Homework 13

蒋翌坤 20307100013

1. Let $X^TX = PDP^T$, where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$ and P is orthogonal. Then,

$$\begin{aligned}
\mathbb{E}(\left\|\hat\beta\right\|^2) &= \mathbb{E}(\hat\beta^T\hat\beta) = \mathbb{E}(y^TX(X^TX)^{-2}X^Ty) \\
&= (X\beta)^TX(X^TX)^{-2}X^TX\beta + \sigma^2\mathrm{tr}(X(X^TX)^{-2}X^T) \\
&= \beta^T\beta + \sigma^2\mathrm{tr}(PD^{-1}P^T) = \|\beta\|^2 + \sigma^2\sum_{j=1}^p \lambda_j^{-1}
\end{aligned}$$

□

2.

$$\begin{aligned}
\mathrm{MSE}(\tilde\beta) &= \mathbb{E}\left((\tilde\beta-\beta)^T(\tilde\beta-\beta)\right) = \mathbb{E}\left(\tilde\beta^T\tilde\beta - \beta^T\tilde\beta - \tilde\beta^T\beta + \beta^T\beta\right) \\
&= \mathrm{tr}(\mathrm{Cov}(\tilde\beta)) + \mathbb{E}(\tilde\beta)^T\mathbb{E}(\tilde\beta) - 2\beta^T\mathbb{E}(\tilde\beta) + \beta^T\beta \\
&= \mathrm{tr}(\mathrm{Cov}(\tilde\beta)) + \left\|\mathbb{E}(\tilde\beta) - \beta\right\|^2
\end{aligned}$$

In addition,

$$\begin{aligned}
\mathbb{E}\left(\left\|P(\tilde\beta-\beta)\right\|\right) &= \mathbb{E}\left((\tilde\beta-\beta)^TP^TP(\tilde\beta-\beta)\right) = \mathbb{E}\left((\tilde\beta-\beta)^T(\tilde\beta-\beta)\right) \\
&= \mathbb{E}\left(\left\|\tilde\beta-\beta\right\|^2\right)
\end{aligned}$$

□

3. (a) Note that $\beta = \Phi_1\alpha_1 + \Phi_2\alpha_2$

$$\begin{aligned}
\mathbb{E}(\bar\beta) &= \mathbb{E}(\Phi_1\Lambda_1^{-1}\Phi_1^TX^Ty) = \Phi_1\Lambda_1^{-1}\Phi_1^TX^TX(\Phi_1\alpha_1 + \Phi_2\alpha_2) \\
&= \Phi_1\Lambda_1^{-1}\Phi_1^T\Phi\Lambda\Phi^T(\Phi_1\alpha_1 + \Phi_2\alpha_2) \\
&= \Phi_1\mathrm{diag}(I_r, 0)\alpha_1 + \Phi_1\mathrm{diag}(I_r, 0)\mathrm{diag}(0, I_{n-r})\alpha_2 = \Phi_1\alpha_1
\end{aligned}$$

(b)

$$\begin{aligned}
\mathrm{MSE}(\bar\beta) &= \mathrm{tr}(\mathrm{Cov}(\bar\beta)) + \left\|\mathbb{E}(\bar\beta) - \beta\right\|^2 \\
&= \sigma^2\mathrm{tr}(\Phi_1\Lambda_1^{-1}\Phi_1^TX^T(\Phi_1\Lambda_1^{-1}\Phi_1^TX^T)^T) + \|\Phi_2\alpha_2\|^2 \\
&= \sigma^2\mathrm{tr}(\Lambda_1^{-1}) + \|\alpha_2\|^2 \\
&= \sigma^2\sum_{j=1}^r \lambda_j^{-1} + \|\alpha_2\|^2
\end{aligned}$$

□