Bell Inequality

If one notices carefully, the thought experiment given in the last section is based on the assumptions known as locality and realism. Therefore, before the derivation of Bell inequality we shall look at the formal definitions of these concepts.

Locality According to the principle of locality, *"A physical process occurring at one place cannot have any influence on other processes occurring at locations outside its light cone"*. This principle is based on special relativity which does not allow any type of information transfer greater than the speed of light.

Realism The concept of realism can be stated as, *"physical objects and their properties pre-exist without the influence of an observer"*. Therefore, it means that physical quantities have pre-defined values which are independent of the measurement process. In this case, one can also identify pre-existing properties of the objects as element of reality in EPR arguments.

Now, to derive the Bell inequality we shall approximately follow, Bell's 1971 argument [26] given in [27]. Consider a source which emits a system consisting of two-component that we call particles for simplicity. Here, we are only concerned with properties of the particles that form a 2-D quantum system and therefore, the possible outcomes of some measurement are represented as ± 1. Suppose after emission each of these particle comes into the possession of two parties that we name as Alice and Bob. Moreover, consider that their measurement devices can perform measurements with different settings. These settings could be different directions for spin measurements in case of spin system but we can keep the discussion general. We denote these settings as A_i and B_i for first and second particle respectively.

Now, suppose that quantum mechanics is not complete and there exist some other variables, which may determine the properties of the particle under consideration. These variables are usually called hidden variables, since these are unknown to us now but could be discovered in the future. We represent all such variables collectively with a parameter λ. These variables could be characteristic features of the source generating the two particles and may also belong to a probability space Λ, (i.e. $\lambda \in \Lambda$) from which it is sampled through some probability distribution $P(\lambda)$, during each emission. Therefore,

$$\int_{\Lambda} P(\lambda)\, d\lambda = 1 \tag{2.20}$$

where $P(\lambda) \geq 0$. Then, the outcomes of both Alice and Bob measurement, will be a consequence of their settings and the sampled parameter λ, which will also be responsible for any kind of correlation between the outcomes.

Suppose further that the two particles fly apart and arrive at the two different laboratories, where Alice and Bob are ready to measurement with their devices as described above. We denote Alice measurement outcome by $a(A_0, \lambda) = \pm 1$, since it can depend on the controllable parameter (measurement's setting) A_1, and hidden variables given by λ. Similarly, Bob outcome is denoted by $b(B_0, \lambda) = \pm 1$. Note that, here we have utilized the locality assumption, because, we assumed that Alice's outcome $a(A_0, \lambda)$ is dependent only on her own setting A_0 but not on Bob's setting B_0, which should be true according to the principle of locality if Alice and Bob are space-like separated while choosing their settings. The same is true for Bob also. It should also be noted here that we are assuming a deterministic hidden variable model, since parameters $a(A_0, \lambda)$ and $b(B_0, \lambda)$ are not probabilities. Now, the correlation between the measurements of Alice and Bob can be represented by a correlation function of the form

$$E(A_0, B_0) = \int_{\Lambda} a(A_0, \lambda)\, b(B_0, \lambda)\, P(\lambda)\, d\lambda \tag{2.21}$$

Suppose that Alice and Bob both perform measurement with two settings denoted as A_0 and A_1 for Alice and B_0 and B_1 for Bob. Then

$$\begin{aligned}
E(A_0,B_0)-E(A_0,B_1) &= \int_\Lambda [a(A_0,\lambda)\, b(B_0,\lambda) - a(A_0,\lambda)\, b(B_1,\lambda)]\ P(\lambda)\, d\lambda \\
&= \int_\Lambda a(A_0,\lambda)\, b(B_0,\lambda)\,[1 \pm a(A_1,\lambda)\, b(B_1,\lambda)]\ P(\lambda)\, d\lambda \\
&- \int_\Lambda a(A_0,\lambda)\, b(B_1,\lambda)\,[1 \pm a(A_1,\lambda)\, b(B_0,\lambda)]\ P(\lambda)\, d\lambda
\end{aligned}$$

A careful reader might notice here that we have quietly used the assumption of realism by considering that the parameters λ do not depend on the setting one choose i.e. say, Alice can choose between setting A_0 or A_1 without affecting λ[1]. Also note that since, $a(A_0,\lambda) = \pm 1$ and $b(B_0,\lambda) = \pm 1$, Therefore, $|a(A_0,\lambda)| = 1$ and $|b(B_0,\lambda)| = 1$. But for our derivation we will use even weaker restriction given as,

$$|a(A_0,\lambda)| \leq 1 \qquad\qquad |b(B_0,\lambda)| \leq 1 \tag{2.22}$$

and, of course, the outcomes of the setting represented by A_1 and B_1 are also assumed to have similar restrictions. Now using these restriction and the triangle inequality[2] we can write

$$\begin{aligned}
|E(A_0,B_0)-E(A_0,B_1)| &\leq \left|\int_\Lambda a(A_0,\lambda)\, b(B_0,\lambda)\,[1 \pm a(A_1,\lambda)\, b(B_1,\lambda)]\ P(\lambda)\, d\lambda\right| \\
&+ \left|\int_\Lambda a(A_0,\lambda)\, b(B_1,\lambda)\,[1 \pm a(A_1,\lambda)\, b(B_0,\lambda)]\ P(\lambda)\, d\lambda\right| \\
&\leq \int_\Lambda [1 \pm a(A_1,\lambda)\, b(B_1,\lambda)]\ P(\lambda)\, d\lambda \\
&+ \int_\Lambda [1 \pm a(A_1,\lambda)\, b(B_0,\lambda)]\ P(\lambda)\, d\lambda \\
&= 2 \pm [E(A_1,B_1) + E(A_1,B_0)]
\end{aligned}$$

Note that here we have used the fact that

$$\begin{aligned}
&[1 \pm a(A_1,\lambda)\, b(B_1,\lambda)] \geq 0 \\
\text{and}\quad &[1 \pm a(A_1,\lambda)\, b(B_0,\lambda)] \geq 0
\end{aligned}$$

meaning that these are positive numbers for which $|x| = x, \forall\, x \in \mathbb{R}^+$ is true. Now since for any choice of the sign, the right most side of the inequality is greater or equal to the left side, so we can write it as

$$|E(A_0,B_0)-E(A_0,B_1)| + |E(A_1,B_1)+E(A_1,B_0)| \leq 2 \tag{2.23}$$

[1] We also assume that they have *free will* to choose the settings they want and there is nothing which force them to choose certain setting during a run

[2] The form of triangle inequality used is $|x+(-y)| \leq |x| + |-y| \ \ \forall\, x \in \mathbb{R}$.

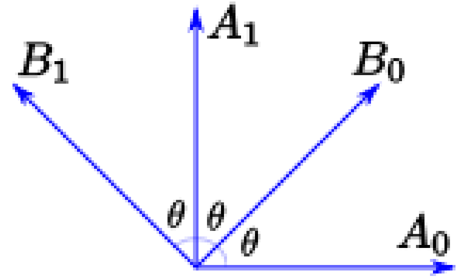

Figure 2.2: A simplest possibility for the four setting that Alice and Bob can choose to violate Bell inequality

Now, once again with the application of triangle inequality we arrive to the final result,

$$|E(A_0,B_0) - E(A_0,B_1) + E(A_1,B_1) + E(A_1,B_0)| \leq 2 \tag{2.24}$$

This can also be written in the standard form as

$$-2 \leq E(A_0,B_0) - E(A_0,B_1) + E(A_1,B_1) + E(A_1,B_0) \leq 2 \tag{2.25}$$

To see if quantum mechanics violates this inequality, consider the state given in Eq. (2.19) and define Alice and Bob settings A_0 and B_0 as directions in the Bloch sphere given by $\hat{\sigma}_{A_0} = \vec{r}_{A_0}.\vec{\sigma}$ and $\hat{\sigma}_{B_0} = \vec{r}_{B_0}.\vec{\sigma}$ respectively. It can be shown that

$$E(A_0,B_0) = \langle\psi|\hat{\sigma}_{A_0} \otimes \hat{\sigma}_{B_0}|\psi\rangle = -cos\ (\theta_{A_0B_0}) \tag{2.26}$$

This will be easy to follow, if one notices that the state given in Eq. (2.19), is rotationally invariant and therefore one can rotate his reference frame such that $\vec{r}_{A_0}$ becomes the z-direction. Now, to calculate all the parameters in expression (2.24), consider the four setting A_0, A_1, B_0 and B_1 define as in Fig. 2.2. In this case expression (2.24) becomes

$$|-cos\ \theta + cos\ 3\theta - cos\ \theta - cos\ \theta| \leq 2. \tag{2.27}$$

The reader can check that for $\theta = 45°$ we will get $2\sqrt{2}$, much higher than what can be obtained by a local realist hidden variables theory. Note that this is the maximum violation that quantum mechanics can offer and it is known as Tsirelson bound [28]. Since the invention of Bell inequalities, numerous experiment have violated it, some can be found here [9–11; 13–15].

The proof that quantum mechanics is in conflict with local hidden variables assumption is known as Bell theorem and this particular form of the Bell inequality is called as CHSH–J. Clauser, M. Horne, A. Shimony and R. A. Holt–inequality [6].