Homework 4

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Problem 1

(a)

Let $x = \theta$ and $y = a\theta^2$. On the x-y plane, (x, y) forms a parabola. So, the parameter space does not contain a two-dimensional open set.

(b)

The sample's joint distribution is

$$\begin{aligned} f(\mathbf{x}, \theta) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi a\theta^2}} \exp\left\{-\frac{(x_i - \theta)^2}{2a\theta^2}\right\} \\ &= \frac{1}{(\sqrt{2\pi a\theta^2})^n} \exp\left\{-\frac{\sum_{i=1}^{n}(x_i - \theta)^2}{2a\theta^2}\right\} \\ &= \frac{1}{(\sqrt{2\pi a\theta^2})^n} \exp\left\{-\frac{(n-1)S^2 - n(\bar{X} - \theta)^2}{2a\theta^2}\right\} \end{aligned}$$

So $T = (\bar{X}, S^2)$ is a sufficient statistic for θ. However, let $g(x, y) = ax^2 - y$, we have $\mathrm{E}(g(T)) = a\theta^2 - a\theta^2 = 0$, so the family of distributions is not complete.

Problem 2

(a)

The sample's joint distribution is

$$f(\mathbf{x}, \theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} \mathbb{I}(0 < x_i < 1) = \prod_{i=1}^{n} \mathbb{I}(0 < x_i < 1) \cdot \theta^n \exp\left\{(\theta - 1) \sum_{i=1}^{n} \log x_i\right\}$$

We can see that $\sum_{i=1}^{n} \log X_i$ is a sufficient statistic for θ. However, $\sum_{i=1}^{n} X_i$ can not be written as a function of $\sum_{i=1}^{n} \log X_i$ and vice versa, so $\sum_{i=1}^{n} X_i$ is not a sufficient statistic for θ.

(b)

$f(x, \theta) = \theta \exp\left\{(\theta - 1) \log x\right\}$, so f is from an exponential family, $\theta > 0$, so the parameter space contains one-dimensional open set. Therefore, $T = \sum_{i=1}^{n} \log x_i$ is a complete sufficient statistic for θ.

Problem 3

$$
\begin{aligned}
\mathrm{E}(X) &= \sum_{k=1}^{\infty} -\frac{k}{\ln(1-p)}\frac{p^k}{k} = \sum_{k=1}^{\infty} -\frac{p^k}{\ln(1-p)} = -\frac{p}{(1-p)\ln(1-p)} \\
&\Rightarrow \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X} = -\frac{\hat{p}}{(1-\hat{p})\ln(1-\hat{p})} \\
&\Rightarrow (\hat{p}-1)\ln(1-\hat{p}) = \frac{\hat{p}}{\bar{X}} \\
\text{Take derivative} &\Rightarrow \ln(1-\hat{p}) + 1 = \frac{1}{\bar{X}} \\
&\Rightarrow \hat{p} = 1 - e^{\frac{1}{\bar{X}}-1}
\end{aligned}
$$

Problem 4

Let $Y = \frac{X-\mu}{\sigma} \sim N(0,1)$, then $P(X>1) = P(Y > \frac{1-\mu}{\sigma}) = 1 - \Phi(\frac{1-\mu}{\sigma})$ where Φ is the CDF of standard normal distribution. We know $\hat{\mu} = \bar{X}$ and $\hat{\sigma}^2 = S^2$, so $P\widehat{(X>1)} = 1 - \Phi(\frac{1-\bar{X}}{S})$

Problem 5

Note that $x_1, \ldots, x_n$ are iid from $f(x)$, so the probability of $x_1 < \cdots < x_n$ is $\frac{1}{n!}$ of all the permutations. Then, we have

$$
\begin{aligned}
\int_{a<x_1<\cdots<x_n<b} \cdots \int f(x_1)\cdots f(x_n)dx_1\cdots dx_n &= \frac{1}{n!}\int_{x_1,\ldots,x_n\in(a,b)} \cdots \int f(x_1)\cdots f(x_n)dx_1\cdots dx_n \\
&= \frac{1}{n!}\prod_{i=1}^{n}\int_a^b f(x)dx \\
&= \frac{1}{n!}[F(b)-F(a)]^n
\end{aligned}
$$

$\square$