Homework 2

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1. Note that $\frac{1}{n}\sum_{i=1}^{n}\hat{y} = \frac{1}{n}\sum_{i=1}^{n}\hat{a}_0 + \hat{b}_0 x_i = \hat{a}_0 + \hat{b}_0\bar{x} = \bar{y}$, so

$$\begin{aligned}\sum_{i=1}^{n}(y_i-\bar{y})^2 &= \sum_{i=1}^{n}(y_i-\hat{y}_i+\hat{y}_i-\bar{y})^2\\ &= \sum_{i=1}^{n}(y_i-\hat{y}_i)^2+\sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2+2\sum_{i=1}^{n}(y_i-\hat{y}_i)(\hat{y}_i-\bar{y})\\ &= \sum_{i=1}^{n}(y_i-\hat{y}_i)^2+\sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2\end{aligned}$$

Therefore, $\mathrm{SST} = \mathrm{SSE}+\mathrm{SSR}$

2. Proof of $t = \frac{\hat{b}\sqrt{L_{xx}}}{\hat{\sigma}} = \frac{\sqrt{n-2}r}{\sqrt{1-r^2}}$

$$t = \frac{\hat{b}\sqrt{L_{xx}}}{\hat{\sigma}} = \frac{\hat{b}\sqrt{L_{xx}}}{\sqrt{\frac{\sum_{i=1}^{n}(y_i-\hat{y}_i)^2}{n-2}}} = \frac{\hat{b}\sqrt{(n-2)L_{xx}}}{\sqrt{L_{yy}-\sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2}} = \frac{\sqrt{n-2}r}{\sqrt{1-\sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2/L_{yy}}} = \frac{\sqrt{n-2}r}{\sqrt{1-r^2}}$$

Proof of $F = t^2$

$$F = \frac{\mathrm{SSR}/1}{\mathrm{SSE}/(n-2)} = \frac{(n-2)\mathrm{SSR}}{\mathrm{SSE}} = \frac{n-2}{\mathrm{SST}/\mathrm{SSR}-1} = \frac{n-2}{1/r^2-1} = \frac{(n-2)r^2}{1-r^2} = \frac{\hat{b}^2L_{xx}}{\hat{\sigma}^2} = t^2$$

3. Proof of $\mathrm{Var}(e_i) = (1-h_{ii})\sigma^2$

$$\begin{aligned}\mathrm{Var}(e_i) &= \mathrm{Var}(y_i-\hat{a}-\hat{b}x_i)\\ &= \mathrm{Var}(y_i)+\mathrm{Var}(\hat{a})+\mathrm{Var}(\hat{b}x_i)-2\mathrm{Cov}(y_i,\hat{a})-2\mathrm{Cov}(y_i,\hat{b}x_i)+2\mathrm{Cov}(\hat{a},\hat{b}x_i)\\ &= \sigma^2+\Big\{\frac{1}{n}+\frac{\bar{x}^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\Big\}\sigma^2+x_i^2\frac{\sigma^2}{L_{xx}}\\ &\quad -2\sigma^2\sum_{i=1}^{n}\Big\{\frac{1}{n}-\frac{\bar{x}(x_i-\bar{x})}{L_{xx}}\Big\}-2x_i\sigma^2\sum_{i=1}^{n}\frac{x_i-\bar{x}}{L_{xx}}-2x_i\sigma^2\frac{\bar{x}}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\\ &= \Big(1-\frac{1}{n}-\frac{(x_i-\bar{x})^2}{L_{xx}}\Big)\sigma^2\\ &= (1-h_{ii})\sigma^2\end{aligned}$$

Proof of $\mathbb{E}(\hat{\sigma}^2) = \sigma^2$

$$\begin{aligned}\mathbb{E}(\hat{\sigma}^2) &= \frac{1}{n-2}\sum_{i=1}^{n}\mathbb{E}(e_i^2) = \frac{1}{n-2}\sum_{i=1}^{n}\mathrm{Var}(e_i) \qquad \text{since } \mathbb{E}(e_i)=0\\ &= \frac{\sigma^2}{n-2}\sum_{i=1}^{n}(1-h_{ii}) = \frac{\sigma^2}{n-2}\Big\{n-\sum_{i=1}^{n}(\frac{1}{n}+\frac{(x_i-\bar{x})^2}{L_{xx}})\Big\}\\ &= \frac{\sigma^2}{n-2}(n-\frac{n}{n}-1) = \sigma^2\end{aligned}$$

4. From previous result, we know $F = t^2 = \frac{(n-2)r^2}{1-r^2}$. Therefore, $r^2 = \frac{t^2}{t^2+n-2} = \frac{F}{F+n-2}$

5.(a) $x'_i = 2x_i, \bar{x}' = 2\bar{x}$

$$\tilde{b} = \frac{\sum_{i=1}^n (x'_i - \bar{x}')(y_i - \bar{y})}{\sum_{i=1}^n (x'_i - \bar{x}')^2} = \frac{1}{2}\frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{1}{2}\hat{b}$$

$$\tilde{a} = \bar{y} - \tilde{b}\bar{x}' = \bar{y} - \hat{b}\bar{x} = \hat{a}$$

(b) $x'_i = x_i + 2, \bar{x}' = \bar{x} + 2$

$$\tilde{b} = \frac{\sum_{i=1}^n (x'_i - \bar{x}')(y_i - \bar{y})}{\sum_{i=1}^n (x'_i - \bar{x}')^2} = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \hat{b}$$

$$\tilde{a} = \bar{y} - \tilde{b}\bar{x}' = \bar{y} - \hat{b}(\bar{x}+2) = \hat{a} - 2\hat{b}$$