Homework 3

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1.(a) $A(A-I)=0 \Rightarrow 0=|A(A-I)|=|A||A-I| \Rightarrow |A|=0$ or $|A-I|=0$. If $|A|=0$, then 0 is obviously an eigenvalue. If $|A-I|=0$, then 1 is obviously an eigenvalue. So, A has at least one eigenvalue 0 or 1.

□

(b) Let x be the eigenvector corresponding to the eigenvalue λ. It has been proven that λ can be 0 or 1 previously at (a). If $\lambda \neq 0, 1$, then $Ax = \lambda x \neq \lambda^2 x = AAx \Rightarrow A \neq AA$, which is a contradiction. So, λ must be 0 or 1.

□

2. Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ and $p(\lambda) = |\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$. We know that λ_i are to solution to $p(\lambda)=0$, so $p(\lambda) = \prod_{i=1}^n (\lambda - \lambda_i)$. Let $\lambda = 0$, we get $p(0) = \prod_{i=1}^n (0-\lambda_i) = (-1)^n \prod_{i=1}^n \lambda_i$. Also, $p(0) = |-A| = (-1)^n |A|$ So, $|A| = \prod_{i=1}^n \lambda_i$.

Consider the coefficient of λ^{n-1}. From $p(\lambda) = \prod_{i=1}^n (\lambda - \lambda_i)$, the coefficient of λ^{n-1} is $-(\sum_{i=1}^n \lambda_i)$. Expanding the determinant, we have $|\lambda I - A| = \begin{vmatrix} \lambda - a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & \lambda - a_{nn} \end{vmatrix} = (\lambda - a_{11}) \ldots (\lambda - a_{nn}) + q(\lambda)$, where $q(\lambda)$ is a function of λ with max degree of $n-2$, so the coefficient of λ^{n-1} is $-(\sum_{i=1}^n a_{ii}) = -\mathrm{tr}(A)$. Therefore, we have $\sum_{i=1}^n \lambda_i = \mathrm{tr}(A)$

□

3. We have $BAx = \lambda x$. Then, $ABAx = A\lambda x = \lambda Ax$. So, λ is also an eigenvalue of AB with the corresponding eigenvector being Ax.

If λ is not an eigenvalue of BA. Then, $\forall x \neq 0, \lambda BAx \neq \lambda x$. So, $\lambda ABAx \neq \lambda Ax$. Here, Ax is also any vectors not equal to 0, So, λ is not an eigenvalue of AB.

□