Homework 10

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1. We have $X^TX = \sum_{k=1}^n x_k x_k^T$, $X_{-i}^T X_{-i} = \sum_{k\neq i} x_k x_k^T = X^TX - x_i x_i^T$. Note that

$$(X_{-i}^T X_{-i})^{-1} = (X^TX)^{-1} + \frac{(X^TX)^{-1} x_i x_i^T (X^TX)^{-1}}{1 - x_i^T (X^TX)^{-1} x_i}$$

because

$$\begin{aligned}(X_{-i}^T X_{-i})(X_{-i}^T X_{-i})^{-1} &= (X^TX)(X^TX)^{-1} - x_i x_i^T (X^TX)^{-1} \\ &\quad + \frac{(X^TX)(X^TX)^{-1} x_i x_i^T (X^TX)^{-1} - x_i x_i^T (X^TX)^{-1} x_i x_i^T (X^TX)^{-1}}{1 - x_i^T (X^TX)^{-1} x_i} \\ &= I - x_i x_i^T (X^TX)^{-1} + \frac{x_i (1 - x_i^T (X^TX)^{-1} x_i) x_i^T (X^TX)^{-1}}{1 - x_i^T (X^TX)^{-1} x_i} \\ &= I\end{aligned}$$

Note that $X_{-i}^T y_{-i} = \sum_{k\neq i} x_k y_k = \sum_{k=1}^n x_k y_k - x_i y_i = X^T y - x_i y_i$. So,

$$\begin{aligned}\hat\beta_{-i} &= (X_{-i}^T X_{-i})^{-1} X_{-i}^T y_{-i} \\ &= \Big((X^TX)^{-1} + \frac{(X^TX)^{-1} x_i x_i^T (X^TX)^{-1}}{1 - x_i^T (X^TX)^{-1} x_i}\Big)\Big(X^T y - x_i y_i\Big) \\ &= \hat\beta + \frac{(X^TX)^{-1} x_i x_i^T \hat\beta - (1 - x_i^T (X^TX)^{-1} x_i)(X^TX)^{-1} x_i y_i - (X^TX)^{-1} x_i x_i^T (X^TX)^{-1} x_i y_i}{1 - x_i^T (X^TX)^{-1} x_i} \\ &= \hat\beta + \frac{(X^TX)^{-1} x_i x_i^T \hat\beta - (X^TX)^{-1} x_i y_i}{1 - x_i^T (X^TX)^{-1} x_i} \\ &= \hat\beta - \frac{(X^TX)^{-1} x_i e_i}{1 - h_{ii}}\end{aligned}$$

$$D_i = \Big(\frac{e_i}{1-h_{ii}}\Big)^2 \frac{1}{p\hat\sigma^2} \Big((X^TX)^{-1} x_i\Big)^T X^T X \Big((X^TX)^{-1} x_i\Big) = \Big(\frac{e_i}{1-h_{ii}}\Big)^2 \frac{1}{p\hat\sigma^2} h_{ii} = \frac{h_{ii} r_i^2}{p(1-h_{ii})}$$

□

2. Note that $\mathrm{Var}(e_j) = \sigma^2(1-h_{jj})$, $\frac{e_j^2}{\sigma^2(1-h_{jj})} \sim \chi_1^2$ and $\frac{e^Te}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^n e_i^2 \sim \chi_{n-p}^2$.

   Also, $\frac{1}{\sigma^2}(e^Te - \frac{e_j^2}{(1-h_{jj})}) \sim \chi_{n-p-1}^2$

   We have

$$F_j = \frac{(n-p-1) r_j^2}{n-p-r_j^2} = \frac{e_j^2}{\sigma^2(1-h_{jj})} \Bigg/ \frac{\sum_{i=1}^n e_i^2 - \frac{e_j^2}{1-h_{jj}}}{(n-p-1)\sigma^2} \sim F(1, n-p-1)$$

□