# 14.31/14.310 Lecture 9

# Probability---moments of a distribution

Ok, back to probability now.

Where were we? Ah, yes, talking about moments of distributions, expectation, in particular.

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Well, we can obviously figure out how Y is distributed---we know how to do that---and then use that distribution to compute, say, $E(Y)$.

There may be an easier way---it can be shown that

$$E(Y) = E(g(X)) = \int y f_Y(y)\,dy = \int g(x) f_X(x)\,dx$$

# Probability---St. Petersburg paradox

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So let's calculate them:

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No one would be willing to pay me an infinite amount to play this game.

I would guess that I wouldn't have any takers at $20, and that's a lot less than infinity.

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Economists know that people have diminishing marginal utility of money. In other words, their valuation of additional money decreases as the amount of money they have increases.

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So this is only a paradox unless you know a little bit of economics.

# Probability---properties of expectation

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Really, what if the X's aren't independent? Yes, really, they don't have to be independent.

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   $Y = a_1X_1 + a_2X_2 + \ldots + a_nX_n + b$
5. $E(XY) = E(X)E(Y)$ if $X, Y$ independent

# Probability---another moment: variance

In addition to describing the location, or center, of a distribution of a random variable, we often would like to describe how spread out it is.  There's a moment for that, variance.

$$\mathrm{Var}(X) = E[(X-\mu)^2]$$

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Note that variance is an expectation, so many of its properties will follow from that.

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In other words, shift a distribution and its variance doesn't change. Shrink or spread out a distribution and its variance changes by the square of the multiplicative factor.

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Ah, here we actually need independence.

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This last property can provide a handy way to compute variance.

# Probability---standard deviation

Often it's convenient for the measure of dispersion to have the same units as the random variable. For this reason, we define <u>standard deviation</u>.

$$SD(X) = \sigma = \sqrt{Var(X)} = \sqrt{\sigma^2}$$

# Probability---variance of a function

Since variance is an expectation, we can apply the results of expectation of a function of a random variable to get variance of a function of a random variable.

So if $Y = r(X)$,

$$\mathrm{Var}(Y) = E(Y^2) - E(Y)^2 = E(r(X)^2) - E(r(X))^2$$

$$= \int r(x)^2 f_X(x)\,dx - \left[\int r(x) f_X(x)\,dx\right]^2$$

# Probability---conditional expectation

A <u>conditional expectation</u> is the expectation of a conditional distribution. In other words,

$E(Y|X) = \int y f_{Y|X}(y|x)dy$

Note that $E(Y|X)$ is a function of X, and, therefore, a random variable. $E(Y|X=x)$ is just a number.

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<u>Thm</u> $E(E(Y|X)) = E(Y)$ "Law of Iterated Expectations"

# Probability---conditional variance

The definition of <u>conditional variance</u> follows from that of variance and conditional expectation.

<u>Thm</u> $\mathrm{Var}(E(Y|X)) + E(\mathrm{Var}(Y|X)) = \mathrm{Var}(Y)$

"Law of Total Variance"

# Probability---two laws

"Law of Iterated Expectations"

$E(E(Y|X)) = E(Y)$

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$Var(E(Y|X)) + E(Var(Y|X)) = Var(Y)$

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May seem a little mysterious, not clear how they're useful.

# Probability---example

A former student of mine started an innovation incubator in NYC. Suppose he's been doing this for a few years and has kept track of the number of patents produced every year in his incubator. He knows that $E(N) = 2$ and $Var(N) = 2$.

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Suppose there are 5 patents this year. What is the probability that 3 are commercial successes?

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$S|N=n \sim B(n,.2)$,

so $P(S=3|N=5) = 5!/(3!2!).2^3(1-.2)^2$

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How do we get this? Compute the expectation of a Bernoulli random variable and add it up n times

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$$E(S) = E(E(S|N)) = E(Np) = .2E(N) = .4$$

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What is the (unconditional) variance of number of commercial successes? Can use the Law of Total Variance.

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$$\mathrm{Var}(S) = \mathrm{Var}(E(S|N)) + E(\mathrm{Var}(S|N))$$

$$= \mathrm{Var}(Np) + E(Np(1-p))$$

# Probability---example

What is the (unconditional) variance of number of commercial successes?

$$\begin{aligned}
Var(S) &= Var(E(S|N)) + E(Var(S|N)) \\
&= Var(Np) + E(Np(1-p)) \\
&= .2^2 Var(N) + .2(1-.2)E(N) = .4
\end{aligned}$$

# Probability---covariance and correlation

We now have moments to describe the location, or center, of a distribution of a random variable and how spread out that distribution is. We are often interested in the relationship between random variables, and we have a moment of joint distributions to describe one aspect of that relationship, <u>covariance</u>.

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

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We often denote $Cov(X,Y)$ with $\sigma_{XY}$

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$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

And we have a standardized version, correlation.

$$\rho(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]/\sqrt{Var(X)}\sqrt{Var(Y)}$$

# Probability---covariance and correlation

$$\rho(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]/\sqrt{Var(X)}\sqrt{Var(Y)}$$

We say that X&Y are "positively correlated" if $\rho > 0$.

We say that X&Y are "negatively correlated" if $\rho < 0$.

We say that X&Y are "uncorrelated" if $\rho = 0$.

# Probability---properties of covariance

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6. $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y)$

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6. $Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$
7. $|\rho(X,Y)| <= 1$
8. $|\rho(X,Y)| = 1$ iff $Y = aX + b,\ a \neq 0$

# Probability---a preview of regression

We have two random variables, X&Y.

$EX = \mu_X$, $\mathrm{Var}X = \sigma_X^2$

$EY = \mu_Y$, $\mathrm{Var}Y = \sigma_Y^2$

$\rho_{XY} = \mathrm{Cov}(X,Y)/(\sigma_X \sigma_Y)$

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We know that, if $\rho_{XY} = 1$ then $Y = a + bX$, $b > 0$, and if
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If $|\rho_{XY}| < 1$, then we can write $Y = \alpha + \beta X + V$.

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We know that, if $\rho_{XY} = 1$ then $Y = a + bX$, $b > 0$, and if $\rho_{XY} = -1$ then $Y = a + bX$, $b < 0$.

If $|\rho_{XY}| < 1$, then we can write $Y = \alpha + \beta X + V$.

V is another random variable, but what can we say about it?

# Probability---a preview of regression

What we can say about $V$ depends on how we define $\alpha$ & $\beta$.

Let $\beta = \rho_{XY}\sigma_Y/\sigma_X$

Let $\alpha = \mu_Y - \beta\mu_X$

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We then call $\alpha$ & $\beta$ "regression coefficients," and think of $\alpha + \beta X$ as the part of $Y$ "explained by" $X$ and $V$ as the "unexplained" part.

# Probability---inequalities

Two inequalities involving moments of distributions and tail probabilities often come in handy:

## Markov Inequality

X is a random variable that is always non-negative.
Then for any t > 0, P(X>=t) <= E(X)/t.

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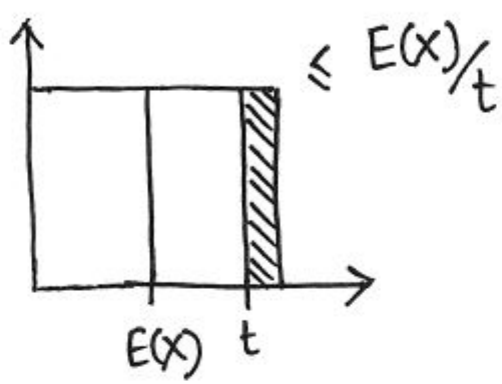


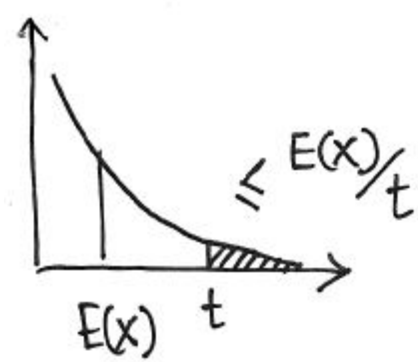

# Probability---inequalities

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X is a random variable for which Var(X) exists. Then for any t>0, P(|X-E(X)| >= t) <= Var(X)/$t^2$.

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