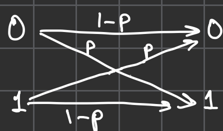
Quantum Error Correction : Class 2

Recap of 14th November :

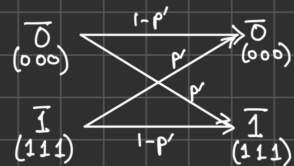
- Classical Error correction using repetition: encode messages by repeating 0's and 1's. Decode by a majority vote.

Before Error correction



Binary symmetric channel

After Error correction:



$$p' = 3p^2(1-p) + (1-p)^3$$

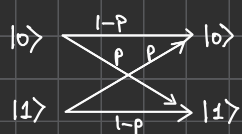
where p' is the logical error rate for the repetition code.

- Quantum Error correction

Note: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

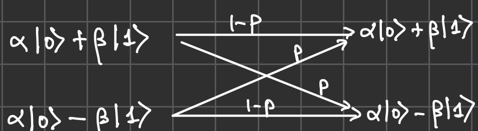
Two types of noise: Bit flip and Phase flip:

NOISE:



Bit flip channel

$$\mathcal{E}(\rho) = (1-p)\rho + pX\rho X$$



$$\mathcal{E}(\rho) = (1-p)\rho + pZ\rho Z$$

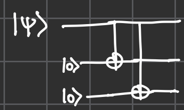
Phase flip channel: flips the relative sign between $|0\rangle$ and $|1\rangle$.

ENCODING:

$|0\rangle \mapsto |\bar{0}\rangle = |000\rangle$

$|1\rangle \mapsto |\bar{1}\rangle = |111\rangle$

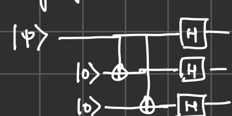
Encoding for the Bit flip code



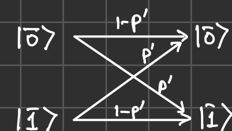
$|0\rangle \mapsto |\bar{0}\rangle = |{+}{+}{+}\rangle$

$|1\rangle \mapsto |\bar{1}\rangle = |{-}{-}{-}\rangle$

Encoding for the phase flip code



LOGICAL ERROR RATE:



where $p' = 3p^2(1-p) + (1-p)^3$ is the logical error rate of the Bit flip code.

Once again, the logi-cal error rate is the same:

$$p' = 3p^2(1-p) + (1-p)^3.$$

Systematic method of doing Quantum Error Correction:

- Bit flip code:

  $|\bar{0}\rangle = |000\rangle$
  $|\bar{1}\rangle = |111\rangle$

  Recall: These are eigenstates with eigenvalue $+1$ of $Z_1 Z_2 = Z \otimes Z \otimes \mathbb{1}$ and $Z_2 Z_3 = \mathbb{1} \otimes Z \otimes Z$.

  HW: Verify this.

- Bit flip channel: $\mathcal{E}(\rho) = (1-p)\,\rho + p X \rho X$ (apply 'X' on any qubit with probability 'p').
- Consider the effect of all of the single-qubit 'X' errors:

Error: E

$|\bar{\varphi}\rangle = \alpha|000\rangle + \beta|111\rangle$

$\xrightarrow{\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}} \alpha|000\rangle + \beta|111\rangle := |\bar{\varphi}\rangle$

$\xrightarrow{X\otimes\mathbb{1}\otimes\mathbb{1}} \alpha|100\rangle + \beta|011\rangle := |\varphi_1\rangle$

$\xrightarrow{\mathbb{1}\otimes X\otimes\mathbb{1}} \alpha|010\rangle + \beta|101\rangle := |\varphi_2\rangle$

$\xrightarrow{\mathbb{1}\otimes\mathbb{1}\otimes X} \alpha|001\rangle + \beta|110\rangle := |\varphi_3\rangle$

Before Errors Occur, we know $Z_1 Z_2 |\bar{\varphi}\rangle = (+1)|\bar{\varphi}\rangle$
$Z_2 Z_3 |\bar{\varphi}\rangle = (+1)|\bar{\varphi}\rangle$

$Z_1 Z_2 (E|\bar{\varphi}\rangle) = (+1)|\bar{\varphi}\rangle$	$Z_2 Z_3 (E|\bar{\varphi}\rangle) = (+1)|\bar{\varphi}\rangle$
$Z_1 Z_2 |\varphi_1\rangle = -|\varphi_1\rangle$	$Z_2 Z_3 |\varphi_1\rangle = +|\varphi_1\rangle$
$Z_1 Z_2 |\varphi_2\rangle = -|\varphi_2\rangle$	$Z_2 Z_3 |\varphi_2\rangle = -|\varphi_2\rangle$
$Z_1 Z_2 |\varphi_3\rangle = |\varphi_3\rangle$	$Z_2 Z_3 |\varphi_3\rangle = -|\varphi_3\rangle$

Every error X_1, X_2, X_3 results in a pair of eigenvalues (measurement outcomes) for Z_1Z_2, Z_2Z_3:

E		measurement of Z_1Z_2 on $E|\bar{\varphi}\rangle$	measurement of Z_2Z_3 on $E|\bar{\varphi}\rangle$	error-syndrome
$\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}$	$\longrightarrow$	$+1$	$+1$	0 0
$X\otimes\mathbb{1}\otimes\mathbb{1}$	$\longrightarrow$	-1	$+1$	1 0
$\mathbb{1}\otimes X\otimes\mathbb{1}$	$\longrightarrow$	-1	-1	1 1
$\mathbb{1}\otimes\mathbb{1}\otimes X$	$\longrightarrow$	$+1$	-1	0 1

- Measuring Z_1Z_2, Z_2Z_3 tells us what the error syndrome is.
  - $\rightarrow$ Measurement does not collapse the logical information because it cannot distinguish between $|\bar{0}\rangle$ and $|\bar{1}\rangle$.

- STEP 1: SYNDROME EXTRACTION (Also called Error detection)

- STEP 2: DECODING (RECOVERY)

- Given the error syndrome, what is the most likely error that could have occurred.

Syndrome	What errors can cause this syndrome?	Which one is the most likely?
0 0	$\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$	
0 1	$\mathbb{1} \otimes \mathbb{1} \otimes X$	
1 0	$X \otimes \mathbb{1} \otimes \mathbb{1}$	
1 1	$\mathbb{1} \otimes X \otimes \mathbb{1}$	

- For every Quantum Error Correcting Code, we will consider these two steps.
- HW: Compute the error-syndromes for the Z_1, Z_2, Z_3 errors on the phase-flip code

CORRECTING BOTH BIT and PHASE ERRORS: SHOR'S CODE

Suppose we have both bit and phase flip errors:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \xrightarrow{1-p} \alpha|0\rangle + \beta|1\rangle$$ No error

$$\xrightarrow{p/3} \alpha|1\rangle + \beta|0\rangle$$ Bit flip error: X

$$\xrightarrow{p/3} \alpha|0\rangle - \beta|1\rangle$$ Phase flip: Z

$$\xrightarrow{p/3} \alpha|1\rangle - i\beta|0\rangle$$ Bit and Phase flip: Y

$$\mathcal{E}(\rho) = (1-p)\,\rho + \frac{p}{3} X \rho X + \frac{p}{3} Z \rho Z + \frac{p}{3} Y \rho Y$$: Depolarizing channel.

Recall that we can correct Bit and Phase flip separately using QEC codes:

(i) $|0\rangle \longrightarrow |000\rangle$, $|1\rangle \longrightarrow |111\rangle$ — Corrects Bit flip

(ii) $|0\rangle \longmapsto |+++\rangle$, $|1\rangle \longmapsto |---\rangle$ — Corrects phase flips

QUESTION: Can we combine these two strategies to correct X and Z errors?

$$|0\rangle \longrightarrow |{+}{+}{+}\rangle = (|0\rangle+|1\rangle)(|0\rangle+|1\rangle)(|0\rangle+|1\rangle)/\sqrt{8}$$

$$\underbrace{|1\rangle \longrightarrow |{-}{-}{-}\rangle = (|0\rangle-|1\rangle)(|0\rangle-|1\rangle)(|0\rangle-|1\rangle)/\sqrt{8}}_{\boxed{\text{Step 1: Protect against phase flip } (Z)}}$$

$$|0\rangle \mapsto |000\rangle$$
$$|1\rangle \mapsto |111\rangle$$

Step 2: Protect against Bit flips

$$|0\rangle \longrightarrow |{+}{+}{+}\rangle = (|0\rangle+|1\rangle)(|0\rangle+|1\rangle)(|0\rangle+|1\rangle)$$

$$\underbrace{|1\rangle \longrightarrow |{-}{-}{-}\rangle = (|0\rangle-|1\rangle)(|0\rangle-|1\rangle)(|0\rangle-|1\rangle)}_{\text{Step 1: Protect against phase flip } (Z)}$$

$$|\bar{0}\rangle = \frac{(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)}{\sqrt{8}}$$

$$|\bar{1}\rangle = \frac{(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)}{\sqrt{8}}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \mapsto |\bar{\psi}\rangle = \alpha|\bar{0}\rangle + \beta|\bar{1}\rangle$$

- Hence the Shor's code is given by:

$$|0\rangle \mapsto |\bar{0}\rangle = \frac{(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)}{\sqrt{8}}$$

$$|1\rangle \mapsto |\bar{1}\rangle = \frac{(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)}{\sqrt{8}}$$

We are using $n=9$ physical qubits to encode $k=1$ logical qubit.

- What are the operators (similar to Z_1Z_2, Z_2Z_3) for which $|\bar{0}\rangle, |\bar{1}\rangle$ are eigenstates? Measuring these operators will specify the syndromes for QEC.

(i) NOTE: Every subset of 3 qubits is like the phase flip code:

$$Z \otimes Z \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \quad : \quad Z_1 Z_2$$
$$1 \otimes Z \otimes Z \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \quad : \quad Z_2 Z_3$$
$$1 \otimes 1 \otimes 1 \otimes Z \otimes Z \otimes 1 \otimes 1 \otimes 1 \otimes 1 \quad : \quad Z_4 Z_5$$
$$1 \otimes 1 \otimes 1 \otimes 1 \otimes Z \otimes Z \otimes 1 \otimes 1 \otimes 1 \quad : \quad Z_5 Z_6$$

$$\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes Z\otimes Z\otimes\mathbb{1} \quad : Z_7 Z_8$$
$$\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes \mathbb{1}\otimes Z\otimes Z \quad : Z_8 Z_9$$

Idea: If we flip the sign of two qubits : $|1\rangle \mapsto -|1\rangle$, in the same block, the state is unchanged.

(ii) If we flip all the $|0\rangle$'s in a block to $|1\rangle$'s then $|\bar{1}\rangle \mapsto -|\bar{1}\rangle$, but if we flip all $|0\rangle$'s <u>in a pair of</u> blocks, then $|\bar{1}\rangle \mapsto |\bar{1}\rangle$, and $|\bar{0}\rangle \mapsto |\bar{0}\rangle$.

$$X\otimes X\otimes X\otimes X\otimes X\otimes X\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1} \quad : X_1X_2X_3X_4X_5X_6$$
$$\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes X\otimes X\otimes X\otimes X\otimes X \quad : X_4X_5X_6X_7X_8X_9$$

- Hence, there are in total 8 operators for which $\alpha|\bar{0}\rangle+\beta|\bar{1}\rangle$ is an eigenstate.

Z_1Z_2

Z_2Z_3

Z_4Z_5

Z_5Z_6

Z_7Z_8

Z_8Z_9

$X_1X_2X_3X_4X_5X_6$

$X_4X_5X_6X_7X_8X_9$

- Let us look at the effect of Errors on the Shor Code, and syndromes:

$|\psi\rangle = \alpha|0\rangle+\beta|1\rangle$

$\downarrow$

$\alpha|\bar{0}\rangle+\beta|\bar{1}\rangle = |\bar{\psi}\rangle$

X_1: $\left.\begin{aligned}&\alpha(|100\rangle+|011\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)\\ &+\beta(|100\rangle-|011\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)\end{aligned}\right\} = |\psi_1\rangle$

$Z_1Z_2|\psi_1\rangle = -|\psi_1\rangle$

$Z_2Z_3|\psi_1\rangle = |\psi_1\rangle$

$\vdots$

Syndrome of $|\psi_1\rangle$ = 10000000

Z_2: $\alpha(|000\rangle-|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)$
$+\beta(|000\rangle+|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)$

X_4: $\alpha(|000\rangle+|111\rangle)(|100\rangle+|011\rangle)(|000\rangle+|111\rangle)$
$+\beta(|000\rangle-|111\rangle)(|100\rangle-|011\rangle)(|000\rangle-|111\rangle)$

Z_6: $\left.\begin{aligned}&\alpha(|000\rangle+|111\rangle)(|000\rangle-|111\rangle)(|000\rangle+|111\rangle)\\ &+\beta(|000\rangle-|111\rangle)(|000\rangle+|111\rangle)(|000\rangle-|111\rangle)\end{aligned}\right\} |\psi_6\rangle$:

$Z_1Z_2|\psi_6\rangle = |\psi_6\rangle$

$Z_4Z_5|\psi_6\rangle = |\psi_6\rangle$

$Z_5Z_6|\psi_6\rangle = |\psi_6\rangle$

$X_1X_2X_3X_4X_5X_6|\psi_6\rangle = -|\psi_6\rangle$

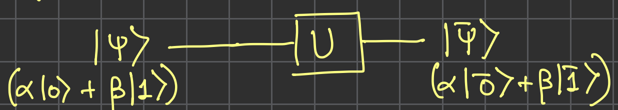
X_7: $\alpha(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|100\rangle+|011\rangle)+\ldots$

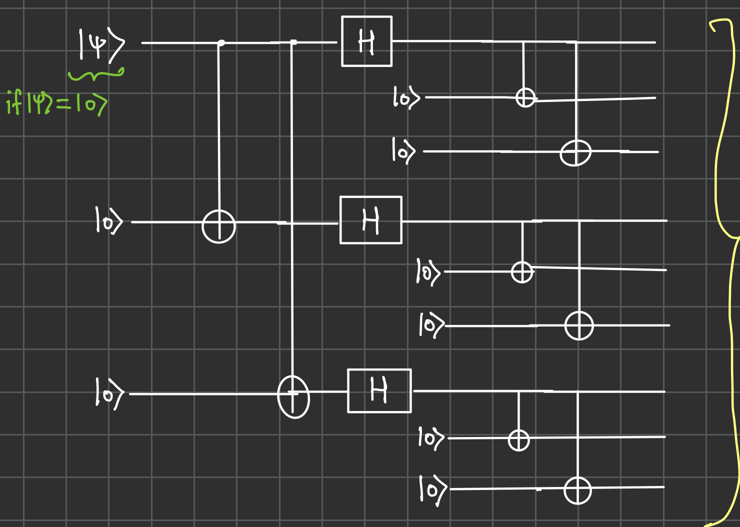
$\vdots$

Error	Z_1Z_2	Z_2Z_3	Z_4Z_5	Z_5Z_6	Z_7Z_8	Z_8Z_9	$X_1X_2X_3X_4X_5X_6$	$X_4X_5X_6X_7X_8X_9$	Syndrome
X_1	-1	1	1	1	1	1	1	1	10000000
Z_2									
X_4									
Z_6	1	1	1	1	1	1	-1	1	00000010
X_7									
⋮									

- Decoding: Given the error syndrome, determine the error that caused it.
- Table of syndromes and the most likely error: Decoder.

Preparing the encoded states: Encoding circuit : U





$$|\bar{\psi}\rangle = \frac{(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)}{\sqrt{8}}$$

Projector onto the code space:

$$P = |\bar{0}\rangle\langle\bar{0}| + |\bar{1}\rangle\langle\bar{1}|$$

Check: $P|\bar{\psi}\rangle = |\bar{\psi}\rangle$ for encoded states and $P|\psi\rangle = 0$ for any states not in Shor Code

THE 5-QUBIT CODE

- The Shor's code uses 9 qubits to encode a single logical qubit to protect against any single qubit errors?
- Can we still correct all single qubit errors using less than 9 qubits in a QEC?

$$|0\rangle \to |\bar{0}\rangle = \frac{1}{4}\Big[|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle - |11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle - |10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle\Big]$$

$$|1\rangle \to |\bar{1}\rangle = \frac{1}{4}\Big[|11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle + |10101\rangle - |00100\rangle - |11001\rangle - |00111\rangle - |00010\rangle - |11100\rangle - |00001\rangle - |10000\rangle - |01110\rangle - |10011\rangle - |01000\rangle + |11010\rangle\Big]$$

The encoded states : $|\Psi\rangle = \alpha|\bar{0}\rangle + \beta|\bar{1}\rangle$ are eigenstates with +1 eigenvalue of:

X Z Z X I
I X Z Z X
X I X Z Z
Z X I X Z

(Note: they are cyclic permutations)

- The 5-qubit code can correct all single qubit errors.
- Question: What are the single qubit errors and their syndromes? We will see this in our next class.