作业 1

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2.8

$$\begin{aligned}\mathrm{Cov}(W_t, W_{t-k}) &= \mathrm{Cov}(\sum_{i=1}^{n} c_i Y_{t-i+1}, \sum_{j=1}^{n} c_j Y_{t-k-j+1}) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \mathrm{Cov}(Y_{t-i+1}, Y_{t-k-j+1}) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \gamma_{k+j-i}\end{aligned}$$

可以发现，$\mathrm{Cov}(W_t, W_{t-k})$ 只与两个时刻差 k 有关，此外二阶矩和 $\{Y_t\}$ 一样有限的，均值为常数 $\mu\sum_{i=1}^{n} c_i$，因此 $\{W_t\}$ 是平稳的。

2.17

$$\begin{aligned}\mathrm{Var}(\bar{Y}) &= \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(Y_i) + \frac{2}{n^2}\sum_{k=1}^{n-1}\sum_{j=1}^{n-k}\mathrm{Cov}(Y_k, Y_{k+j}) = \frac{1}{n^2}n\gamma_0 + \frac{2}{n^2}\sum_{k=1}^{n-1}(n-k)\gamma_k \\ &= \frac{\gamma_0}{n} + \frac{2}{n}\sum_{k=1}^{n-1}(1-\frac{k}{n})\gamma_k = \frac{1}{n}\sum_{k=-n+1}^{n-1}(1-\frac{|k|}{n})\gamma_k\end{aligned}$$

2.18

a

$$\begin{aligned}\sum_{t=1}^{n}(Y_t-\bar{Y})^2 + n(\bar{Y}-\mu)^2 &= \sum_{t=1}^{n}(Y_t^2 - 2Y_t\bar{Y} + \bar{Y}^2) + n(\bar{Y}^2 - 2\bar{Y}\mu + \mu^2) \\ &= \sum_{t=1}^{n}Y_t^2 - 2\bar{Y}\sum_{t=1}^{n}Y_t + \bar{Y}\sum_{t=1}^{n}Y_t + \bar{Y}\sum_{t=1}^{n}Y_t - 2\mu\sum_{t=1}^{n}Y_t + n\mu^2 \\ &= \sum_{t=1}^{n}Y_t^2 - 2\mu\sum_{t=1}^{n}Y_t + n\mu^2 = \sum_{t=1}^{n}(Y_t-\mu)^2\end{aligned}$$

b

$$\begin{aligned}\mathrm{E}(S^2) &= \frac{1}{n-1}\Big(\mathrm{E}(\sum_{t=1}^{n}(Y_t-\mu)^2) - n\mathrm{E}((\bar{Y}-\mu)^2)\Big) \\ &= \frac{1}{n-1}\sum_{t=1}^{n}\mathrm{Var}(Y_t) - \frac{n}{n-1}\mathrm{Var}(\bar{Y}) = \frac{n}{n-1}\gamma_0 - \frac{n}{n-1}\mathrm{Var}(\bar{Y}) \\ &= \frac{n}{n-1}\Big(\gamma_0 - \frac{\gamma_0}{n} - \frac{2}{n}\sum_{k=1}^{n-1}(1-\frac{k}{n})\gamma_k\Big) = \gamma_0 - \frac{2}{n-1}\sum_{k=1}^{n-1}(1-\frac{k}{n})\gamma_k\end{aligned}$$

c

当 $k \geq 1$ 时，$\gamma_k = 0$，于是 $\mathrm{E}(S^2) = \gamma_0$

2.24

令 γ_k 为 $\{X_t\}$ 的自协方差函数，于是有

$$\begin{aligned}\mathrm{Cov}(Y_t, Y_{t-k}) &= \mathrm{Cov}(X_t, X_{t-k}) + \mathrm{Cov}(X_t, e_{t-k}) + \mathrm{Cov}(e_t, X_{t-k}) + \mathrm{Cov}(e_t, e_{t-k}) \\ &= \gamma_k, \quad k \geq 1\end{aligned}$$

$\mathrm{E}(Y_t) = \mathrm{E}(X_t) = \mu$，二阶矩和 $\{X_t\}$ 一样有限，因此，$\{Y_t\}$ 也是平稳的。

$$\mathrm{Corr}(Y_t, Y_{t-k}) = \frac{\mathrm{Cov}(Y_t, Y_{t-k})}{\mathrm{Var}(Y_t)} = \frac{\gamma_k}{\gamma_0 + \sigma_e^2} = \frac{\rho_k}{1 + \sigma_e^2/\gamma_0} = \frac{\rho_k}{1 + \sigma_e^2/\sigma_X^2}$$

4.1

$\theta_1 = \dfrac{1}{2}, \theta_2 = -\dfrac{1}{4}$，于是 $\rho_0 = 1, \rho_1 = \dfrac{-1/2 - 1/8}{1 + 1/4 + 1/16} = -\dfrac{10}{21}, \rho_2 = \dfrac{1/4}{1 + 1/4 + 1/16} = \dfrac{4}{21}$, $\rho_k = 0, k \geq 3$

4.3

$\rho_1 = \dfrac{-\theta}{1 + \theta^2}$，令 $f(\theta) = \dfrac{-\theta}{1 + \theta^2}$，$\dfrac{df(\theta)}{d\theta} = \dfrac{\theta^2 - 1}{(1 + \theta^2)^2} = 0 \Rightarrow \theta = 1, -1$，于是找到了 ρ_1 的最值点: $\theta = 1, \min \rho_1 = -0.5; \theta = -1, \max \rho_1 = 0.5$

4.8

$\phi_1 = 0, \lambda_1 = \sqrt{\phi_2}, \lambda_2 = -\sqrt{\phi_2}$，于是 ϕ_2 需要满足以下条件: $\begin{cases} \sqrt{\phi_2} < 1 \\ |\phi_2| < 1 \end{cases} \Rightarrow \phi_2 \in (-1, 1)$