Homework 7

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Problem 1

Let $S_p^2 = \frac{10S_X^2+20S_Y^2}{11+21-2} = 1.22$, 95% CI for $\mu_1 - \mu_2$ is $\left((\bar{X}-\bar{Y}) \pm S_p\sqrt{\frac{11+21}{11\times 21}}t_{11+21-2,0.975}\right) = (-1.64, 0.04)$

Problem 2

Voting for candidate A or not can be treated as a Bernoulli random variable. Let X_i be the i-th voter's vote, $X_i \sim \text{Ber}(p)$, where p is the probability of voting for candidate A. Here, $n = 1000$ and $\sum_{i=1}^n X_i = 450 \sim \text{Bin}(n,p)$. Note that $\text{Bin}(n,p)$ is approximately $N(np, np(1-p))$, so $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i \sim N(p, \frac{p(1-p)}{n})$, $\frac{\bar{X}-p}{\sqrt{p(1-p)/n}} \sim N(0,1)$

We use $\sqrt{\frac{\bar{X}(1-\bar{X})}{n}}$ to estimate $\sqrt{\frac{p(1-p)}{n}}$, so 95% CI for p is $\left(\bar{X} \pm 1.96\sqrt{\frac{\bar{X}(1-\bar{X})}{n}}\right) = (0.43, 0.46)$.

Therefore, 95% confidence interval for the true supporting rate of candidate A is $(0.43, 0.46)$. The confidence interval does not cover 0.5.

Problem 3

Let $\bar{X} = \frac{1}{m}\sum_{i=1}^m X_i, \bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i, S_X^2 = \frac{1}{m-1}\sum_{i=1}^m (X_i - \bar{X})^2, S_Y^2 = \frac{1}{n-1}\sum_{i=1}^n (Y_i - \bar{Y})^2$. Suppose $\sigma_2^2 = \lambda\sigma_1^2 = \sigma^2$, $\bar{Y} - \bar{X} \sim N(b-a, \sigma^2(\frac{1}{\lambda m} + \frac{1}{n}))$, so $\frac{\bar{Y}-\bar{X}-(b-a)}{\sigma\sqrt{\frac{1}{\lambda m}+\frac{1}{n}}} \sim N(0,1)$

Let $S_p^2 = \frac{\lambda(m-1)S_X^2 + (n-1)S_Y^2}{m+n-2}$, note that $\frac{\lambda(m-1)S_X^2}{\sigma^2} = \frac{(m-1)S_X^2}{\sigma_1^2} \sim \chi^2_{m-1}$, $\frac{(n-1)S_Y^2}{\sigma^2} \sim \chi^2_{n-1}$, so $\frac{\lambda(m-1)S_X^2+(n-1)S_Y^2}{\sigma^2} \sim \chi^2_{m+n-2}$, $\frac{S_p^2}{\sigma^2} \sim \frac{\chi^2_{m+n-2}}{m+n-2}$

We can replace σ with S_p, we have $\frac{\bar{Y}-\bar{X}-(b-a)}{S_p\sqrt{\frac{1}{\lambda m}+\frac{1}{n}}} = \frac{\bar{Y}-\bar{X}-(b-a)}{\sigma\sqrt{\frac{1}{\lambda m}+\frac{1}{n}}} \Bigg/ \sqrt{\frac{S_p^2}{\sigma^2}} \sim t_{m+n-2}$, so $1-\alpha$ CI for $b-a$ is $\left(\bar{Y} - \bar{X} \pm S_p\sqrt{\frac{1}{\lambda m} + \frac{1}{n}}t_{m+n-2,1-\frac{\alpha}{2}}\right)$

Problem 4

$\bar{X} \sim N(\mu, \frac{16}{n})$, so $\frac{\bar{X}-\mu}{4/\sqrt{n}} \sim N(0,1)$, let $z_\alpha = \Phi^{-1}(\alpha)$, we have $P(z_{\alpha_1} < \frac{\bar{X}-\mu}{4/\sqrt{n}} < z_{1-\alpha_2}) = 1 - \alpha_1 - \alpha_2$ for some α_1, α_2. Let $\alpha_1 + \alpha_2 = \alpha$, so $1-\alpha$ CI for μ is $\left(\bar{X} - \frac{4}{\sqrt{n}}z_{1-\alpha_2}, \bar{X} - \frac{4}{\sqrt{n}}z_{\alpha_1}\right)$, its length is $\frac{4}{\sqrt{n}}(z_{1-\alpha_2} - z_{\alpha_1})$. To get minimal n, we first need to maximize $(z_{1-\alpha_2} - z_{\alpha_1})$.

$z_{1-(\alpha-\alpha_1)} - z_{\alpha_1} = -\Phi^{-1}(\alpha-\alpha_1) - \Phi^{-1}(\alpha_1) \le -2\sqrt{\Phi^{-1}(\alpha-\alpha_1)\Phi^{-1}(\alpha_1)}$, equation holds iff $\Phi^{-1}(\alpha - \alpha_1) = \Phi^{-1}(\alpha_1) \Rightarrow \alpha_1 = \frac{\alpha}{2}$

So, $\frac{4}{\sqrt{n}}(z_{1-\frac{\alpha}{2}} - z_{\frac{\alpha}{2}}) = \frac{8}{\sqrt{n}}z_{1-\frac{\alpha}{2}} \le L$, $n \ge \frac{64}{L^2}(z_{1-\frac{\alpha}{2}})^2$

Therefore, minimal n is $\lceil\frac{64}{L^2}(z_{1-\frac{\alpha}{2}})^2\rceil$, where $\lceil\cdot\rceil$ is the ceiling function.