Homework 1

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Problem 1

We have $f_{X_{(1)},X_{(n)}}(x,y) = n(n-1)[F(y)-F(x)]^{n-2}f(x)f(y)$

Let $u = y-x, v = \frac{1}{2}(x+y)$, then $|J| = 1$

$$f_{R,V}(u,v) = n(n-1)[F(v+\frac{u}{2}) - F(v-\frac{u}{2})]^{n-2}f(v-\frac{u}{2})f(v+\frac{u}{2})$$

Here, $F(x) = \frac{x}{a}, f(x) = \frac{1}{a}$ for $0 < x < a$. Therefore, the joint pdf of R and V are

$$f_{R,V}(u,v) = \begin{cases} n(n-1)[\frac{v+\frac{u}{2}}{a} - \frac{v-\frac{u}{2}}{a}]^{n-2}\frac{1}{a}\frac{1}{a} = \frac{n(n-1)}{a^n}u^{n-2}, 0 < u < a, 0 < v < a \\ 0, \text{otherwise} \end{cases}$$

Problem 2

Draw two samples from $N(0,a)$ as X, Y, order them, we get $X_{(1)}, X_{(2)}$. Therefore, $Z = X_{(1)}$. So, the distribution of Z is the same as $X_{(1)}$:

$$f_Z(z) = 2[1-F(z)]^{2-1}f(z) = 2f(z)[1-F(z)]$$

where $F, f \sim N(0,a)$

Problem 3

$Y = d\tan X$, where $X \sim U(0,\pi/2)$, d is a constant. Therefore, the pdf of Y is

$$f_Y(y) = \frac{2}{\pi}\cdot\frac{1}{1+\frac{y^2}{d^2}}\cdot\frac{1}{d} = \frac{2}{\pi d(1+\frac{y^2}{d^2})} = \frac{2}{\pi d(1+\frac{y^2}{d^2})}, 0 < y < \infty$$

$$\begin{aligned} \mathrm{E}(Y) &= \int_0^\infty y\cdot\frac{2}{\pi d(1+\frac{y^2}{d^2})}dy = \frac{2}{\pi}\int_0^\infty \frac{\frac{y}{d}}{1+\frac{y^2}{d^2}}dy \\ &\overset{u=y/d}{=} \frac{2d}{\pi}\int_0^\infty \frac{u}{1+u^2}du = \frac{2d}{\pi}\cdot\frac{1}{2}\log(1+u^2)\Big|_0^\infty = \infty \end{aligned}$$

Problem 4

a

Let $W_1 = X_1^2/\sigma^2 \sim \chi_1^2, W_2 = X_2^2/\sigma^2 \sim \chi_1^2$, we have $Y_1 = \sigma^2(W_1+W_2), Y_2^2 = \frac{W_1}{W_1+W_2}$
$W_1 = Y_1Y_2^2/\sigma^2, W_2 = Y_1(1-Y_2^2)/\sigma^2$,

$$|J| = \begin{vmatrix} \frac{Y_2^2}{\sigma^2} & \frac{2Y_1Y_2}{\sigma^2} \\ \frac{1-Y_2^2}{\sigma^2} & -\frac{2Y_1Y_2}{\sigma^2} \end{vmatrix} = \frac{2Y_1|Y_2|}{\sigma^4}$$

The joint distribution of W_1, W_2 is

$$f_{W_1,W_2}(w_1,w_2) = \frac{1}{2}\cdot\frac{1}{\sqrt{w_1w_2}}\cdot\frac{1}{\Gamma(1/2)^2}\exp(-\frac{w_1+w_2}{2})$$

Then, the joint distribution of Y_1, Y_2 is

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= \frac{1}{2}\cdot\frac{\sigma^2}{|y_1y_2|\sqrt{1-y_2^2}}\cdot\frac{1}{\Gamma(1/2)^2}\exp(-\frac{y_1}{2\sigma^2})\cdot\frac{2y_1|y_2|}{\sigma^4} \\ &= \frac{1}{\sigma^2}\exp(-\frac{y_1}{2\sigma^2})\cdot\frac{1}{\Gamma(1/2)^2}\cdot\frac{1}{\sqrt{1-y_2^2}} \end{aligned}$$

b

$\frac{Y_1}{\sigma^2} = X_1^2/\sigma^2 + X_2^2/\sigma^2 \sim \chi_2^2$, so

$$f_{Y_1}(y_1) = \frac{1}{\sigma^2}\exp(-\frac{y_1}{2\sigma^2}), y_1 > 0$$

$$\begin{aligned} f_{Y_2}(y_2) = \int_0^\infty f_{Y_1,Y_2}(y_1,y_2)dy_1 &= \frac{1}{\Gamma(1/2)^2}\cdot\frac{1}{\sqrt{1-y_2^2}}\int_0^\infty\frac{1}{\sigma^2}\exp(-\frac{y_1}{2\sigma^2})dy_1 \\ &= \frac{1}{\Gamma(1/2)^2}\cdot\frac{1}{\sqrt{1-y_2^2}} \end{aligned}$$

$$f_{Y_1,Y_2}(y_1,y_2) = f_{Y_1}(y_1)f_{Y_2}(y_2)$$

Therefore, Y_1, Y_2 are independent.

A geometric interpretation is that consider a right triangle whose two sides are X_1 and X_2 ($X_1, X_2 > 0$), then Y_1 is the square of the hypotenuse, and Y_2 is the sine of the angle opposite to X_1. Since Y_1 and Y_2 are independent, we know that the length of the hypotenuse and the angle are independent.

Problem 5

Let $\delta = \sqrt{\sum_{i=1}^n a_i^2}$, $\mathbf{C} = \begin{pmatrix} \frac{a_1}{\delta} & \cdots & \frac{a_n}{\delta} \\ c_{21} & \cdots & c_{2n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}$, and $\mathbf{C}$ orthogonal. Let $\mathbf{W} = \mathbf{CX}$. We have

$$\mathbf{W}^T\mathbf{W} = \mathbf{X}^T\mathbf{C}^T\mathbf{CX} = \mathbf{X}^T\mathbf{X}$$

$$W_1 = \frac{a_1}{\delta}X_1 + \cdots + \frac{a_n}{\delta}X_n = \frac{1}{\delta}\sum_{i=1}^n a_iX_i$$

$$\mathrm{E}(W_1) = \frac{1}{\delta}\sum_{i=1}^n a_i\mathrm{E}(X_i) = \frac{1}{\sqrt{\sum_{i=1}^n a_i^2}}\sum_{i=1}^n a_i^2 = \sqrt{\sum_{i=1}^n a_i^2}$$

$$\mathrm{Var}(W_1) = \frac{1}{\delta^2}\sum_{i=1}^n a_i^2\mathrm{Var}(X_i) = \frac{1}{\sum_{i=1}^n a_i^2}\sum_{i=1}^n a_i^2 = 1$$

Therefore, $W_1 \sim N(\sqrt{\sum_{i=1}^n a_i^2}, 1)$

When $i \geq 2$,

$$\mathrm{E}(W_i) = \mathrm{E}(\sum_{k=1}^n c_{ik} X_k) = \sum_{k=1}^n c_{ik} a_k = \delta \sum_{k=1}^n c_{ik} \frac{a_k}{\delta} = 0$$

$$\mathrm{Var}(W_i) = \sum_{k=1}^n c_{ik}^2 = 1$$

Therefore, $W_i \sim N(0,1)$, $W_2^2 + \cdots + W_n^2 \sim \chi^2_{n-1}$.

Let $Y = W_1 \sim N(\sqrt{\sum_{i=1}^n a_i^2}, 1), Z = W_2^2 + \cdots + W_n^2 \sim \chi^2_{n-1}$, we have

$$X_1^2 + \ldots X_n^2 = W_1^2 + W_2^2 + \cdots + W_n^2 = Y^2 + Z$$

Y depends on W_1 and Z depends on $W_2, \ldots, W_n$, and $W_1, W_2, \ldots, W_n$ are independent. Therefore, Y and Z are independent.