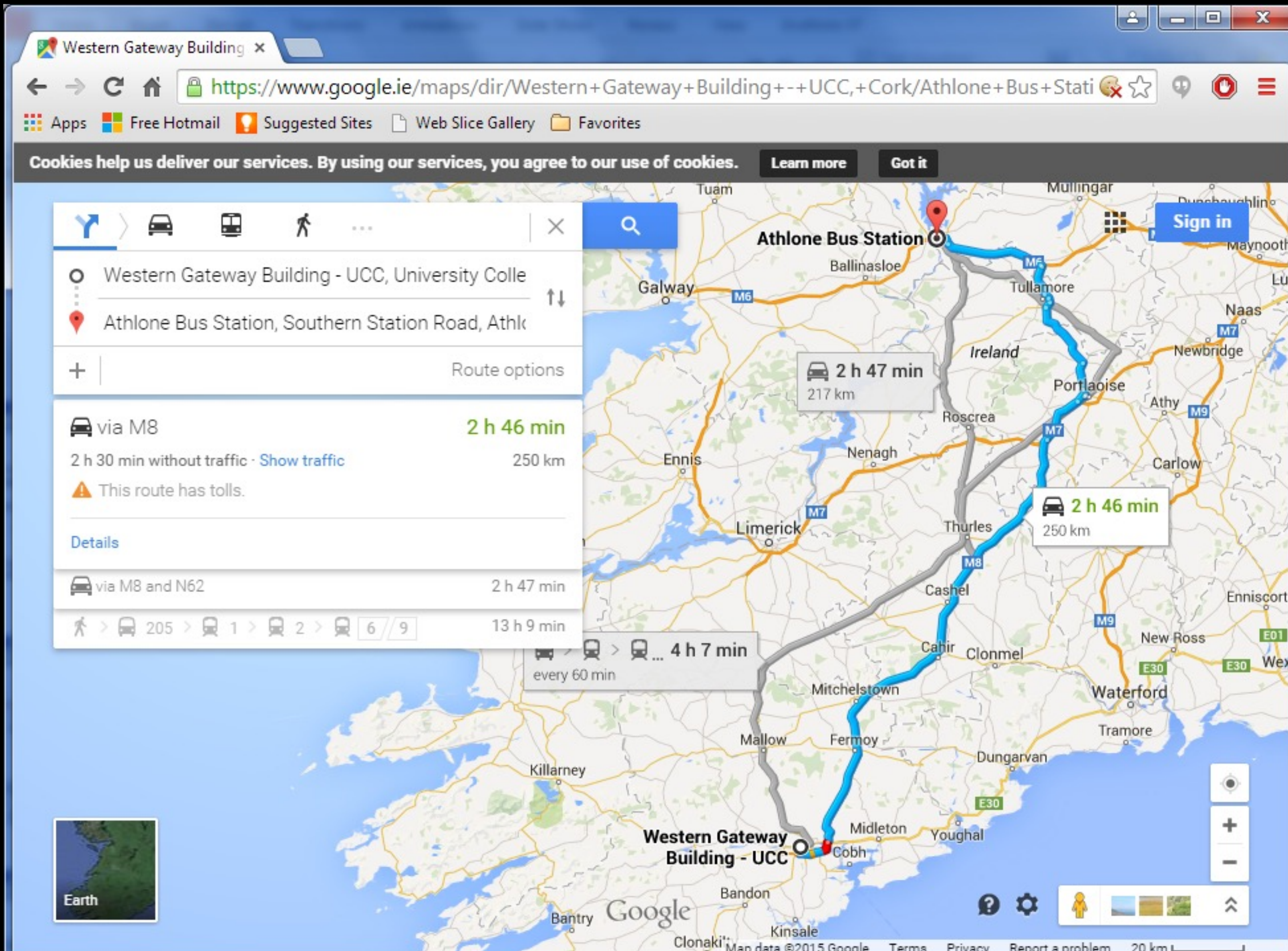
The Graph ADT and implementations

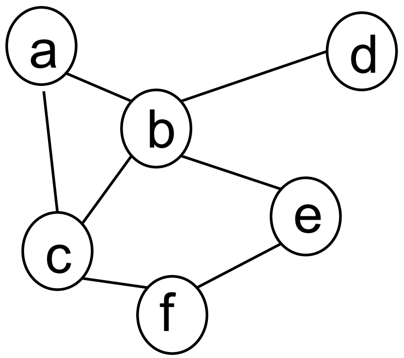


Graphs

A graph is an abstract representation of the relationships between a set of objects.

We saw graphs in CS1113:

- social network graphs, call graphs, route map graphs, ...
- graph properties
- shortest path algorithms
- spanning trees



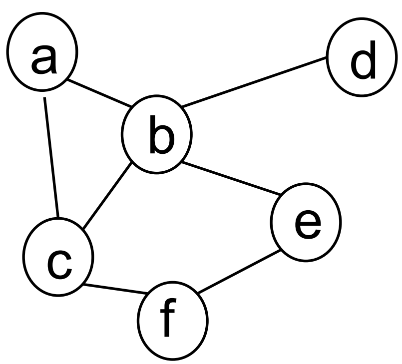
But it was all on paper. We didn't see how to

- implement them efficiently
- implement efficient algorithms for processing them

A *simple graph* is a pair (V,E), where V is a set of *vertices*, and E is a set of *edges*, and each edge is a set {x,y}, where x and y are vertices in V.

The *degree* of a vertex x is the number of edges that contain x.

Two vertices x and y are *adjacent* if there is an edge {x,y}.
Edge {x,y} is *incident* on x (and incident on y).
The *neighbours* of a vertex x are all other vertices adjacent to x



V = {a,b,c,d,e,f}

E = { {a,b}, {a,c}, {b,c}, {b,d}, {b,e}, {c,f}, {e,f} }

degree(b) = 4

neighbours(b) = {a,c,d,e}

In a *multigraph* E is a bag of edges, and so there may be multiple edges {x,y} in E for the same pair of vertices x and y.

In a *directed* graph, each edge is an ordered pair (x,y).

For a directed graph,

- the *out-degree* of a vertex is the number of edges with x as the first element of the pair
- the *in-degree* of a vertex .... x as the second element.

A *weighted* graph has a function w from E to some set, defining a *weight* with the edge.

We can also associate *labels* from some set L with either vertices or edges.

Designing the ADTs

We will maintain the graph as a complex data structure, which is composed of vertices and edges.

What methods should a Graph Abstract Data Type offer?

- this is not asking how to implement the data structure
- there are many different ways to implement graphs
- before we start, we need to be clear about what questions we will ask the graph, what commands we will issue, and what we want the data structure to give us in return

E.g. What are your vertices? Is this vertex linked to this other one? What is the weight of that vertex? What is the weight of the edge between those two vertices? How many edges link to that vertex?
Add a new edge to connect these two vertices. Remove that vertex. ...

Vertex and Edge ADTs

Vertex

element(): returns the label of the vertex

Edge

vertices(): returns the pair of vertices the edge is incident on

opposite(x): if the edge is incident on x, return the other vertex

element(): return the label of the edge

(Undirected) Graph ADT

vertices():	return a list of all vertices
edges():	return a list of all edges
num_vertices():	return the number of vertices
num_edges():	return the number of edges
get_edge(x,y):	return the edge between x and y (if it exists)
degree(x):	return the degree of vertex x
get_edges(x):	return a list of all edges incident on x
add_vertex(elt):	add a new vertex with element = elt
add_edge(x, y, elt)	a new edge between x and y, with element elt
remove_vertex(x):	remove vertex and all incident edges
remove_edge(e):	remove edge e

Implementing the ADT

The main operations will be retrieving vertices and edges. Updating will be relatively rare.

The main question is how to store and retrieve the edges. There are 4 main options:

1. a list of edges

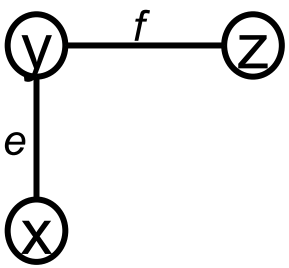
2. adjacency list:
   - for each vertex, store a list of the edges incident on it

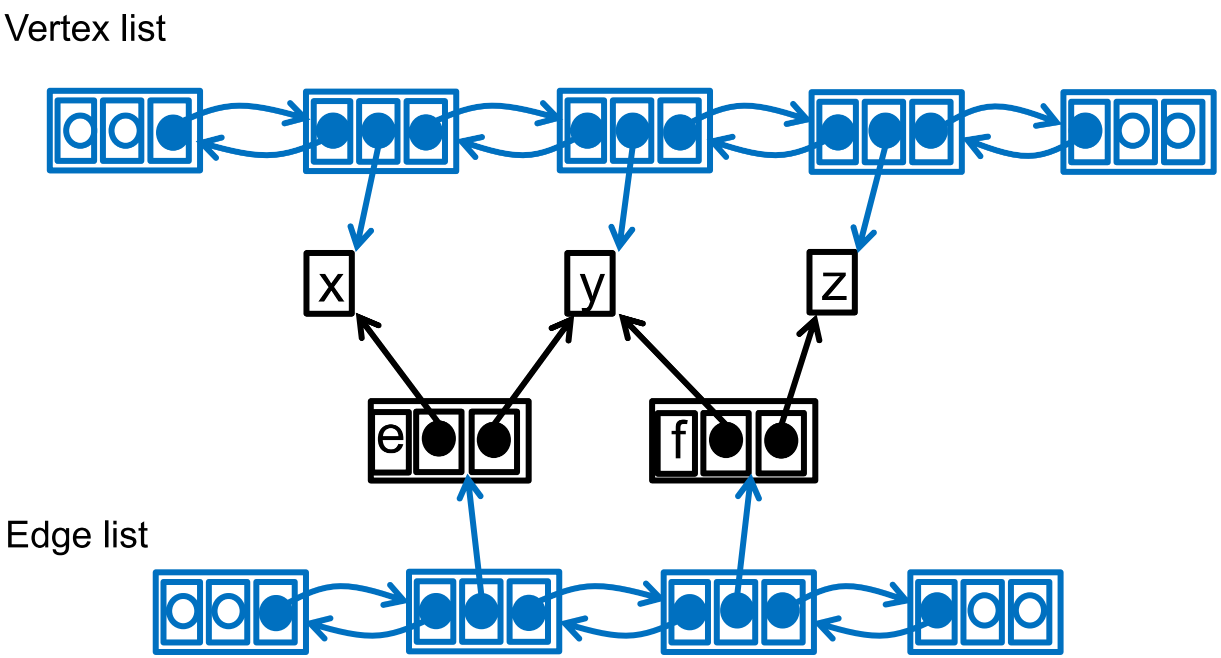
3. adjacency map:
   - for each vertex, store a map of the edges incident on it, using the other vertex as the key

4. adjacency matrix:
   - maintain a 2D array, where matrix[i][j] contains a reference to the edge {i,j} (ie the edge between the ith and jth vertices)

Edge List

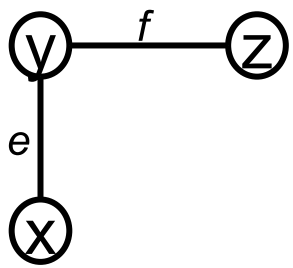
Maintain the vertices and edges in unordered linked lists.





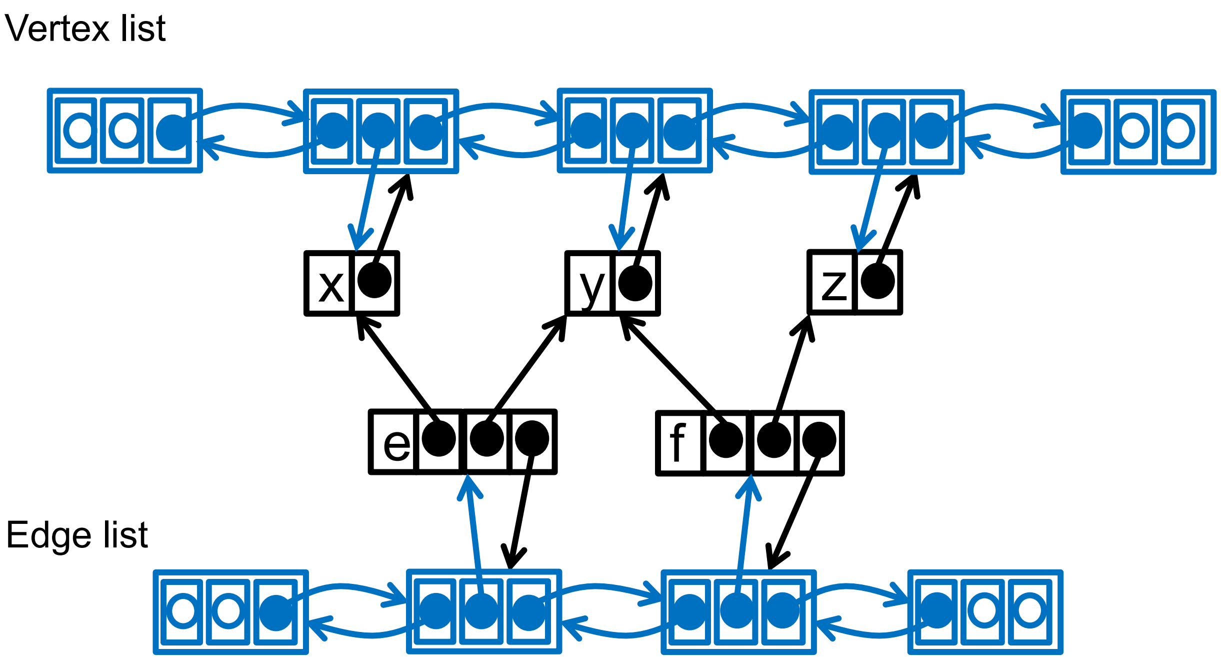
Edge List (improved)

Maintain the vertices and edges in unordered linked lists.



each vertex maintains a reference back to the list elt

each edge maintains a reference back to the list elt



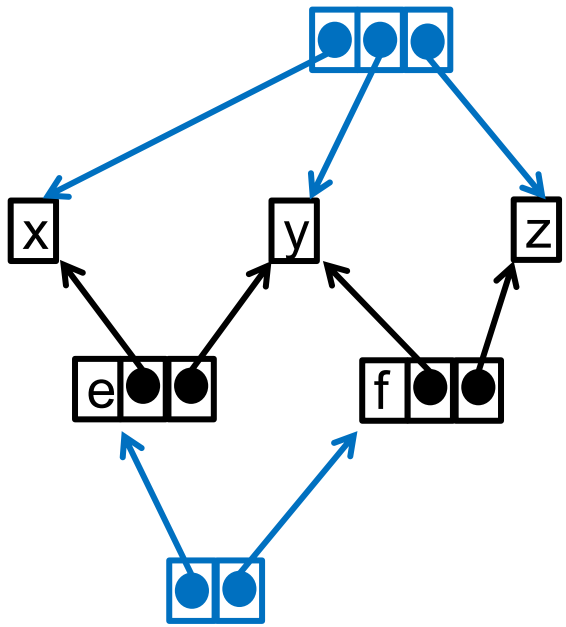
Edge List: complexity

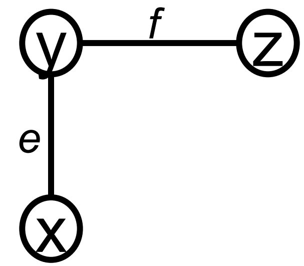
If n is the number of vertices, and m is the number of edges,

Space complexity:	O(n + m)
get_edge(x,y):	O(m) – must check each edge
degree(x):	O(m) – must check each edge
get_edges(x):	O(m) – must check each edge
add_vertex(elt):	O(1)
add_edge(x, y, elt):	O(1)
remove_edge(e):	O(1)
remove_vertex(x):	O(m) - must check each edge

Would it make a difference if we sorted the lists?

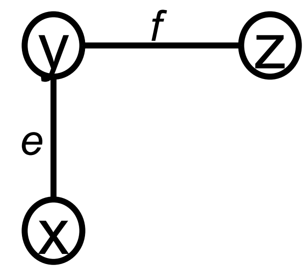
Now going to sketch the implementation
like this, to make the sketch more compact

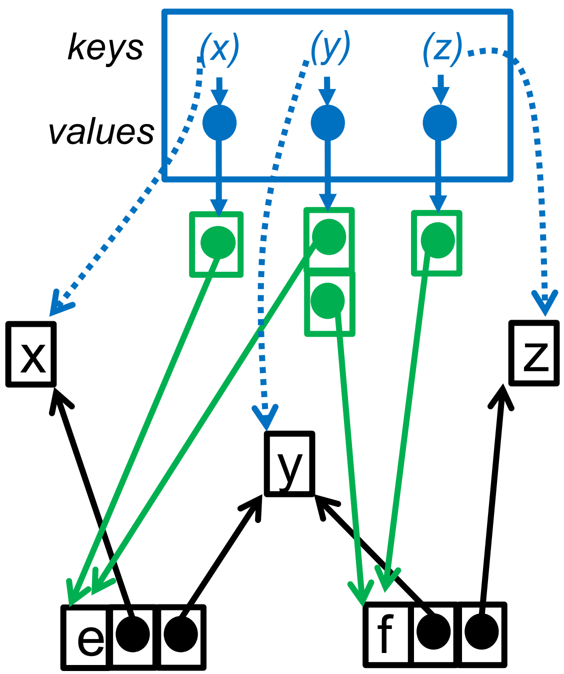


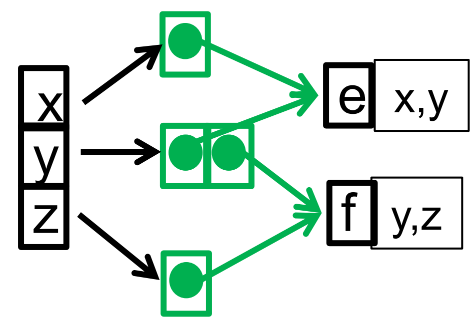


Adjacency List

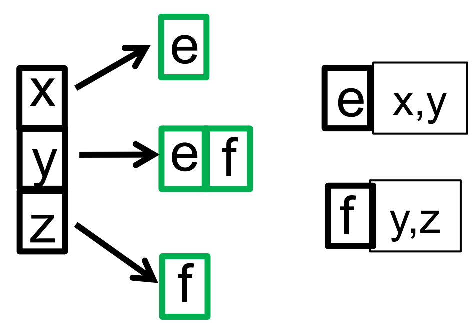
Maintain a data structure of vertices.
- simplest to use a dictionary
- key is reference to vertex object
- value is reference to list of edges incident on that vertex







This is the simpler sketch to understand, as long as we remember what references we are hiding ...



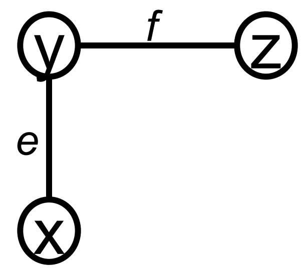
Adjacency List: complexity

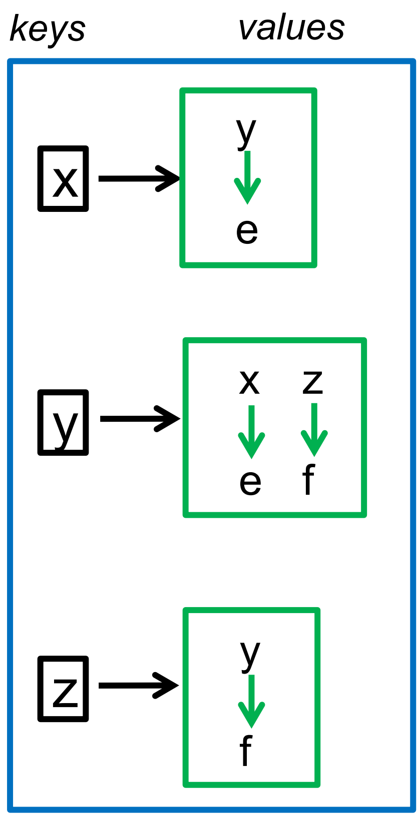
If n is the number of vertices, and m is the number of edges,

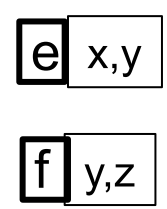
Space complexity:	O(n + m)	
get_edge(x,y):	O(min(degree(x), degree(y)))	
degree(x):	O(1)	
get_edges(x):	O(degree(x))	(assuming we copy the list, rather than return the actual adjacency list ...)
add_vertex(elt):	O(1)	
add_edge(x, y, elt):	O(1)	
remove_edge(e):	O(degree(x) + degree(y))	
remove_vertex(x):	O(n)	(O(degree(x) to identify the edges on x, but then removing those edges depends on the degree of the other vertex in each case, since we have to remove the edge from that other vertex's adjacency list ...)

Adjacency map

Maintain a data structure of vertices (e.g. dictionary)
Each vertex (key) maintains (value) a hash-map where the other vertices are the keys, and the incident edges are the values.





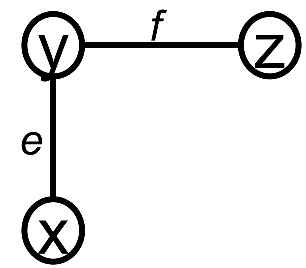


Adjacency Map: complexity

If n is the number of vertices, and m is the number of edges,

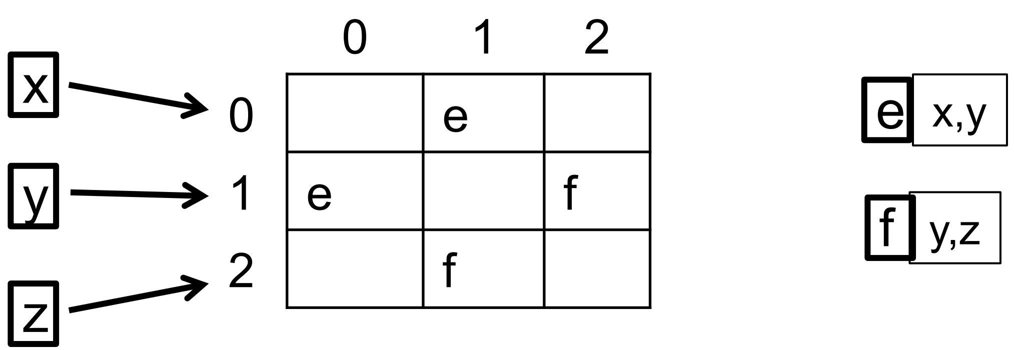
Space complexity:	O(n + m)	
		worst case
get_edge(x,y):	O(1) *expected*	O(min(degree(x), degree(y)))
degree(x):	O(1)	
get_edges(x):	O(degree(x))	
add_vertex(elt):	O(1)	
add_edge(x, y, elt):	O(1) *expected*	
remove_edge(e):	O(1) *expected*	
remove_vertex(x):	O(degree(x))	

Adjacency matrix



Associate a unique integer in 0 to n-1 with each vertex

Maintain a 2D array, where cell[i][j] contains a reference to the edge between i and j



Adjacency matrix: complexity

If n is the number of vertices, and m is the number of edges,

Space complexity:	$O(n^2)$	Wasteful for sparse graphs
get_edge(x,y):	$O(1)$	
degree(x):	$O(n)$	
get_edges(x):	$O(n)$	
add_vertex(elt):	$O(n^2)$	
add_edge(x, y, elt):	$O(1)$	
remove_edge(e):	$O(1)$	
remove_vertex(x):	$O(n^2)$	

Summary

	edge list	adjacency list	adjacency map	adjacency matrix
Space	O(n + m)	O(n + m)	O(n + m)	$O(n^2)$
get_edge	O(m)	O(min(deg(x), deg(y)))	O(1) *expected*	O(1)
degree	O(m)	O(1)	O(1)	O(n)
get_edges(x)	O(m)	O(deg(x))	O(deg(x))	O(n)
add_vertex	O(1)	O(1)	O(1)	$O(n^2)$
add_edge	O(1)	O(1)	O(1) *expected*	O(1)
remove_edge	O(1)	O(1)	O(1) *expected*	O(1)
remove_vertex	O(m)	O(deg(x))	O(deg(x))	$O(n^2)$

Adding to the underlying structures may change some of these complexities

Next lecture

Graph traversals