Balanced BSTs (AVL Trees)

Different Binary Search Trees from same data

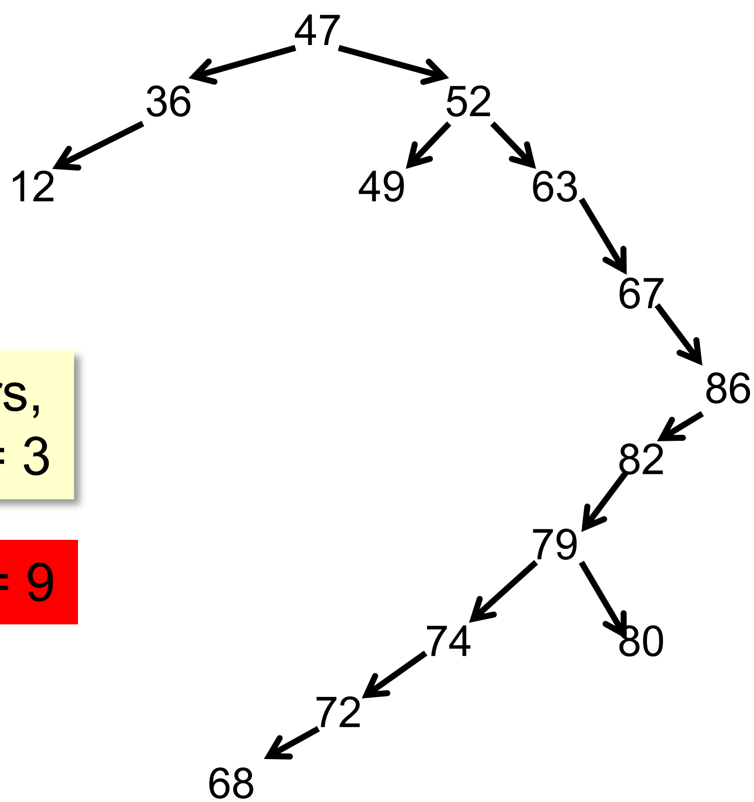
Rotating nodes in Binary Search Trees

Balanced Binary Search Trees
(or AVL Trees)

BST structure affects search …

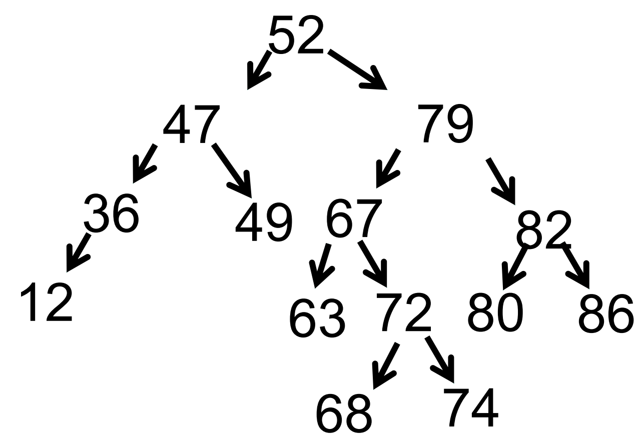
Our Binary Search Tree methods for adding and removing nodes tried to do as little work as possible for each individual operation.

But the structure of the BST has a significant impact on search -- search in a BST has complexity O(height of tree).



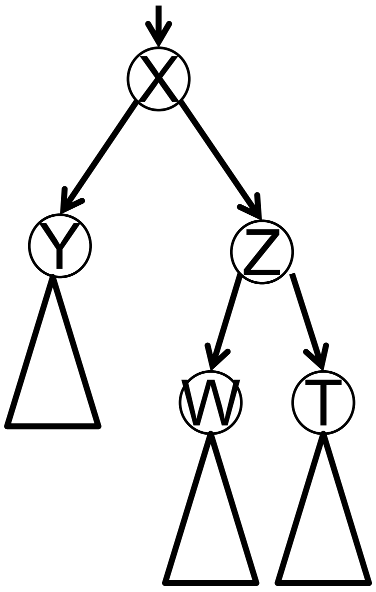
14 numbers, so $\lfloor \log n \rfloor = 3$

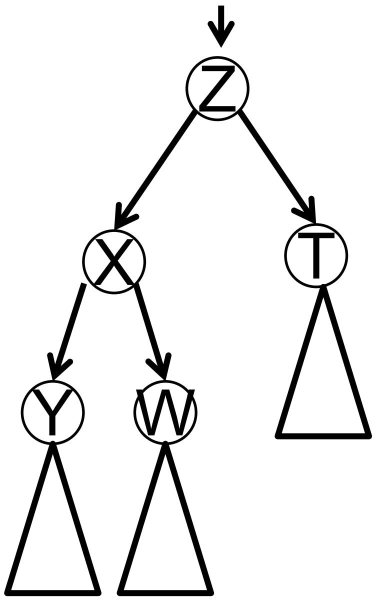
but depth = 9



this BST has the same content, but its height is only 4

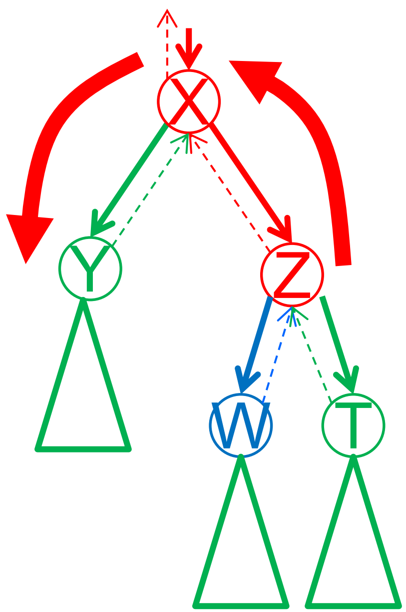
Alternative BSTs

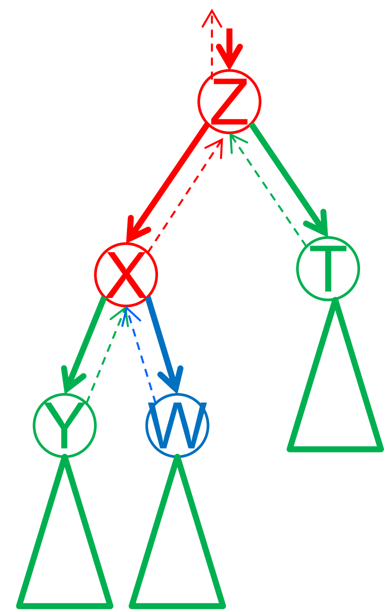




Y (and subtree) < X < W (and subtree) <Z < T (and subtree)

Rotation





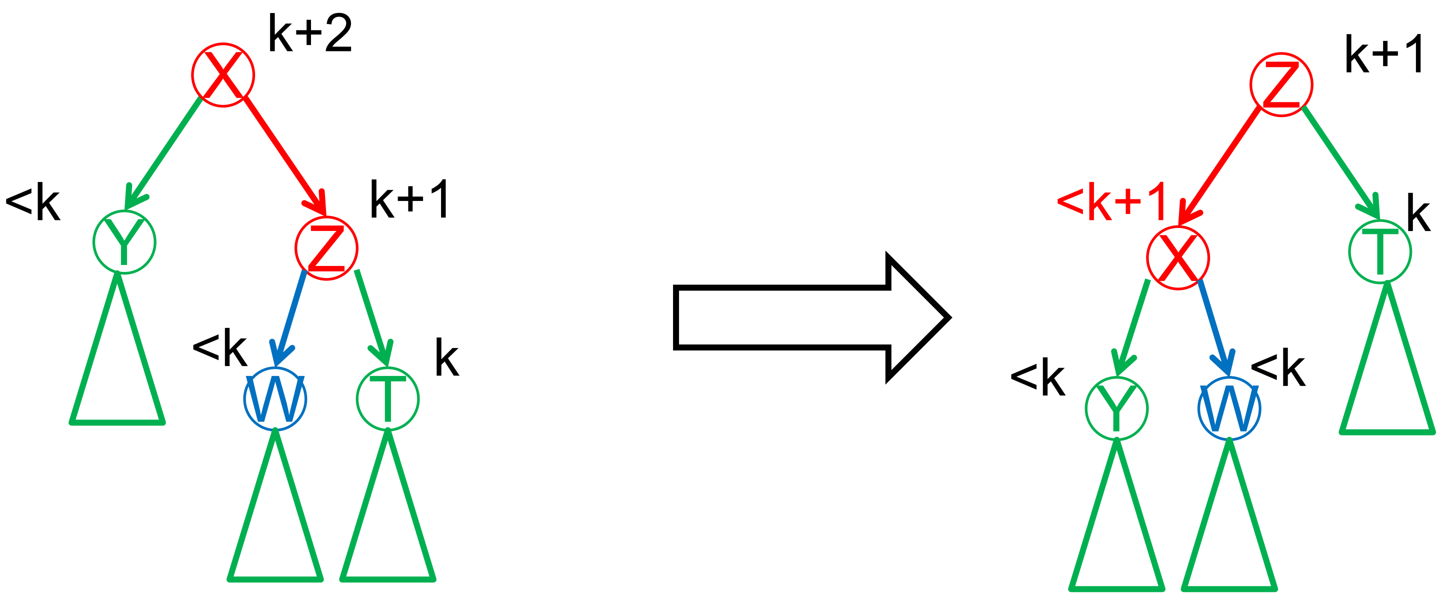
X.rightchild = Z.leftchild
Z.leftchild.parent = X
Z.leftchild = X
Z.parent = X.parent
X.parent.?child = Z
X.parent = Z

O(1) operations

also symmetric version

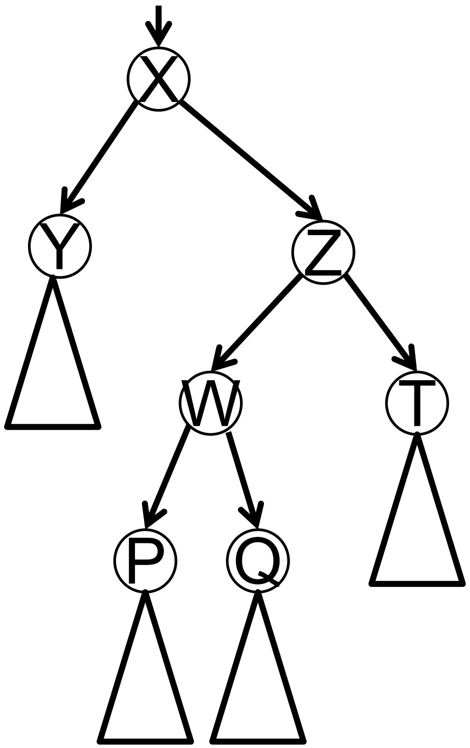
with appropriate
checks for None
and maybe for
outer Tree object

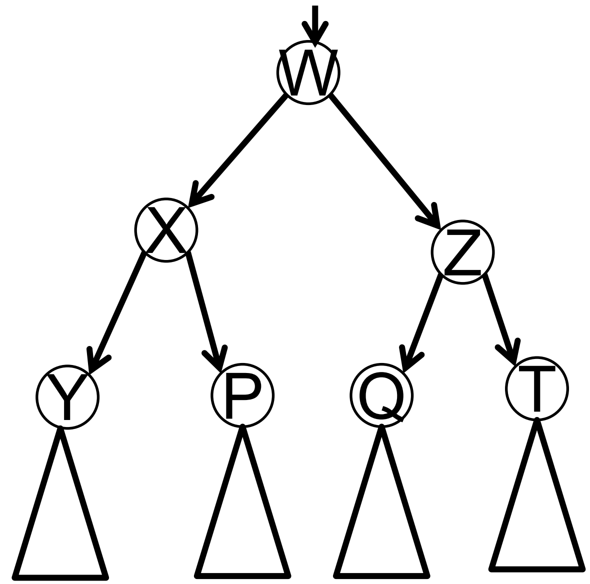
Rotation: height changes



If T causes Z's height, and Z's height - Y's height $\geq$ 2
then *X is unbalanced*, and this rotation will reduce the overall height of the tree

Alternative BSTs (ii)

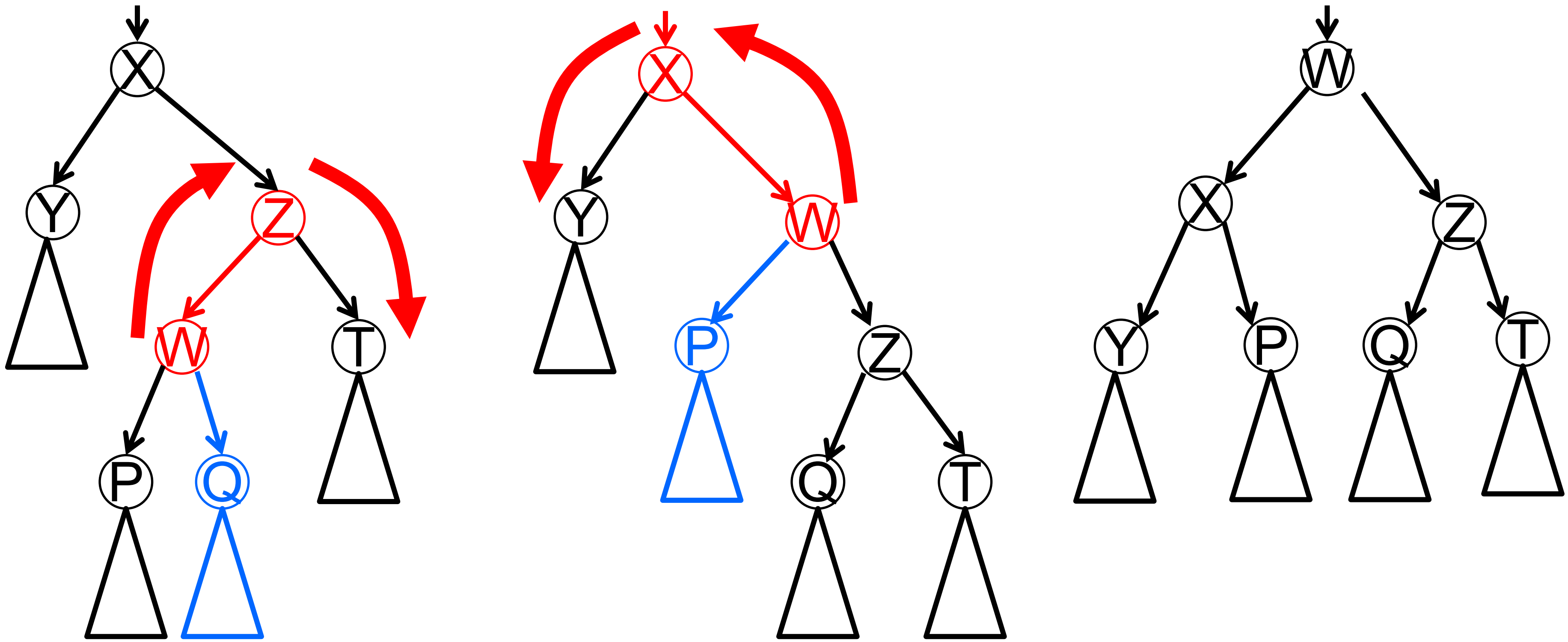




Y < X < P < W < Q < Z < T

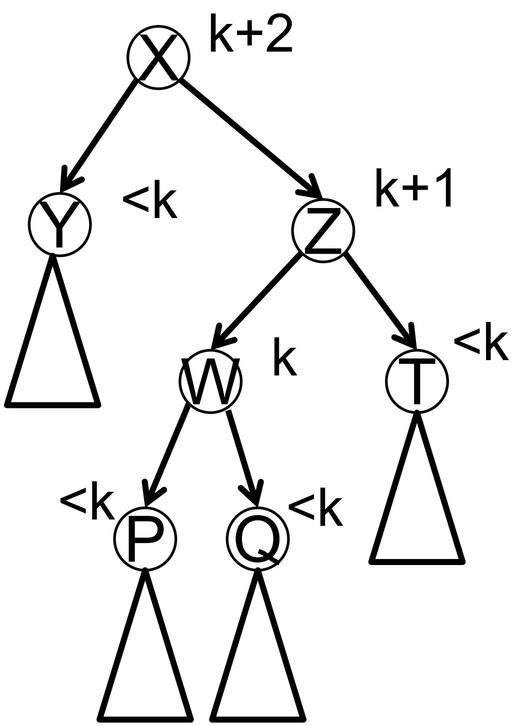
Double Rotation

drawing now without the parent links, to simplify the sketch ...

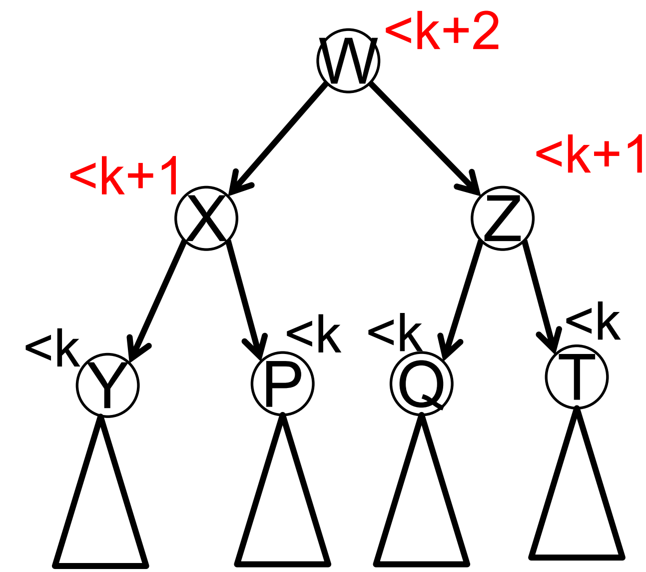


Two rotations, so also O(1)

Double Rotation: height changes



If W's height == T's height
no change to height
of tree, but new
tree has better
balance

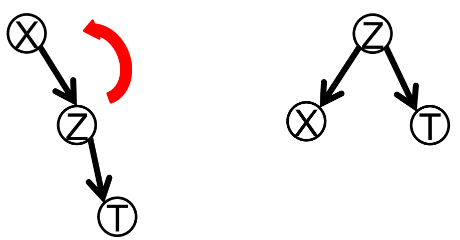


If W causes Z's height, and W’s height > T’s height, and Z's height - Y's height $\geq$ 2 then *X is unbalanced,* and this double rotation will reduce the overall height of the tree

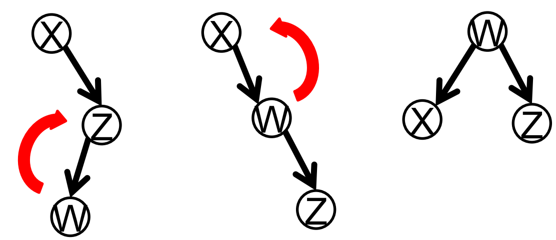
Simplified restructuring rules

A node X is *unbalanced* if children Y and Z are such that $|h_Z - h_Y| \geq 2$
or if X has only one child, and $h_x \geq 2$

If node X is unbalanced with higher rightchild Z
and Z's rightchild is its higher child
(i.e. height is caused by a straight line)
then rotate Z into X (from the right)

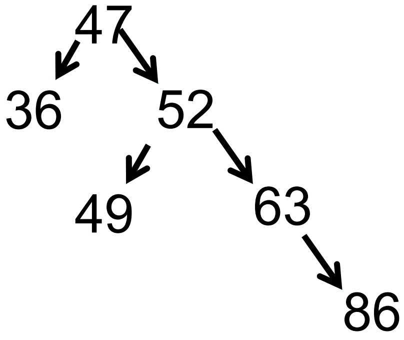


If node X is unbalanced with higher rightchild Z
and Z's leftchild W is its higher child (or equal)
(i.e. height is caused by a zig-zag)
then rotate W into Z (from the left)
rotate W into X (from the right)

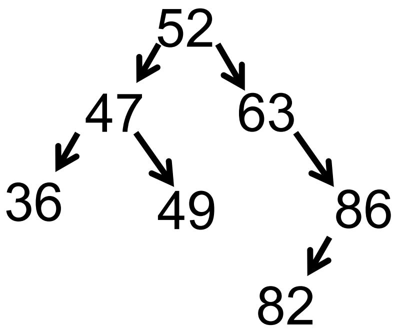


(and symmetric versions)

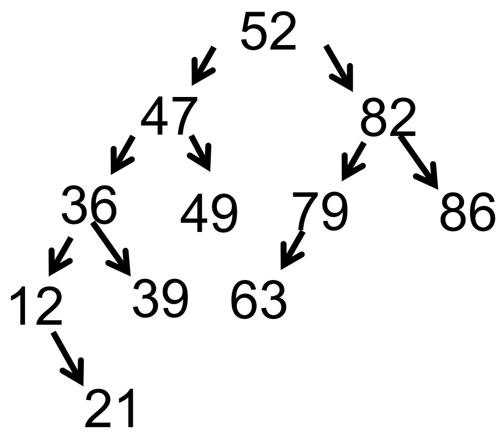
Restructure?



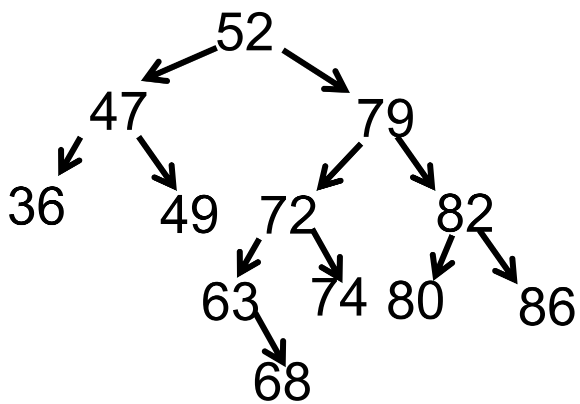
Restructure?



Restructure?



Restructure?



AVL Trees

An *AVL* Tree (or a *balanced* tree) is a Binary Search Tree in which:

- no node is unbalanced
- each time an item is added or removed from the tree, it is rebalanced by applying rotations.

Note that only ancestors of the (added / deleted / moved) node can become unbalanced, so we only need to search *up* the tree after each change.

Georgy Adelson-Velsky and Evgenii Landis (1962). "An algorithm for the organization of information". *Proceedings of the USSR Academy of Sciences* (in Russian) 146: 263–266.

AVL Trees: rebalance

After each *addition* of an item using the normal BST operation, start at the parent of the new item and rebalance it

maintain a height variable for each BSTNode

```
rebalance()
   update the height
   if node is unbalanced
      restructure the node
      if node had a parent before restructuring
         parent.rebalance()
   else
      if height changed and node has a parent
         parent.rebalance()
```

AVL Trees: rebalance (ii)

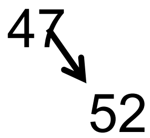
After each *removal* of an item using the normal BST operation,

```
if removed node was leaf
   leaf.parent.rebalance()
else if removed node was semileaf
   semileaf.parent.rebalance()
else   #internal, moved 'biggest' up, removed biggest
   node = the original parent of biggest
   node.rebalance()
```

add 47,52,36,63,49,67,86,82,79,80,12,74,72,68

47

add 47,52,36,63,49,67,86,82,79,80,12,74,72,68



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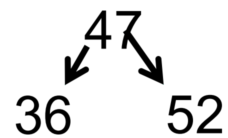

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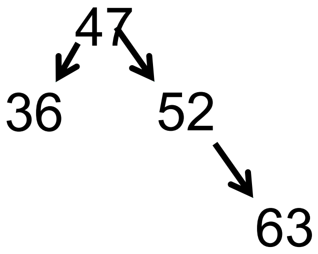

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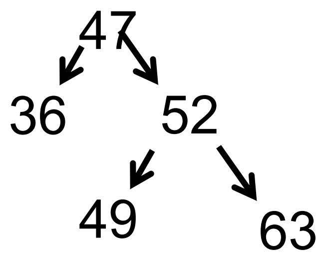

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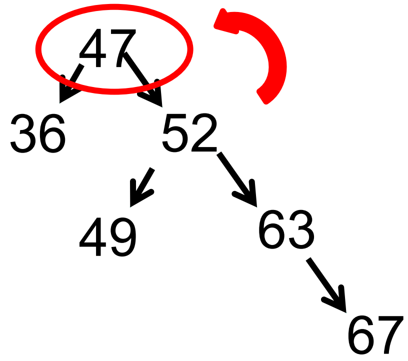

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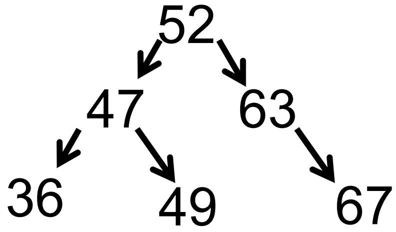

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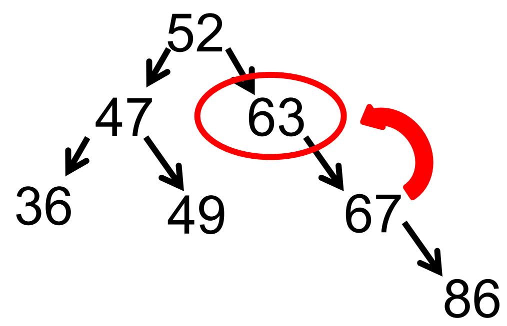

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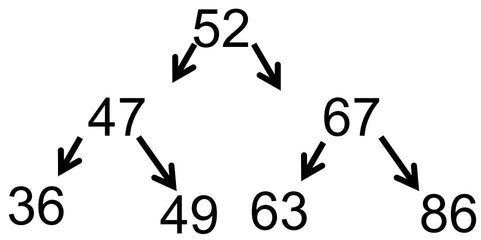

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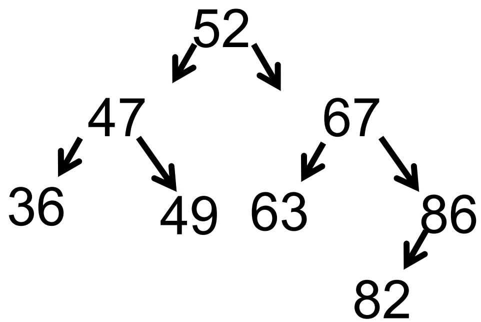

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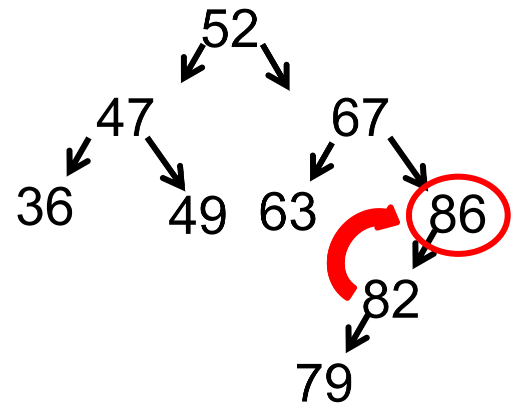

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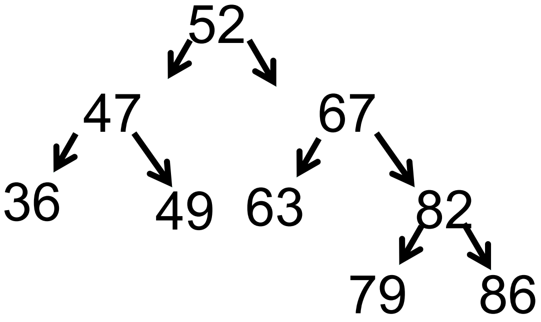

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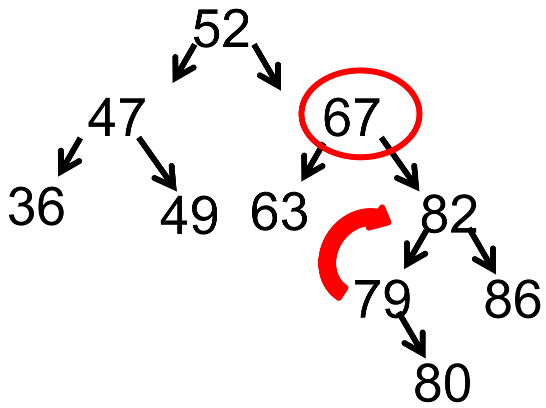

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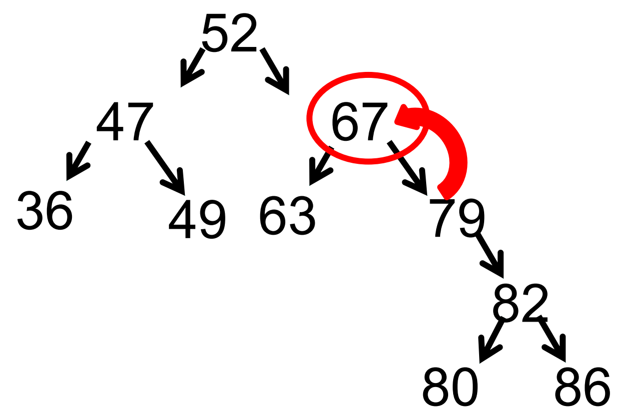

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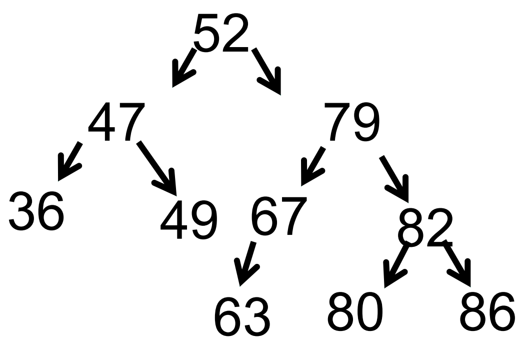

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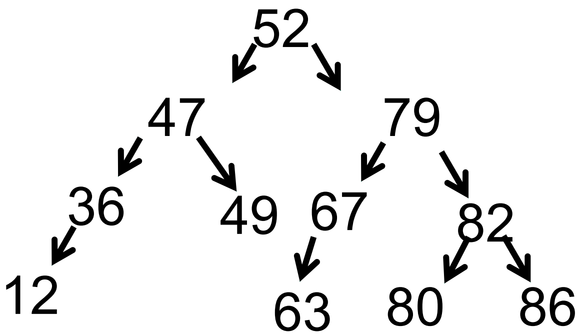

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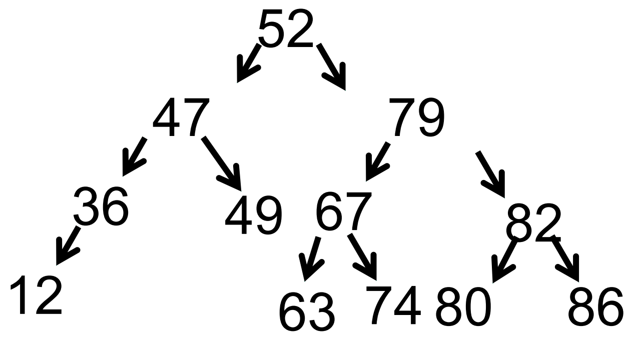

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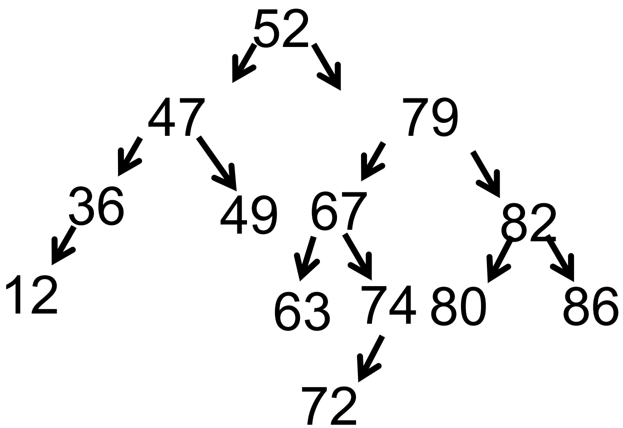

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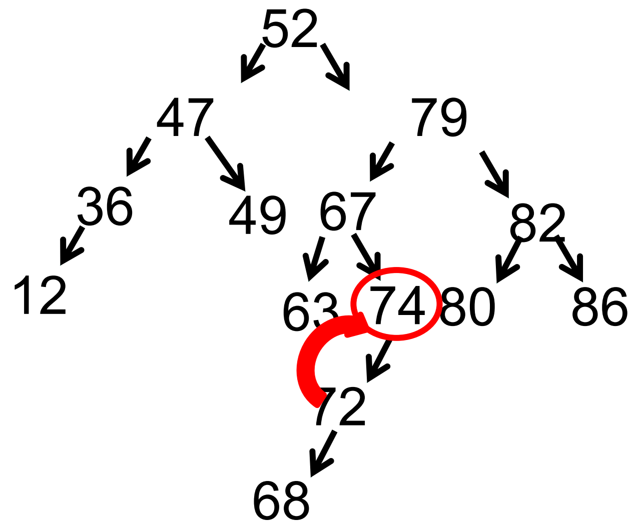

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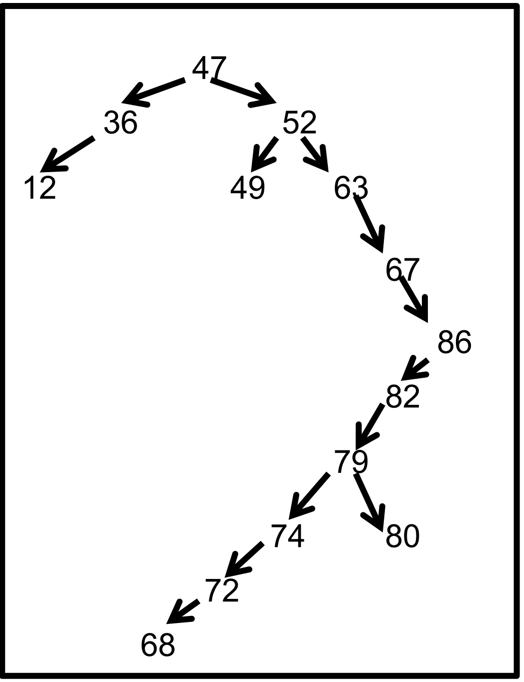
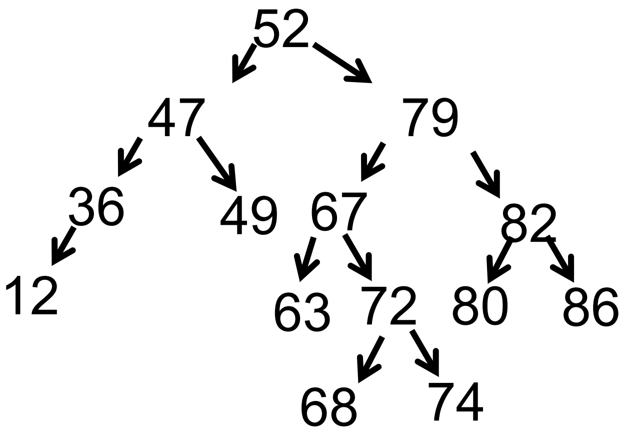
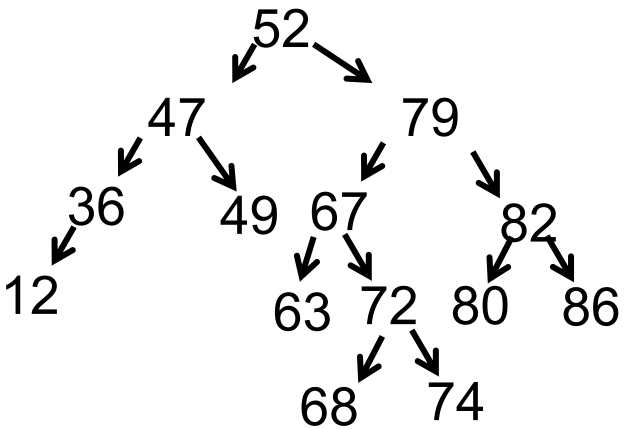
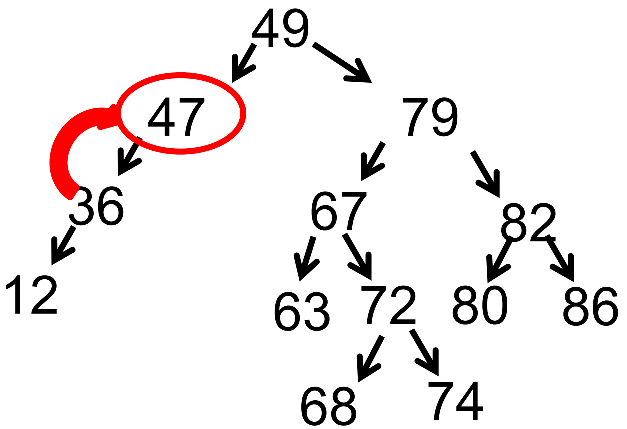




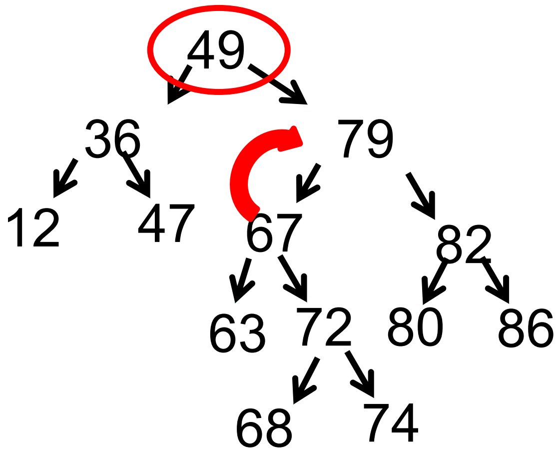
remove 52



remove 52



remove 52



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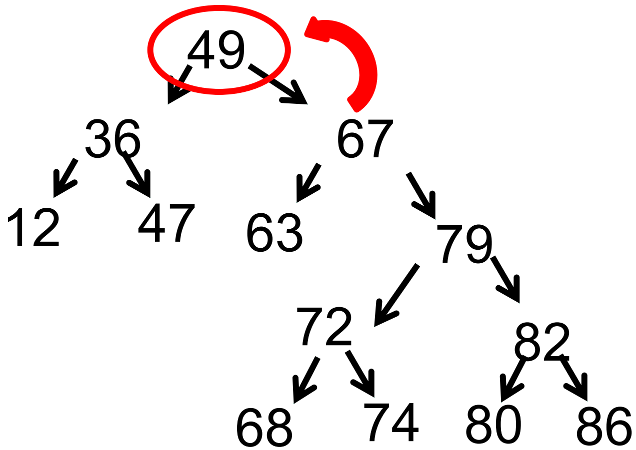

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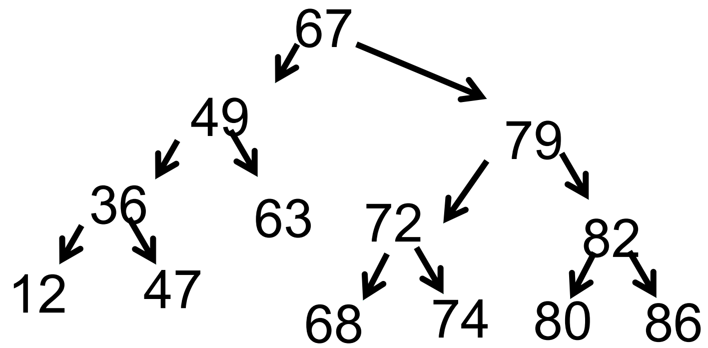


AVL Tree Properties

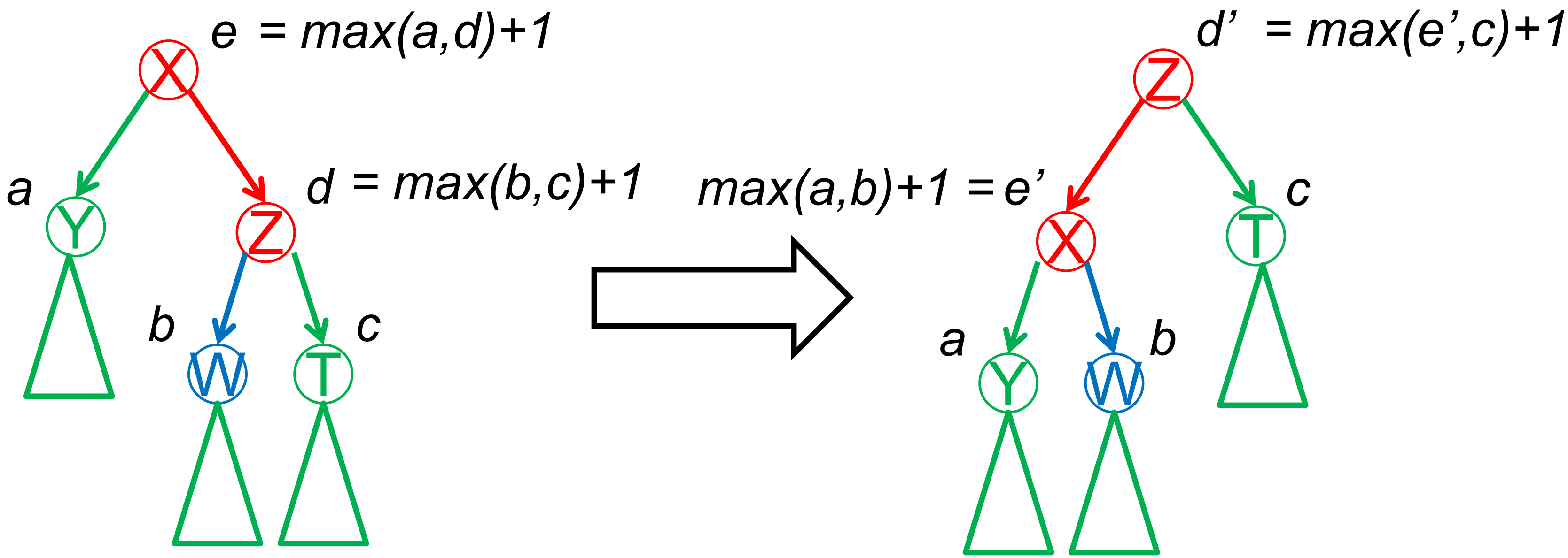
the height of an AVL Tree for n items is O(log n)

will be proved later

search is O(log n)
add is O(log n)
delete is O(log n)
find_min, find_max is O(log n)
traversing all nodes is O(n)

Derivation of height change for single rotation

If e > d’, then this rotation reduces the height of the tree



height e = max(a+1, b+2, c+2)

height d’ = max(a+2, b+2, c+1)

We need max(a+1, b+2, c+2) > max(a+2, b+2, c+1)

Cases: (i) a+1 was the max. No good, since a+2 is the new max, and e < d’

(ii) b+2 was the max. No good, since new max ≥ b+2, and e ≤ d’

(iii) c+2 was the max. If c+2 ≤ a+2, No good, since e ≤ d’

So (iii) is only option, and c+2 > a+2, and c+2 > b+2. i.e. c>a and c>b

Next lecture

more ADTs