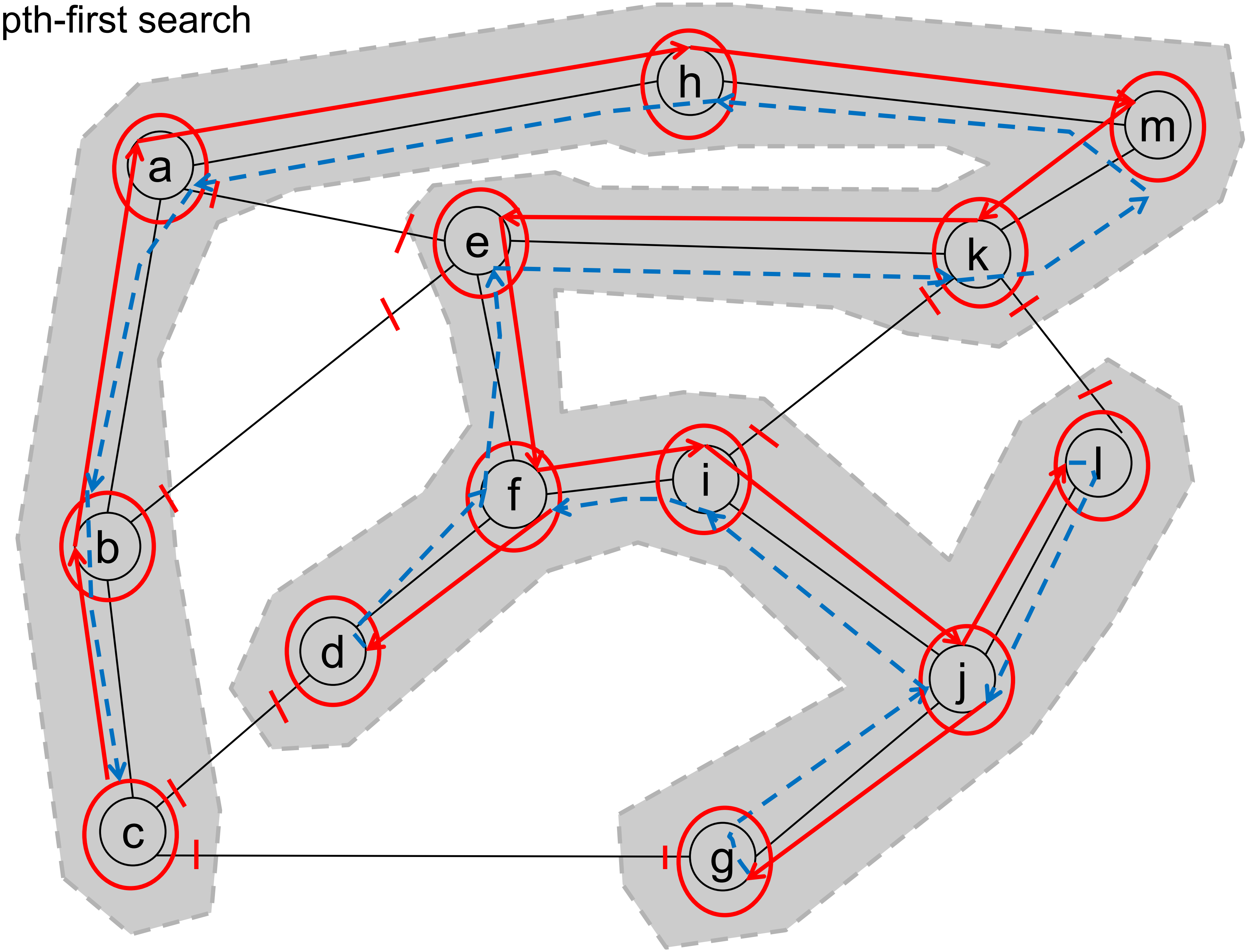
Breadth First Search
Directed Graphs
Transitive Closure

Class exercise:

For a simple undirected graph with *n* vertices, what is the maximum number of edges?

Depth-first search



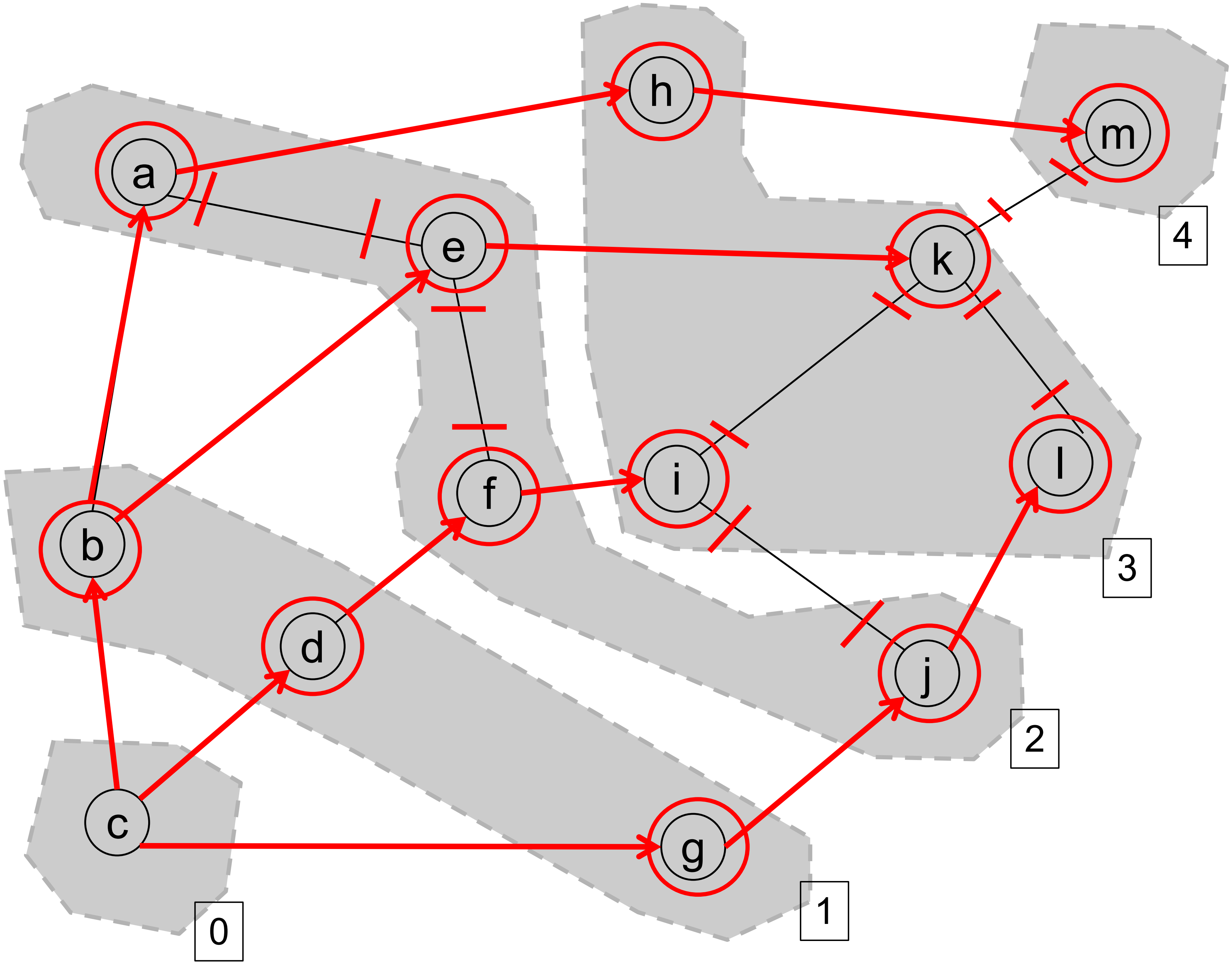
Breadth-first Traversal

Depth-first search was easy to implement, and is fast.

But some vertices which are very close to the start vertex are not discovered until late in the traversal.

The paths in the DFS spanning tree can be very long, even for vertices that are adjacent to the root in the graph.

Breadth-first traversal instead visits all vertices that are 1 hop away from the start before those that are 2 hops away, etc.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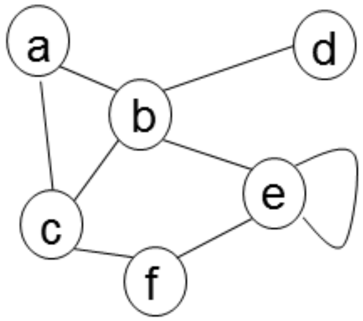
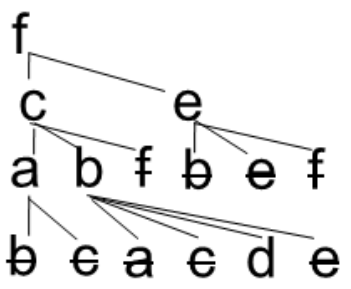
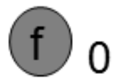
Finding the shortest path (informal)

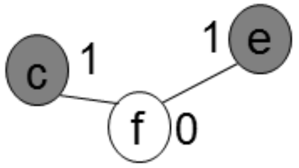
We start at the start vertex. We look at all vertices that are adjacent that we haven't seen before. If one of them is the end vertex, we stop; else we mark as being 1 hop away. We then take each 1-hop vertex in turn, and look at all vertices that are adjacent to them (ignoring any we have seen before). If one of them is the end vertex, we stop; else we mark as 2 hops. We then take each 2-hop vertex in turn, ... and so on. Either we find the end vertex via the shortest path, or the start vertex cannot be connected to the end vertex.

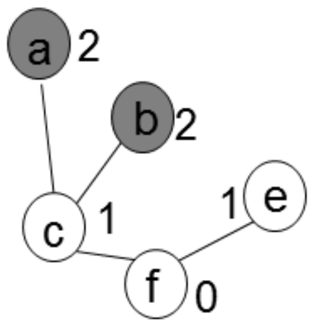
Example: find the shortest path from f to d

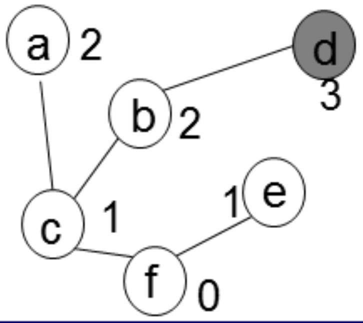












```
    def breadthfirstsearch(self, v):
```

Informal description:

Use a dictionary as before to record ‘marked’ vertices.
Process each level in turn. The first level contains just v.
For each item in level i, if it has a neighbour x that has not been marked, add x to the next level, and mark it.
Return the ‘marked’ dictionary.

Breadth First Search: properties

1. Breadth-first search computes, for each vertex, the path with the fewest number of edges from the start vertex

2. Breadth-first search has worst-case time complexity O(n+m)

Proof:
(by a similar argument to that for Depth-first search – each vertex is expanded at most once, and each edge is examined at most twice)

Directed Graphs

The ADT for Directed graphs is essentially the same as for undirected, except we treat the order of the vertices in an Edge as significant,

```
Edge
        get_start()
        get_end()
```

and we require the following methods for the Graph:

in_degree(x):	return the in-degree of vertex x
out_degree(x):	return the out-degree of vertex x
get_in_edges(x):	return a list of all edges pointing in to x
get_out_edges(x):	return a list of all edges pointing away from x

Class exercise:

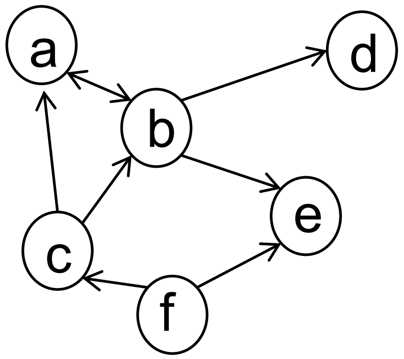
For a simple *directed* graph with *n* vertices, what is the maximum number of edges?

Directed graph: traversals

The basic algorithms for depth-first and breadth-first can be easily adapted to find all vertices reachable from a start vertex in a directed graph.

But note that a path from X to Y does not imply that there is a path from Y to X

The definition of a connected component no longer applies.



CS1112

Notation: if R is a relation $R \subseteq AxA$, we will say $R^{(2)} = R^oR$, $R^{(3)} = R^oR^{(2)}$, etc and so $R^{(n)} = R^oR^{(n-1)}$.

Let A be a set s.t. $|A| = n$, and let R be a relation $R \subseteq AxA$
The transitive closure of R is $R \cup R^{(2)} \cup \ldots \cup R^{(n-1)}$

Example: $A = \{1,2,3,4\}$
$R = \{(1,2), (2,4)\}$
transitive closure of $R = \{ (1,2), (2,4), (1,4)\}$

$R = \{(1,2),(2,4)\}$
$R^{(2)} = R^oR = \{(1,4)\}$
$R^{(3)} = R^oR^{(2)} = \{ \}$

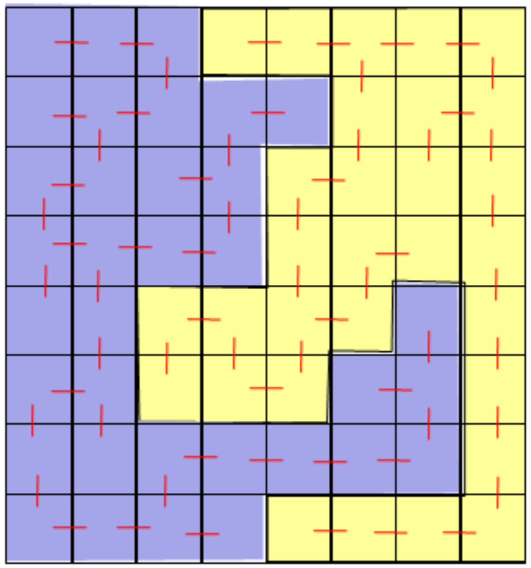
i.e. add in to R every pair (x,y) needed to make R transitive

transitive closure of $R = R \cup R^{(2)} \cup R^{(3)} = \{(1,2),(2,4),(1,4)\}$

CS1112

Example: consider the problem of finding a path for a robot from one area of a factory to another.

Or analysing secure zones in an airport – which areas are reachable from which other areas, without going through security control?



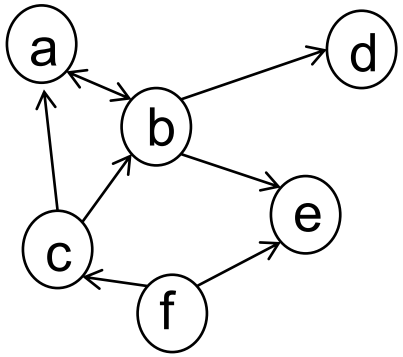
The red lines indicate doors that open between zones.

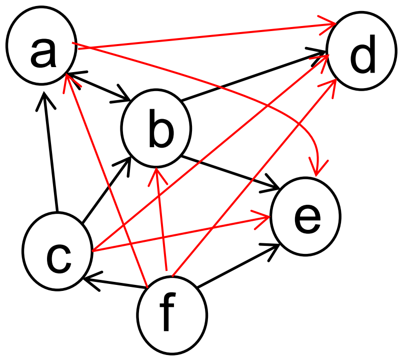
The transitive closure is the set of rooms that can be reached from each starting point.

Transitive Closure of a graph

Given a graph G = (V,E), the transitive closure G* = (V,t(E))

where an edge (x,y) ∈ t(E) if and only if there is a path from x to y in G





What is the transitive closure of this graph?

Computing the transitive closure

```
Version 1 (pseudocode)
Input: graph G = (V,E)
Output: graph G* = (V,t(E))

G* = a copy of G (same vertices, set of new edge copies)
for each vertex v in G
    tree = depthfirstsearch(G, v)
    for each x in tree
        add edge v-x into G*
```

Analysis: a single iteration round the for loop has one run of depthfirstsearch (O(n+m)), and we may add up to n edges into G*, so O(n+m) for one iteration. There are n vertices in G, so O(n(n+m))

Transitive closure (v2)

To simplify notation, if $G = (V,E)$ and $e \in E$, we say $e \in G$

For a given graph G,
assign a unique number from 1 to *n* to each vertex, and define:

$G^{(0)} = G$
$G^{(1)} = G^{(0)} + \{(x,y): (x,v_1) \in G^{(0)} \text{ and } (v_1,y) \in G^{(0)}\}$
...
$G^{(i)} = G^{(i-1)} + \{(x,y): (x,v_i) \in G^{(i-1)} \text{ and } (v_i,y) \in G^{(i-1)}\}$, for $i>0$

$G^{(i)}$ contains as edges all pairs (x,y) where there is a path from x to y in G containing only intermediate vertices in $\{v_1, v_2, ..., v_i\}$.

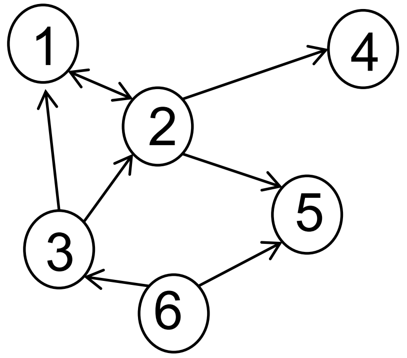
For a graph with *n* vertices, the transitive closure $G^* = G^{(n)}$.

The Floyd Warshall Algorithm to compute transitive closure

```
FloydWarshall (pseudocode)
Input: A directed graph G = (V,E), with |V| = n
Output: G*

$G^{(0)}$ = G
for k from 1 to n
    $G^{(k)} = G^{(k-1)}$
    for all pairs $(v_i,v_j)$ s.t. $i \neq j$, $i\neq k$, $j\neq k$
        if $(v_i,v_k) \in G^{(k-1)}$ and $(v_k,v_j) \in G^{(k-1)}$
            add$(v_i,v_j)$ to $G^{(k)}$
return $G^{(n)}$
```

Analysis: each inner for loop considers (*n*-1)*(*n*-2) pairs, checks if they are in a graph, and maybe adds them to a graph. We do the outer loop *n* times. So, if we can check and add edges in O(1), FW runs in worst case time $O(n^3)$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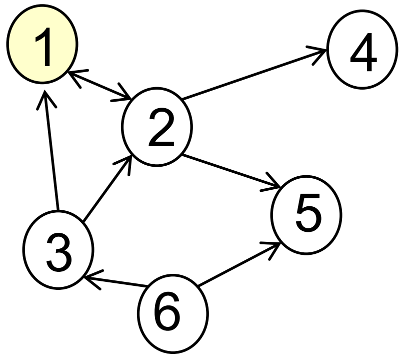
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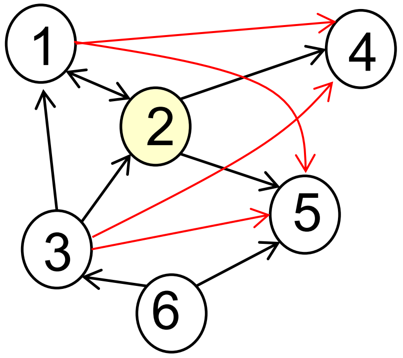
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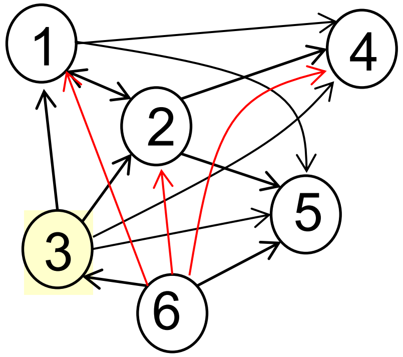
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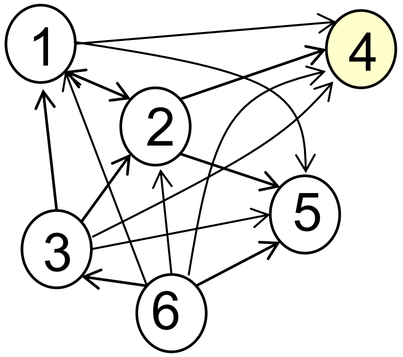
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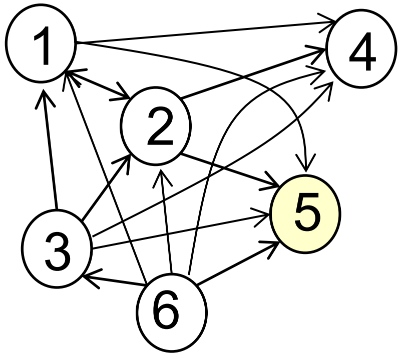
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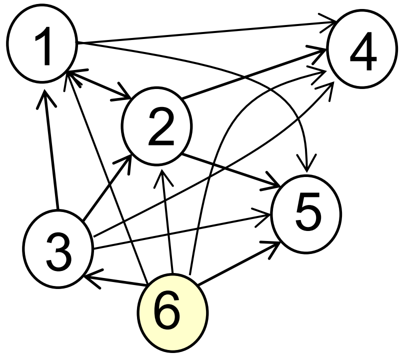
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The first transitive closure algorithm has time complexity $O(n(n+m))$.
Floyd Warshall has time complexity $O(n^3)$.

For dense graphs, the number of edges $m = O(n^2)$, and so the two algorithms have the same complexity. In these cases,
Floyd Warshall is often faster (the lower order terms are better).

For sparse graphs, the number of edges is $O(n)$, and so
Floyd Warshall has a higher time complexity.

Also, Floyd Warshall performs faster when the graph is represented as an adjacency matrix, while transitive closure v1 performs better when the graph is an adjacency list or adjacency map.

```
def floydwarshall(self):
    gstar = copy.deepcopy(self)
    vs = gstar.vertices()
    n = len(vs)
    for k in range(n):
        for i in range(n):
            if (i != k and gstar.get_edge(vs[i],vs[k]) != None):
                for j in range(n):
                    if (i != j and k != j
                    and gstar.get_edge(vs[k],vs[j]) is not None):
                        if gstar.get_edge(vs[i],vs[j]) == None:
                            gstar.add_edge(vs[i],vs[j],1)
    return gstar
```

Next lecture

No in-person lectures on 27th or 28th February

One lecture video and pdf will be posted on

Adaptable priority queues

which will be needed for the Assignment (Version 2).

Next in-person lecture will be on Tuesday 5th March.