Homework 8

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1. We know $\frac{\text{SSE}}{\sigma^2} \sim \chi^2(n-p)$, $\text{SST} = \sum_{i=1}^n (y_i - \bar{y})^2$, y_i is i.i.d from normal with variance σ^2, so $\frac{\text{SST}}{\sigma^2} \sim \chi^2(n-1)$. Therefore, $\text{SSR} = \text{SST} - \text{SSE} \sim \chi^2(p-1)$,

$$\frac{(n-p)/(p-1)}{R^{-2}-1} = \frac{\text{SSR}/(p-1)}{\text{SSE}/(n-p)} = \frac{\frac{\text{SSR}}{\sigma^2}/(p-1)}{\frac{\text{SSE}}{\sigma^2}/(n-p)} \sim F(p-1, n-p)$$

$\square$

2.(a) Note that $Z \perp \epsilon$, so $X \perp \epsilon$

$$\begin{aligned}
\mathbb{E}(\hat{\beta}) &= \mathbb{E}\Big[\Big(\sum_{i=1}^n X_i X_i^T\Big)^{-1} \sum_{k=1}^n X_k Y_k\Big] = \mathbb{E}\Big[\Big(\sum_{i=1}^n X_i X_i^T\Big)^{-1} \sum_{k=1}^n X_k (X_k^T \beta + \epsilon_k)\Big] \\
&= \mathbb{E}\Big[\Big(\sum_{i=1}^n X_i X_i^T\Big)^{-1} \sum_{k=1}^n X_k X_k^T \beta\Big] + \mathbb{E}\Big[\Big(\sum_{i=1}^n X_i X_i^T\Big)^{-1} \sum_{k=1}^n X_k\Big]\mathbb{E}(\epsilon_k) \\
&= \beta
\end{aligned}$$

Note that $\mathbb{E}(\epsilon_k \epsilon_k^T) = \mathbb{E}(\epsilon_k^2) = \sigma^2$, $\mathbb{E}(\epsilon_k \epsilon_j^T) = \mathbb{E}(\epsilon_k \epsilon_j) = \mathbb{E}(\epsilon_k)\mathbb{E}(\epsilon_j) = 0, \forall k \neq j$

$$\begin{aligned}
\text{Cov}(\hat{\beta}) &= \text{Cov}\Big[\Big(\sum_{i=1}^n X_i X_i^T\Big)^{-1} \sum_{k=1}^n X_k (X_k^T \beta + \epsilon_k)\Big] = \text{Cov}\Big[\beta + \Big(\sum_{i=1}^n X_i X_i^T\Big)^{-1} \sum_{k=1}^n X_k \epsilon_k\Big] \\
&= \text{Cov}\Big[\Big(\sum_{i=1}^n X_i X_i^T\Big)^{-1} \sum_{k=1}^n X_k \epsilon_k\Big] \\
&= \sum_{k=1}^n \text{Cov}\Big[\Big(\sum_{i=1}^n X_i X_i^T\Big)^{-1} X_k \epsilon_k\Big] + \sum_{k \neq j} \text{Cov}\Big[\Big(\sum_{i=1}^n X_i X_i^T\Big)^{-1} X_k \epsilon_k, \Big(\sum_{i=1}^n X_i X_i^T\Big)^{-1} X_j \epsilon_j\Big] \\
&= \sum_{k=1}^n \mathbb{E}\Big[\Big(\sum_{i=1}^n X_i X_i^T\Big)^{-1} X_k \epsilon_k \epsilon_k^T X_k^T \Big(\sum_{i=1}^n X_i X_i^T\Big)^{-1}\Big] + \sum_{k \neq j} \mathbb{E}\Big[\Big(\sum_{i=1}^n X_i X_i^T\Big)^{-1} X_k \epsilon_k \epsilon_j^T X_j^T \Big(\sum_{i=1}^n X_i X_i^T\Big)^{-1}\Big] \\
&= \sigma^2 \mathbb{E}\Big[\Big(\sum_{i=1}^n X_i X_i^T\Big)^{-1} \sum_{k=1}^n \Big(X_k X_k^T\Big)\Big(\sum_{i=1}^n X_i X_i^T\Big)^{-1}\Big] \\
&= \sigma^2 \mathbb{E}\Big[\Big(\sum_{i=1}^n X_i X_i^T\Big)^{-1}\Big] = \sigma^2 \Big\{\mathbb{E}\Big(\sum_{i=1}^n X_i X_i^T\Big)\Big\}^{-1} = \sigma^2 \Big\{n\Sigma\Big\}^{-1} \\
&= \frac{1}{n}\sigma^2 \Sigma^{-1}
\end{aligned}$$

So,

$$\mathbb{E}(n^{1/2}(\hat{\beta} - \beta)) = 0, \quad \text{Var}(n^{1/2}(\hat{\beta} - \beta)) = \sigma^2 \Sigma^{-1}, \quad n^{1/2}(\hat{\beta} - \beta) \xrightarrow{d} N(0, \sigma^2 \Sigma^{-1})$$

$\square$

(b) $\theta = \mathbb{E}(Y) = \mathbb{E}(X^T \beta) = \mu^T \beta$, $\mathbb{E}(\hat{\theta}) = \mu^T \mathbb{E}(\hat{\beta}) = \mu^T \beta = \theta$, so $\hat{\theta}$ is unbiased.

$\square$

(c) $n^{1/2}\hat{\beta} \xrightarrow{d} N(n^{1/2}\beta, \sigma^2\Sigma^{-1})$, so $n^{1/2}\hat{\theta} = n^{1/2}\mu^T\hat{\beta} \xrightarrow{d} N(n^{1/2}\mu^T\beta, \sigma^2\mu^T\Sigma^{-1}\mu)$

$$\tau^2 = \mathrm{Var}(n^{1/2}\hat{\theta}) = \sigma^2\mu^T\Sigma^{-1}\mu$$

Note that

$$\begin{aligned}
\mu^T\Sigma^{-1}\mu &= \begin{pmatrix} 1 \\ \mu_Z \end{pmatrix}^T \begin{pmatrix} 1 & \mu_Z \\ \mu_Z & \mathbb{E}(Z^2) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \mu_Z \end{pmatrix} \\
&= \mathrm{Var}(Z)^{-1} \begin{pmatrix} 1 \\ \mu_Z \end{pmatrix}^T \begin{pmatrix} \mathbb{E}(Z^2) & -\mu_Z \\ -\mu_Z & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \mu_Z \end{pmatrix} \\
&= 1
\end{aligned}$$

So, $\tau^2 = \sigma^2$

$$\mathrm{Var}(n^{1/2}\bar{Y}) = n\mathrm{Var}(\bar{Y}) = \frac{1}{n}\mathrm{Var}(\sum_{i=1}^{n} \gamma_0 + \gamma_1 Z_i + \epsilon_i) = \mathrm{Var}(Z) + \sigma^2 \geq \tau^2$$

The equation holds iff $\mathrm{Var}(Z) = 0$, i.e. Z is some known constants.