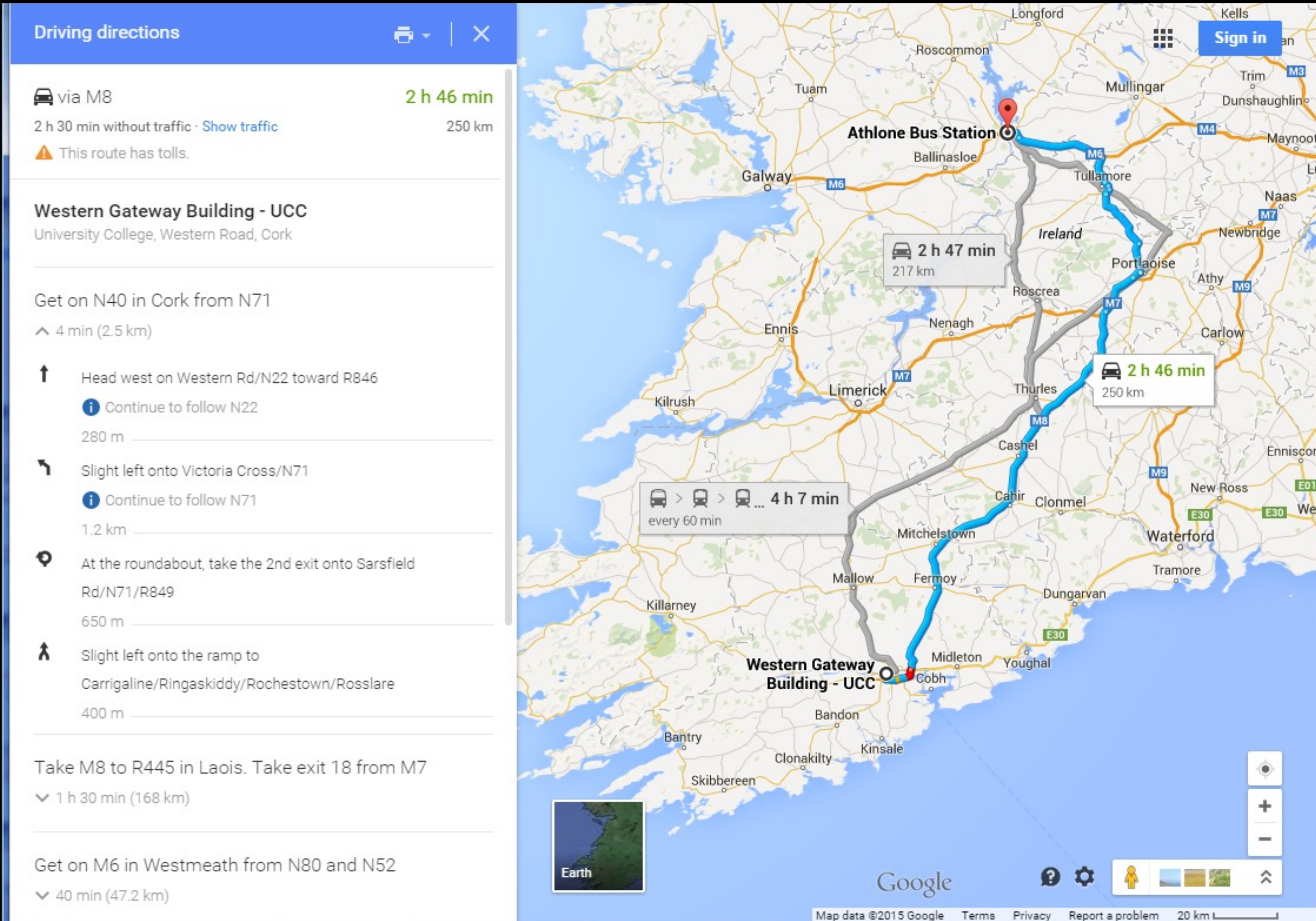
Shortest Paths in Weighted Graphs



A *path* from v_0 to v_k in a simple graph G is a sequence of vertices

$$<v_0, v_1, v_2, ..., v_k>$$

such that (v_i,v_{i+1}) is a oriented edge in G for each i from 0 to k-1.
If G is a *directed* graph, then the orientation of the edge must be the same as the direction.

The length of the path is the number of vertices - 1.

Equivalently, a path from v_0 to v_k in a graph is a sequence of oriented edges

$$<(v_0, v_1), (v_1, v_2), ..., (v_{k-2}, v_{k-1}), (v_{k-1}, v_k)>$$

The length of the path is the number of edges.

In a weighted graph, we have a function *w* which maps from the set of edges to some other set, assigning a *weight* to each edge.

$$w : E \rightarrow S$$

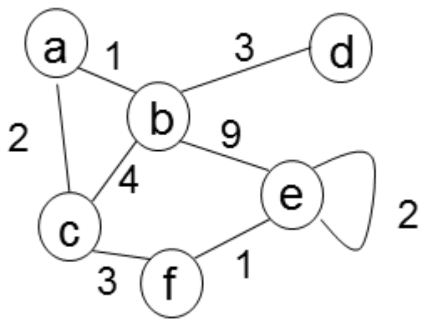
If the output set S is the integers, or other numerical set, then *w* can be viewed as the cost of travelling the edge, or distance across the edge, or time required to cross the edge, etc.

If we have a path made up of weighted edges where the weights are from some numerical set, then we can define the cost of the path.

If P is a path from v_0 to v_k, then the cost of the path is

$$\sum_{i=0}^{k-1} w(v_i, v_{i+1})$$

Path costs



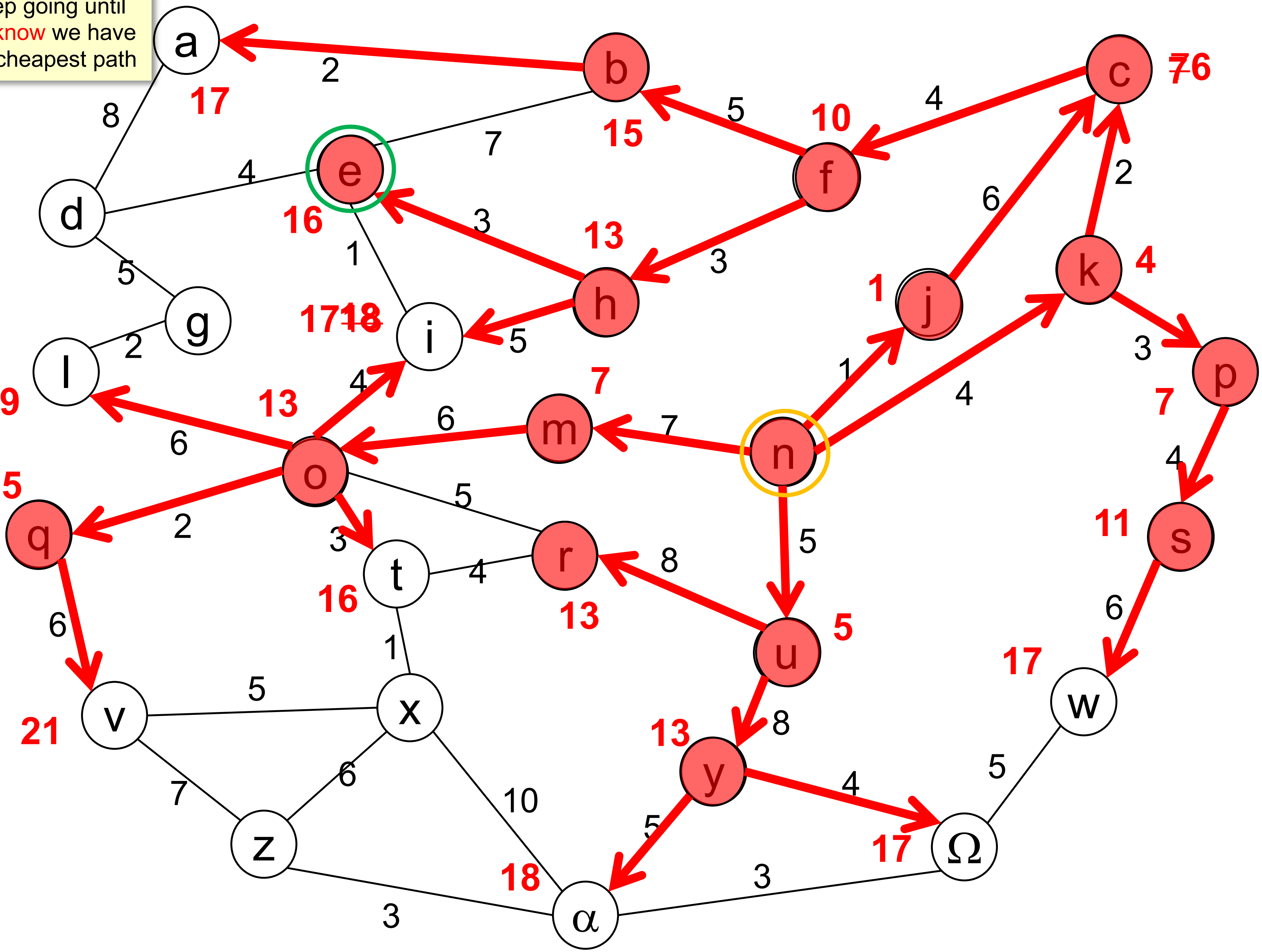
G = (V,E), where
V = {a,b,c,d,e,f}
E = {({a,b},1), ({a,c},2), ({b,c},4), ({b,d},3), ({b,e},9), ({c,f},3), ({e,e},2), ({e,f},1)}

Path <a,b,c,f>:	<(a,b),	(b,c),	(c,f)>		Total cost
Edge costs:	1	4	3		8

What is the cost of <f,e,e,b>?

Can we write an algorithm that computes the cheapest path between any pair of vertices?

keep going until we know we have the cheapest path



Adapt the algorithm from CS1113 so that we continue until we have found the shortest path from our start vertex to all other vertices that are reachable

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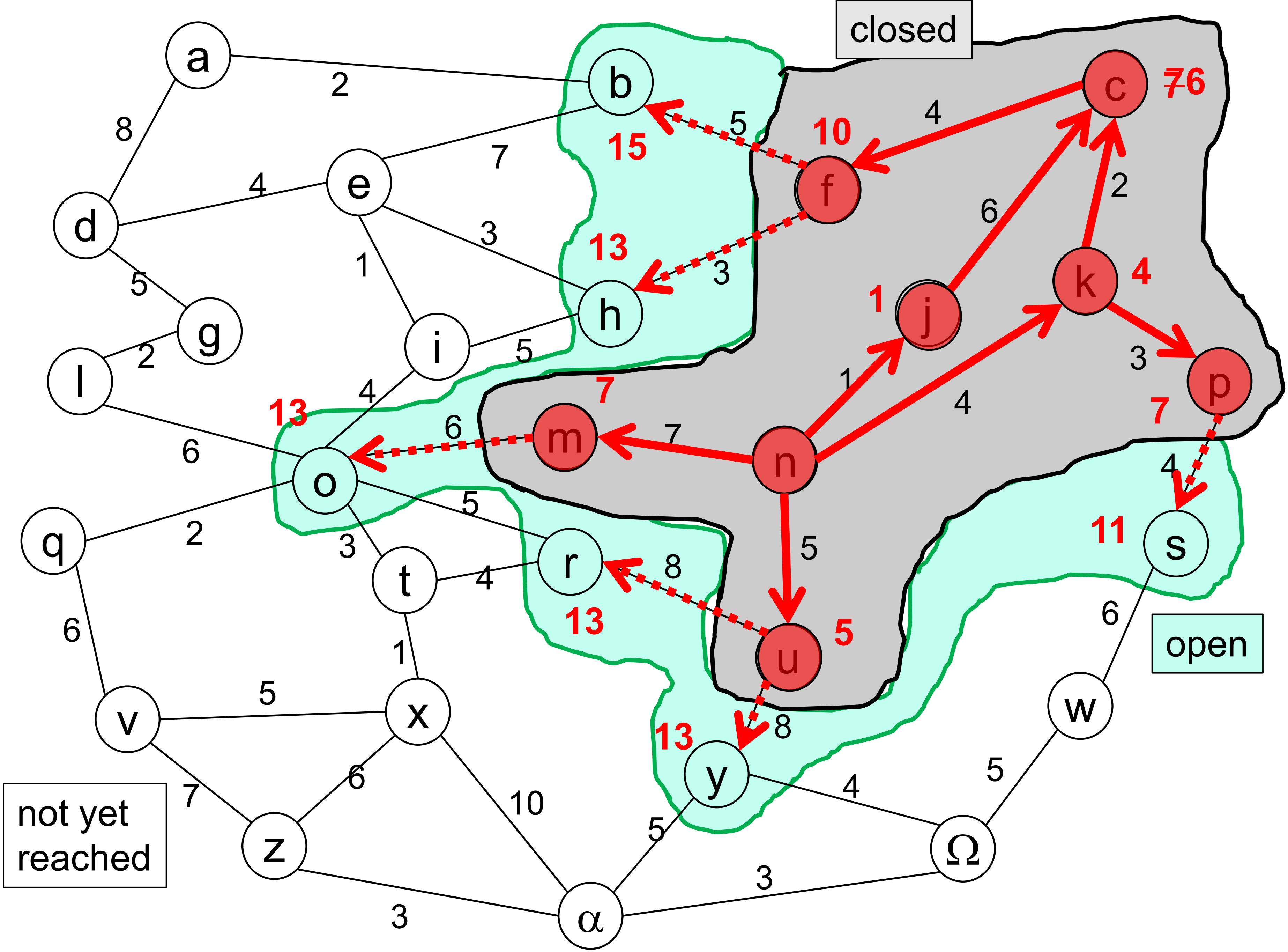
while we still have open vertices
    pick the open vertex with lowest cost
    expand from that open vertex to all neighbours not yet closed
    for each neighbour we reached
        add the cost of the edge to the neighbour onto the cost for the
                            open vertex to get a new path cost to the neighbour
        if the path cost is lower than the previous one, or neighbour is new
            update the cost of the neighbour to be the path cost
    close the current vertex
```

How should we implement this in Python, using our ADT implementations?

What we need to do ...	Which data structures?
Maintain the original graph in an implementation that allows fast lookups of the neighbours of a given vertex	Adjacency map or adjacency list implementation
Final output should be a structure we can query which has, for each vertex, a cost and the previous vertex on its shortest path	Dictionary with vertices as keys, and values are (cost, predecessor) pairs
Maintain the path costs for all open vertices in a structure we can query, obtain the minimum cost vertex efficiently, and update with new costs efficiently	APQ for open vertices, key is the cost, value is the vertex Dictionary of locations for accessing elements in the APQ



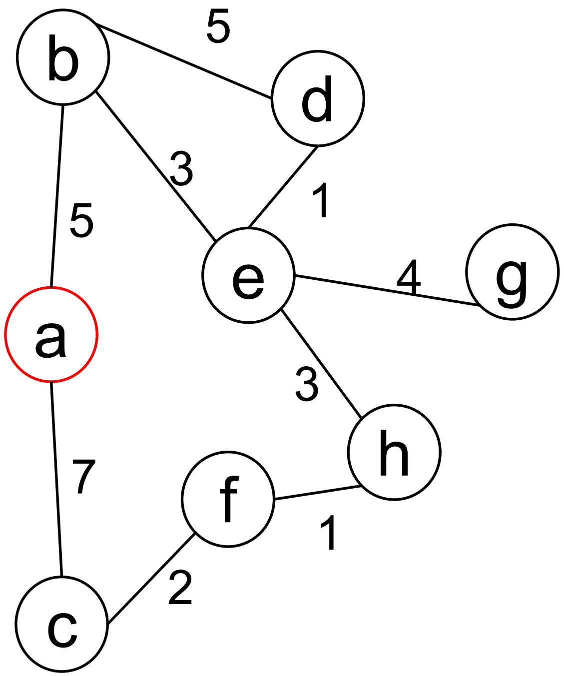
closed
open
not yet reached
a
b
c
d
e
f
g
h
i
j
k
l
m
n
o
p
q
r
s
t
u
v
w
x
y
z
α
Ω
~~7~~6
10
15
13
1
4
7
13
7
5
13
11
13

dijkstra(*s*): find all shortest paths from *s*   #pseudocode

open starts as an empty APQ
locs is an empty dictionary (keys are vertices, values are location in *open*)
closed starts as an empty dictionary
preds starts as a dictionary with value for *s* = None

add *s* with APQ key 0 to *open*, and add s:(elt returned from APQ) to *locs*
while *open* is not empty
    remove the min element *v* and its cost (key) from *open*
    remove the entry for *v* from *locs* and *preds* (which returns predecessor)
    add an entry for *v*:(cost, predecessor) into *closed*
    for each edge *e* from *v*
        w is the opposite vertex to *v* in *e*
        if *w* is not in *closed*
            newcost is *v*'s key plus *e*'s cost
            if *w* is not in *locs //i.e. not yet added into open*
                add *w:v* to *preds*,
                add *w:newcost* to *open*
                add *w:(elt* returned from *open)* to *locs*
            else if *newcost* is better than *w*'s *oldcost*
                update *w:v* in *preds*, update w's cost in *open* to *newcost*
return *closed*

Run Dikstra to compute shortest paths from *a* to all other vertices



closed:

preds:

open:

locs:

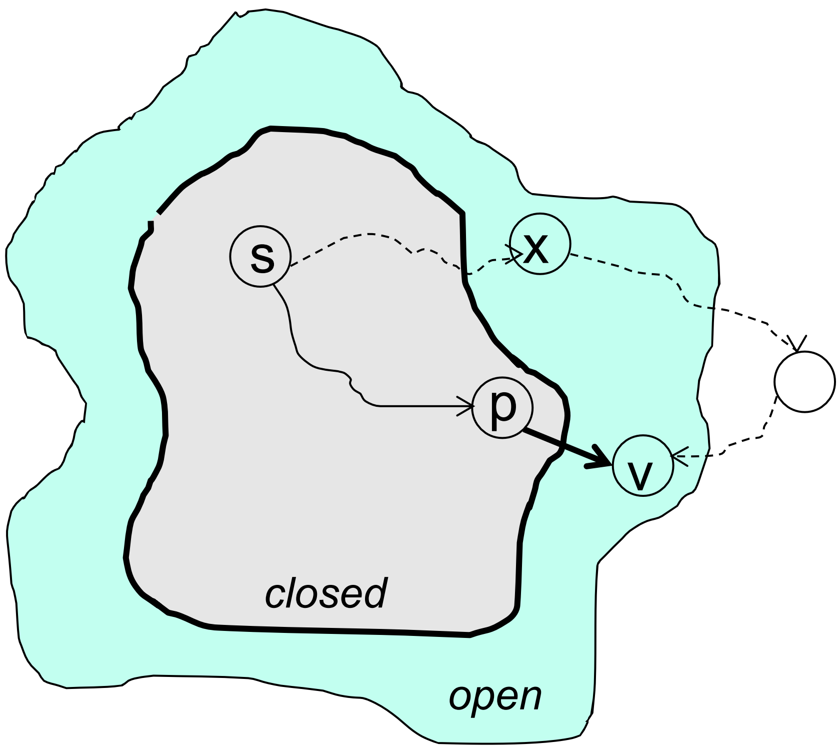
Is Dijkstra's algorithm correct?

As long as the graph does not contain a negative weight, when a vertex v is added to 'closed', its path cost is the cost of the shortest path from s to v

Proof:

3. The path from *s* to *x* must be less than the path from *s* to *v* (since this is supposed to be a shorter path to *v* than the one via *p*). But *v* was the item in *open* with the lowest path cost.

Contradiction.



2. the supposed shorter path. *x* is the first vertex on the 'shorter path' not already in *closed*. *x* must be in *open* (since *x*'s predecessor is in *closed*, and we expanded one hop out from every vertex in *closed*), so we know a path cost from *s* to *x*.

1. About to add *v* into *closed*.

(so *v* is the item in *open* with lowest discovered path cost from *s*) Suppose there is another path to *v* in the full graph that is shorter than this current one via *p* ...

Is Dijkstra's algorithm correct?

As long as the graph does not contain a negative weight, when a vertex *v* is added to '*closed*', its path cost is the cost of the shortest path from *s to v*

Proof:

We can recreate the path by stepping backwards over the predecessors. Let P be this path, and all vertices in P (except *v*) must be in *closed*. Suppose the path cost of P is not the lowest possible for *v*. Let Q be a path with a lower cost. If all the vertices in Q are in closed, then the last vertex before *v* in Q would have led to *v* with Q's cost when it was closed. Contradiction.
Therefore there must be a vertex in Q not in closed. Let *x* be the first vertex in Q not in closed. Since the cost of Q is less than the cost of P, the cost of the path to *x* must be less than the cost of P. But then *x* should have been removed from '*open*' before *v*. Contradiction.

Therefore the path P from *s* to *v* must be the shortest path.

Worst case running time of Dijkstra

Each vertex added into *open* at most once, and taken out of *open* at most once. So do the outer loop *n* times.
Each time round the loop, remove min from the APQ.
Delete an entry from the *locs* dictionary (hash map)
Add an entry into the *closed* dictionary
Go round the inner loop once per edge (amortised)
Inside inner loop: add to APQ or update it, and update the *predecessors* dictionary

So we have:

- n additions to the APQ
- n removals of min key item from APQ
- n deletions from the dictionary
- n additions to a dictionary
- m updates to a dictionary
- m updates (or additions) to the APQ

```
dijkstra(s): find all shortest paths from s

open starts as an empty APQ
locs is an empty dictionary
closed starts as an empty dictionary
preds is a dict with value for s = None

add s with APQ key 0 to open
add s:(elt returned from APQ) to locs
while open is not empty
   remove min element v and its cost from open
   remove entry for v from locs  and preds
   add entry for v:(cost, pre) into closed
   for each edge e from v
      w is the opposite vertex to v  in e
      if w is not in closed
         newcost is v's key plus e's cost
         if w is not in locs //i.e. not yet inopen
            add w:v to preds,
            add w:newcost to open
            add w:(elt returned from open) to locs
         else if newcost is better than w's oldcost
            update w:v in preds
            update w's cost in open to newcost
return closed
```

Dictionary updates are O(1) expected (and could be O(1) if we used a list of fixed length), so the complexity will depend on the APQ.

Dijkstra APQ: heap vs unsorted

Complexity of each operation

Heap APQ		Unsorted APQ
O(log n)	n additions to the APQ	O(1)
O(log n)	n removals of min key item from APQ	O(n)
O(log n)	m updates (or additions) to the APQ	O(1)

Heap APQ

so total is O(n log n + m log n)
which is O((n+m)log n).

If graph is dense: $O(n^2 \log n)$
If graph is sparse: O(n log n)

Unsorted APQ

so total is $O(n + n^2 + m)$
which is $O(n^2 + m)$.

If graph is dense: $O(n^2)$
If graph is sparse: $O(n^2)$

If number of edges $m << n^2/\log n$, then prefer the heap APQ

If number of edges $m >> n^2/\log n$, then prefer the unsorted list APQ

Next lecture

Further graph algorithms