Homework 11

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1. Note that X^TX is PD because X has full rank and for any y, $y^TX^TXy = (Xy)^T(Xy) > 0$, so $\lambda_{\min}(X^TX) > 0$.

We have $\left|\lambda I - X^TX\right| = 0 \Leftrightarrow \left|(\lambda + k)I - (X^TX + kI)\right| = 0$. For any $\lambda + k$ that is an eigenvalue of $X^TX + kI$, λ is an eigenvalue of X^TX. $\lambda_{\min}(X^TX + kI) = \lambda_{\min}(X^TX) + k > \lambda_{\min}(X^TX)$.

Therefore, for any $k > 0$, $\lambda_{\min}(X^TX + kI) > \lambda_{\min}(X^TX) > 0$.

2. Let $X^TX = QDQ^T$ where Q is orthogonal and D is diagonal. Then $X^TX + kI = Q(D + kI)Q^T$. So,

$$\hat{\beta}_k = (X^TX + kI)^{-1}X^TX\hat{\beta} = Q(D + kI)^{-1}Q^TQDQ^T\hat{\beta} = Q(D + kI)^{-1}DQ^T\hat{\beta}$$

The diagonal element of $(D + kI)^{-1}D$ is strictly smaller than 1. Let the maximum diagonal element of $(D + kI)^{-1}D$ be d. So, for any j, $\hat{\beta}_{k[j]} = \sum_{i=1}^{p}(Q(D + kI)^{-1}DQ^T)_{[ji]}\hat{\beta}_{[i]} < d\hat{\beta}_{[j]} < \hat{\beta}_{[j]}$. Therefore, $\left\|\hat{\beta}_k\right\| < \left\|\hat{\beta}\right\|$

No. it is not possible that there exists some j such that $\mathbb{P}(\hat{\beta}_{k[j]} = 0) > 0$. Note that $\hat{\beta}_k$ is a linear transformation of $y \sim N(X\beta, \sigma^2 I)$, so $\hat{\beta}_{k[j]}$ follows some normal distribution, which is a continuous distribution. The probability of the random variable equals to some constant is zero. So, $\mathbb{P}(\hat{\beta}_{k[j]} = 0) = 0$.