Homework 6

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Problem 1

$T = \sum_{i=1}^{n} X_i \sim \Gamma(n\alpha, \lambda)$, so $\mathrm{E}(\frac{\bar{X}}{\alpha}) = \frac{1}{n\alpha} \cdot \frac{n\alpha}{\lambda} = \frac{1}{\lambda}$.

We know Gamma distribution has PDF $f(x, \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$. Note that α is known, so $x^{\alpha-1}$ is only a function of x, so it is from an exponential family with only $-\lambda x$ as the exponential term. Therefore, $\sum_{i=1}^{n} X_i$ is complete sufficient. $\frac{\bar{X}}{\alpha}$ can be rewritten as a function of T: $\frac{\bar{X}}{\alpha} = \phi(T) = \frac{T}{n\alpha}$. Thus, $\frac{\bar{X}}{\alpha}$ is the best unbiased estimator of $\frac{1}{\lambda}$.

□

Problem 2

Let $S = \frac{1}{\sqrt{n}} \left(\sum_{i=1}^{n} X_i^2 \right)^{\frac{1}{2}}$, $Q = \frac{nS^2}{\sigma^2} \sim \chi^2_n$,

$$
\begin{aligned}
\mathrm{E}(S) = \mathrm{E}(\sqrt{\frac{Q\sigma^2}{n}}) &= \sqrt{\frac{\sigma^2}{n}} \int_0^\infty \sqrt{q} f_q dq \\
&= \sqrt{\frac{\sigma^2}{n}} \int_0^\infty \sqrt{q} \frac{1}{2^{n/2}\Gamma(n/2)} q^{n/2-1} e^{-q/2} dq \\
&= \sqrt{\frac{\sigma^2}{n}} \frac{1}{\sqrt{2}\Gamma(n/2)} \int_0^\infty \left(\frac{q}{2}\right)^{(n+1)/2-1} e^{-q/2} dq \\
&= \frac{\sqrt{2}\Gamma((n+1)/2)}{\Gamma(n/2)} \frac{\sigma}{\sqrt{n}}
\end{aligned}
$$

So,

$$
\mathrm{E}(\hat{\sigma}) = \frac{\Gamma(n/2)}{\sqrt{2}\Gamma((n+1)/2)} \frac{\sqrt{2}\Gamma((n+1)/2)}{\Gamma(n/2)} \sigma = \sigma
$$

Note that $N(0, \sigma^2)$ is from exponential family: $f(x, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \exp\left\{-\frac{x^2}{\sigma^2}\right\}$, so $\sum_{i=1}^{n} X_i^2 = nS^2$ is complete sufficient.

$\hat{\sigma}$ can be written as a function of S^2, so $\hat{\sigma}$ is UMVUE for σ.

Problem 3

Suppose there exists an unbiased estimator $g(x)$ for $\frac{1}{\theta}$, then

$$
\mathrm{E}(g(x)) = \int_0^\theta g(x) \frac{1}{\theta} dx = \frac{1}{\theta} \int_0^\theta g(x) dx = \frac{1}{\theta}
$$

So, $\int_0^\theta g(x)dx = 1$. Suppose there is a differentiable function $G(x)$ $(x \geq 0)$ such that $\frac{d}{dx}G(x) = g(x)$, then $\int_0^\theta g(x)dx = G(\theta) - G(0) = 1$, $G(\theta) = 1 + G(0), \forall \theta > 0$. The equation only holds when $G(x) = \begin{cases} a, x = 0 \\ 1 + a, x > 0 \end{cases}$ for some constant a. However, such $G(x)$ leads to $g(x) = 0, \forall x > 0$, so $\mathrm{E}(g(x)) = 0$, which is a contradiction. Therefore, there does not exist an unbiased estimator for $\frac{1}{\theta}$.

□

Problem 4

Sample's joint distribution is $f_a(\mathbf{x}) = e^{na - \sum_{i=1}^n x_i} I_{(a,\infty)}(x_{(1)}) = e^{-\sum_{i=1}^n x_i} e^{na} I_{(a,\infty)}(x_{(1)})$. So, $X_{(1)}$ is sufficient for a.

$F_X(x) = 1 - e^{-(x-a)}, x > a$, so the distribution of $X_{(1)}$ is $f_{X_{(1)}}(x) = n[1 - F_X(x)]^{n-1} f_X(x) = n[e^{-(x-a)}]^{n-1} e^{-(x-a)} = ne^{-n(x-a)}, x > a$

$$0 = \mathrm{E}(g(X_{(1)})) = \int_a^\infty g(x) ne^{-n(x-a)} dx$$

So, $\forall a$, $\int_a^\infty g(x)e^{-nx}dx = 0$. For some $b > a$, we have

$$\int_a^\infty g(x)e^{-nx}dx = \int_a^b g(x)e^{-nx}dx + \int_b^\infty g(x)e^{-nx}dx = 0$$

Differentiating both sides with respect to a, we have $g(x)e^{-nx} = 0$. So, $g(x) \equiv 0$. Therefore, $X_{(1)}$ is complete sufficient for a.

$\mathrm{E}(X_{(1)}) = \int_a^\infty xne^{-n(x-a)}dx = a + \frac{1}{n}$. Let $T = X_{(1)} - \frac{1}{n}$, $\mathrm{E}(T) = a$, so T is UMVUE for a.