Homework 2

蒋翌坤 20307100013

Problem 1

We know that $\bar{X} \sim N(\mu, \sigma^2/n)$. Since X_{n+1} is independent of $X_1, \ldots, X_n$, it is independent of $\bar{X}$, so $X_{n+1} - \bar{X} \sim N(0, \frac{n+1}{n}\sigma^2)$. We also know that $\frac{nS_n^2}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$, $X_{n+1}, \bar{X}$ are also independent of S_n. Therefore,

$$\frac{X_{n+1} - \bar{X}}{S_n}\sqrt{\frac{n-1}{n+1}} = \frac{X_{n+1} - \bar{X}}{\sqrt{\frac{n+1}{n}\sigma^2}} \Bigg/ \sqrt{\frac{nS_n^2}{(n-1)\sigma^2}} \sim \frac{N(0,1)}{\sqrt{\chi^2_{n-1}/(n-1)}} \sim t_{n-1}$$

Problem 2

Let $Y_i = \frac{X_i}{\sigma_i} \sim N(0,1)$ and $\alpha_i = \dfrac{1}{\sigma_i}$, we have

$$\begin{aligned}
\xi = \sum_{i=1}^n \frac{(X_i - Z)^2}{\sigma_i^2} &= \sum_{i=1}^n \Big(Y_i - \frac{\alpha_i}{\sum_{j=1}^n \alpha_j^2}\sum_{j=1}^n \alpha_j Y_j\Big)^2 \\
&= \sum_{i=1}^n \Big(Y_i^2 - 2\frac{\alpha_i Y_i}{\sum_{j=1}^n \alpha_j^2}\sum_{j=1}^n \alpha_j Y_j + \Big(\frac{\alpha_i}{\sum_{j=1}^n \alpha_j^2}\sum_{j=1}^n \alpha_j Y_j\Big)^2\Big) \\
&= \sum_{i=1}^n Y_i^2 - 2\frac{\sum_{i=1}^n \alpha_i Y_i}{\sum_{j=1}^n \alpha_j^2}\sum_{j=1}^n \alpha_j Y_j + \frac{\sum_{i=1}^n \alpha_i^2}{(\sum_{j=1}^n \alpha_j^2)^2}\Big(\sum_{j=1}^n \alpha_j Y_j\Big)^2 \\
&= \sum_{i=1}^n Y_i^2 - 2\frac{(\sum_{i=1}^n \alpha_i Y_i)^2}{\sum_{i=1}^n \alpha_i^2} + \frac{(\sum_{i=1}^n \alpha_i Y_i)^2}{\sum_{i=1}^n \alpha_i^2} \\
&= \sum_{i=1}^n Y_i^2 - \frac{(\sum_{i=1}^n \alpha_i Y_i)^2}{\sum_{i=1}^n \alpha_i^2}
\end{aligned}$$

Let $\delta = \sqrt{\sum_{i=1}^n \alpha_i^2}$, $\mathbf{A} = \begin{pmatrix} \frac{\alpha_1}{\delta} & \frac{\alpha_2}{\delta} & \cdots & \frac{\alpha_n}{\delta} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ and $\mathbf{A}$ is orthogonal. Let $\mathbf{W} = \mathbf{AY}$, where $\mathbf{Y} = (Y_1, \ldots, Y_n)'$ and $\mathbf{W} = (W_1, \ldots, W_n)'$. Therefore, we have $\mathbf{W}'\mathbf{W} = \mathbf{Y}'\mathbf{A}'\mathbf{A}\mathbf{Y} = \mathbf{Y}'\mathbf{Y}$, and $W_1^2 = \Big(\frac{\sum_{i=1}^n \alpha_i Y_i}{\delta}\Big)^2 = \frac{(\sum_{i=1}^n \alpha_i Y_i)^2}{\sum_{i=1}^n \alpha_i^2}$. So,

$$\xi = \sum_{i=1}^n Y_i^2 - W_1^2 = \mathbf{Y}'\mathbf{Y} - W_1^2 = \mathbf{W}'\mathbf{W} - W_1^2 = \sum_{i=2}^n W_i^2$$

When $i, j \geq 2, i \neq j$, we have

$$\mathrm{E}(W_i) = \sum_{j=1}^n a_{ij}\mathrm{E}(Y_j) = 0, \quad \mathrm{Var}(W_i) = \sum_{j=1}^n a_{ij}^2 \mathrm{Var}(Y_j) = \sum_{j=1}^n a_{ij}^2 = 1, \quad \mathrm{Cov}(W_i, W_j) = \sum_{k=1}^n a_{ik}a_{jk} = 0$$

Here, W_i's are multivariate normal, so covariance equals to 0 implies that W_i's are independent. Therefore, $\xi = \sum_{i=2}^n W_i^2 \sim \chi^2_{n-1}$.

Problem 3

Here, let $\hat{\lambda} = \bar{X}$. Using CLT, we have $\hat{\lambda} \xrightarrow{d} N(\lambda, \frac{\lambda}{n})$. So, $\frac{\hat{\lambda}-\lambda}{\sqrt{\lambda/n}} \xrightarrow{d} N(0,1)$. Then, using LLN, we know that $\frac{\lambda}{\hat{\lambda}} \xrightarrow{p} 1$ and $\sqrt{\frac{\lambda}{\hat{\lambda}}} \xrightarrow{p} 1$. Using Slusky's theorem, we have $\dfrac{\bar{X}-\lambda}{\sqrt{\bar{X}/n}} = \dfrac{\hat{\lambda}-\lambda}{\sqrt{\hat{\lambda}/n}} = \dfrac{\hat{\lambda}-\lambda}{\sqrt{\lambda/n}} \cdot \sqrt{\dfrac{\lambda}{\hat{\lambda}}} \xrightarrow{d} N(0,1)$. $\square$

Problem 4

(a) We can differenciate $\int f(x|\boldsymbol{\theta})dx = \int h(x)c(\boldsymbol{\theta}) \exp\Big(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\Big)dx$ with respect to θ_j. Note that the integral range is not related to $\boldsymbol{\theta}$, so we can take the partial derivative into the integral. Then,

$$\begin{aligned}
\frac{\partial}{\partial\theta_j}\int f(x|\boldsymbol{\theta})dx &= \frac{\partial}{\partial\theta_j}\Big\{c(\boldsymbol{\theta})\int h(x)\exp\big(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\big)dx\Big\} \\
&= \frac{\partial}{\partial\theta_j}c(\boldsymbol{\theta})\int h(x)\exp\Big(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\Big)dx + c(\boldsymbol{\theta})\frac{\partial}{\partial\theta_j}\Big\{\int h(x)\exp\Big(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\Big)dx\Big\} \\
&= \frac{\partial}{\partial\theta_j}c(\boldsymbol{\theta})\int h(x)\exp\Big(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\Big)dx + c(\boldsymbol{\theta})\int h(x)\frac{\partial}{\partial\theta_j}\Big\{\exp\Big(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\Big)\Big\}dx \\
&= \frac{\partial}{\partial\theta_j}c(\boldsymbol{\theta})\int h(x)\exp\Big(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\Big)dx \\
&\quad + \int h(x)c(\boldsymbol{\theta})\exp\Big(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\Big)\Big(\sum_{i=1}^{k}\frac{\partial}{\partial\theta_j}w_i(\boldsymbol{\theta})t_i(x)\Big)dx \\
&= \frac{\partial}{\partial\theta_j}c(\boldsymbol{\theta})\int h(x)\exp\Big(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\Big)dx + \mathrm{E}\Big(\sum_{i=1}^{k}\frac{\partial w_i(\boldsymbol{\theta})}{\partial\theta_j}t_i(X)\Big) \\
&= 0
\end{aligned}$$

We know that

$$\int f(x|\boldsymbol{\theta})dx = \int h(x)c(\boldsymbol{\theta})\exp\Big(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\Big)dx = c(\boldsymbol{\theta})\int h(x)\exp\Big(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\Big)dx = 1$$

Then,

$$\int h(x)\exp\Big(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\Big)dx = \frac{1}{c(\boldsymbol{\theta})}$$

Therefore,

$$\begin{aligned}
\mathrm{E}\Big(\sum_{i=1}^{k}\frac{\partial w_i(\boldsymbol{\theta})}{\partial\theta_j}t_i(X)\Big) &= -\frac{\partial}{\partial\theta_j}c(\boldsymbol{\theta})\int h(x)\exp\Big(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})t_i(x)\Big)dx \\
&= -\frac{\partial}{\partial\theta_j}c(\boldsymbol{\theta})\frac{1}{c(\boldsymbol{\theta})} \\
&= -\frac{\partial}{\partial\theta_j}\log c(\boldsymbol{\theta})
\end{aligned}$$ $\square$

(b) Similar to (a), we take the second partial derivative. Let $g(x,\boldsymbol{\theta}) = \sum_{i=1}^k w_i(\boldsymbol{\theta})t_i(x)$ to simplify a bit.

$$
\begin{aligned}
\frac{\partial^2}{\partial\theta_j^2}\int f(x|\boldsymbol{\theta})dx &= \frac{\partial}{\partial\theta_j}\left\{\frac{\partial}{\partial\theta_j}c(\boldsymbol{\theta})\int h(x)\exp\Big(g(x,\boldsymbol{\theta})\Big)dx + \int h(x)c(\boldsymbol{\theta})\exp\Big(g(x,\boldsymbol{\theta})\Big)\Big(\frac{\partial}{\partial\theta_j}g(x,\boldsymbol{\theta})\Big)dx\right\}\\
&= \frac{\partial}{\partial\theta_j}\left\{\frac{\partial}{\partial\theta_j}c(\boldsymbol{\theta})\int h(x)\exp\Big(g(x,\boldsymbol{\theta})\Big)dx\right\} + \frac{\partial}{\partial\theta_j}\left\{c(\boldsymbol{\theta})\int h(x)\exp\Big(g(x,\boldsymbol{\theta})\Big)\Big(\frac{\partial}{\partial\theta_j}g(x,\boldsymbol{\theta})\Big)dx\right\}\\
&= \frac{\partial^2}{\partial\theta_j^2}c(\boldsymbol{\theta})\int h(x)\exp\Big(g(x,\boldsymbol{\theta})\Big)dx\\
&\quad + \frac{\partial}{\partial\theta_j}c(\boldsymbol{\theta})\int h(x)\exp\Big(g(x,\boldsymbol{\theta})\Big)\Big(\frac{\partial}{\partial\theta_j}g(x,\boldsymbol{\theta})\Big)dx\\
&\quad + \frac{\partial}{\partial\theta_j}c(\boldsymbol{\theta})\int h(x)\exp\Big(g(x,\boldsymbol{\theta})\Big)\Big(\frac{\partial}{\partial\theta_j}g(x,\boldsymbol{\theta})\Big)dx\\
&\quad + c(\boldsymbol{\theta})\int h(x)\Big\{\exp\Big(g(x,\boldsymbol{\theta})\Big)\Big(\frac{\partial}{\partial\theta_j}g(x,\boldsymbol{\theta})\Big)^2 + \exp\Big(g(x,\boldsymbol{\theta})\Big)\Big(\frac{\partial^2}{\partial\theta_j^2}g(x,\boldsymbol{\theta})\Big)\Big\}dx\\
&= \frac{\partial^2}{\partial\theta_j^2}c(\boldsymbol{\theta})\frac{1}{c(\boldsymbol{\theta})} + \frac{2}{c(\boldsymbol{\theta})}\frac{\partial}{\partial\theta_j}c(\boldsymbol{\theta})\mathrm{E}\Big(\sum_{i=1}^k\frac{\partial w_i(\boldsymbol{\theta})}{\partial\theta_j}t_i(X)\Big)\\
&\quad + \mathrm{E}\Big(\Big(\sum_{i=1}^k\frac{\partial w_i(\boldsymbol{\theta})}{\partial\theta_j}t_i(X)\Big)^2\Big) + \mathrm{E}\Big(\sum_{i=1}^k\frac{\partial^2 w_i(\boldsymbol{\theta})}{\partial\theta_j^2}t_i(X)\Big)\\
&= \frac{\partial^2}{\partial\theta_j^2}c(\boldsymbol{\theta})\frac{1}{c(\boldsymbol{\theta})} - \frac{2}{c(\boldsymbol{\theta})}\frac{\partial}{\partial\theta_j}c(\boldsymbol{\theta})\frac{\partial}{\partial\theta_j}\log c(\boldsymbol{\theta}) + \mathrm{E}\Big(\Big(\sum_{i=1}^k\frac{\partial w_i(\boldsymbol{\theta})}{\partial\theta_j}t_i(X)\Big)^2\Big) + \mathrm{E}\Big(\sum_{i=1}^k\frac{\partial^2 w_i(\boldsymbol{\theta})}{\partial\theta_j^2}t_i(X)\Big)\\
&= \frac{\partial^2}{\partial\theta_j^2}\log c(\boldsymbol{\theta}) - \Big(\frac{\partial}{\partial\theta_j}\log c(\boldsymbol{\theta})\Big)^2 + \mathrm{E}\Big(\Big(\sum_{i=1}^k\frac{\partial w_i(\boldsymbol{\theta})}{\partial\theta_j}t_i(X)\Big)^2\Big) + \mathrm{E}\Big(\sum_{i=1}^k\frac{\partial^2 w_i(\boldsymbol{\theta})}{\partial\theta_j^2}t_i(X)\Big)\\
&= 0
\end{aligned}
$$

Then,

$$
\mathrm{E}\Big(\Big(\sum_{i=1}^k\frac{\partial w_i(\boldsymbol{\theta})}{\partial\theta_j}t_i(X)\Big)^2\Big) = \Big(\frac{\partial}{\partial\theta_j}\log c(\boldsymbol{\theta})\Big)^2 - \frac{\partial^2}{\partial\theta_j^2}\log c(\boldsymbol{\theta}) - \mathrm{E}\Big(\sum_{i=1}^k\frac{\partial^2 w_i(\boldsymbol{\theta})}{\partial\theta_j^2}t_i(X)\Big)
$$

Therefore, we have

$$
\begin{aligned}
\mathrm{Var}\Big(\sum_{i=1}^k\frac{\partial w_i(\boldsymbol{\theta})}{\partial\theta_j}t_i(X)\Big) &= \mathrm{E}\Big(\Big(\sum_{i=1}^k\frac{\partial w_i(\boldsymbol{\theta})}{\partial\theta_j}t_i(X)\Big)^2\Big) - \mathrm{E}\Big(\sum_{i=1}^k\frac{\partial w_i(\boldsymbol{\theta})}{\partial\theta_j}t_i(X)\Big)^2\\
&= \Big(\frac{\partial}{\partial\theta_j}\log c(\boldsymbol{\theta})\Big)^2 - \frac{\partial^2}{\partial\theta_j^2}\log c(\boldsymbol{\theta}) - \mathrm{E}\Big(\sum_{i=1}^k\frac{\partial^2 w_i(\boldsymbol{\theta})}{\partial\theta_j^2}t_i(X)\Big) - \Big(\frac{\partial}{\partial\theta_j}\log c(\boldsymbol{\theta})\Big)^2\\
&= -\frac{\partial^2}{\partial\theta_j^2}\log c(\boldsymbol{\theta}) - \mathrm{E}\Big(\sum_{i=1}^k\frac{\partial^2 w_i(\boldsymbol{\theta})}{\partial\theta_j^2}t_i(X)\Big)
\end{aligned}
$$

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