Homework 11

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Problem 1

We have $f(X|\theta) \propto \theta^X (1-\theta)^{n-X}$ and $\pi(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$. So,

$$\pi(\theta|X) \propto \theta^{X+\alpha-1}(1-\theta)^{n-X+\beta-1} \sim \mathrm{Beta}(X+\alpha, n-X+\beta)$$

We have

$$\begin{aligned}
\mathrm{E}(\frac{1}{1-\theta}|X) &= \frac{\Gamma(\alpha+\beta+n)}{\Gamma(X+\alpha)\Gamma(n-X+\beta)} \int_0^1 \frac{1}{1-\theta}\theta^{X+\alpha-1}(1-\theta)^{n-X+\beta-1} d\theta \\
&= \frac{\Gamma(\alpha+\beta+n)}{\Gamma(X+\alpha)\Gamma(n-X+\beta)} \frac{\Gamma(X+\alpha)\Gamma(n-X+\beta-1)}{\Gamma(\alpha+\beta+n-1)} \\
&= \frac{\alpha+\beta+n-1}{n-X+\beta-1} \\
\mathrm{E}(\frac{1}{\theta(1-\theta)}|X) &= \frac{\Gamma(\alpha+\beta+n)}{\Gamma(X+\alpha)\Gamma(n-X+\beta)} \int_0^1 \frac{1}{\theta(1-\theta)}\theta^{X+\alpha-1}(1-\theta)^{n-X+\beta-1} d\theta \\
&= \frac{\Gamma(\alpha+\beta+n)}{\Gamma(X+\alpha)\Gamma(n-X+\beta)} \frac{\Gamma(X+\alpha-1)\Gamma(n-X+\beta-1)}{\Gamma(\alpha+\beta+n-2)} \\
&= \frac{(\alpha+\beta+n-1)(\alpha+\beta+n-2)}{(n-X+\beta-1)(X+\alpha-1)}
\end{aligned}$$

So, the Bayes estimator is

$$\begin{aligned}
\hat{\theta} &= \frac{\mathrm{E}(\frac{\theta}{\theta(1-\theta)}|X)}{\mathrm{E}(\frac{1}{\theta(1-\theta)}|X)} = \frac{\alpha+\beta+n-1}{n-X+\beta-1} \cdot \frac{(n-X+\beta-1)(X+\alpha-1)}{(\alpha+\beta+n-1)(\alpha+\beta+n-2)} \\
&= \frac{X+\alpha-1}{\alpha+\beta+n-2}
\end{aligned}$$

Problem 2

We have $f(\mathbf{X}|\theta) \propto \theta^n e^{-\theta n \bar{X}}$ and $\pi(\theta) \propto \theta^{\beta-1} e^{-\alpha\theta}$. So,

$$\pi(\theta|\mathbf{X}) \propto \theta^{n+\beta-1} e^{-\theta(n\bar{X}+\alpha)} \sim \Gamma(n+\beta, n\bar{X}+\alpha)$$

So, the Bayes estimator for θ is $\hat{\theta} = \mathrm{E}(\theta|\mathbf{X}) = \dfrac{n+\beta}{n\bar{X}+\alpha}$

Let $\lambda = \frac{1}{\theta}$, we have $f(\mathbf{X}|\lambda) \propto \lambda^{-n} e^{-\frac{n\bar{X}}{\lambda}}$, $\lambda \sim \mathrm{IG}(\beta, \frac{1}{\alpha}), \pi(\lambda) \propto \theta^{-(\beta+1)} e^{-\frac{1}{\alpha\theta}}$. So,

$$\pi(\lambda|\mathbf{X}) \propto \lambda^{-(n+\beta+1)} e^{-\frac{n\bar{X}+1/\alpha}{\theta}} \sim \mathrm{IG}(n+\beta, n\bar{X}+\frac{1}{\alpha})$$

So, the Bayes estimator for $\frac{1}{\theta}$ is $\widehat{\dfrac{1}{\theta}} = \mathrm{E}(\lambda|\mathbf{X}) = \dfrac{n\bar{X}+\frac{1}{\alpha}}{n+\beta-1}$

Problem 3

The main message of the paper is that p-value is meaningless under large data because it uses group mean when testing, thus it always shows significance under large n. D-value is introduced to deal with this problem. It tells how strongly the populations from two groups can be discriminated on the individual basis and does not have a tendency to increase or decrease with the sample size.