Homework 3

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Problem 1

Sample's joint distribution is

$$f_{\mathbf{X}}(\mathbf{x}|\theta) = \prod_{i=1}^{n} e^{i\theta - x_i}\mathbb{I}(x_i > i\theta) = e^{-\sum_{i=1}^{n} x_i} e^{\frac{n(n+1)}{2}\theta} \prod_{i=1}^{n} \mathbb{I}(\frac{x_i}{i} > \theta) = \underbrace{e^{-\sum_{i=1}^{n} x_i}}_{h(\mathbf{x})} \cdot \underbrace{e^{\frac{n(n+1)}{2}\theta}\mathbb{I}(\min \frac{x_i}{i} > \theta)}_{g(T|\theta)}$$

So, $f_{\mathbf{X}}$ can be factored into $h(\mathbf{x})$ and $g(T|\theta)$. Thus, $T = \min \frac{x_i}{i}$ is a sufficient statistic for θ.

$\square$

Problem 2

To prove the independence of $\bar{X}$ and $X_{(n)} - X_{(1)}$, we just need to prove the independence of $\sum_{k=1}^{n} X_k$ and $X_i - X_j$ for $\forall i \neq j$.

We know that X_i's are jointly normal, so $\sum_{k=1}^{n} X_k$ and $X_i - X_j$ are bivariate normal. $\mathrm{Cov}(\sum_{k=1}^{n} X_k, X_i - X_j) = \mathrm{Var}(X_i) - \mathrm{Var}(X_j) = 0$. Therefore, $\sum_{k=1}^{n} X_k$ and $X_i - X_j$ are independent for $\forall i \neq j$. Thus, $\bar{X}$ and $X_{(n)} - X_{(1)}$ are independent.

$\square$

Problem 3

Sample's joint distribution is

$$f(\mathbf{x}|\theta) = \prod_{i=1}^{n} \frac{1}{2} e^{-|x_i - \theta|} = \frac{1}{2^n} e^{-\sum_{i=1}^{n} |x_i - \theta|}$$

For any samples $\mathbf{X}$ and $\mathbf{Y}$, we have

$$\frac{f(\mathbf{x}|\theta)}{f(\mathbf{y}|\theta)} = \frac{e^{-\sum_{i=1}^{n} |x_i - \theta|}}{e^{-\sum_{i=1}^{n} |y_i - \theta|}} = e^{\sum_{i=1}^{n} |y_i - \theta| - \sum_{i=1}^{n} |x_i - \theta|}$$

We can see that $\frac{f(\mathbf{x}|\theta)}{f(\mathbf{y}|\theta)}$ is independent of θ iff $X_{(i)} = Y_{(i)}$ for $i = 1, \ldots, n$. So, $T = (X_{(1)}, \ldots, X_{(n)})$ are a minimal sufficient statistic for θ, i.e., further reduction is not possible.

$\square$

Problem 4

Let $Y = \log X$, $f_Y(y|\alpha) = \alpha e^{y(\alpha-1)} e^{-e^{y\alpha}} e^{y} = \alpha e^{y\alpha} e^{-e^{y\alpha}}$.

Let $Z = \alpha Y$. $f_Z(z|\alpha) = \alpha e^{z} e^{-e^{z}} \frac{1}{\alpha} = e^{z} e^{-e^{z}}$. We can see that Z is independent of α.

So, $\frac{\log X_1}{\log X_2} = \frac{Z_1}{Z_2}$ is independent of α. Therefore, $\frac{\log X_1}{\log X_2}$ is an ancillary statistic.

$\square$