $P = -(\partial F/\partial V) = T(\partial S/\partial V)$ . That is the entire contribution to  $P$  is due to the entropy, not internal energy. In other words, the origin of pressure in ideal gas is entropic. (I will let you speculate what this means.)

Finally, we would like to calculate specific heats at constant pressure and volume. Rewriting (1.72) using the equation of state we get

$$S(T, P, N) = \frac{d-1}{2}N \lg(2\pi mT) + N \lg \frac{NT}{P} + \frac{d-1}{2}N. \quad (1.74)$$

So, using the following formulas for the specific heats for constant volume and pressure:

$$\begin{aligned} C_V &= \left. \frac{\delta Q}{dT} \right|_V = T \left. \frac{\partial S}{\partial T} \right|_V = \frac{(d-1)N}{2}, \\ C_P &= \left. T \frac{\partial S}{\partial T} \right|_P = \frac{(d+1)N}{2}, \end{aligned} \quad (1.75)$$

we find the *Myer's law*:

$$\kappa = \frac{C_P}{C_V} = \frac{d+1}{d-1}. \quad (1.76)$$

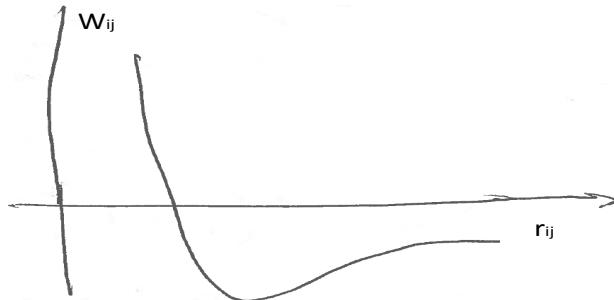
## Van der Waals gas

- Interacting gas. To go beyond the ideal gas law we can add two-particle interaction to the Hamiltonian,

$$H = \sum_i \frac{\vec{p}_i^2}{2m} + \sum_{i < j} W_{ij}, \quad W_{ij} = W(r_{ij}), \quad r_{ij} = |\vec{r}_i - \vec{r}_j|, \quad (1.77)$$

for example using the Lennard–Jones potential as a model

$$W_{ij} \propto \frac{A}{r_{ij}^{12}} - \frac{B}{r_{ij}^6} : \quad (1.78)$$



You shall derive the first correction to ideal gas law in your homework. For

dense gas, one has to use the so called *cluster expansion*, obtaining so a series of corrections to the ideal gas law of the form ( $n = N/V$ )

$$P = nT \left[ 1 + nB_2(T) + n^2 B_3(T) + n^3 B_4(T) + \dots \right]. \quad (1.79)$$

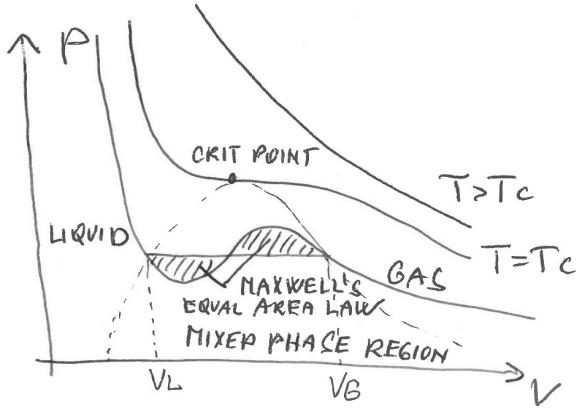
It turns out, however, that most of the gases can be described by the following miraculous equation:

- Van der Waals equation. Derived in 1873 by Johannes Diderik van der Waals, awarding him a Nobel Prize in 1910, the two-parametric Van der Waals equation reads

$$\left( P + \frac{a}{v^2} \right) (v - b) = T. \quad (1.80)$$

Parameter  $a$  measures the attraction between particles ( $a > 0$ ) and  $b$  corresponds to the “volume of fluid particles”. Many of the gases can be fitted by these two parameters.

Plotting the isotherms in the  $P - v$  diagram, we notice that, while for sufficiently high temperatures these behave the same way as isotherms of ideal gas, there exists a critical temperature  $T_c$  where an inflection point develops. Let's call this inflection point a *critical point*; it is characterized by  $(P_c, v_c, T_c)$ . Below  $T_c$  we observe that isotherms oscillate, as displayed in the following figure.



As noted by Maxwell, such oscillations are unphysical and in fact correspond to liquid/gas *change of phase*. The phase transition happens at a constant pressure  $P_0$  obtained, for a given isotherm, by the Maxwell's equal area law:

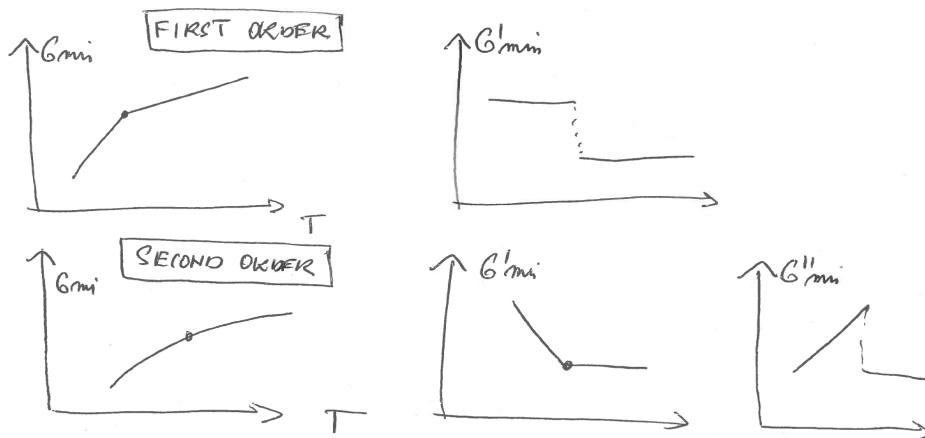
$$P_0(v_g - v_l) = \int_{v_l}^{v_g} P dv. \quad \Leftrightarrow \quad \oint v dP = 0. \quad (1.81)$$

Historically, it was a short way from Van der Waals equation to description of phase transitions.

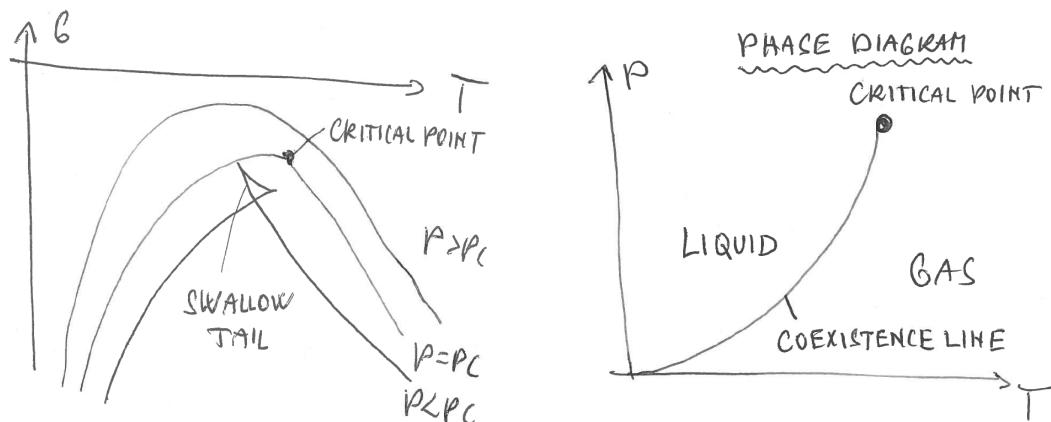
## Phase transitions

In thermodynamics, the phase transitions are characterized by the Gibbs free energy  $G$ , namely by the surface  $G = G(P, T)$  in the 3d ( $G, P, T$ )-space. The surfaces for two different phases will in general intersect in a line along which the Gibbs are equal. Along the line, the two phases are in equilibrium, while away from it the phase with lower Gibbs is the stable one. (Although in principle one can have *metastable states* as supersaturated vapour or superheated liquid.)

One can distinguish various types of phase transitions based on the ‘analyticity’ of the global minimum of  $G$ . Namely, if  $G_{\min}$  is continuous but its first derivative suffers a jump, we observe a *first order phase transition*. Such transitions require *latent heat*. If  $G$  is continuous and so is its first derivative but second derivative has a jump we observe a *second order phase transition* and so on. Often second and higher order phase transitions are called *continuous phase transitions*.



The Gibbs free energy for the Van der Waals fluid is displayed on the l.h.s. of the next figure:



For  $P > P_c$  this is nice and smooth and only one phase exists. However, for  $P < P_c$ , we observe a characteristic *swallow tail* and the corresponding first order

phase transition. This corresponds to the liquid/gas phase transition. Note also that at a critical point the swallow tail disappears and the phase transition is of second order.

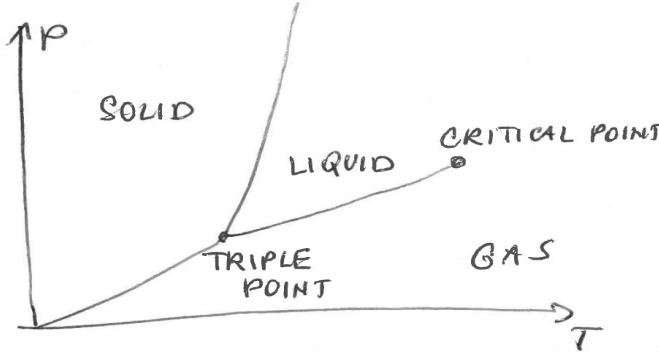
One can plot the corresponding  $P - T$  phase diagram, where the coexistence line corresponds to the ‘coexistence of two phases’, see right figure above.

The slope of this line is governed by the Clausius–Clapeyron equation

$$\left. \frac{dP}{dT} \right|_{\text{coexistence}} = \frac{\Delta s}{\Delta v} = \frac{s_g - s_l}{v_g - v_l}, \quad (1.82)$$

where  $L$  stands for the latent heat. Similarly the critical point (where the phase transition is of second order) is described by the Ehrenfest equations, which we shall study in tutorial. As we shall see later, second order phase transitions possess many interesting properties, including certain *universality*.

Physical systems may display a variety of phases and phase transitions. The most ‘typical’ is the solid/liquid/gas phase transition, see figure.

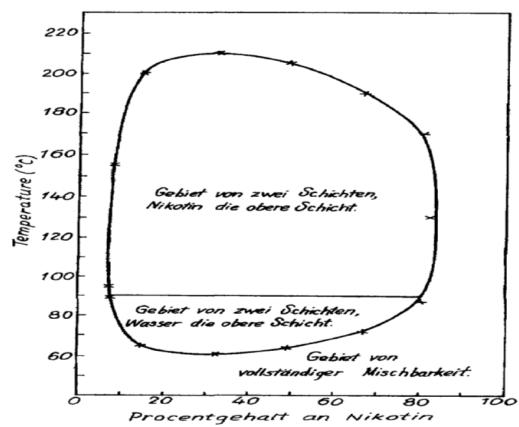


We have also the following interesting possibility.

## Reentrant phase transitions

**Definition.** A system undergoes a reentrant phase transition if a monotonic variation of any thermodynamic quantity results in two (or more) phase transitions such that the final state is macroscopically similar to the initial state.

This phenomenon was first observed by Hudson in 1904 in the nicotine/water mixture.



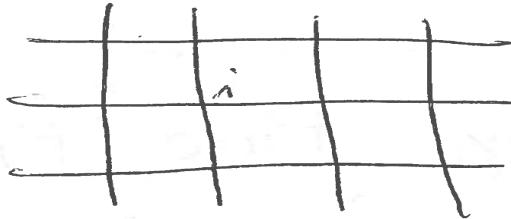
Since then it was observed in multicomponent fluid systems, gels, ferroelectrics, liquid crystals, binary gases, and more recently higher-dimensional black hole spacetimes.

# Chapter 2: Microscopic Theory of Phases and Phase Transitions

## 2.1 Introduction

- Matter organizes itself in phases that are characterized by different macroscopic properties. For example,
  - *solid/liquid/gas*: solid is characterized by “rigidity to shear”, liquid does not have rigidity to shear but has a well defined interface, while gas does not.
  - *ferromagnet/paramagnet*: ferromagnet has macroscopic magnetization.
  - *superconductor/normal matter*: superconductor has zero electrical resistance and demonstrates expulsion of magnetic flux fields.
- Appearance of macroscopic properties distinguishing phases is a consequence of ordering. For example, rigidity corresponds to ordering of atoms in a crystal lattice, ferromagnetic phase corresponds to ordering of atomic magnetic moments (alignment).
- Canonical Example: Ising model

This is a simple mathematical model of ferromagnetism invented by W. Lenz in 1920. Magnetic dipole moments of atomic spins are organized in a  $d$ -dimensional cubic lattice, with sites denoted by  $i$ ,



and are described by discrete variables  $\sigma_i = \pm 1$ . Each spin interacts with its neighbours through the Hamiltonian

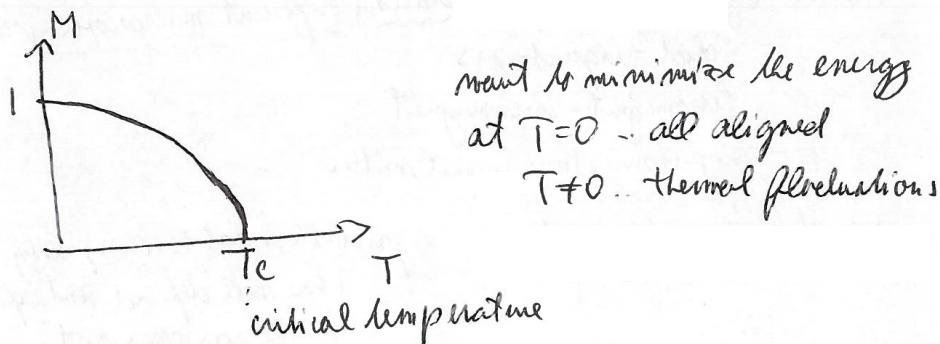
$$H = -\frac{1}{2} \sum_{ij} J_{ij} \sigma_i \sigma_j , \quad (2.1)$$

where the minus sign indicates that the spins want to align their dof. The one-dimensional model was solved by Ising in 1925 and demonstrates no phase transitions, contrary to its two-dimensional cousin (Onsager 1944). In dimensions

greater than four, the phase transition is exactly described by the mean field theory. We shall use this model as a canonical example to learn about phase transitions.

- 2 phases: Ferromagnetic (FM) and paramagnetic (PM) are distinguished by order parameter: magnetization

$$M = \langle \sigma_i \rangle = \begin{cases} = 0 & \text{PM} \\ \neq 0 & \text{FM} \end{cases} \quad (2.2)$$



Since the magnetization changes continuously as we cross  $T_c$ , this is a continuous phase transition. Note the non-analytic behavior!

- Note also that while  $H$  is invariant under  $\sigma_i \rightarrow -\sigma_i$ , in the FM phase this symmetry is spontaneously broken. (System “randomly chooses” up or down.) This is true more generally: the existence of order parameter is tied to spontaneously broken symmetry.
- Let us now do the following simple exercise: We add external magnetic field  $B$  to a finite system with  $N$  sites:

$$H = -\frac{1}{2} \sum_{ij} J_{ij} \sigma_i \sigma_j - B \sum_i \sigma_i. \quad (2.3)$$

We start with  $B > 0$  and ask what happens to  $M = \frac{1}{N} \sum \sigma_i$ . Let's calculate the probability of finding the magnetization up or down:

$$r = \frac{P_{M>0}}{P_{M<0}} = \frac{e^{-NBM/T}}{e^{NBM/T}} = e^{-\frac{2NBM}{T}}. \quad (2.4)$$

Consider now two different limits:

- Keep  $N$  finite and take  $B \rightarrow 0$ . Then we find  $r \rightarrow 1$ . This means that over long time both states are equally happening and the average magnetization is zero:  $M = 0$ .

- First take the thermodynamic limit  $N \rightarrow \infty$  and then  $B \rightarrow 0$ . Then we find  $r = 0$ . This means that the system will remain in the state defined by previous  $B$  indefinitely and  $M$  is now “well defined”.

This brings us to the following important conclusion: *Phases and transitions in between them are defined sharply only for infinite systems.* (Macroscopic systems have long timescales.)

## 2.2 Mean field theory

The idea of the mean field theory (MFT) is to replace all multi-body interactions with an effective one body interaction with an average *mean (molecular) field*, reducing any multi-body problem into an effective one-body problem. This effectively represents the “zeroth-order” expansion of the Hamiltonian in fluctuations: a MFT system has no fluctuations, all multi-body interactions were replaced by an interaction with a mean field. Interestingly, as we shall see later the prediction of this (very crude) approximation becomes exact for systems in more than four dimensions.

- Let us demonstrate this explicitly on the Ising model. We do the following expansion in fluctuations:

$$\begin{aligned}\sigma_i &= \sigma_i - \langle \sigma_i \rangle + \langle \sigma_i \rangle = \sigma_i - M + M, \\ \sigma_i \sigma_j &= [(\sigma_i - M) + M][(\sigma_j - M) + M] = \underbrace{(\sigma_i - M)(\sigma_j - M)}_{\epsilon^2 \approx 0} + M(\sigma_i + \sigma_j) - M^2.\end{aligned}$$

The Hamiltonian now reads

$$H = -\frac{1}{2} \sum_{ij} J_{ij} M(\sigma_i + \sigma_j) + \frac{1}{2} \sum_{ij} J_{ij} M^2. \quad (2.5)$$

We further denote by

$$J = \sum_j J_{ij}. \quad (2.6)$$

This is independent of  $i$  as we have uniform lattice and we assume it to be finite (which it is provided  $J_{ij}$  are sufficiently short range). Putting all together and using the symmetry of  $J_{ij}$  we thus obtained the following simplified MFT Ising Hamiltonian:

$$H = -MJ \sum_i \sigma_i + \frac{1}{2} NJM^2, \quad (2.7)$$

which is a “non-interacting” Hamiltonian in some external mean field  $M$  (obtained by throwing away the fluctuations).

- We now switch on the statistical physics machinery and calculate:

$$\begin{aligned}
 Z &= \sum_{\{\sigma_i\}} e^{-\beta H} = \sum_{\sigma_1=\pm 1} \cdots \sum_{\sigma_N=\pm 1} e^{-\beta H} = \left( \prod_{i=1}^N \sum_{\sigma_i=\pm 1} \right) e^{-\beta H} \\
 &= \left( \prod_{i=1}^N \sum_{\sigma_i=\pm 1} \right) \exp(\beta M J \sum_j \sigma_j - \frac{1}{2} \beta N J M^2) \\
 &= e^{-\frac{N J M^2}{2T}} \prod_{i=1}^N \left( \sum_{\sigma_i=\pm 1} e^{\frac{M J \sigma_i}{T}} \right) = e^{-\frac{N J M^2}{2T}} [2 \cosh(M J / T)]^N, \quad (2.8)
 \end{aligned}$$

giving

$$F = -T \lg Z = \frac{N J M^2}{2} - NT \lg [2 \cosh(M J / T)]. \quad (2.9)$$

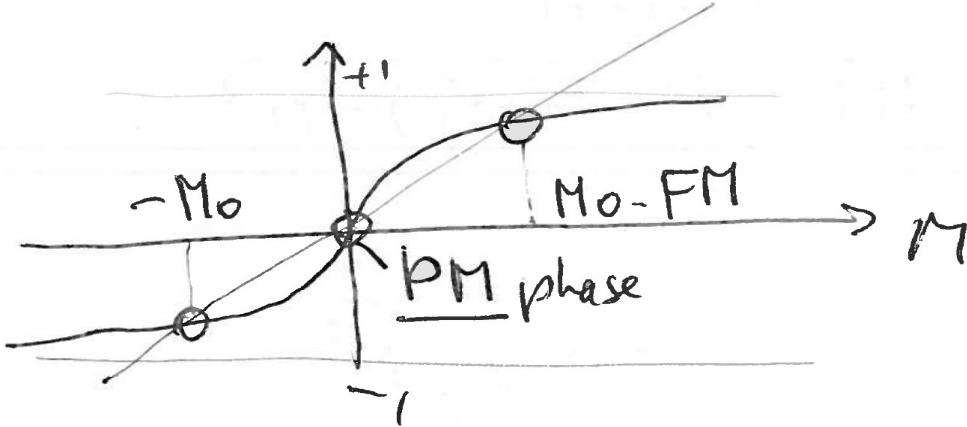
To find the actual  $M$  in equilibrium we have to minimize  $F$  w.r.t.  $M$ :

$$\frac{\partial F}{\partial M} = N J M - N J \tanh(M J / T) = 0, \quad (2.10)$$

that is

$$M = \tanh(M J / T). \quad (2.11)$$

This is an implicit relation for  $M = M(T)$  that can be solved graphically:



The FM phase at  $M_0 \neq 0$  occurs when the slope of r.h.s.  $[\tanh(M J / T)]$  in the origin is bigger than the slope of l.h.s. ( $M$ ). This depends on the  $J/T$  ratio. Expanding  $\tanh x \approx x + \dots$ , we find:

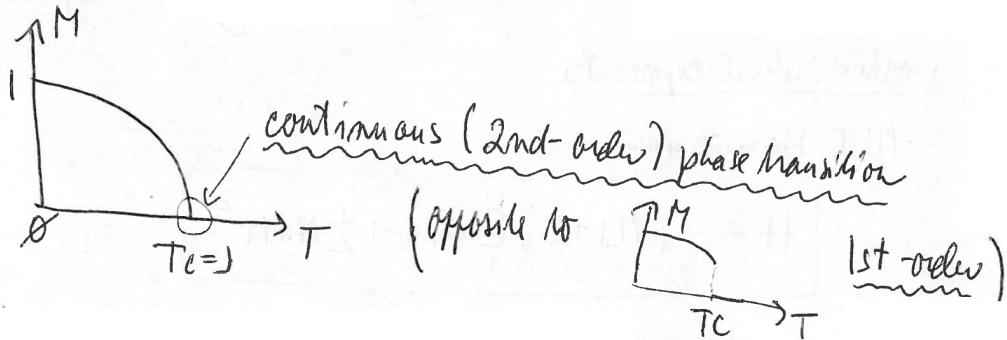
$$\begin{aligned}
 T < J &\quad M = \pm M_0, \\
 T > J &\quad M = 0,
 \end{aligned}$$

giving the critical temperature

$$T_c = J. \quad (2.12)$$

Obviously, the critical temperature depends on how strongly the spins interact.

- The studied phase transition is continuous (2nd-order) phase transition (order parameter changes continuously).

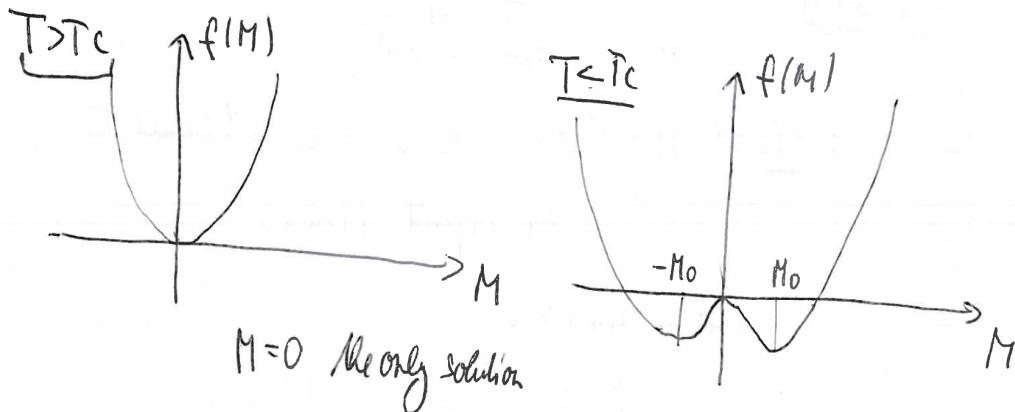


Such phase transitions are characterized by (universal) critical exponents. Critical exponents describe behavior of fundamental physical quantities in the vicinity of the critical point.

- For example, Critical exponent  $\beta$  describes the behavior of the magnetization near the critical point. Let us calculate it. Since our phase transition is continuous,  $M$  is small near  $T_c$ . So the free energy per lattice site reads

$$\begin{aligned} f &= \frac{F}{N} = \frac{JM^2}{2} - T \lg[2 \cosh(MJ/T)] \\ &\approx \frac{JM^2}{2} - \frac{1}{2}T \left(\frac{MJ}{T}\right)^2 + \frac{1}{12}T \left(\frac{MJ}{T}\right)^4 + O(M^6) \\ &= \frac{T_c}{2} \left(1 - \frac{T_c}{T}\right) M^2 + \frac{T_c^4}{12T^3} M^4 + \dots \end{aligned} \quad (2.13)$$

We sketch the behavior of  $f$  in the next figure



We then have

$$\frac{\partial f}{\partial M} = T_c \left(1 - \frac{T_c}{T}\right) M + \frac{T_c^4}{3T^3} M^3 = 0 \Rightarrow M \approx \pm \sqrt[3]{3 \left(1 - \frac{T}{T_c}\right)}. \quad (2.14)$$

Introducing

$$t = \frac{T - T_c}{T_c}, \quad (2.15)$$

we thus found that near the critical point

$$M \propto (-t)^\beta, \quad \beta = \frac{1}{2}. \quad (2.16)$$

Here  $\beta$  is the critical exponent for magnetization, or more generally, a critical exponent for the order parameter. Note that the MFT prediction of  $\beta$  does not depend on and take into account dimensionality of the system.

- Other critical exponents. Let us next calculate the critical exponent for susceptibility,

$$\chi = \left( \frac{\partial M}{\partial B} \right)_T \Big|_{B=0}, \quad (2.17)$$

reflecting the ‘reaction’ of the system to an external magnetic field  $B$  (whether the material is ‘attracted into’ or ‘repelled out of’ magnetic field). The MFT Hamiltonian then reads

$$H = -(B_J + B) \sum_i \sigma_i + \frac{1}{2} N J M^2, \quad \text{where } B_J = M J \quad (2.18)$$

is called “molecular field”. This gives the modified free energy

$$F = \frac{N J M^2}{2} - N T \lg \left[ 2 \cosh \left( \frac{M J + B}{T} \right) \right] \Rightarrow M = \tanh \left( \frac{M J + B}{T} \right). \quad (2.19)$$

We know that for  $T > T_c$  and when  $B = 0$  we have  $M = 0$ . So we expect that at small  $B$ ,  $M \propto B$ . Since  $\tanh x \approx x - x^3/3$ , we have

$$M \approx \frac{M J + B}{T} - \frac{1}{3} \left( \frac{M J + B}{T} \right)^3. \quad (2.20)$$

Differentiating w.r.t.  $B$  we then have

$$\chi = \frac{\partial M}{\partial B} \approx \left( \chi \frac{J}{T} + \frac{1}{T} \right) \left[ 1 - \left( \frac{M J + B}{T} \right)^2 \right]_{B=0} = \left( \chi \frac{T_c}{T} + \frac{1}{T} \right) \left[ 1 - \frac{M^2 T_c^2}{T^2} \right]. \quad (2.21)$$

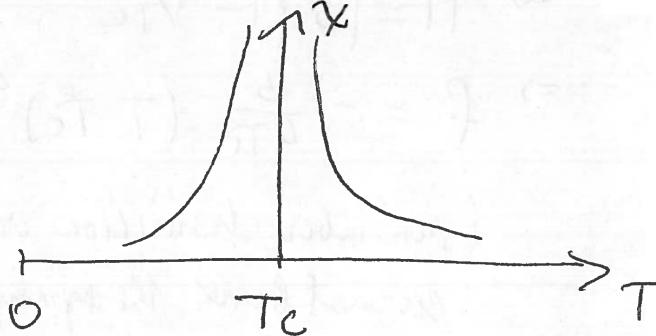
For  $T > T_c$ , we have  $M = 0$  and hence we get

$$T > T_c : \quad \chi = \frac{1}{T} \frac{1}{1 - T_c/T} = \frac{1}{T - T_c} \propto t^{-\gamma} \Rightarrow [\gamma = 1]. \quad (2.22)$$

On the other hand, for  $T < T_c$  we found, see (2.14),  $M^2 \approx 3(1 - T/T_c)$ , and hence

$$T < T_c : \quad \chi = \frac{1}{2} \frac{3T - 2T_c}{T_c(T_c - T)} \approx \frac{1}{2} \frac{1}{T_c - T} \propto |t|^{-\gamma'} \quad \Rightarrow \quad [\gamma' = 1]. \quad (2.23)$$

So we concluded that susceptibility critical exponents are  $\gamma = \gamma' = 1$ .



- We next want to know how the system responds exactly at  $T = T_c$ . Using again the expansion (2.20) and plugging  $T = T_c$ , we have

$$M = \left( \frac{3B}{T_c} \right)^{1/3} \propto B^{1/\delta} \quad \Rightarrow \quad [\delta = 3]. \quad (2.24)$$

$\delta$  is called field critical exponent.

- Let us finally calculate the specific heat (thermodynamic response function), determined from

$$\boxed{C_V = T \left( \frac{\partial S}{\partial T} \right)_V}. \quad (2.25)$$

This quantity is simple to measure and allows one to “detect” the presence of a phase transition. (One even does not need to know what the order parameter is.) We have

$$S = - \left( \frac{\partial F}{\partial T} \right)_V \quad \Rightarrow \quad \boxed{c_V = - \frac{T}{N} \left( \frac{\partial^2 F}{\partial T^2} \right)_V = -T \left( \frac{\partial^2 f}{\partial T^2} \right)_V}. \quad (2.26)$$

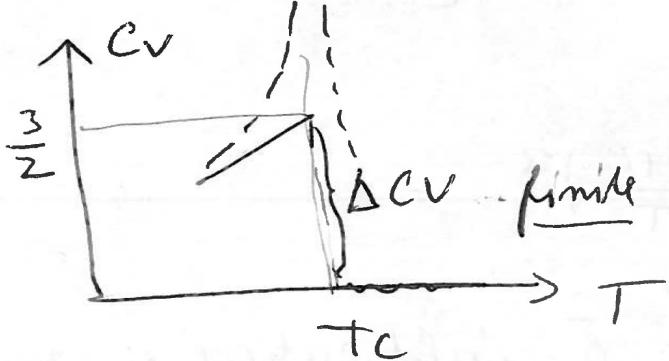
Using the expression (2.13) for  $f$ , let us first consider the case  $T < T_c$ . Then we have the expression (2.14) for  $M$ , to find

$$f = \frac{T_c}{2} \left( 1 - \frac{T_c}{T} \right) M^2 + \frac{T_c^4}{12T^3} M^4 + \dots = -\frac{3}{4T_c} (T - T_c)^2. \quad (2.27)$$

(Remember that the transition occurs only at  $B = 0$  otherwise the symmetry is already broken.) On the other hand for  $T > T_c$  we have  $M = 0$  and hence  $f = 0$ . So we find

$$c_V = -T \left( \frac{\partial^2 f}{\partial T^2} \right)_V = \begin{cases} \frac{3}{2} \frac{T}{T_c} \propto |t|^{-\alpha'} & \text{for } T < T_c \\ 0 \propto (t)^{-\alpha} & \text{for } T > T_c \end{cases} \Rightarrow \begin{cases} \alpha' = 0 \\ \alpha = 0 \end{cases}, \quad (2.28)$$

The fact that  $\alpha$  has a finite jump is an artifact of MFT. In reality we observe divergent behavior, with the specific heat critical exponents  $\alpha$  and  $\alpha'$ . In MFT we have found  $\alpha = 0 = \alpha'$ .



Critical exponents  $\alpha, \beta, \gamma, \delta$  fully characterize universal properties of continuous phase transitions.

- Landau's theory. MFT results are very robust and follow from general symmetry considerations.

– All we need to know is the order parameter and the symmetries:

$$M = \langle \sigma_i \rangle, \quad \sigma_i \rightarrow -\sigma_i \quad \Rightarrow \quad f(M) = f(-M). \quad (2.29)$$

– Near  $T_c$  we must have

$$f \approx rM^2 + uM^4, \quad u > 0, \quad (2.30)$$

the latter being required for stability. We further postulate that  $r$  changes sign at  $T_c$ , i.e.,  $r \propto T - T_c$ . As we shall see in tutorial this is all we need to know. This is the idea of ingenious *Landau's theory of phase transitions*.

- It turns out that diverse systems that occur in nature and are seemingly very different may have the same critical exponents. Transitions that are characterized by the same set of critical exponents belong to the same universality class.

MFT explains the universality of critical exponents. In fact, it predicts that there is only one universality class, characterized by

$$\alpha = 0, \quad \beta = \frac{1}{2}, \quad \gamma = 1, \quad \delta = 3.$$

(2.31)

Such critical exponents are almost never observed and critical points characterized by different critical exponents exist in nature.

- Experimentally, critical exponents depend on:

\* dimensionality of space  $d$

- \* nature of order parameter (number of components:  $M, \vec{M}, \dots$ )
- \* symmetry of the Hamiltonian
- Reason as to why the MFT fails is that it treats order parameter as a constant. We rather want to treat it as a TD variable that can fluctuate.

## 2.3 From Ising model to field theory

- Hubbard–Stratonovich transformation. We want to recast the problem of calculating the partition function for the Ising model,

$$Z = \sum_{\{\sigma_i\}} e^{\frac{1}{2T} \sum_{ij} J_{ij} \sigma_i \sigma_j}, \quad H = -\frac{1}{2} \sum_{ij} J_{ij} \sigma_i \sigma_j, \quad (2.32)$$

as a ‘field theory’. This is achieved by applying the so called Hubbard–Stratonovich transformation, which is nothing more than a generalization of the well known Gaussian integral:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\varphi e^{-\frac{1}{2}a\varphi^2 + b\varphi} = \frac{1}{\sqrt{a}} e^{\frac{b^2}{2a}} \quad a > 0, \quad (2.33)$$

or it’s multidimensional version:

$$\frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{\infty} d\varphi_1 \dots d\varphi_N e^{-\frac{1}{2}\varphi_i A_{ij} \varphi_j + \gamma_i \varphi_i} = \frac{1}{\sqrt{\det A}} e^{\frac{1}{2} \gamma_i A_{ij}^{-1} \gamma_j}, \quad (2.34)$$

or

$$e^{\frac{1}{2} \gamma_i A_{ij} \gamma_j} = \frac{1}{\sqrt{\det A}} \frac{1}{(2\pi)^{N/2}} \int_{-\infty}^{\infty} d\vec{\varphi} e^{-\frac{1}{2} \varphi_i A_{ij}^{-1} \varphi_j + \gamma_i \varphi_i}. \quad (2.35)$$

Identifying

$$A_{ij} = J_{ij}/T, \quad \gamma_i = \sigma_i, \quad (2.36)$$

we have

$$\begin{aligned} Z &= \frac{T^{N/2}}{\sqrt{\det J}} \frac{1}{(2\pi)^{N/2}} \sum_{\{\sigma_i\}} \int_{-\infty}^{\infty} d\vec{\varphi} e^{-\frac{T}{2} \varphi_i J_{ij}^{-1} \varphi_j + \sigma_i \varphi_i} = |\varphi_i \rightarrow \varphi_i/T| \\ &= \frac{1}{\sqrt{\det J}} \frac{1}{(2\pi T)^{N/2}} \sum_{\{\sigma_i\}} \int_{-\infty}^{\infty} d\vec{\varphi} e^{-\frac{1}{2T} \varphi_i J_{ij}^{-1} \varphi_j + \frac{1}{T} \sigma_i \varphi_i}. \end{aligned} \quad (2.37)$$

In this expression, the pre-factor is a smooth function of  $T$  that gives additional term to  $F = -T \log Z$  which has nothing to do with criticality. From now on let us set it equal one, to have

$$Z = \sum_{\{\sigma_i\}} \int_{-\infty}^{\infty} D\varphi e^{-\frac{1}{2T} \varphi_i J_{ij}^{-1} \varphi_j + \frac{1}{T} \sigma_i \varphi_i}. \quad (2.38)$$

Note that the term  $+\frac{1}{T}\sigma_i\varphi_i$  in the exponential describes “interaction of spin  $\sigma_i$  with some fluctuating field  $\varphi$  (rather than a constant  $M$  in the MFT description). Using the trick

$$\sum_{\sigma_i=\pm 1} e^{\frac{1}{T}\varphi_i\sigma_i} = e^{\varphi_i/T} + e^{-\varphi_i/T} = e^{\log[2 \cosh \varphi_i/T]}, \quad (2.39)$$

we finally have

$$Z = \int D\varphi e^{-S[\varphi]}, \quad S[\varphi] = \frac{1}{2T} \varphi_i J_{ij}^{-1} \varphi_j - \sum_i \log[2 \cosh \varphi_i/T]. \quad (2.40)$$

- Note. MFT corresponds to the *saddle point approximation*. Namely, let  $\bar{\varphi}_i, \dots, \bar{\varphi}_N$  be the configuration that minimizes  $S$ . Then

$$\frac{\partial S}{\partial \varphi_i} \Big|_{\varphi_i=\bar{\varphi}_i} = 0 = \frac{1}{T} \sum_j J_{ij}^{-1} \bar{\varphi}_j - \frac{1}{T} \tanh \bar{\varphi}_i/T \quad \Rightarrow \quad \bar{\varphi}_i = \sum_j J_{ij} \tanh \bar{\varphi}_j/T. \quad (2.41)$$

Since all sites are equivalent, we have  $\bar{\varphi}_1 = \bar{\varphi}_2 = \dots = \bar{\varphi}$ . We also denote by  $J = \sum_j J_{ij}$ , assuming this is finite. So we get

$$\bar{\varphi} = J \tanh \bar{\varphi}/T, \quad (2.42)$$

which is the MFT equation upon identifying  $\bar{\varphi} = MJ$ .

- Spin-spin correlation function. Starting from (2.40) one can calculate the 2-point correlation function, given by

$$G_{ij} = (\sigma_i - \langle \sigma_i \rangle)(\sigma_j - \langle \sigma_j \rangle) = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle. \quad (2.43)$$

The trick to calculate this is to consider spatially dependent magnetic field  $B_i$ ,

$$H = -\frac{1}{2} \sum_{ij} J_{ij} \sigma_i \sigma_j - \sum_i B_i \sigma_i,$$

and calculate

$$G_{ij} = -T \left( \frac{\partial^2 F}{\partial B_i \partial B_j} \right)_{B_i=0}. \quad (2.44)$$

Even in MFT this calculation is rather lengthy. For this reason we skip it here and only state the result

$$G_{ij} = \dots = \frac{a^3 T}{2\pi\kappa} \frac{e^{-|\vec{r}_i - \vec{r}_j|/\xi}}{|\vec{r}_i - \vec{r}_j|^{d-2}}, \quad \xi = \sqrt{\frac{\kappa}{2(T - T_c)}}, \quad (2.45)$$

where  $\kappa$  stands for the spin stiffness, see later.

More generally, the correlation function in  $d$  dimensions takes the following form:

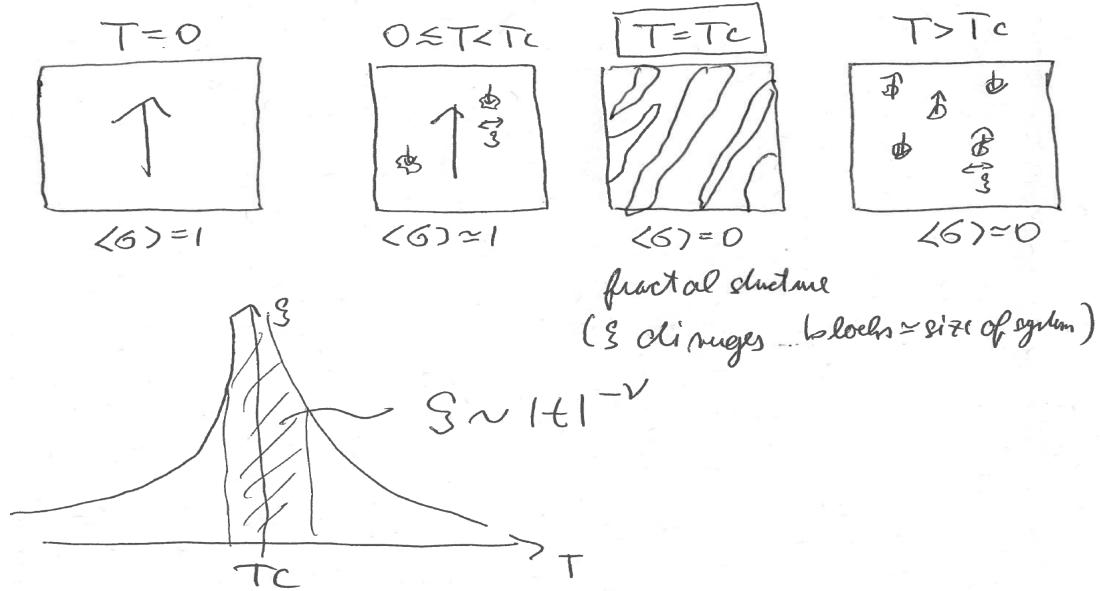
$$G(r) \propto \frac{e^{-r/\xi}}{r^{d-2+\eta}}. \quad (2.46)$$

Here  $\xi$  is called the correlation length. It has the form

$$\xi \propto |t|^{-\nu}. \quad (2.47)$$

and diverges at  $T = T_c$ . As the correlation length becomes infinite at the critical point, there is no length scale and the theory is described by conformal field theory (CFT). The critical exponent  $\eta$  is called the anomalous dimension (it is related to the fractal dimension of the geometry at the critical point).

The following picture displays what happens with spins of the spin system as the temperature decreases from right to left:



The set of critical exponents  $\alpha, \beta, \gamma, \delta, \nu, \eta$  completely describes the behavior of the system near the critical point. However, as you shall see in your tutorial, there are several equalities, named after famous people, relating these exponents. In consequence only two such exponents are typically independent. We have seen that the prediction of MFT is that

$$\alpha = 0, \quad \beta = \frac{1}{2}, \quad \gamma = 1, \quad \delta = 3, \quad \eta = 0, \quad \nu = \frac{1}{2}. \quad (2.48)$$

These also satisfy the mentioned equalities.

- Continuum limit. Let us now perform a continuum limit, employing the following idea: Near the critical point ordered spin block have fractal structure. This allows us to “zoom out”: we still see the ‘same picture’ but loose information about the underlying lattice.

We have

$$\log[2 \cosh \varphi_i/T] = \log 2 + \frac{1}{2} \left( \frac{\varphi_i}{T} \right)^2 - \frac{1}{12} \left( \frac{\varphi_i}{T} \right)^4 + \dots, \quad (2.49)$$

keeping the first few terms of the Taylor expansion. [Note that we need the last term with the minus sign for stability...]

- Let us now look at the term  $\varphi_i J_{ij}^{-1} \varphi_j$  using the lattice Fourier transform. This is

$$\varphi_i = \frac{1}{N} \sum_{\vec{q}} \varphi(\vec{q}) e^{i\vec{q} \cdot \vec{r}_i}, \quad J_{ij} = \frac{1}{N} \sum_{\vec{q}} J(\vec{q}) e^{i\vec{q} \cdot (\vec{r}_i - \vec{r}_j)}. \quad (2.50)$$

We shall also need the “lattice orthogonality relations”:

$$\frac{1}{N} \sum_i e^{-i(\vec{k} - \vec{k}') \cdot \vec{r}_i} = \delta_{\vec{k}', \vec{k}}, \quad \frac{1}{N} \sum_{\vec{q}} e^{-i\vec{q} \cdot (\vec{r}_i - \vec{r}_j)} = \delta_{ij}. \quad (2.51)$$

So we have

$$\begin{aligned} \sum_{ij} \varphi_i J_{ij}^{-1} \varphi_j &= \sum_{ij} \frac{1}{N} \sum_q \varphi(q) e^{i\vec{q} \cdot \vec{r}_i} \frac{1}{N} \sum_{q'} J^{-1}(q') e^{i\vec{q}' \cdot (\vec{r}_i - \vec{r}_j)} \frac{1}{N} \sum_{q''} \varphi(q'') e^{i\vec{q}'' \cdot \vec{r}_j} \\ &= |\text{sum over } i \text{ and } j| = \frac{1}{N} \sum_{q, q', q''} \varphi(q) J^{-1}(q') \varphi(q'') \delta_{q, -q'} \delta_{q', q''} \\ &= \frac{1}{N} \sum_q \varphi(-q) J^{-1}(q) \varphi(q). \end{aligned} \quad (2.52)$$

- What is  $J(q)$ ? Since long distances imply small  $q$ , we shall have  $qR \ll 1$ , with  $R$  denoting the typical range of interactions  $R \propto a$ . So we have

$$\begin{aligned} J(q) &= \frac{1}{N} \sum_{ij} J_{ij} e^{-i\vec{q} \cdot (\vec{r}_i - \vec{r}_j)} \\ &\approx \frac{1}{N} \sum_{ij} J_{ij} \left[ 1 - \underbrace{i\vec{q} \cdot (\vec{r}_i - \vec{r}_j)}_{=0 \text{(since } S_{ij} A_{ij}=0\text{)}} - \frac{1}{2} [\vec{q} \cdot (\vec{r}_i - \vec{r}_j)]^2 + \dots \right] \\ &\approx \underbrace{\frac{1}{N} \sum_{ij} J_{ij}}_{\frac{1}{N} NJ = J} - \frac{1}{2N} \underbrace{\sum_{ij} J_{ij} [\vec{q} \cdot (\vec{r}_i - \vec{r}_j)]^2}_{R^2 J N |q|^2 / d} \\ &\approx J - \frac{1}{2} \kappa |q|^2, \end{aligned} \quad (2.53)$$

where  $\kappa \approx R^2 J/d$  is called spin stiffness.

- So we have

$$\begin{aligned}
 S_0[\varphi] &= \frac{1}{2T^2} \left[ \sum_{ij} T \varphi_i J_{ij}^{-1} \varphi_j - \sum_i \varphi_i^2 \right] \\
 &= \frac{1}{2T^2 N} \sum_q \varphi(-q) \left[ \underbrace{\frac{T}{J(q)} - 1}_{\frac{T}{T_c} - 1 + \frac{R^2 T}{2d T_c} q^2 \approx t + \frac{R^2 q^2}{2d}} \right] \varphi(q) \\
 &= \left| \varphi \rightarrow \frac{T_c \sqrt{2d}}{Ra^{d/2}} \varphi, r = \frac{2dt}{R^2} \right| = \frac{1}{2} \frac{1}{Na^d} \sum_q \varphi(-q) (r + q^2) \varphi(q) \\
 &= \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \varphi(-q) (r + q^2) \varphi(q) \\
 &= \frac{1}{2} \int d^d x \left[ (\nabla \varphi)^2 + r \varphi^2 \right], \tag{2.54}
 \end{aligned}$$

which is a massive Klein–Gordon theory in imaginary time. For the interaction part, we simply have

$$S_{int}[\varphi] = \frac{1}{12T^4} \sum_i \varphi_i^4 = \underbrace{\frac{1}{12T^4}}_{u/4!} \underbrace{\frac{1}{a^d} \sum_i \varphi_i^4}_{\int d^d x} = \int d^d x \frac{u}{4!} \varphi^4. \tag{2.55}$$

So putting everything together, we arrived at

$$\boxed{S[\varphi] = \int d^d x \left[ \frac{1}{2} (\nabla \varphi)^2 + \frac{r}{2} \varphi^2 + \frac{u}{4!} \varphi^4 \right].} \tag{2.56}$$

The important fact to note from the derivation is that  $r$  changes sign at the transition ( $r \propto t$ ) and that  $u > 0$  is required for the stability. This is called the Landau–Ginsburg functional. Its symmetry  $\varphi \rightarrow -\varphi$  reflects the original symmetry of the Ising model  $\sigma \rightarrow -\sigma$ . As we shall see later, higher powers of  $\varphi$  or  $\nabla$  are irrelevant.

## Dimensional analysis

Working in the “units of energy” where

$$\boxed{[k] = \dim(k) = +1, [x] = -1,} \tag{2.57}$$

and from the fact that  $[S] = 0$ , we have

$$\boxed{[S] = 0 = \underbrace{[dx^d]}_{-d} + \underbrace{[\nabla^2]}_{+2} + [\varphi^2] \Rightarrow [\varphi] = \frac{d-2}{2}.} \tag{2.58}$$

Similarly, we find

$$[r] = 2, \quad [u] = 4 - d \equiv \epsilon. \quad (2.59)$$

As we shall see 4 is the upper critical dimension and  $\epsilon$  denotes variations from it.

The coupling  $g$  is called

$$\begin{aligned} \text{relevant : } & [g] > 0 \\ \text{marginal : } & [g] = 0 \\ \text{irrelevant : } & [g] < 0. \end{aligned} \quad (2.60)$$

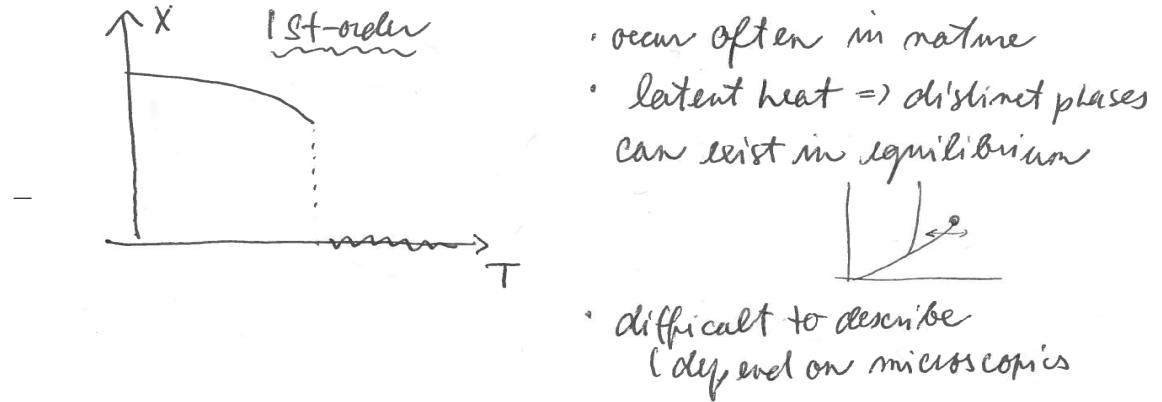
As we shall see in a tutorial, in  $d = 4$  only terms up to  $\varphi^4$  are relevant (e.g.  $+v\varphi^6$ ,  $v$  is irrelevant).

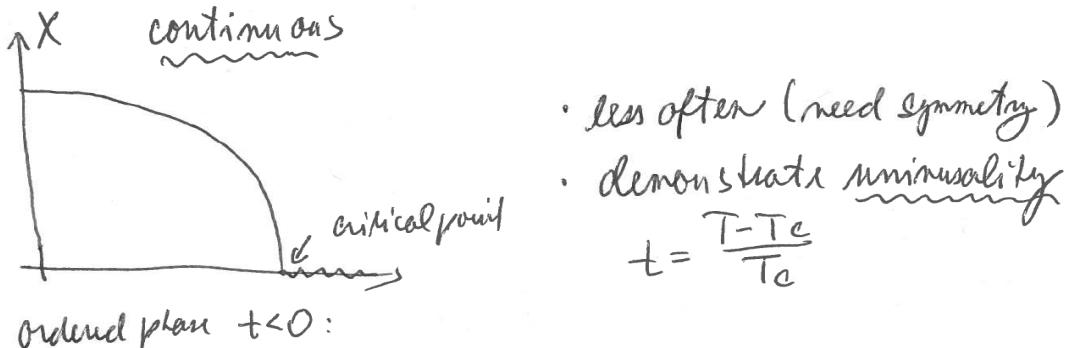
Statement: one cannot do perturbation theory in  $u$  for  $d < 4$ , nor in  $\delta\varphi$ . We have to be more clever and employ the renormalization procedure.

## 2.4 First Look at Renormalization

### Goal

- Goal. understand continuous phase transitions near the critical point.
  - Phase transitions: 1st-order, continuous, infinite-order (topological), liquid-glass, quantum ( $T = 0$ ),...
  - Order parameter  $X$  is associate with the symmetry breaking: magnetization  $M$ ,  $\Delta\rho = \rho_l - \rho_g, \dots$





- Critical point is characterized by critical exponents. For example for the ferromagnet we define

$$C \propto |t|^{-\alpha}, \quad M \propto |t|^{\beta}, \quad \chi \propto |t|^{-\gamma}, \quad H \propto M^{\delta},$$

$$G(r) \propto \frac{a^{-r/\xi}}{r^{d-2+\eta}}, \quad \xi \propto |t|^{-\nu}. \quad (2.61)$$

- Universality classes: have the same critical exponents. It turns out that only two such exponents are independent, they obey the following equalities:

$$\text{Rushbrook:} \quad \alpha + 2\beta + \gamma = 2, \quad (2.62)$$

$$\text{Griffiths:} \quad \alpha + \beta(\delta + 1) = 2, \quad (2.63)$$

$$\text{Fisher*}: \quad \gamma = \nu(2 - \eta), \quad (2.64)$$

$$\text{Josephson*} \quad \alpha = 2 - \nu d. \quad (2.65)$$

Note that equalities denoted by \* involve hyperscaling assumption:  $L^p \Rightarrow \xi^p$ , which is not always valid.

- MFT can qualitatively describes continuous phase transitions, but gives (in  $d < 4$ ) wrong critical exponents. Aim is to “include the effect of fluctuations” and obtain correct critical exponents, using the method of renormalization.
- To do this we shall employ the Landau–Ginsburg functional (2.55) and ‘calculate’ the corresponding partition function.

### Wilsonian (momentum space) renormalization

... is a “formal way of *coarse graining*”. We want to do the integral

$$Z = \int D\varphi e^{-S[\varphi]} \quad (2.66)$$

in *small steps*, avoiding divergences at small  $k$ .

- We have

$$\varphi(\vec{x}) = \int_0^\Lambda \frac{d^d k}{(2\pi)^d} \varphi(\vec{k}) e^{i\vec{k}\cdot\vec{x}}. \quad (2.67)$$

Here,  $\Lambda \propto 1/a$  is the high-momentum cutoff.<sup>1</sup>

- Let us fix  $b > 1$  and split the integration as

$$\int_0^\Lambda = \underbrace{\int_0^{\Lambda/b}}_{\text{slow modes}} + \underbrace{\int_{\Lambda/b}^\Lambda}_{\text{fast modes}}, \quad (2.68)$$

and define the decomposition into slow,  $\varphi_<$ , and fast,  $\varphi_>$  modes as

$$\boxed{\varphi = \varphi_< + \varphi_>,} \quad (2.69)$$

where

$$\varphi_<(k) = \begin{cases} \varphi(k) & 0 < k < \Lambda/b \\ 0, & k > \Lambda/b \end{cases} \quad \text{slow}, \quad (2.70)$$

$$\varphi_>(k) = \begin{cases} 0 & 0 < k < \Lambda/b \\ \varphi(k), & k > \Lambda/b \end{cases} \quad \text{fast}. \quad (2.71)$$

Trick: integrate fast only.

- Decomposing the action into the ‘free’ and ‘interaction’ parts

$$\boxed{S = S_0 + S_{int},} \quad (2.72)$$

we have

$$\begin{aligned} S_0[\varphi] &= \frac{1}{2} \int d^d x \left[ (\nabla \varphi)^2 + r \varphi^2 \right] = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \varphi(-k) \varphi(k) (k^2 + r) \\ &= \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} |\varphi(k)|^2 (k^2 + r) = S_0[\varphi_<] + S_0[\varphi_>], \end{aligned} \quad (2.73)$$

$$\begin{aligned} S_{int} &= \frac{u}{4!} \int d^d x \varphi^4 = \frac{u}{4!} \int \frac{d^d k_1}{(2\pi)^d} \dots \frac{d^d k_4}{(2\pi)^d} \varphi(k_1) \dots \varphi(k_4) (2\pi)^d \delta(k_1 + k_2 + k_3 + k_4) \\ &= S_{int}[\varphi_<, \varphi_>]. \end{aligned} \quad (2.74)$$

The  $\delta$ -function, that emerged from the  $x$ -integral, corresponds to the conservation of 4-particle momentum during the interaction.

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<sup>1</sup>In condensed matter physics this is very natural, given by the existence of lattice. What about QFT? Is it Lorentz Invariant at all scales? Is the spacetime naturally discrete? (C.f. minimal volume in LQG or T-duality in ST.)

- We want to integrate over fast modes:

$$Z = \int D\varphi_{>} D\varphi_{<} e^{-S[\varphi_{<}, \varphi_{>}]} = \int D\varphi_{<} e^{-\tilde{S}[\varphi_{<}]}, \quad (2.75)$$

where the new action,  $\tilde{S}[\varphi_{<}]$  must have the same form as  $S[\varphi]$  (generated new terms are irrelevant), that is

$$\begin{aligned} \tilde{S}[\varphi_{<}] &= \frac{1}{2} \int_0^{\Lambda/b} \frac{d^d k}{(2\pi)^d} |\varphi_{<}(k)|^2 [\tilde{r}_2 k^2 + \tilde{r}_0] \\ &\quad + \frac{\tilde{r}_4}{4!} \int_0^{\Lambda/b} \frac{d^d k_1}{(2\pi)^d} \cdots \frac{d^d k_4}{(2\pi)^d} \varphi_{<}(k_1) \dots \varphi_{<}(k_4) (2\pi)^d \delta(k_1 + k_2 + k_3 + k_4). \end{aligned} \quad (2.76)$$

Next, we want to bring this to exactly the ‘same action’ as before. To do this we set

$$\tilde{k} = kb, \quad \tilde{\varphi}(\tilde{k}) = \varphi_{<}(k)/z. \quad (2.77)$$

The latter is called the wave function renormalization. Dropping tildes, we have

$$\begin{aligned} S[\varphi] &= \frac{1}{2} \int_0^\Lambda \frac{d^d k}{(2\pi)^d} \left[ \underbrace{b^{-d} \tilde{r}_2 b^{-2} z^2}_{1} k^2 + \underbrace{b^{-d} \tilde{r}_0 z^2}_{\tilde{r}} \right] |\varphi(k)|^2 \\ &\quad + \frac{1}{4!} \underbrace{\tilde{r}_4 b^{-4d} b^d z^4}_{\tilde{u}} \int_0^\Lambda \frac{d^d k_1 \dots d^d k_4}{(2\pi)^{4d}} \varphi(k_1) \dots \varphi(k_4) (2\pi)^d \delta(\sum k_i) \end{aligned} \quad (2.78)$$

In the second line, the middle  $b^d$  comes from the  $\delta$ -function. Writing further

$$\tilde{r}_2 = b^\eta, \quad (2.79)$$

where  $\eta$  is called anomalous dimension. (As we shall see it is the same anomalous dimension—one of the critical exponents.) The first requirement  $b^{-d} \tilde{r}_2 b^{-2} z^2 = 1$  then implies

$$z = b^{\frac{d+2-\eta}{2}}, \quad \Rightarrow \quad \varphi(k) = b^{-\frac{d+2-\eta}{2}} \varphi_{<}(k), \quad (2.80)$$

while the other requirements lead to

$$\tilde{r} = b^{2-\eta} \tilde{r}_0, \quad \tilde{u} = b^{4-2\eta-d} \tilde{r}_4. \quad (2.81)$$

So after  $\tilde{k} = kb$  and wave function renormalization, we get

$$S[\varphi] = \int d^d x \left[ \frac{1}{2} (\nabla \varphi)^2 + \frac{\tilde{r}}{2} \varphi^2 + \frac{\tilde{u}}{4!} \varphi^4 \right], \quad (2.82)$$

where  $\tilde{r}, \tilde{u}$  are new coupling constants.

- Effectively

$$(r, u) \rightarrow (\tilde{r}, \tilde{u}). \quad (2.83)$$

We do it repeatedly. Let us denote the fixed point by  $(r^*, u^*)$ .

In each step the correlation length changes as

$$\xi = \xi(r, u) \rightarrow \xi(\tilde{r}, \tilde{u}) = \xi/b. \quad (2.84)$$

At a fixed point we must have

$$\xi(r^*, u^*) = \frac{1}{b} \xi(r^*, u^*), \quad (2.85)$$

which implies

$$\xi(r^*, u^*) = \begin{cases} 0 & \text{phase } (T \neq T_c) \\ \infty & \text{critical point } (T = T_c) \end{cases} \quad (2.86)$$

Depending at which  $T$  I start, I either flow to a fixed point with  $\xi = 0$  ( $T \neq T_c$ ) or  $\xi = \infty$  (at  $T = T_c$ ).

## Explicit calculation

- We demanded

$$Z = \int D\varphi_{<} D\varphi_{>} e^{-S[\varphi_{<}, \varphi_{>}]} = \int D\varphi_{<} e^{-\tilde{S}[\varphi_{<}]}, \quad (2.87)$$

where we have integrated over fast modes. This means that

$$e^{-\tilde{S}[\varphi_{<}]} = e^{-S_0[\varphi_{<}]} \underbrace{\frac{\int D\varphi_{>} e^{-S_0[\varphi_{>}]} e^{-S_{int}[\varphi_{<}, \varphi_{>}]}}{\int D\varphi_{>} e^{-S_0[\varphi_{>}]}}} \underbrace{\int D\varphi_{>} e^{-S_0[\varphi_{>}]}}_{Z_0}. \quad (2.88)$$

So we have

$$\boxed{\tilde{S}[\varphi_{<}] = S_0[\varphi_{<}] - \log \langle e^{-S_{int}} \rangle_{\text{fast}}} - \log Z_0. \quad (2.89)$$

Since the last term is just a constant, we throw it away.

- We shall now employ the cumulant expansion (see Appendix B). Let  $\Omega$  be a random variable. Then

$$\boxed{\langle e^\Omega \rangle = e^{\langle \Omega \rangle + \frac{1}{2} [\langle \Omega^2 \rangle - \langle \Omega \rangle^2] + \dots}} \quad (2.90)$$

So we have

$$\tilde{S}[\varphi_{<}] = S_0[\varphi_{<}] + \langle S_{int} \rangle_{\text{fast}} - \frac{1}{2} \left[ \langle S_{int}^2 \rangle_{\text{fast}} - \langle S_{int} \rangle_{\text{fast}}^2 \right] + \dots \quad (2.91)$$

•

$$\begin{aligned}
 S_{int} &= \frac{u}{4!} \int_{k_1 \dots k_4} [\varphi_<(1) + \varphi_>(1)] \times \dots \times [\varphi_<(4) + \varphi_>(4)] \delta(\sum k_i) (2\pi)^d \\
 &= \frac{u}{4!} \int_0^\Lambda \delta(\ ) (2\pi)^d \left[ \underbrace{\varphi_<(1) \dots \varphi_<(4)}_A + \underbrace{\varphi_>(1) \dots \varphi_>(4)}_B \right. \\
 &\quad + 4 \times \underbrace{\varphi_<(1)\varphi_>(2)\varphi_>(3)\varphi_>(4)}_C + 4 \times \underbrace{\varphi_>(1)\varphi_<(2)\varphi_<(3)\varphi_<(4)}_D \\
 &\quad \left. + 6 \times \underbrace{\varphi_<(1)\varphi_<(2)\varphi_>(3)\varphi_>(4)}_E \right]. \tag{2.92}
 \end{aligned}$$

Integration over fast modes of the term  $B$  gives a constant, while at the same time the term  $A$  need not be integrated.

- Since

$$\langle \cdot \rangle_{\text{fast}} = \frac{1}{Z_0} \int D\varphi_> e^{-S_0[\varphi_>]} (\cdot), \tag{2.93}$$

we have

$$\begin{aligned}
 \langle \varphi_>(3)\varphi_>(4) \rangle_{\text{fast}} &= \frac{1}{Z_0} \int D\varphi_> \varphi_>(k_3)\varphi_>(k_4) e^{-\frac{1}{2} \int_{\Lambda/b}^\Lambda \frac{d^d k}{(2\pi)^d} (k^2 + r) |\varphi_>(k)|^2} \\
 &= G_0(k_3)(2\pi)^d \delta(k_3 + k_4), \tag{2.94}
 \end{aligned}$$

using the result from QFT, where

$$G_0(k) = \frac{1}{k^2 + r}, \tag{2.95}$$

is the propagator of fast modes (see Appendix B for more details).

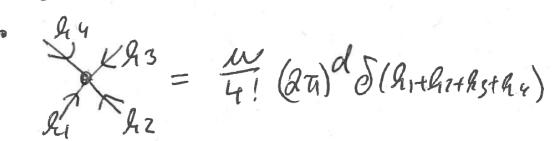
At the same time

$$\boxed{\langle \text{odd number of } \varphi_> \text{'s} \rangle_{\text{fast}} = 0.} \tag{2.96}$$

(We are effectively using Wick's theorem from QFT, which is valid more generally, for any 'Gaussian integration'.)

- We can now introduce the diagrammatic notation:

Feynman diagrams

-   $\frac{w}{4!} (2\pi)^d \delta(k_1 + k_2 + k_3 + k_4)$

--- fast field

— slow field

- $S_{int} = \text{---} + \text{---} + 4 \times \text{---} + 4 \times \text{---} + 6 \times \text{---}$

- process of averaging  $\cong$  connecting 2 fast modes

$$\langle \text{---} \rangle = \text{---}$$

Note

$$\langle \text{---} \rangle = \text{---} = G_0(k_3) (2\pi)^d \delta(k_3 + k_4)$$

So we have

- $\langle S_{int} \rangle = \text{---} + 8 + 6 \times \text{---}$

keeps as is      gives a #

$$6 \times \text{---} = \frac{w}{4} \int_0^\Lambda \prod_{k_1 \dots k_4} (2\pi)^d \delta(k_1 + k_2 + k_3 + k_4) \varphi_{<}(k_1) \varphi_{<}(k_2) \times \\ \times (2\pi)^d G_0(k_3) \delta(k_3 + k_4)$$

$$= \left| \begin{array}{l} k_1 = -k_2 = k \\ k_3 = -k_4 = q \end{array} \right| = \frac{w}{4} \int_{kq}^\Lambda |\varphi_{<}(k)|^2 G_0(q)$$

↑ fast mode

We can now calculate the integral over the fast modes:

$$\begin{aligned} I &= \int_{\Lambda/b}^\Lambda \frac{d^d q}{(2\pi)^d} G_0(q) = \int_{\Lambda/b}^\Lambda \frac{S_d}{(2\pi)^d} \frac{q^{d-1} dq}{q^2 + r} = |b = e^{\Delta l}, \Delta l \text{ is small}| \\ &\approx \frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{r + \Lambda^2} \Delta l, \end{aligned} \tag{2.97}$$

where we treated  $r$  as small. Namely, treating the interval  $(b - a)$  as small, we approximated  $\int_a^b f(x) dx \approx (b - a)f(b)$ .

- So we have obtained

$$\begin{aligned}\tilde{S}[\varphi_<] &= \frac{1}{2} \int_k \left[ \underbrace{1 \times}_{\tilde{r}_2} k^2 + \underbrace{\left( r + \frac{u}{2} I \right)}_{\tilde{r}_0} \right] |\varphi_<(k)|^2 \\ &\quad + \frac{1}{4!} \underbrace{u}_{\tilde{r}_4} \int_{k_1 \dots k_4} (2\pi)^d \delta(\sum k_i) \varphi_<(1) \varphi_<(2) \varphi_<(3) \varphi_<(4). \quad (2.98)\end{aligned}$$

Comparing this to (2.76), and using (2.81), we get

$$\begin{aligned}\tilde{r}_2 &= 1 = b^\eta \quad \Rightarrow \quad \boxed{\eta = 0}, \\ \tilde{r} &= b^{2-\eta} \tilde{r}_0 = b^2 \left( r + \frac{u}{2} I \right), \\ \tilde{u} &= b^{4-2\eta-d} \tilde{r}_4 = b^{4-d} u.\end{aligned}\quad (2.99)$$

So we have obtained the following RG recursion relations:

$$\boxed{\tilde{r} = b^2 \left( r + \frac{u}{2} I \right), \quad \tilde{u} = b^{4-d} u.} \quad (2.100)$$

- Can write these in DE form by writing  $b = e^{\Delta l}$  and

$$\begin{aligned}\tilde{r} &= \left( r + \frac{u}{2} I \right) b^2 \approx \left( r + \frac{u}{2} \frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{r + \Lambda^2} \Delta l \right) (1 + 2\Delta l + \dots) \\ &= r + 2r\Delta l + \frac{u}{2} \frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{r + \Lambda^2} \Delta l + \dots,\end{aligned}\quad (2.101)$$

and so

$$\begin{aligned}\frac{dr}{dl} &\approx \frac{\tilde{r} - r}{\Delta l} = 2r + \frac{u}{2} \frac{S_d}{(2\pi)^d} \frac{\Lambda^d}{r + \Lambda^2}, \\ \frac{du}{dl} &\approx (4-d)u = \epsilon u.\end{aligned}\quad (2.102)$$

- More generally, we obtain a relation

$$\boxed{\frac{dX}{dl} = \beta_X}, \quad (2.103)$$

where  $\beta_X$  is called the  $\beta$ -function. The fixed point then corresponds to  $\boxed{\beta_X = 0}$ . Note that in the leading order the  $\beta$ -function has the following form (see tutorial):

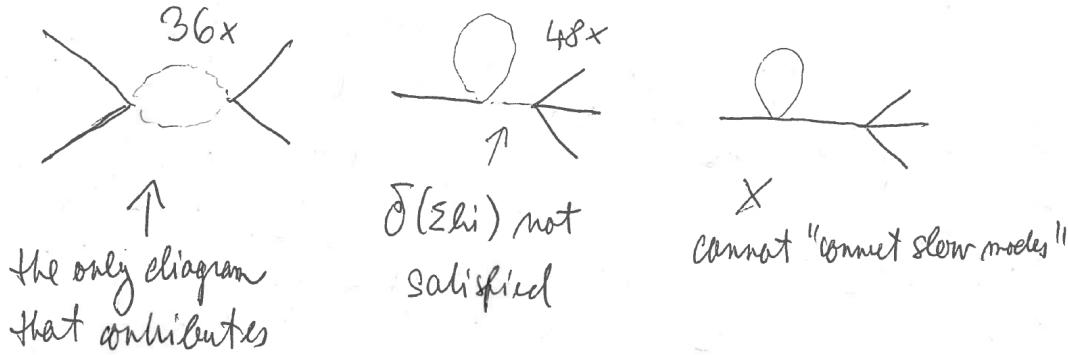
$$\beta_X = [\dim X] X + \dots \quad (2.104)$$

As this is usually the dominant term,  $[\dim X]$  tells me if  $X$  decreases or increases, that is, (ir)relevance of  $X$ .

- To get the first correction to  $\frac{du}{dl}$ , we need to calculate the 2nd cumulant. This is given by connected diagrams of the type

$$\frac{1}{2} [\langle \sin \varphi \rangle - \langle \sin \varphi \rangle^2] = 48 - \text{circle} + 72 - \text{circle} \\ + 72 - \text{circle} + 36 > \cancel{X} + 48 - \text{circle}$$

Since diagrams with 2 external legs (circled above) only contribute to  $\frac{dr}{dl}$ , we rather concentrate on diagrams with four external slow legs:



Only the first type of diagram contributes. More specifically, we have

$$h_1 + h_2 + q_1 + q_2 = 0 \quad q_1 = q \\ h_3 + h_4 - q_1 - q_2 = 0 \quad q_2 = h_3 + h_4 - q$$

Each such diagram gives the following contribution:

$$\left(\frac{u}{4!}\right)^2 \int_{k_1 \dots k_4} (2\pi)^d \delta(k_1 + k_2 + k_3 + k_4) \varphi_{<}(k_1) \varphi_{<}(k_2) \varphi_{<}(k_3) \varphi_{<}(k_4) I_2, \quad (2.105)$$

where we have denoted

$$\begin{aligned} I_2 &= \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} G_0(q) G_0(k_1 + k_2 + q) \\ &\approx \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} G_0^2(q) = \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{(r + q^2)^2} \approx \frac{S_d}{(2\pi)^d} \frac{\Lambda^d \Delta l}{(r + \Lambda^2)^2}, \end{aligned} \quad (2.106)$$

where in the second step we have used the fact that  $k_1, k_2 \ll q$ . So in this

approximation we have

$$\begin{aligned}\tilde{S}[\varphi_<] &= \frac{1}{2} \int_k (k^2 + r + \frac{u}{2} I_1) |\varphi_<(k)|^2 + \left( \frac{u}{4!} - 36 \left( \frac{u}{4!} \right)^2 I_2 \right) \times \\ &\quad \times \int_{k_1 \dots k_4} \varphi_<(k_1) \dots \varphi_<(k_4) (2\pi)^d \delta(k_1 + k_2 + k_3 + k_4).\end{aligned}\quad (2.107)$$

Comparing to (2.76) and using that  $\tilde{r}_2 = 1$ , and hence still  $\eta = 0$ , we have from (2.81)

$$\tilde{u} = \tilde{r}_4 b^{4-2\eta-d} = \left( u - 36 \frac{u^2 I_2}{4!} \right) b^{4-d} = \left( u - \frac{3}{2} u^2 I_2 \right) b^{4-d}.\quad (2.108)$$

Denoting by  $K_d = \frac{S_d}{(2\pi)^d} \Lambda^d$  we hence found the following RG equations:

$$\boxed{\frac{dr}{dl} = \beta_r = 2r + \frac{u}{2} \frac{K_d}{r + \Lambda^2}, \quad \frac{du}{dl} = \beta_u = \epsilon u - \frac{3}{2} u^2 \frac{K_d}{(r + \Lambda^2)^2}.}\quad (2.109)$$

## Fixed points in small $\epsilon$ -expansion

- For our model, the fixed points are given by

$$\frac{dr}{dl} = 0 = \frac{du}{dl}.$$

To analyze these, we shall work in the  $\epsilon = (4-d)$ -expansion, keeping only the leading order in  $\epsilon$  terms (up to  $O(\epsilon)$  where necessary).

In this approximation, we find the following two fixed points (neglecting  $r$  in denominators of the  $\beta$ -functions which are of higher-order in  $\epsilon$ ):

$$\begin{aligned}(r^*, u^*) &= (0, 0) && \text{Gaussian fixed point,} \\ (r^*, u^*) &= \left( -\frac{\epsilon}{6} \Lambda^2, \frac{2}{3} \frac{\Lambda^4}{K_d} \epsilon \right) && \text{Wilson–Fisher fixed point.}\end{aligned}\quad (2.110)$$

- Gaussian fixed point. (MFT critical point):  $(r^*, u^*) = (0, 0)$ .

Let's linearize (2.109) around the critical point (Taylor expand  $\beta$ -functions):

$$\delta r = r - r^*, \quad \delta u = u - u^*. \quad (2.111)$$

$$\frac{d}{dl} \begin{pmatrix} \delta r \\ \delta u \end{pmatrix} = \begin{pmatrix} \frac{\partial \beta_r}{\partial r} & \frac{\partial \beta_r}{\partial u} \\ \frac{\partial \beta_u}{\partial r} & \frac{\partial \beta_u}{\partial u} \end{pmatrix}_{r=r^*, u=u^*} \begin{pmatrix} \delta r \\ \delta u \end{pmatrix}. \quad (2.112)$$

Specifically:  $(r^*, u^*) = (0, 0)$  and we get the following constant matrix  $M$ :

$$M = \begin{pmatrix} 2 & \frac{K_d}{2\Lambda^2} \\ 0 & \epsilon \end{pmatrix}. \quad (2.113)$$

As always with a matrix, we seek its eigenvalues,  $\lambda$ , given by  $\det(M - \lambda I) = 0$ , and its eigenvectors,  $v$ , determined from  $(M - \lambda I)v = 0$ . We find

$$\lambda_t = 2, \quad \lambda_{\tilde{u}} = \epsilon, \quad (2.114)$$

(note the labels) and

$$v_t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_{\tilde{u}} = \begin{pmatrix} 1 \\ \frac{2(\epsilon-2)\Lambda^2}{K_d} \end{pmatrix}. \quad (2.115)$$

This means that in terms of the new ‘couplings’  $t$  and  $\tilde{u}$ , the  $\beta$ -functions become ‘diagonal’ and can be solved by exponentials:

$$\begin{aligned} \frac{dt}{dl} &= \lambda_t t \quad \Leftrightarrow \quad t = t_0 e^{\lambda_t l} = t_0 e^{2l}, \\ \frac{d\tilde{u}}{dl} &= \lambda_{\tilde{u}} \tilde{u} \quad \Leftrightarrow \quad \tilde{u} = \tilde{u}_0 e^{\lambda_{\tilde{u}} l} = \tilde{u}_0 e^{\epsilon l}. \end{aligned} \quad (2.116)$$

So we get the following possibilities:

- $\epsilon < 0$  ( $d > 4$ )
  - G ... stable fixed point
  - stable in  $\tilde{n}$ -direction
  - always unstable in  $n$ -direction (deviations from  $T_c$  always take me away from crit. point)
  - G describes MFT critical point
    - $n$ .. relevant (takes away from crit. point)
    - $\tilde{n}$ .. irrelevant (deviations decay back to crit. point)
- $\epsilon > 0$  (eg  $d=3$ )
  - G unstable
  - (MFT fails below  $d=4$ )
- $\epsilon = 0$  ( $d=4$ ) ... to see what happens, need to go to higher orders.
- Wilson–Fisher fixed point.  
The Wilson–Fisher fixed point occurs at

$$(r^*, u^*) = \left( -\frac{\epsilon}{6}\Lambda^2, \frac{2}{3}\frac{\Lambda^4}{K_d}\epsilon \right). \quad (2.117)$$

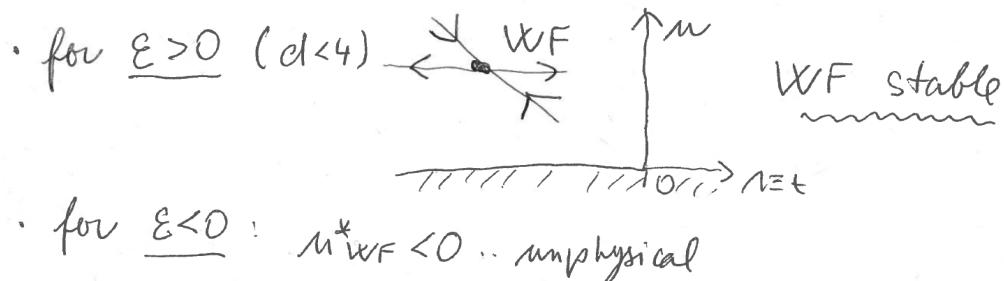
For  $\epsilon > 0$  ( $d < 4$ ) it has  $r^* < 0$ . Hence this critical point occurs at lower temperatures than predicted by MFT ( $r = 0$ ). Fluctuations made ordering more difficult. At the same time for  $\epsilon < 0$ , this critical point occurs in the negative  $u$ -plane and hence is ‘unphysical’ in  $d > 4$ .

Expanding again to the leading order in  $\epsilon$  we find

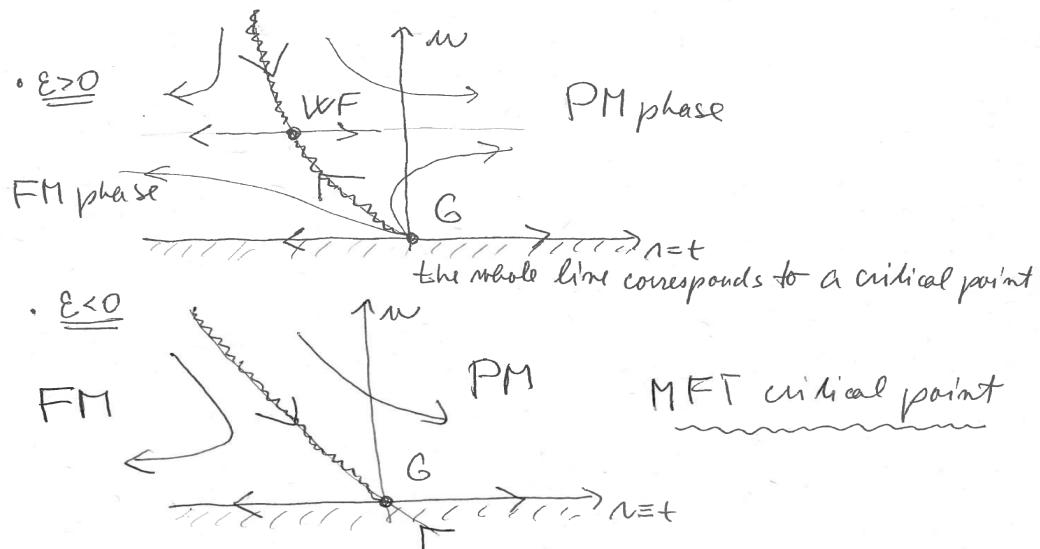
$$M = \begin{pmatrix} 2 - \frac{\epsilon}{3} & \frac{K_d}{2\Lambda^2} \\ 0 & -\epsilon \end{pmatrix}. \quad (2.118)$$

The corresponding eigenvalues and eigenvectors are now

$$\lambda_r = 2 - \frac{\epsilon}{3}, \quad \lambda_{\tilde{u}} = -\epsilon, \quad v_t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_{\tilde{u}} = \begin{pmatrix} 1 \\ -\frac{4(3+\epsilon)\Lambda^2}{3K_d} \end{pmatrix}. \quad (2.119)$$



- Summary: Depending on whether  $\epsilon$  is positive or negative, we have the following RG flows:



## Critical exponents (triumph)

- Close to the critical point, under RG flow in “time”  $l$  we have

$$\begin{aligned}\tilde{k} &= kb, \quad \tilde{x} = x/b, \quad b = e^l, \\ \tilde{\varphi}(\tilde{x}) &= \int \frac{d^d \tilde{k}}{(2\pi)^d} \tilde{\varphi}(\tilde{k}) e^{i\tilde{k}\cdot\tilde{x}} = \int \frac{d^d k}{(2\pi)^d} b^d \underbrace{b^{-\frac{d+2-\eta}{2}} \varphi_<(k)}_{\text{wave function renormalization}} e^{ik\cdot x} = b^{\frac{d-2+\eta}{2}} \varphi(x).\end{aligned}\quad (2.120)$$

Here we have used the wave function renormalization (2.80). At the same time the couplings  $(t, u)$  evolve according to

$$\tilde{t} = te^{\lambda_t l}, \quad \tilde{u} = ue^{\lambda_u l}. \quad (2.121)$$

- Let us recall that the correlation function is defined as

$$G(x) = \langle \varphi(x)\varphi(0) \rangle - \langle \varphi(x) \rangle \langle \varphi(0) \rangle. \quad (2.122)$$

We have

$$G(\tilde{x}, \tilde{t}, \tilde{u}) = G(x/b, te^{\lambda_t l}, ue^{\lambda_u l}) = b^{d-2+\eta} G(x, t, u), \quad (2.123)$$

or

$$G(x, t, u) = b^{-(d-2+\eta)} G(x/b, te^{\lambda_t l}, ue^{\lambda_u l}). \quad (2.124)$$

The second equality in (2.123) follows from the fact that  $G$  is quadratic in  $\phi$ 's and they both of these get renormalized.

- Let us start at a fixed point  $t = 0 = u$ . Then

$$G(x, 0, 0) = b^{-(d-2+\eta)} G(x/b, 0, 0) \Rightarrow G(x, 0, 0) \propto \frac{1}{x^{d-2+\eta}}. \quad (2.125)$$

So  $\eta$  is the anomalous dimension from before.

- Let's move away from the fixed point in  $t$ -direction

$$G(x, t, 0) = b^{-(d-2+\eta)} G(x/b, te^{\lambda_t l}, 0). \quad (2.126)$$

The last relation holds for all  $b$ . In particular, we can choose  $b = x$ , in which case

$$G(x, t, 0) = x^{-(d-2+\eta)} \underbrace{G(1, tx^{\lambda_t}, 0)}_{\text{want } F(x/\xi)}. \quad (2.127)$$

This implies that

$$x^{\lambda_t} t = (xt^{1/\lambda_t})^{\lambda_t} = (x/\xi)^{\lambda_t} \Rightarrow \xi \propto |t|^{-\frac{1}{\lambda_t}}, \quad (2.128)$$

and so

$$\nu = \frac{1}{\lambda_t} = \frac{1}{2 - \frac{\epsilon}{3}} \approx \frac{1}{2} + \frac{\epsilon}{12}. \quad (2.129)$$

For example, for the 3d Ising model  $\epsilon = 1$  and we have found  $\nu = \frac{7}{12} \approx 0.58$ , which is to be compared to exact (numerical) result:  $\nu \approx 0.63$ .

- Let us now calculate the susceptibility. This is calculated as

$$\boxed{\chi = \int d^d x G(x).} \quad (2.130)$$

To motivate this formula we put our field in a magnetic field, getting the following Landau–Ginsburg functional:

$$S = \int d^d x \left[ \frac{1}{2} (\nabla \varphi)^2 + \frac{r}{2} \varphi^2 + \frac{u}{4!} \varphi^4 - B \varphi \right]. \quad (2.131)$$

So we get  $G(x - x') = \frac{\delta \langle \varphi(x) \rangle}{\delta B(x')}$ . Specifically, for a uniform  $B$ , we get susceptibility:

$$\chi = \frac{\partial \langle \varphi \rangle}{\partial B} \Big|_{B=0} = \frac{\partial}{\partial B} \Big|_{B=0} \frac{1}{Z} \int D\varphi \varphi(x) e^{-S[\varphi]} = \int d^d x G(x). \quad (2.132)$$

In any case, we obtain

$$\begin{aligned} \chi &= \int d^d x G(x) = \int d^d x x^{-(d-2+\eta)} F(x/\xi) = |y = x/\xi| \\ &= \xi^d \int d^d y \xi^{-(d-2+\eta)} y^{-(d-2+\eta)} F(y) \propto \xi^{2-\eta}. \end{aligned} \quad (2.133)$$

Therefore

$$\chi \propto |t|^{-\nu(2-\eta)} \Rightarrow \boxed{\gamma = \nu(2-\eta)} = 1 + \frac{\epsilon}{6}. \quad (2.134)$$

For  $\epsilon = 1$  we then have  $\gamma \approx 1.17$  which is to be compared with  $\gamma \approx 1.24$  of the exact numerical calculation.

- Using scaling relations or similar arguments (see tutorial) we also find

$$\begin{aligned} \beta &\approx \frac{1}{2} - \frac{\epsilon}{6} \approx 0.33 & \text{exact: } 0.325, \\ \alpha &\approx \frac{\epsilon}{6} \approx 0.17 & \text{exact: } 0.11. \end{aligned} \quad (2.135)$$

- So we have the following table of critical exponents for the 3d Ising model ( $\epsilon = 1$ ):

Critical Exponent	MFT approximation	RG: $\epsilon$ -expansion	Numerics
$\alpha$	0	$\epsilon/6 \approx 0.17$	0.110
$\beta$	$1/2$	$1/2 - \epsilon/6 \approx 0.33$	0.326
$\gamma$	1	$1 + \epsilon/6 \approx 1.17$	1.237
$\delta$	3	$3 + \epsilon \approx 4$	4.790
$\nu$	$1/2$	$1/2 + \epsilon/12 \approx 0.58$	0.630
$\eta$	0	0	0.036

- Conclusion.

- RG allows for analytic computation of critical exponents.
- Explains universality of continuous phase transitions.
- Provides conceptual foundations for the way we model condensed matter systems. (Starting from a complicated many body interacting microscopic Hamiltonian, the RG flow makes many of the interactions irrelevant for the ‘large scale physics’ and the problem simplifies. This for example justifies why the approximation of *non-interacting electrons* works so well.)
- Can be generalized to more order parameters.

## 2.5 Systems with continuous symmetry

### Vector model

- So far we have discussed the Ising universality class. This is characterized by
  - order parameter is a scalar  $\varphi(\vec{x})$
  - discrete  $Z_2$  symmetry  $\varphi \rightarrow -\varphi$ .

The same universality class applies for example to the VdW gas. (Order parameter which is given by the density of fluid minus the density of liquid, is scalar and has the same symmetry.) It is the aim of this section to go beyond this universality class. Namely, to look at the systems with continuous symmetry.

- Vector model has a bigger symmetry. It describes a ferromagnet in a  $d$ -dimensional lattice whose spins are modeled by  $n$ -component vectors  $\vec{S}_i = (S_i^1, S_i^2, \dots, S_i^n)$  and the following Hamiltonian:<sup>2</sup>

$$H = -\frac{1}{2} \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j, \quad |\vec{S}_i| = 1, \quad (2.137)$$

Contrary to the Ising’s  $Z_2$  discrete symmetry, the  $n$ -vector model has a continuous  $O(n)$  symmetry (all spins in the lattice can be rotated by an arbitrary “global” rotation).

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<sup>2</sup> A special case of  $n$ -vector model is the  $n = 3$  Heisenberg model;  $\vec{S}_i = (S_i^x, S_i^y, S_i^z)$ . The Heisenberg model can be reduced to the Ising model by the following trick: One considers the following Hamiltonian:

$$H = -\frac{1}{2} \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j - K \sum_i (S_i^z)^2. \quad (2.136)$$

By setting  $K$  sufficiently large, we effectively choose the spins to point only in the  $z$ -direction and recover the Ising model.

- The corresponding Landau–Ginsburg functional, describing  $n$  order parameters  $\vec{\varphi} = (\varphi_1, \dots, \varphi_n)$  with  $O(n)$  continuous symmetry reads

$$S[\vec{\varphi}] = \int d^d x \left[ \frac{1}{2} (\nabla \vec{\varphi}) \cdot (\nabla \vec{\varphi}) + \frac{r}{2} \vec{\varphi} \cdot \vec{\varphi} + \frac{u}{4} (\vec{\varphi} \cdot \vec{\varphi})^2 \right]. \quad (2.138)$$

Near the upper critical dimension  $d = 4$ , one can set up small  $\epsilon = (4 - d)$ -expansion. The corresponding behavior is qualitatively similar to that of the Ising model, though critical exponents now depend on the number of components  $n$  (see Homework 3). However, as we shall see near  $d = 2$  physics becomes very different!

- Let us write

$$\vec{\varphi}(x) = \rho(x) \vec{n}(x), \quad (2.139)$$

where  $\rho$  stands for the magnitude and  $\vec{n}$  is a unit vector. Of course we have  $\rho = \sqrt{\vec{\varphi} \cdot \vec{\varphi}}$  and  $\vec{n} \cdot \vec{n} = 1$ . Then we have

$$\begin{aligned} \nabla \varphi^a &= n^a \nabla \rho + \rho \nabla n^a, \\ (\nabla \vec{\varphi}) \cdot (\nabla \vec{\varphi}) &= (\nabla \varphi^a)(\nabla \varphi^a) = (\nabla \rho)^2 + 2\rho \nabla \rho \underbrace{n^a \nabla n^a}_0 + \rho^2 (\nabla n^a)^2 \\ &= (\nabla \rho)^2 + \rho^2 (\nabla n^a)(\nabla n^a). \end{aligned} \quad (2.140)$$

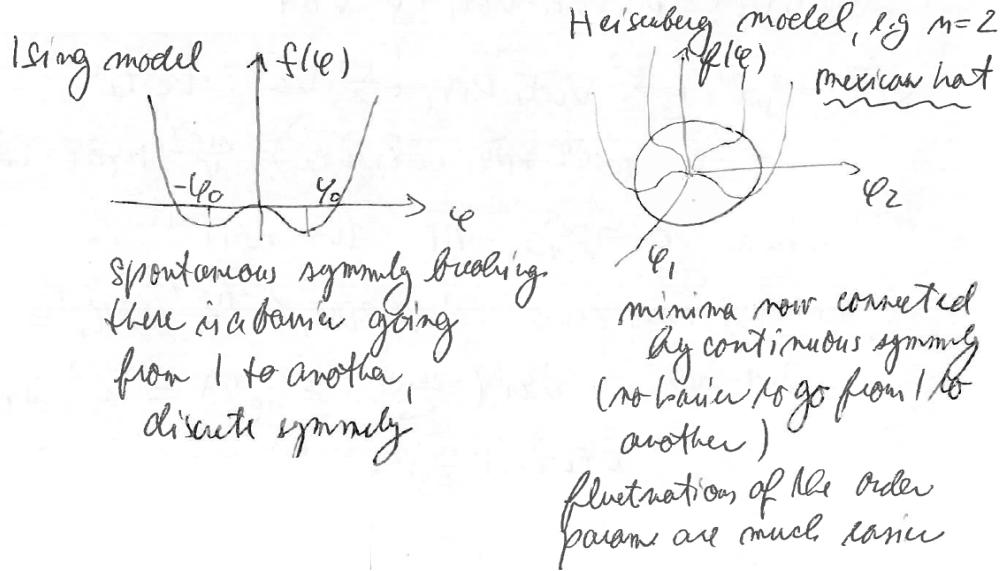
Therefore the functional now reads

$$S[\rho, \vec{n}] = \int d^d x \left[ \frac{1}{2} (\nabla \rho)^2 + \frac{1}{2} \rho^2 (\nabla n^a)(\nabla n^a) + \frac{r}{2} \rho^2 + \frac{u}{4} \rho^4 \right]. \quad (2.141)$$

- Let us first perform the MFT theory analysis. Then both  $\rho$  and  $\vec{n}$  are uniform and we have to minimize  $f(\rho) = \frac{r}{2}\rho^2 + \frac{u}{4}\rho^4$ . The derivative  $f'(\rho) = 0$  gives

$$\rho = \rho_0 = \begin{cases} \sqrt{-r/u} & r < 0, \\ 0 & r > 0, \end{cases} \quad (2.142)$$

while  $\vec{n}$  is arbitrary. So we have found the following important differences between the Ising and Heisenberg models:



- Let us now consider small perturbations around MFT solution. Since in MFT  $\vec{n}$  is arbitrary, we can pick one, say  $\vec{n} = (1, 0, 0, \dots, 0)$ , and write

$$\vec{\varphi}(x) = \underbrace{\rho_0 \vec{n}}_{\text{MFT}} + \underbrace{\rho_0 \delta \vec{\varphi}}_{\text{fluctuation}}, \quad \delta \vec{\varphi} = \vec{n} \delta \varphi_1 + \delta \vec{\varphi}_\perp, \quad (2.143)$$

where  $\delta \vec{\varphi}_\perp$  describes a fluctuation in transverse directions:  $\vec{n} \cdot \delta \vec{\varphi}_\perp = 0$ . Then we have

$$\begin{aligned} \vec{\varphi} \cdot \vec{\varphi} &= (\rho_0 \vec{n} + \rho_0 \delta \vec{\varphi}) \cdot (\rho_0 \vec{n} + \rho_0 \delta \vec{\varphi}) = \rho_0^2 + 2\rho_0^2 \underbrace{\vec{n} \cdot \delta \vec{\varphi}}_{\delta \varphi_1} + \rho_0^2 \delta \vec{\varphi} \cdot \delta \vec{\varphi} \\ &= \rho_0^2 [1 + 2\delta \varphi_1 + (\delta \varphi_1)^2 + (\delta \vec{\varphi}_\perp)^2], \\ (\vec{\varphi} \cdot \vec{\varphi})^2 &= \rho_0^4 [1 + 4\delta \varphi_1 + 6(\delta \varphi_1)^2 + 2(\delta \vec{\varphi}_\perp)^2], \\ \nabla \varphi^a \nabla \varphi^a &= \rho_0^2 \nabla \delta \varphi_1 \nabla \delta \varphi_1 + \rho_0^2 \nabla \delta \varphi_\perp^a \nabla \delta \varphi_\perp^a. \end{aligned} \quad (2.144)$$

Hence we have

$$\begin{aligned} S[\vec{\varphi}] &= \int d^d x \left[ \frac{\rho_0^2}{2} (\nabla \delta \varphi_1)^2 + \frac{\rho_0^2}{2} (\nabla \delta \vec{\varphi}_\perp)^2 + \frac{r}{2} \rho_0^2 [1 + 2\delta \varphi_1 + (\delta \varphi_1)^2 + (\delta \vec{\varphi}_\perp)^2] \right. \\ &\quad \left. + \frac{u}{4} \rho_0^4 [1 + 4\delta \varphi_1 + 6(\delta \varphi_1)^2 + 2(\delta \vec{\varphi}_\perp)^2] \right]. \end{aligned} \quad (2.145)$$

Since we used the MFT theory solution as a zeroth-order approximation, we have  $\rho_0^2 = -r/u$ . This then implies that

$$\frac{r}{2} \rho_0^2 2\delta \varphi_1 + \frac{u}{4} \rho_0^4 4\delta \varphi_1 = 0, \quad (2.146)$$

and the first-order terms cancel (as they should as we are expanding around the minimum). At second-order, we have

$$\begin{aligned} (\delta\varphi_1)^2 \left( \frac{r}{2}\rho_0^2 + \frac{3}{2}u\rho_0^2 \right) &= \frac{r^2}{u}(\delta\varphi_1)^2 = \rho_0^2|r|(\delta\varphi_1)^2, \\ \delta\varphi_{\perp}^a \delta\varphi_{\perp}^a \left( \frac{r}{2}\rho_0^2 + \frac{u\rho_0^4}{2} \right) &= 0. \end{aligned} \quad (2.147)$$

Putting all this together, we have found

$$S[\delta\vec{\varphi}] = \int d^d x \frac{\rho_0^2}{2} \left[ (\nabla\delta\varphi_1)^2 + (\nabla\delta\vec{\varphi}_{\perp})^2 + 2|r|(\delta\varphi_1)^2 \right]. \quad (2.148)$$

Note that as a consequence of  $O(n)$  symmetry, there is no ‘mass term’ for  $\vec{\varphi}_{\perp}$ . Such modes behave as massless and describe the so called goldstone bosons. Goldstone modes arise when continuous symmetry is spontaneously broken. In the Fourier picture,

$$\delta\varphi_1(x) = \frac{1}{V} \sum_k \delta\varphi_1(k) e^{i\vec{k}\cdot\vec{x}}, \quad \delta\varphi_{\perp}^a(x) = \frac{1}{V} \sum_k \delta\varphi_{\perp}^a(k) e^{i\vec{k}\cdot\vec{x}}, \quad (2.149)$$

the last formula becomes

$$S[\delta\vec{\varphi}] = \frac{\rho_0^2}{2V} \sum_k \left[ k^2 |\delta\varphi_1(k)|^2 + k^2 |\delta\vec{\varphi}_{\perp}(k)|^2 + 2|r||\delta\varphi_1|^2 \right]. \quad (2.150)$$

Obviously, the goldstone modes do not cost any energy in the limit  $k \rightarrow 0$ .

- Goldstone modes appear generically in nature. For example, another example of goldstone modes in condensed matter systems are the *acoustic phonons* generated by the formation of a crystal, breaking the continuous translation symmetry of space. Their energy is  $\epsilon_k = \hbar c_s k$ , where  $c_s$  stands for the sound velocity, which vanishes as  $k \rightarrow 0$ . Also in particle physics the massless particles emerge from the symmetry breaking.
- Correlation functions.

- For the massive mode we obtain the usual behavior. Namely, since  $\langle\varphi\rangle = \rho_0 \vec{n}$  we have  $\langle\delta\varphi_1\rangle = 0$  and hence

$$\begin{aligned} G_1(\vec{x} - \vec{x}') &= \langle \delta\varphi_1(\vec{x}) \delta\varphi_1(\vec{x}') \rangle = \frac{1}{V^2} \sum_k \underbrace{\langle \delta\varphi_1(k) \delta\varphi_1(-k) \rangle}_{\frac{V}{\rho_0^2} \frac{1}{k^2 + 2|r|}} e^{i\vec{k}\cdot(\vec{x} - \vec{x}')} \\ &\sim \frac{1}{|\vec{x} - \vec{x}'|^{d-2}} e^{-\frac{|\vec{x} - \vec{x}'|}{\xi}}, \quad \xi = \frac{1}{\sqrt{2|r|}}. \end{aligned} \quad (2.151)$$

Note that we can read the form of the ‘propagator’ directly from the momentum functional (2.150). As we approach  $T_c$ ,  $r \rightarrow 0$  and  $\xi \rightarrow \infty$  in the way  $\xi \propto 1/|t|^{1/2}$  and our prediction is consistent with the MFT prediction  $\nu = 1/2$  for these modes.

- On the other hand for the perpendicular modes we have

$$G_{\perp}(\vec{x} - \vec{x}') = \langle \delta\varphi_{\perp}^a(\vec{x}) \delta\varphi_{\perp}^a(\vec{x}') \rangle = \frac{1}{V^2} \sum_k \underbrace{\langle \delta\varphi_{\perp}^a(k) \delta\varphi_{\perp}^a(-k) \rangle}_{\frac{V}{\rho_0^2} \frac{n-1}{k^2}} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} . \quad (2.152)$$

This illustrates the fact that the correlation length of goldstone modes is always infinite,  $\xi_{\perp} = \infty$ .

In particular, we have

$$\begin{aligned} G_{\perp}(0) &= \langle \delta\varphi_{\perp}^a(\vec{x}) \delta\varphi_{\perp}^a(\vec{x}) \rangle = \frac{n-1}{\rho_0^2} \frac{1}{V} \sum_k \frac{1}{k^2} \\ &= \frac{n-1}{\rho_0^2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} = \frac{n-1}{\rho_0^2} \frac{S_d}{(2\pi)^d} \int_0^{\Lambda} dk k^{d-3} . \end{aligned} \quad (2.153)$$

The integral diverges (near  $k = 0$ ) for  $d \leq 2$  and converges for  $d > 2$ .

For  $d > 2$  the Goldstone modes give a finite contribution to fluctuations around the (ordered) MFT value. This means that there is an ordered phase characterized by a finite critical temperature,  $0 < T_c < (T_c)_{\text{MFT}}$ , and we observe a phase transition.

However, in  $d \leq 2$  the integral diverges for large scales (small momenta), the MFT breaks down, and there is no ordered phase at any temperature. In other words  $T_c(d) = 0$  for  $d \leq 2$  and only PM phase exists. This is a special case of the following famous theorem:

- Coleman–Mermin–Wagner Theorem. *For systems with continuous symmetry, there is no long range order at any finite temperature for  $d \leq 2$ .*

So in our journey we have encountered two important dimensions: Lower critical dimension  $\boxed{d=2}$  characterized by

$$\boxed{\lim_{d \rightarrow 2+} T_c(d) = 0 ,} \quad (2.154)$$

and upper critical dimension  $\boxed{d = 4}$  above which the MFT gives exact critical exponents and we need not to worry about fluctuations.

## Near $d = 2$ dimensions

- $(2 + \epsilon)$ -expansion. Note that previously, we have done  $4-d = \epsilon$  expansion, treating a deviation from the upper critical dimensions as a small parameter. Alternatively,

one can do an expansion around the lower critical dimension. (Of course, for Ising model nothing special happens in  $d = 2$ .)

- Non-linear sigma model. Let us return back to the functional (2.141). If  $d$  is near two, fluctuations of  $n^a$  are important, but we can ignore fluctuations of  $\rho$  (can use the MFT value). The functional then reduces to

$$S[\vec{n}] = \frac{\rho^2}{2} \int d^d x \nabla n^a \nabla n^a = \frac{1}{2T} \int d^d x \nabla n^a \nabla n^a , \quad (2.155)$$

where the latter can be achieved by ‘redefining the units’. Note that due to the presence of the constraint,  $|\vec{n}| = 1$ , this is not a free Gaussian theory, but rather a complicated theory called the non-linear sigma model.

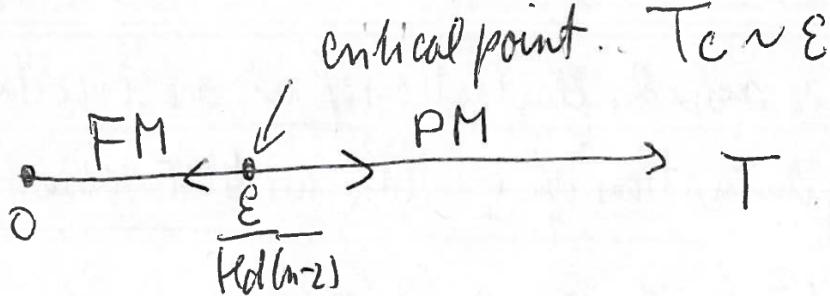
- We can now set up  $d = 2 + \epsilon$  expansion so that  $\lim_{d \rightarrow 2} T_c^* \rightarrow 0$ . This reads

$$\frac{d\tilde{T}}{dl} = -\epsilon \tilde{T} + (n-2)K_d \tilde{T}^2 , \quad \tilde{T} = T \Lambda^\epsilon , \quad K_d = \frac{S_d}{(2\pi)^d} . \quad (2.156)$$

This has two fixed points. The one at  $\tilde{T}^* = 0$  is (as always) stable, while the one at

$$\tilde{T}^* = \frac{\epsilon}{K_d(n-2)} \quad (2.157)$$

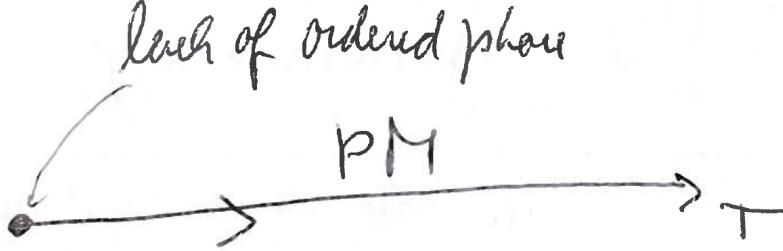
is unstable. So, we have the following RG flow:



- For  $d = 2$ , the RG equations simplify to

$$\frac{d\tilde{T}}{dl} = (n-2)K_2 \tilde{T}^2 . \quad (2.158)$$

This is displayed in the following figure.



This is exactly the same flow as in  $d = 4$  YM theory describing the interaction of quarks. The theory possesses asymptotic freedom. (Strength of interaction between quarks grows as the distance between them increases. One can relate the distance between the quarks with the correlation length. The correlation length goes to zero as we probe larger and larger scales.) This is a first hint on far reaching similarities between gauge theories in  $4d$  and non-linear sigma model in  $d = 2$ , subject nowadays to exhaustive study.

Solving the flow equation, one finds

$$\xi(T) \propto e^{\frac{2\pi}{(n-2)T}}. \quad (2.159)$$

So exactly at  $T = 0$ , the system does order and we have uniform  $\vec{n}$ .

### Nobel prize 2016: Vortices and Kosterlitz–Thouless phase transition

Let us now concentrate on the very interesting special case of the non-linear sigma model when

$$d = 2 = n. \quad (2.160)$$

Then the  $\beta$ -function (2.156) vanishes (in fact to all orders in  $\tilde{T}$ ).

- Writing

$$\vec{n} = \cos \theta \hat{x} + \sin \theta \hat{y}, \quad (2.161)$$

where  $\theta \in (0, 2\pi)$ , automatically solves the constraint  $|\vec{n}|^2 = 1$ . Moreover, we find

$$\nabla \vec{n} = -\sin \theta (\nabla \theta) \hat{x} + \cos \theta (\nabla \theta) \hat{y} \Rightarrow \nabla n^a \nabla n^a = (\nabla \theta)^2, \quad (2.162)$$

and the non-linear sigma model becomes

$$S[\theta] = \frac{1}{2T} \int d^2x (\nabla \theta)^2. \quad (2.163)$$

Again this is not a Gaussian theory as  $\theta$  is compact.

- Pretending that  $\theta$  is a real variable, one can obtain the following correlation function:<sup>3</sup>

$$G(\vec{x}) = \langle e^{i\theta(\vec{x})} e^{-i\theta(0)} \rangle = \dots = \left(\frac{a}{x}\right)^{\frac{T}{2\pi}}, \quad (2.164)$$

where  $a = 1/\Lambda$  is the lattice spacing. Comparing to  $G(x) \propto \frac{1}{x^{d-2+\eta}}$ , we conclude that

$$\eta = \eta(T) = \frac{T}{2\pi}. \quad (2.165)$$

This looks like an infinite line of fixed points with the temperature dependent critical exponents ( $\beta$ -function vanishes for every  $T$ ). However, even though  $\xi \rightarrow \infty$ , we have  $G(x \rightarrow \infty) \rightarrow 0$  and so there is no long range order *in agreement* with the CMW theorem.<sup>4</sup>

However, this cannot be physically right, at large  $T$  we should get independent fluctuations and hence for sufficiently large  $T$ ,  $\xi$  must be finite. (Think of the original microscopic vector model for  $T \gg J$ .) So there must exist a finite  $T_c$  at which

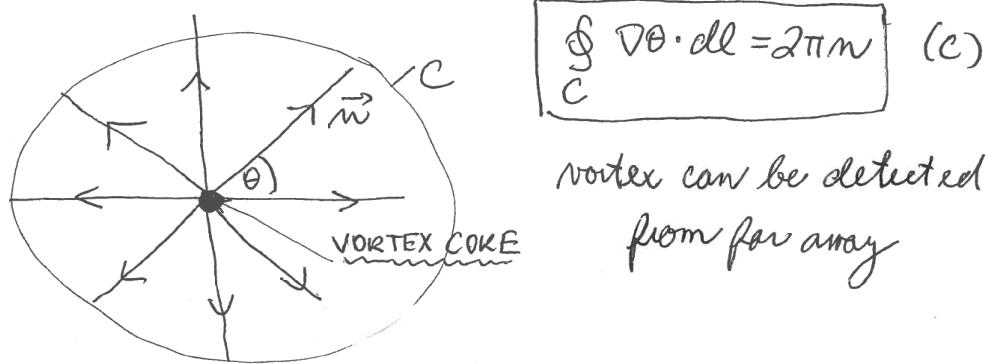
$$G(x) \propto \frac{1}{x^{\eta(T)}} \quad \text{to} \quad G(x) \propto \frac{e^{-x/\xi}}{x^{\eta(T)}}, \quad (2.166)$$

and

$$\xi = \begin{cases} \infty & T < T_c, \\ \text{finite} & T > T_c. \end{cases} \quad (2.167)$$

This cannot be characterized by an order parameter.

- By assuming  $\theta(x)$  to be a non-compact real field, we have implicitly neglected the possibility of having vortex-like topological defects. A vortex is a ‘point’ around which  $\theta$  winds by  $2\pi$  (or any integer multiple of  $2\pi$ ), see figure:



<sup>3</sup>What we really want is the correlation function for  $n$ 's that is why we define  $G$  as exponentials of  $\theta$ 's. Such a correlation function gives  $\langle \psi(\vec{x})\psi^*(0) \rangle$  correlation function for the superfluid, or the  $\langle m(\vec{x})m(0) \rangle$  correlation function for the magnet.

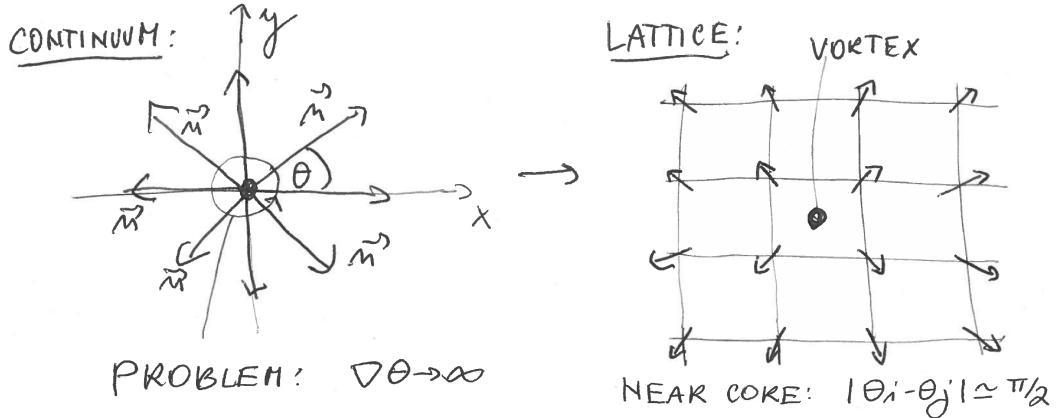
<sup>4</sup>Note that the way we defined our correlation length (2.164) the ‘background value’ is not subtracted. With such a definition, the long range order would be characterized by  $G(x \rightarrow \infty) \rightarrow M^2, M = \langle e^{i\theta} \rangle$ .

Here,  $n$  is called the winding number (the sum of angles around  $C$  is  $2\pi n$ ).

- Problem: One can show that (see later)

$$\nabla\theta \propto \frac{1}{r}. \quad (2.168)$$

This means that near the vortex core,  $\nabla\theta$  diverges, the continuous limit breaks down, and one has to return back to the lattice model, see figure and next point if interested.



- Let us go back to the lattice model (setting  $J = 1$ ):

$$H = - \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j, \quad (2.169)$$

where the sum is for simplicity over nearest neighbors. Since  $S_i^x = \cos \theta_i$  and  $S_i^y = \sin \theta_i$ , this gives

$$H = - \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j). \quad (2.170)$$

How is the continuum limit obtained from here? Let us assume that we are far away from any vortex core.  $\theta_i$  then vary slowly on the scale of lattice spacing and we can Taylor expand as  $\cos(\theta_i - \theta_j) \approx 1 - \frac{1}{2}(\theta_i - \theta_j)^2$  and so

$$H = \text{const.} + \frac{1}{2} \sum_{\langle ij \rangle} (\theta_i - \theta_j)^2 \rightarrow \frac{1}{2} \int d^2x (\nabla\theta)^2. \quad (2.171)$$

Note however that near the core we cannot take the continuum limit as  $\nabla\theta \rightarrow \infty$  whereas in the lattice (see figure) it remains finite;  $|\theta_i - \theta_j| \approx \pi/2$  near the core.

- Let us estimate the energy of a single vortex with winding number  $n$ . We can do this by employing the continuum limit (as it only breaks down in a finite region

in the vicinity of the vortex core). So we want to minimize

$$H = \frac{1}{2} \int d^2x (\nabla\theta)^2, \quad (2.172)$$

subject to a constraint (C). We have

$$\delta H = \frac{1}{2} \int d^2x 2\nabla\delta\theta \cdot \nabla\theta = - \int d^2x \nabla^2\theta \delta\theta = 0, \quad (2.173)$$

where we have integrated by parts, getting:

$$\boxed{\nabla^2\theta = 0, \quad \oint \vec{\nabla}\theta \cdot d\vec{l} = 2\pi n,} \quad (2.174)$$

where the integral is taken over any loop enclosing the origin. Using polar coordinates  $(r, \varphi)$  this has the following solution:

$$\boxed{\theta = n\varphi,} \quad (2.175)$$

Indeed  $\nabla\theta = \frac{n\hat{\varphi}}{r}$  and the statement follows. The energy of a single vortex then reads

$$E = \frac{1}{2} \int d^2x \frac{n^2}{r^2} = \frac{1}{2} \int_0^{2\pi} d\varphi \int_a^L dr \frac{n^2}{r^2} r = \pi n^2 \log(L/a). \quad (2.176)$$

Here  $L$  is the size of the system and we introduced another cutoff  $a$  (removing the center) so that the continuum limit does not break down. So it costs a lot of energy to create a vortex.

- At  $T = 0$  the vortices do not occur. For  $T > 0$  we should minimize the free energy  $F = E - TS$ . If the entropy associated with vortices is high enough, we can have them. Let us estimate this entropy as

$$\begin{aligned} S &= \log(\text{number of states}) \\ &\approx \log(\text{number of places we can place the vortex core}) \\ &\approx \log\left(\frac{L}{a}\right)^2 = 2\log(L/a). \end{aligned} \quad (2.177)$$

Of course,  $n = 1$  vortex has the lowest energy. Then we estimate

$$F = (\pi - 2T) \log(L/a). \quad (2.178)$$

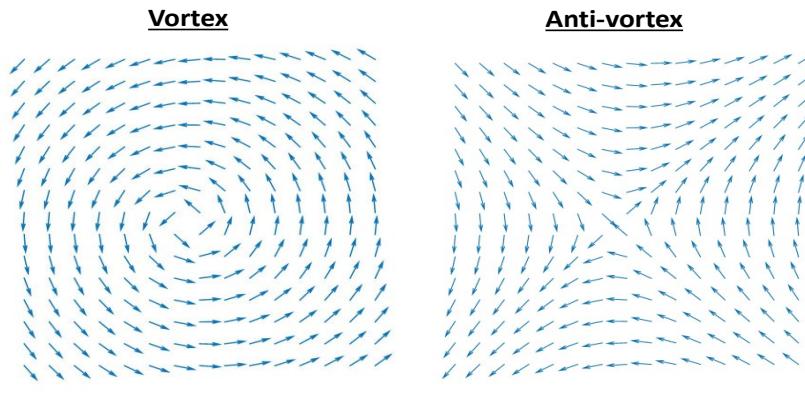
When this becomes negative we can have vortices, giving

$$\boxed{T > T_{\text{KT}} = \frac{\pi}{2}.} \quad (2.179)$$

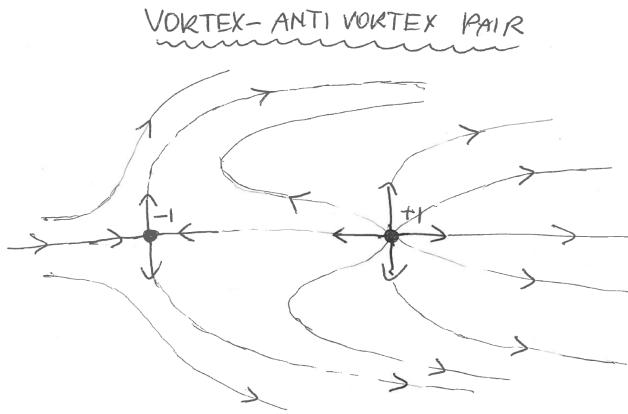
So at  $T_c = T_{\text{KT}}$  we have the Kosterlitz–Thouless vortex/no-vortex phase transition.

- Connecting with (2.167), one can roughly think of the correlation length as the average distance between the free vortices. This is infinite below  $T_K$  and finite above  $T_K$ . The phase transition manifests itself in the change of the character of the correlation function (2.166) but the order parameter is always zero for any  $T$ .
- The described Kosterlitz–Thouless topological phase transition without an order parameter, and its subsequent observation in  $\text{He}^4$  (as a superfluid to normal fluid phase transition), was awarded a Nobel prize in 2016 (Congratulations!).
- Let us finally remark that even below  $T_K$  the vortices do exist. However, they are bound to vortex pairs, see figure, that have total  $n = 0$  and cost significantly lower energy:

$$U \propto J n_i n_j \log(|\vec{x}_i - \vec{x}_j|/a). \quad (2.180)$$



<https://johncarlosbaez.wordpress.com/author/johncarlosbaez/>



Note also that vortices can exist in higher dimensions. For example, in  $d = 3$ , we have a string. They, however, do not play a central role for phase transitions as we have a long range order.  $d = 2$  is a special beast with no long range order and vortices play the central role.

# Appendix A: Bonus Lecture: Black Hole Thermodynamics

## Motivation

Let us study some properties of the Schwarzschild black hole:

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega^2, \quad f = 1 - \frac{2m}{r}, \quad d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2. \quad (\text{A.1})$$

- Asymptotic mass (conserved energy)

Noether's theorem : symmetry  $\leftrightarrow$  conserved quantity  
 described by Killing fields

$$\nabla_{(\mu} k_{\nu)} = \nabla_\mu k_\nu + \nabla_\nu k_\mu = 0$$

$k^M$  geometry "does not change"  
Spec:  $k = \partial_t$   $\uparrow \uparrow^t$  Spacetime is static  
 $\Rightarrow$  energy is conserved

This leads to the following prescription for the *Komar mass*:

$$M = -\frac{1}{8\pi} \int_{S_\infty^2} *dk = m. \quad (\text{A.2})$$

- Black hole horizon is located at  $r_+$  given by  $f(r_+) = 0$ , giving

$$r_+ = 2M. \quad (\text{A.3})$$

This represents a "boundary of the black hole". At the same time it is a surface of infinite blueshift, which is the reason why black hole looks black for the external observer.

Surface of infinite blueshift...black

$$\frac{V_R}{V_t} = \frac{\Delta \gamma_t}{\Delta \gamma_R} = \frac{r=R}{\sqrt{f(r)} \Delta t} = \frac{\sqrt{1 - \frac{2M}{r}}}{\sqrt{1 - \frac{2M}{R}}} \rightarrow 0$$

- Surface gravity. The black hole horizon is a *Killing horizon*: a null surface generated by Killing field  $k = \partial_t$ .

$$k^\mu \nabla_\mu k^\nu / H = \partial^\nu k^\mu / H$$

$\uparrow$   
surface gravity

It can be shown that  $\kappa$  reads

$$\kappa = \frac{f'(r_+)}{2} = \frac{1}{2} \frac{2M}{r_+^2} = \frac{1}{4M} = \frac{1}{2r_+}. \quad (\text{A.4})$$

Note that this ‘coincides’ with Newtonian acceleration evaluated on the black hole horizon:

$$\kappa = \frac{M}{r_+^2} = \frac{M}{(2M)^2} = \frac{1}{4M}. \quad (\text{A.5})$$

- Horizon area. Taking the  $dt = 0 = dr$ , the induced spatial metric ‘on the horizon’ is  $d\gamma^2 = r_+^2 d\Omega^2$ . The area then reads

$$A = \int \sqrt{\det \gamma} d\theta d\varphi = \int r_+^2 \sin \theta d\theta d\varphi = 4\pi r_+^2. \quad (\text{A.6})$$

- Observation: Calculating the following differentials:

$$dM = \frac{dr_+}{2}, \quad dA = 8\pi r_+ dr_+, \quad (\text{A.7})$$

we find that

$$dM = \frac{\kappa}{2\pi} \frac{dA}{4}.$$

(A.8)

## Laws of Black Hole Mechanics

Bardeen, Carter and Hawking (1973) proved the following 4 laws of black hole mechanics. For a stationary, charged, and rotating black hole with mass  $M$ , angular momentum  $J$ , and charge  $Q$ , we have:

- **Zeroth law:** The surface gravity  $\kappa$  is constant on the black hole horizon.
- **First law:**

$$dM = \frac{\kappa}{2\pi} \frac{dA}{4} + \underbrace{\Omega dJ + \Phi dQ}_{\text{work terms}}. \quad (\text{A.9})$$

Here,  $\Omega$  is the angular velocity of the black hole horizon, and  $\Phi$  is its ‘electrostatic potential’.

- **Second law:** Classically, the area of the horizon never decreases.

$$dA \geq 0. \quad (\text{A.10})$$

- **Third law:** It is impossible to reduce  $\kappa$  to zero in a finite number of steps.

We would like to compare these to the laws of thermodynamics. In particular, the first law to

$$dE = TdS + \text{work terms}. \quad (\text{A.11})$$

However, there is a problem: Classical black holes act as ultimate sponges: no heat can flow out, they are at absolute zero temperature. So we cannot have  $\kappa \propto T$ .

## Black Hole Thermodynamics

- Wheeler’s cup of tea: “If you throw a cup o tea to the black hole, where did its entropy go?” Based on analyzing this question Bekenstein proposed

$$S \propto A. \quad (\text{A.12})$$

- Hawking 1974. When quantum effects are taken into account, black holes radiate away as black body with

$$T = \frac{\hbar\kappa}{2\pi k_B}, \quad S = \frac{A}{4\hbar G_N}. \quad (\text{A.13})$$

Derivation used QFT in curved space. Since then the same results have been reproduced by many other approaches, e.g: Euclidean path integral, tunneling, string theory, LQG.

## Euclidean Trick

- Thermal Green functions have periodicity in imaginary time  $[\tau = it]$ :

$$G(\tau) = G(\tau + \beta), \quad \beta = 1/T. \quad (\text{A.14})$$

Conversely, periodicity of  $G$  defines a thermal state. Green functions of quantum fields in the vicinity of black holes have this property (as seen by a static observer). What about gravitational field itself?

- Let us consider the Euclideanized Schwarzschild solution ( $\tau = it$ ):

$$ds^2 = f d\tau^2 + \frac{dr^2}{f} + r^2 d\Omega^2. \quad (\text{A.15})$$

Near the horizon we may expand

$$f = \underbrace{f(r_+)}_0 + \underbrace{(r - r_+)}_{\Delta r} \underbrace{f'(r_+)}_{2\kappa} + \dots = 2\kappa\Delta r. \quad (\text{A.16})$$

Therefore, the near horizon limit of the ‘Euclidean Schwarzschild solution’ takes the following form:

$$ds^2 = 2\kappa\Delta r d\tau^2 + \frac{dr^2}{2\kappa\Delta r} + r_+^2 d\Omega^2. \quad (\text{A.17})$$

We can now introduce a new coordinate  $\rho$  by

$$d\rho^2 = \frac{dr^2}{2\kappa\Delta r} \Leftrightarrow d\rho = \frac{dr}{\sqrt{2\kappa\Delta r}} \Leftrightarrow \Delta r = \frac{\kappa}{2}\rho^2, \quad (\text{A.18})$$

getting

$$ds^2 = \kappa^2\rho^2 d\tau^2 + d\rho^2 + r_+^2 d\Omega^2. \quad (\text{A.19})$$

So if we introduce a new ‘angular coordinate’  $[\psi = \kappa\tau]$ , we get

$$ds^2 = \rho^2 d\psi^2 + d\rho^2 + r_+^2 d\Omega^2, \quad (\text{A.20})$$

and the first two terms just look like polar coordinates on the plane. The original black hole horizon corresponds to  $\rho = 0$ . Since this was non-singular originally, we expect it to be non-singular again. This is achieved by setting (we want to avoid conical singularity)

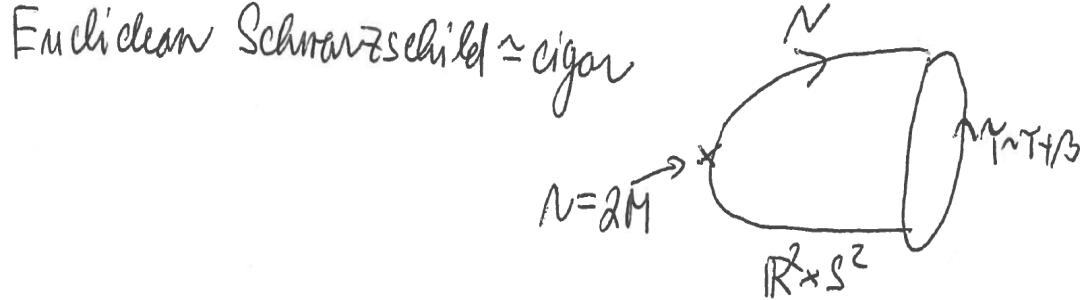
$$\psi \sim \psi + 2\pi \Leftrightarrow \tau \sim \tau + \underbrace{2\pi/\kappa}_\beta \Leftrightarrow T = \frac{\kappa}{2\pi}. \quad (\text{A.21})$$

In particular,

$$T = \frac{1}{8\pi M} \quad (\text{A.22})$$

for the Schwarzschild solution.

- Let us finally note that the Euclidean Schwarzschild solution (A.15) looks like a surface of a *cigar*, where each point corresponds to a suppressed sphere:



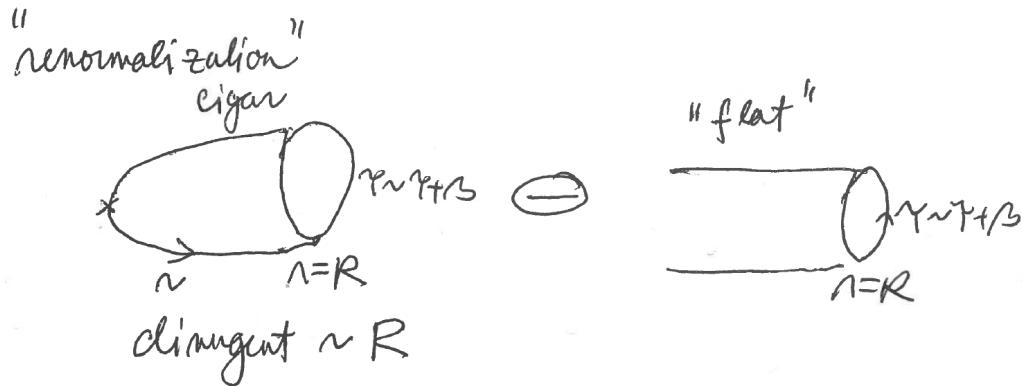
- Partition function. One can calculate the gravitational partition function in the WKB approximation as

$$Z = \int Dg e^{-S_E[g]} \approx e^{-S_E(g_c)}, \quad (\text{A.23})$$

where  $g_c$  stands for the metric(s) describing the classical solution(s). Note that the Euclidean action  $S_E$  consists of two terms: the Einstein–Hilbert action and the Gibbons–Hawking term:

$$S_E = \int_{\Omega} \frac{d^4x \sqrt{g} R}{16\pi G_N} + \int_{\partial\Omega} \frac{d^3x \sqrt{h} K}{8\pi G_N}. \quad (\text{A.24})$$

The second term is needed to ensure well-posed variational principle (it kills the unwanted boundary terms in the case of a compact manifold). Here  $K$  stands for the extrinsic curvature and the second integral is over the boundary. On shell  $R = 0$  and the first term vanishes. It is the second term that determines the value of the action. This is done as follows. We introduce a boundary at  $r = R$ , calculate the contribution of the second term and then let  $R \rightarrow \infty$ . Unfortunately, the second term then diverges and we have to renormalize as displayed in the figure:



So we get

$$S_E = S_E(\text{cigar}) - S_E(\text{flat}) = \frac{\beta M}{2}. \quad (\text{A.25})$$

Free energy then becomes

$$F = -\frac{1}{\beta} \log Z = M/2, \quad (\text{A.26})$$

giving the following entropy:

$$S = -\frac{\partial F}{\partial T} = \left| T = \frac{1}{8\pi M} \right| = \frac{1}{16\pi T^2} = \pi r_+^2 = \frac{A}{4}, \quad (\text{A.27})$$

confirming Bekenstein's result.

## Final Remarks

- Hawking radiation is a kinematic effect. (One needs equivalence principle, vacuum fluctuations, but the Einstein equations are not required.) This opens a possibility for observing this effect in '*analogue systems*', e.g. surface water waves.
- Black holes do not radiate as true black body as some waves 'scatter back' to the horizon. For this reason the corresponding distribution reads

$$\langle n_\omega \rangle = \frac{\Gamma_\omega}{e^{\omega/T} - 1}, \quad (\text{A.28})$$

where  $\Gamma_\omega$  is the so called *greybody factor*.

Correspondingly, the black hole loses mass according to the 'effective' Stefan–Boltzmann law

$$\frac{dM}{dt} \propto -\sigma T^4 A \propto -\frac{1}{M^2}, \quad (\text{A.29})$$

so that it would completely evaporate in ( $M_S$  denoting the mass of the Sun)

$$T \approx \left( \frac{M}{M_S} \right)^3 \times 10^{71} s, \quad (\text{A.30})$$

see tutorial for more details.

- Note also that since

$$T = \frac{1}{8\pi M}, \quad (\text{A.31})$$

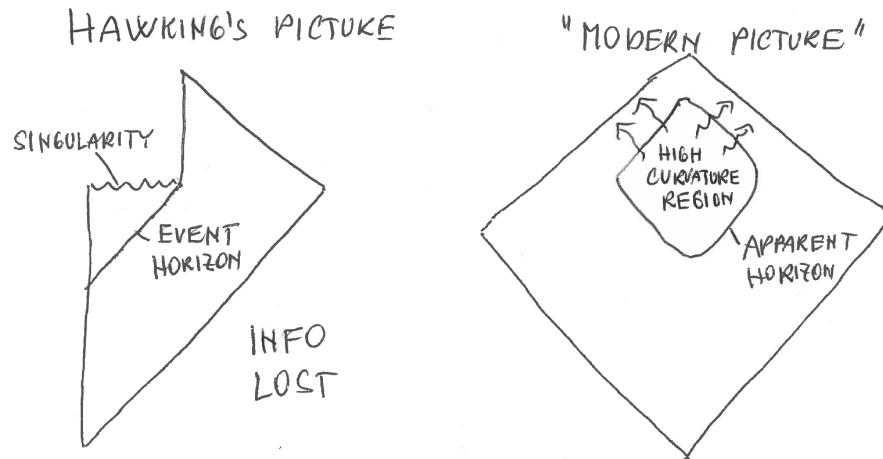
smaller the black hole is the hotter it is. This implies that the evaporation accelerates and towards the end we can observe 'black hole explosions' (CERN?)

This also means that Schwarzschild black hole has a *negative specific heat*:

$$C = T \frac{\partial S}{\partial T} = -\frac{1}{8\pi T^2}. \quad (\text{A.32})$$

(This is quite typical for self-gravitating systems. For example, a satellite as it falls it increases its kinetic energy; a gravothermal catastrophe described by Lynden Bell.)

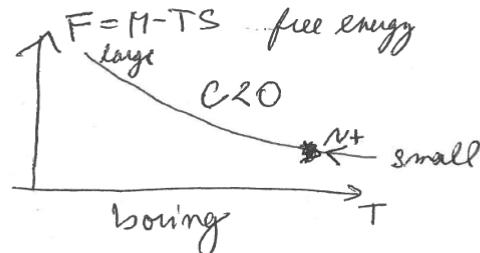
- Black hole information paradox: see the following picture and read Wikipedia:



- Phase transitions.

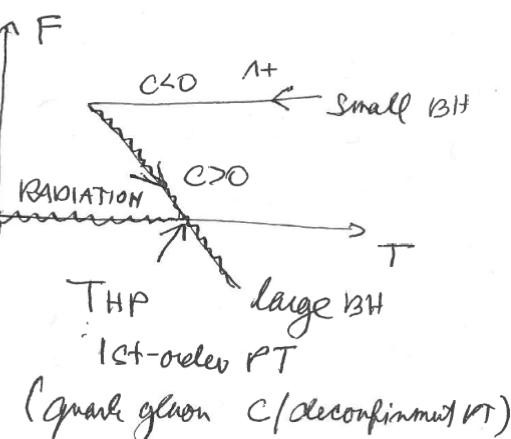
- Schwarzschild

$$f = 1 - \frac{2M}{r}$$



- Schwarzschild-AdS

$$f = 1 - \frac{2M}{r} + \frac{N^2}{r^2}$$



# Appendix B: Some Technical Results

## Wick's Theorem

*Wick's theorem* simplifies the calculation of averages of many fields with respect to a Gaussian statistical weight. The great simplification is that the average of many fields can be expressed as a product of averages of two fields (2-point correlation functions). More explicitly the following is true:

**Wick's Theorem.** *For a free field theory, i.e. a theory with no interactions, for which the partition function has the form of a Gaussian functional integral, we have*

$$\langle \phi_1 \phi_2 \dots \phi_{2n} \rangle = \langle \phi_1 \phi_2 \rangle \dots \langle \phi_{2n-1} \phi_{2n} \rangle + \text{all other possible contractions of two fields}, \quad (\text{B.1})$$

where  $\phi_i$  are Gaussian fields and by “contraction” of two fields we mean taking their average.

Let us prove this statement for a real scalar field  $\phi$ , with action

$$S[\phi] = \frac{1}{2} \sum_{i,j} \phi_i M_{ij} \phi_j \equiv \frac{1}{2} \phi_i M_{ij} \phi_j. \quad (\text{B.2})$$

The partition function reads

$$Z = \int \mathcal{D}\phi e^{-S[\phi]}, \quad \mathcal{D}\phi = \prod_i \frac{d\phi_i}{\sqrt{2\pi}}, \quad (\text{B.3})$$

and the *statistical averages* are calculated through

$$\langle \dots \rangle = \frac{1}{Z} \int \mathcal{D}\phi (\dots) e^{-S[\phi]}. \quad (\text{B.4})$$

- To calculate these averages, we introduce a *source current*  $J$  and a *generating functional*

$$G[J] = \frac{1}{Z} \int \mathcal{D}\phi e^{-S[\phi]+J_i \phi_i} = \frac{1}{Z} \int \mathcal{D}\phi e^{-\frac{1}{2} \phi_i M_{ij} \phi_j + J_i \phi_i}. \quad (\text{B.5})$$

Taking one derivative of the generating functional w.r.t  $J$  gives

$$\frac{\partial G[J]}{\partial J_i} \Big|_{J=0} = \frac{1}{Z} \int \mathcal{D}\phi e^{-S[\phi] + J_k \phi_k} \phi_i \Big|_{J=0} = \langle \phi_i \rangle, \quad (\text{B.6})$$

and two derivatives give

$$\frac{\partial^2 G[J]}{\partial J_i \partial J_j} \Big|_{J=0} = \frac{1}{Z} \int \mathcal{D}\phi e^{-S[\phi] + J_k \phi_k} \phi_j \phi_i \Big|_{J=0} = \langle \phi_i \phi_j \rangle, \quad (\text{B.7})$$

and so on, for  $n$  fields we have

$$\boxed{\frac{\partial^n G}{\partial J_1 \dots \partial J_n} \Big|_{J=0} = \langle \phi_1 \dots \phi_n \rangle}, \quad (\text{B.8})$$

the latter quantity being called the *n-point correlation function*.

- The relation (B.8) is true for any action  $S$  and shows that we can calculate all possible correlation functions as long as we know the form of  $G$  as a functional of  $J$ , i.e., as long as we perform the integrals  $\int \mathcal{D}\phi$  in (B.5). In general this is hard. However, when  $S$  is Gaussian we can use the Hubbard–Stratonovich transformation as follows.

$$\int \mathcal{D}\phi e^{-\frac{1}{2} \phi_i M_{ij} \phi_j + J_i \phi_i} = \frac{1}{(2\pi)^{N/2}} \int \prod_i d\phi_i e^{-\frac{1}{2} \phi_i M_{ij} \phi_j + J_i \phi_i} = \frac{1}{\sqrt{\det M}} e^{\frac{1}{2} J_i M_{ij}^{-1} J_j}. \quad (\text{B.9})$$

In particular, the partition function reads

$$Z = \int \mathcal{D}\phi e^{-\frac{1}{2} \phi_i M_{ij} \phi_j + J_i \phi_i} \Big|_{J=0} = \frac{1}{\sqrt{\det M}} e^{\frac{1}{2} J_i M_{ij}^{-1} J_j} \Big|_{J=0} = \frac{1}{\sqrt{\det M}}. \quad (\text{B.10})$$

And therefore  $G[J]$  is

$$\boxed{G[J] = \frac{1}{Z} \int \mathcal{D}\phi e^{-\frac{1}{2} \phi_i M_{ij} \phi_j + J_i \phi_i} = e^{\frac{1}{2} J_i M_{ij}^{-1} J_j}}. \quad (\text{B.11})$$

- It is now straightforward to calculate the 1-point and 2-point correlation functions. By differentiating  $G$ , we get

$$\begin{aligned} \frac{\partial G[J]}{\partial J_i} &= e^{\frac{1}{2} \text{Tr}(JM^{-1}J)} M_{ij}^{-1} J_j, \\ \frac{\partial^2 G[J]}{\partial J_i \partial J_j} &= e^{\frac{1}{2} \text{Tr}(JM^{-1}J)} [M_{ij}^{-1} + M_{ik}^{-1} J_k M_{j\ell}^{-1} J_\ell]. \end{aligned} \quad (\text{B.12})$$

Hence, all 1-point functions vanish,

$$\langle \phi_i \rangle = e^{\frac{1}{2} \text{Tr}(JM^{-1}J)} M_{ij}^{-1} J_j \Big|_{J=0} = 0, \quad (\text{B.13})$$

whereas the 2-point function is

$$\langle \phi_i \phi_j \rangle = e^{\frac{1}{2} \text{Tr}(JM^{-1}J)} [M_{ij}^{-1} + M_{ik}^{-1} J_k M_{j\ell}^{-1} J_\ell] \Big|_{J=0} = M_{ij}^{-1}. \quad (\text{B.14})$$

The latter quantity is sometimes called the *propagator*.

- The pattern for higher-point correlation functions is now obvious. Every derivative of  $G$  with respect to  $J$  either pulls down a factor of  $M^{-1}J$  or turns a factor of  $M^{-1}J$  into  $M^{-1}$ . Since we are taking the limit  $J = 0$  at the end, all terms of the form  $M^{-1}J$  vanish. Only an even number of fields will leave factors of  $M^{-1}$  (without  $J$ ) at the end. Hence, all the terms in any  $(2n+1)$ -point correlation function will in the end vanish, that is, *all odd correlation functions identically vanish*.

In the case of a  $2n$ -point function, when applying the derivatives, we will get all possible combinations of  $M_{ij}^{-1}$ , with the indices  $i, j$  running from 1 to  $2n$ . Therefore, a  $2n$ -point correlation function will look like

$$\begin{aligned} \langle \phi_{i_1} \dots \phi_{i_{2n}} \rangle &= M_{i_1 i_2}^{-1} \dots M_{i_{2n-1} i_{2n}}^{-1} + \text{all other possible combinations} \\ &= \langle \phi_{i_1} \phi_{i_2} \rangle \langle \phi_{i_3} \phi_{i_4} \rangle \dots \langle \phi_{i_{2n-1}} \phi_{i_{2n}} \rangle + \langle \phi_{i_1} \phi_{i_3} \rangle \langle \phi_{i_2} \phi_{i_4} \rangle \dots \langle \phi_{i_{2n-1}} \phi_{i_{2n}} \rangle \\ &\quad + \text{all other possible contractions}. \end{aligned} \quad (\text{B.15})$$

- In particular, for a 4-point function we get

$$\langle \psi_1 \psi_2 \psi_3 \psi_4 \rangle = \langle \psi_1 \psi_2 \rangle \langle \psi_3 \psi_4 \rangle + \langle \psi_1 \psi_3 \rangle \langle \psi_2 \psi_4 \rangle + \langle \psi_1 \psi_4 \rangle \langle \psi_2 \psi_3 \rangle, \quad (\text{B.16})$$

and the 6-point function reads

$$\begin{aligned} \langle \psi_1 \psi_2 \psi_3 \psi_4 \psi_5 \psi_6 \rangle &= \\ &+ \langle \psi_1 \psi_2 \rangle \langle \psi_3 \psi_4 \rangle \langle \psi_5 \psi_6 \rangle + \langle \psi_1 \psi_3 \rangle \langle \psi_2 \psi_4 \rangle \langle \psi_5 \psi_6 \rangle + \langle \psi_1 \psi_4 \rangle \langle \psi_2 \psi_3 \rangle \langle \psi_5 \psi_6 \rangle \\ &+ \langle \psi_1 \psi_2 \rangle \langle \psi_3 \psi_5 \rangle \langle \psi_4 \psi_6 \rangle + \langle \psi_1 \psi_3 \rangle \langle \psi_2 \psi_5 \rangle \langle \psi_4 \psi_6 \rangle + \langle \psi_1 \psi_5 \rangle \langle \psi_2 \psi_3 \rangle \langle \psi_4 \psi_6 \rangle \\ &+ \langle \psi_1 \psi_2 \rangle \langle \psi_4 \psi_5 \rangle \langle \psi_3 \psi_6 \rangle + \langle \psi_1 \psi_4 \rangle \langle \psi_2 \psi_5 \rangle \langle \psi_3 \psi_6 \rangle + \langle \psi_1 \psi_5 \rangle \langle \psi_2 \psi_4 \rangle \langle \psi_3 \psi_6 \rangle \\ &+ \langle \psi_1 \psi_3 \rangle \langle \psi_4 \psi_5 \rangle \langle \psi_2 \psi_6 \rangle + \langle \psi_1 \psi_4 \rangle \langle \psi_3 \psi_5 \rangle \langle \psi_2 \psi_6 \rangle + \langle \psi_1 \psi_5 \rangle \langle \psi_3 \psi_4 \rangle \langle \psi_2 \psi_6 \rangle \\ &+ \langle \psi_2 \psi_3 \rangle \langle \psi_4 \psi_5 \rangle \langle \psi_1 \psi_6 \rangle + \langle \psi_2 \psi_4 \rangle \langle \psi_3 \psi_5 \rangle \langle \psi_1 \psi_6 \rangle + \langle \psi_2 \psi_5 \rangle \langle \psi_3 \psi_4 \rangle \langle \psi_1 \psi_6 \rangle. \end{aligned} \quad (\text{B.17})$$

Specifically, in the continuum limit and the momentum space we encountered the following average:

$$\langle \phi(k_1) \phi(k_2) \rangle = \frac{1}{Z} \int D\phi \phi(k_1) \phi(k_2) e^{-\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} (k^2 + r) |\phi(k)|^2}. \quad (\text{B.18})$$

Since

$$\int \frac{d^d k}{(2\pi)^d} (k^2 + r) |\phi(k)|^2 = \int d^d k d^d k' \frac{1}{(2\pi)^d} (k^2 + r) \delta(k + k') \phi(k) \phi(k'), \quad (\text{B.19})$$

we can identify

$$M_{kk'} = \frac{1}{(2\pi)^d} \delta(k + k')(k^2 + r) \quad \Leftrightarrow \quad M_{kk'}^{-1} = \frac{1}{k^2 + r} (2\pi)^d \delta(k + k'). \quad (\text{B.20})$$

So we have

$$\begin{aligned} \langle \phi(k_1)\phi(k_2) \rangle &= \frac{1}{Z} \int D\phi \phi(k_1)\phi(k_2) e^{-\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} (k^2 + r) |\phi(k)|^2} \\ &= M_{k_1 k_2}^{-1} = \frac{1}{k_1^2 + r} (2\pi)^d \delta(k_1 + k_2), \end{aligned} \quad (\text{B.21})$$

which is the formula used in the main text.

## Cumulant Expansion

The following result is useful for calculating statistical averages and is true for any random variable  $\Omega$ :

$$\boxed{\langle e^{\Omega} \rangle = e^{\langle \Omega \rangle + \frac{1}{2} [\langle \Omega^2 \rangle - \langle \Omega \rangle^2] + \dots}} \quad (\text{B.22})$$

This result is known as the *cumulant expansion*. Let us show that it is true to second order in  $\Omega$ , by considering the Taylor series expansion of the “generating function”

$$g(t) = \ln \langle e^{t\Omega} \rangle. \quad (\text{B.23})$$

The Taylor expansion of the “generating function” gives

$$g(t) = \sum_n g^{(n)}(0) \frac{t^n}{n!}. \quad (\text{B.24})$$

We calculate the terms to order  $t^2$  (and consequently to order  $\Omega^2$ ),

$$\begin{aligned} g(0) &= \ln \langle e^0 \rangle = 0, \\ g'(0) &= \frac{1}{\langle e^{t\Omega} \rangle} \langle e^{t\Omega} \Omega \rangle \Big|_{t=0} = \langle \Omega \rangle, \\ g''(0) &= -\frac{1}{\langle e^{t\Omega} \rangle^2} \langle e^{t\Omega} \Omega^2 \rangle \Big|_{t=0} + \frac{1}{\langle e^{t\Omega} \rangle} \langle e^{t\Omega} \Omega^2 \rangle \Big|_{t=0} = \langle \Omega^2 \rangle - \langle \Omega \rangle^2. \end{aligned} \quad (\text{B.25})$$

We thus have

$$\ln \langle e^{t\Omega} \rangle = \langle \Omega \rangle t + \frac{1}{2} [\langle \Omega^2 \rangle - \langle \Omega \rangle^2] t^2 + \dots \quad (\text{B.26})$$

In particular, setting  $t = 1$  this gives

$$\ln \langle e^{\Omega} \rangle = \langle \Omega \rangle + \frac{1}{2} [\langle \Omega^2 \rangle - \langle \Omega \rangle^2] + \dots, \quad (\text{B.27})$$

which is (B.22).

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