Homework 9

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Problem 1

We have

$$f(\mathbf{X},\mathbf{Y}|\mu_1,\mu_2,\sigma^2) = (2\pi\sigma^2)^{-\frac{m+n}{2}} \exp\Big\{-\frac{1}{2\sigma^2}\Big(\sum_{i=1}^{m}(X_i-\mu_1)^2 + \sum_{j=1}^{n}(Y_j-\mu_2)^2\Big)\Big\}$$

Under H_0, let $\mu_1 = \mu_2 = \mu$, we can treat $X_1,\dots,X_m,Y_1,\dots,Y_n$ as one sample i.i.d from $N(\mu,\sigma^2)$, so $\hat{\mu} = \frac{\sum_{i=1}^m X_i + \sum_{j=1}^n Y_j}{m+n}$, $\hat{\sigma}^2_{H_0} = \frac{1}{m+n}\Big(\sum_{i=1}^m (X_i-\hat{\mu})^2 + \sum_{j=1}^n (Y_j-\hat{\mu})^2\Big)$. Therefore,

$$L_{\Theta_0} = (2\pi e \hat{\sigma}^2_{H_0})^{-\frac{m+n}{2}}$$

Unrestricted MLE is $\hat{\mu}_1 = \bar{X}, \hat{\mu}_2 = \bar{Y}, \hat{\sigma}^2 = \frac{1}{m+n}\Big(\sum_{i=1}^m (X_i-\bar{X})^2 + \sum_{j=1}^n (Y_j-\bar{Y})^2\Big)$. Therefore,

$$L_{\Theta} = (2\pi e \hat{\sigma}^2)^{-\frac{m+n}{2}}$$

So, $W = -2\log\lambda = (m+n)\log\frac{\hat{\sigma}^2_{H_0}}{\hat{\sigma}^2} = (m+n)\log\frac{\sum_{i=1}^m (X_i-\hat{\mu})^2 + \sum_{j=1}^n (Y_j-\hat{\mu})^2}{\sum_{i=1}^m (X_i-\bar{X})^2 + \sum_{j=1}^n (Y_j-\bar{Y})^2} \sim \chi^2_1$.

At level α, we reject H_0 when $W > \chi^2_{1,1-\alpha}$

Problem 2

We have

$$f(\mathbf{X},\mathbf{Y}|\mu,\theta) = \mu^m(\prod_{i=1}^m X_i)^{\mu-1}\theta^n(\prod_{j=1}^n Y_j)^{\theta-1}$$

Under H_0, let $\mu = \theta = a$, we have $\log f(\mathbf{X},\mathbf{Y}|a) = (m+n)\log a + (a-1)(\sum_{i=1}^m \log X_i + \sum_{j=1}^n \log Y_j)$, so $\hat{a} = -\frac{m+n}{\sum_{i=1}^m \log X_i + \sum_{j=1}^n \log Y_j}$. Therefore,

$$L_{\Theta_0} = \hat{a}^{m+n}(\prod_{i=1}^m X_i \prod_{j=1}^n Y_j)^{\hat{a}-1} = \hat{a}^{m+n}\exp\Big\{-\Big(m+n+\sum_{i=1}^m \log X_i + \sum_{j=1}^n \log Y_j\Big)\Big\}$$

Unrestricted MLE is $\hat{\mu} = -\frac{m}{\sum_{i=1}^m \log X_i}, \hat{\theta} = -\frac{n}{\sum_{j=1}^n \log Y_j}$. Therefore,

$$L_{\Theta} = \hat{\mu}^m\hat{\theta}^n\exp\Big\{-\Big(m+n+\sum_{i=1}^m \log X_i + \sum_{j=1}^n \log Y_j\Big)\Big\}$$

So, $W = -2\log\lambda = 2\log\frac{\hat{\mu}^m\hat{\theta}^n}{\hat{a}^{m+n}} = 2\Big(m\log\frac{m}{\sum_{i=1}^m X_i} + n\log\frac{n}{\sum_{j=1}^n Y_j} - (m+n)\log\frac{m+n}{\sum_{i=1}^m X_i + \sum_{j=1}^n Y_j}\Big) \sim \chi^2_1$.

At level α, we reject H_0 when $W > \chi^2_{1,1-\alpha}$

Problem 3

1.

Let $Z = X_1 + X_2$, we have $f_Z(z) = \begin{cases} z - 2\theta, & 2\theta < z < 2\theta + 1 \\ 2\theta + 2 - z, & 2\theta + 1 \le z < 2\theta + 2 \\ 0, & \text{otherwise} \end{cases}$

$P_{H_0}(X_1 > 0.95) = 1 - 0.95 = 0.05$; $P_{H_0}(X_1 + X_2 > C) = \begin{cases} 1, & C \le 0 \\ 1 - \frac{1}{2}C^2, & 0 < C \le 1 \\ \frac{1}{2}C^2 - 2C + 2, & 1 \le C < 2 \\ 0, & C \ge 2 \end{cases}$

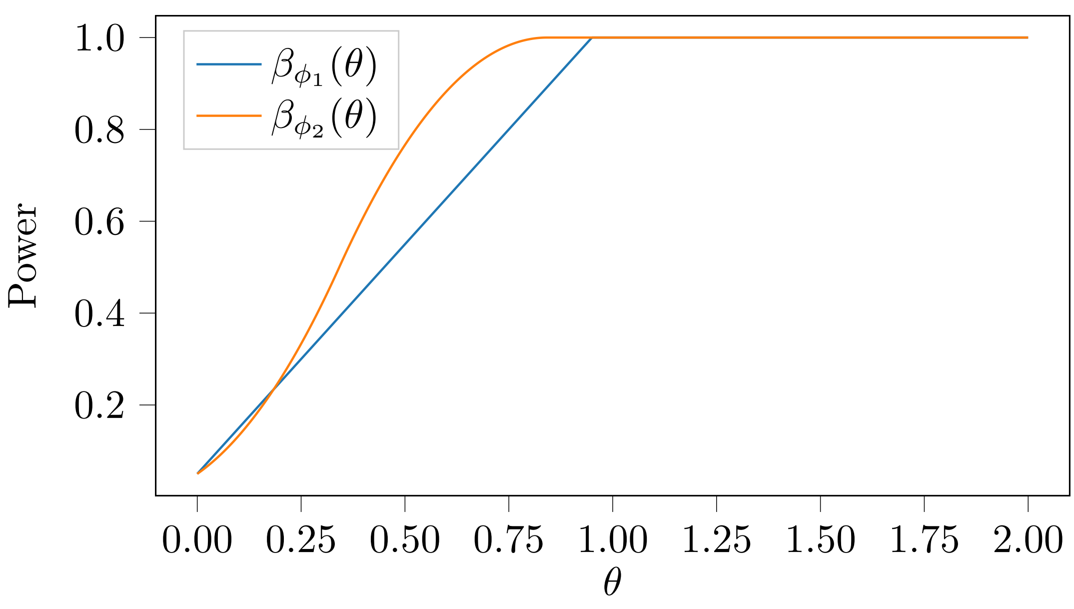
We want $\frac{1}{2}C^2 - 2C + 2 = 0.05$, So, $C = 2 - \sqrt{0.1} \approx 1.68$

2.

$\beta_{\phi_1}(\theta) = P_\theta(X_1 > 0.95) = \begin{cases} 1, & 0.95 \le \theta \\ \theta + 0.05, & \theta < 0.95 < \theta + 1 \\ 0, & 0.95 \ge \theta + 1 \end{cases}$

$\beta_{\phi_2}(\theta) = P_\theta(X_1 + X_2 > C) = \begin{cases} 1, & C < 2\theta \\ 1 - \frac{1}{2}(C - 2\theta)^2, & 2\theta \le C < 2\theta + 1 \\ \frac{1}{2}(2\theta + 2 - C)^2, & 2\theta + 1 \le C < 2\theta + 2 \\ 0, & C \ge 2\theta + 2 \end{cases}$ $(C \approx 1.68)$

The graph of two power functions is as follows.



3.

ϕ_2 is not more powerful than ϕ_1. When $\theta = 0.1$, $\beta_{\phi_2} \approx 0.13$ while $\beta_{\phi_1} = 0.15 > \beta_{\phi_2}$, so ϕ_2 is not more powerful than ϕ_1.

4.

Under H_0, $L_{\Theta_0} = f(X_1, X_2) = X_1 X_2 \mathbb{I}(0 < X_1, X_2 < 1)$

Unrestricted MLE is $\hat{\theta} = \max\{X_1, X_2\} - 1$. $L_{\Theta_1} = (X_1 - \hat{\theta})(X_2 - \hat{\theta})\mathbb{I}(\hat{\theta} < X_1, X_2 < \hat{\theta} + 1)$

Therefore,

$$\lambda = \frac{L_{\Theta_0}}{L_\Theta} = \frac{X_1 X_2 \mathbb{I}(0 < X_1, X_2 < 1)}{(X_1 - \hat{\theta})(X_2 - \hat{\theta})\mathbb{I}(\hat{\theta} < X_1, X_2 < \hat{\theta} + 1)}$$

We reject H_0 when λ is smaller than some constant c such that $\mathrm{E}_{H_0}(\phi) = \alpha$ where $\phi = \begin{cases} 1, & \lambda < c \\ r, & \lambda = c \\ 0, & \lambda > c \end{cases}$.

In this way, we can get a more powerful test.