November 15, 2023

Logistical information:

(a) Name: Pavithran Iyer (PAVI). Work at Xanadu Quantum Technologies.

(b) Email: pavithran.sridhar@gmail.com

(c) Time zone: Toronto (GMT − 5 hrs), IST — 10 hrs 30 min

(d) 8 lectures

(i) Classical error correction and introducing the bit and phase flip quantum error correcting codes.

(ii) The Shor's code and the 5-qubit code, which correct arbitrary errors on a single qubit.

(iii) Knill-Laflamme error correction conditions. Error discretization theorem.

(iv) Stabilizer formalism for quantum error correction. Structure of the stabilizer group with respect to a stabilizer group. Errors, syndromes and logical operators.

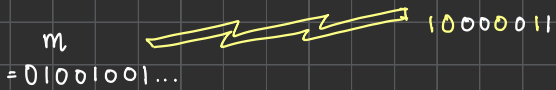
(v) The decoding problem. Logical error rate and the idea of a threshold.

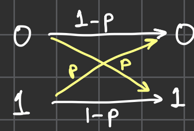
(vi) Fault tolerant schemes for syndrome extraction. Shor's scheme for syndrome extraction.

(vii) Steane error correction scheme and Knill's quantum error correction.

(viii) Family of quantum error correcting codes of increasing physical qubits.

Correcting Errors in classical information:





Binary symmetric channel

Simplest Solution: repeat every bit 'n' times (say $n=3$)

$$0 \mapsto 000$$
$$1 \mapsto 111$$

$$m \mapsto \underbrace{000}_{}\ \underbrace{111}_{}\ \underbrace{000}_{}\ \underbrace{000}_{}\ \underbrace{111}_{}\ \underbrace{000}_{}\ \underbrace{000}_{}\ \underbrace{111}_{}$$
$$\overline{m} = \ \ \overline{0}\ \ \ \overline{1}\ \ \ \overline{0}\ \ \ \overline{0}\ \ \ \overline{1}\ \ \ \overline{0}\ \ \ \overline{0}\ \ \ \overline{1}$$

Idea: If some of the bits in the repeated sequence are flipped, we will use the others to guess its original value. MAJORITY VOTE

$$000 \to 000,\quad 000 \xrightarrow{p} 010,\quad 000 \xrightarrow{p} 001,\quad 000 \xrightarrow{p} 100$$

$$111 \to 111,\quad 111 \xrightarrow{p} 110,\quad 111 \xrightarrow{p} 101,\quad 111 \xrightarrow{p} 011$$

By seeing that the "majority" are 0, we know that all these come from 000.

majority are 1, so they come from 111.

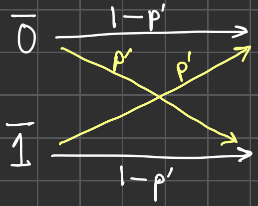
What have we achieved by this process?

- Although information can get corrupted with probability p, encoded information gets corrupted with probability $\ll p$.
- Compute the probability of error correction method failing.

$$000 \mapsto \left.\begin{matrix}000\\001\\010\\100\end{matrix}\right\} \to \overline{0} = 000 \text{ with probability } (1-p)^3 + 3p(1-p)^2$$

$$\left.\begin{matrix}110\\011\\101\\111\end{matrix}\right\} \to \overline{1} = 111 \text{ with probability } 3p^2(1-p) + p^3$$

NOTE: $\mathbb{P}(\text{information corruption}): p \mapsto 3p^2(1-p)+p^3$ ✓



where $p' = 3p^2(1-p) + p^3$

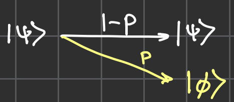
Terminology: p' = probability of QEC to fail.
$\equiv$ LOGICAL ERROR RATE

Homework: • Calculate the logical error rates for n repetitions? If not, then for $n=5$?
• When (for what p) is it beneficial to encode?)

QUANTUM CASE:

Recall that
- Information is encoded in Quantum states: $|\psi\rangle\langle\psi| = \rho$
- Error is any operation that maps states to states: $\mathcal{E}(\rho) \equiv$ state after the error.

Inspired from the Classical solution:



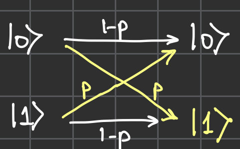
Some noise that maps $|\psi\rangle$ to $|\phi\rangle$ with probability 'p' and leaves it invariant with probability $(1-p)$.

Can we repeat every state in our register? $|\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle \otimes \ldots \otimes |\psi\rangle$?

$\to$ No: No cloning theorem. Only orthogonal states can be cloned. $|0\rangle \mapsto |000\rangle$, $|1\rangle \mapsto |111\rangle$

$\to$ We need a different method of encoding with respect to a basis.

Quantum Bit flip channel: Correcting for only 'X' errors



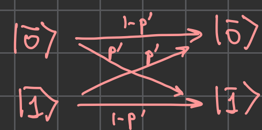
Applies a 'X' with probability p:
Bit flip channel
$\mathcal{E}(\rho) = (1-p)\,\rho + p\,X\rho X$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$\downarrow$

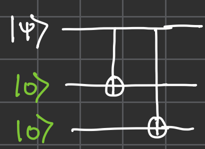
$\underbrace{|\bar{\psi}\rangle}_{\text{Encoding of } |\psi\rangle} = \alpha\underbrace{|000\rangle}_{|\bar{0}\rangle = |000\rangle} + \beta\underbrace{|111\rangle}_{|\bar{1}\rangle = |111\rangle}$

Terminology:
Encoded states: $|\bar{0}\rangle = |000\rangle$, $|\bar{1}\rangle = |111\rangle$
Bit flip code: Span$\{|\bar{0}\rangle, |\bar{1}\rangle\}$.



Logical error rate:
$p' = 3p^2(1-p) + p^3$.

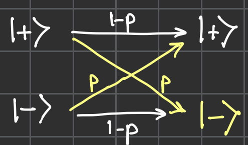
Encoding circuit for 3-qubit bit flip code :



$$|0\rangle \mapsto |000\rangle$$
$$|1\rangle \mapsto |111\rangle$$

Remark: $|\bar{0}\rangle$ and $|\bar{1}\rangle$ are also eigenstates of Z_1Z_2, Z_2Z_3

Quantum Phase flip channel : correcting for only 'Z' errors.



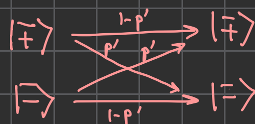
Applies a 'Z' with probability p:

Bit flip channel

$$\mathcal{E}(\rho) = (1-p)\,\rho + p\, Z \rho Z$$

$$\underbrace{|\psi\rangle}_{\downarrow} = \alpha \underbrace{|+\rangle}_{\downarrow} + \beta|-\rangle$$

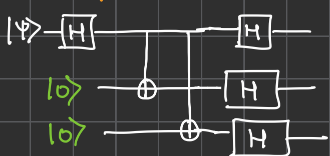
$$\underbrace{|\bar{\psi}\rangle}_{\text{Encoding of } |\psi\rangle} = \alpha \underbrace{|+++\rangle}_{|\bar{+}\rangle = |+++\rangle} + \beta \underbrace{|---\rangle}_{|\bar{-}\rangle = |---\rangle}$$



Logical error rate:

$$p' = 3p^2(1-p) + p^3.$$

Encoding circuit for 3-qubit phase flip code :



$$|+\rangle \mapsto |+++\rangle$$
$$|-\rangle \mapsto |---\rangle$$

Remark: $|\bar{+}\rangle$ and $|\bar{-}\rangle$ are also eigenstates of X_1X_2, X_2X_3.