Homework 7

蒋翌坤 20307100013

1. $\hat{\beta}, e$ are both linear functions of y. Since $y \sim N(X\beta, \sigma^2 I)$, so $\hat{\beta}, e$ are jointly normal. $\text{Cov}(\hat{\beta}, e) = (X^T X)^{-1} X^T \text{Cov}(y)(I-H)^T = \sigma^2 (X^T X)^{-1} X^T (I - X(X^T X)^{-1} X^T) = 0$. So $\hat{\beta} \perp e$

□

2. We already know $\hat{\beta}, e$ are normal.

$\mathbb{E}(\hat{\beta}) = \beta$, $\text{Cov}(\hat{\beta}) = (X^T X)^{-1} X^T \sigma^2 I ((X^T X)^{-1} X^T)^T = \sigma^2 (X^T X)^{-1}$, so $\hat{\beta} \sim N(\beta, \sigma^2 (X^T X)^{-1})$

$e = (I-H)y = (I-H)(X\beta + \epsilon) = (I-H)\epsilon$, $e^T e = \epsilon^T (I-H)\epsilon$. $rank(I) = n$ and $rank(H) = p$, so $rank(I-H) = n-p$. Here $(I-H)^2 = (I-H)$, $\epsilon \sim N(0, \sigma^2 I)$, then $\frac{\epsilon}{\sigma} \sim N(0, I)$, so $\frac{e^T e}{\sigma^2} = \frac{\epsilon^T (I-H)\epsilon}{\sigma^2} \sim \chi^2(n-p)$

□

3. $D = A(X^T X)^{-1} A^T = AX^{-1}(AX^{-1})^T$. For any nonzero vector $x \in \mathbb{R}^m$,

$x^T D x = x^T A X^{-1} (AX^{-1})^T x = ((AX^{-1})^T x)^T (AX^{-1})^T x \geq 0$.

For $x^T D x$ to be zero, $(AX^{-1})^T = 0 \Rightarrow A = AX^{-1}X = 0$. However, $rank(A) = m \neq 0$. Therefore, $x^T D x > 0$, and D is positive definite.

□

4. We have $\sum_{i=1}^n \hat{y}_i = \sum_{i=1}^n \sum_{j=1}^p x_{ij} \hat{\beta}_j = \sum_{j=1}^p \hat{\beta}_j \sum_{i=1}^n x_{ij} = \sum_{j=1}^p \hat{\beta}_j n \bar{x}_j$. Note that we get $\hat{\beta}_j$ under the constraint $0 = \sum_{i=1}^n x_{ic}(y_i - \sum_{j=1}^p \hat{\beta}_j x_{ij})$ for $c = 1, 2, \ldots, p$, and $x_{i1} = 1$. We have $n\bar{y} = n \sum_{j=1}^p \hat{\beta} \bar{x}_j$. So, $\sum_{i=1}^n \hat{y}_i = n\bar{y}$, $\sum_{i=1}^n (y_i - \hat{y}_i) = 0$ and $\sum_{j=1}^p \hat{\beta}_j \sum_{i=1}^n x_{ij}(y_i - \hat{y}_i) = \sum_{i=1}^p \hat{y}_i (y_i - \hat{y}_i) = 0$. Therefore,

$$
\begin{aligned}
\sum_{i=1}^n (y_i - \bar{y})^2 &= \sum_{i=1}^n (y_i - \hat{y}_i + \hat{y}_i - \bar{y})^2 \\
&= \sum_{i=1}^n (\hat{y}_i - \bar{y})^2 + \sum_{i=1}^n (y_i - \hat{y}_i)^2 + 2\sum_{i=1}^n (y_i - \hat{y}_i)(\hat{y}_i - \bar{y}) \\
&= \sum_{i=1}^n (\hat{y}_i - \bar{y})^2 + \sum_{i=1}^n (y_i - \hat{y}_i)^2 + 2\Big(\sum_{i=1}^n \hat{y}_i (y_i - \hat{y}_i) - \bar{y}(\sum_{i=1}^n (y_i - \hat{y}_i))\Big) \\
&= \sum_{i=1}^n (\hat{y}_i - \bar{y})^2 + \sum_{i=1}^n (y_i - \hat{y}_i)^2
\end{aligned}
$$

□

5. $\mathbb{E}(A\hat{\beta}) = u$ and $\text{Cov}(A\hat{\beta}) = A\sigma^2 (X^T X)^{-1} A^T = \sigma^2 D$, so $A\hat{\beta} \sim N(u, \sigma^2 D)$. So, $(A\hat{\beta} - u)^T \frac{D^{-1}}{\sigma^2} (A\hat{\beta} - u) \sim \chi^2(m)$. $\frac{\text{SSE}}{\sigma^2} = \frac{e^T e}{\sigma^2} \sim \chi^2(n-p)$. Also, SSE only depends on e and $(A\hat{\beta} - u)^T D^{-1} (A\hat{\beta} - u)$ only depends on $\hat{\beta}$. They are independent. Therefore,

$$
\frac{(A\hat{\beta} - u)^T D^{-1} (A\hat{\beta} - u)/m}{\text{SSE}/(n-p)} = \frac{(A\hat{\beta} - u)^T \frac{D^{-1}}{\sigma^2} (A\hat{\beta} - u)/m}{\frac{\text{SSE}}{\sigma^2}/(n-p)} \sim \frac{\chi^2(m)/m}{\chi^2(n-p)/(n-p)} \sim F(m, n-p)
$$

□