Homework 1

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1. Yes. A probable biological interpretation is that children will inherit genes from their parents, and genes are the main factor that determines the height of a person. Therefore, the height of a child is positively related to the height of his parents as is shown by the "regression" trend.

2. To get the OLSE of b_0, we need to find the solution of the following equation:

$$\hat{b}_0 = \arg\min_{b_0} \sum_{i=1}^{n} (y_i - b_0 x_i)^2$$

Taking the derivative of b_0, plugging $\hat{b}_0$ to replace b_0 and letting the derivative to be 0, we have $\sum_{i=1}^{n} x_i(y_i - \hat{b}_0 x_i) = 0$. Therefore, the OLSE of b_0 is $\hat{b}_0 = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$

3.(a)

$$\frac{\partial \log L(a_0, b_0, \sigma^2)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (y_i - a_0 - b_0 x_i)^2 = 0 \Rightarrow \sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - a_0 - b_0 x_i)^2$$

$$\Rightarrow \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{a}_0 - \hat{b}_0 x_i)^2$$

(b) The difference between the assumption for the OLSE and MLE of (a_0, b_0) is that in MLE, we assume that y_i and ε_i are independent from each other and $\varepsilon_i \sim N(0, \sigma^2)$, while in OLSE, we only assume that $\mathrm{Cov}(\varepsilon_i, \varepsilon_j) = 0, i \neq j$, there is no independent or normality assumption.

4.

$$\mathbb{E}(\hat{a}) = \mathbb{E}(\bar{y}) - \mathbb{E}(\hat{b})\bar{x} = a + b\bar{x} - b\bar{x} = a$$

$$\begin{aligned}
\mathrm{Var}(\hat{a}) &= \mathrm{Cov}(\bar{y} - \hat{b}\bar{x}, \bar{y} - \hat{b}\bar{x}) \\
&= \mathrm{Var}(\bar{y}) + \bar{x}^2 \mathrm{Var}(\hat{b}) - 2\bar{x}\mathrm{Cov}(\frac{1}{n}\sum_{i=1}^{n} y_i, \sum_{i=1}^{n} \frac{x_i - \bar{x}}{\sum_{i=1}^{n}(x_i - \bar{x})^2} y_i) \\
&= \frac{\sigma^2}{n} + \bar{x}^2 \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} - 2\bar{x} \sum_{i=1}^{n} \frac{x_i - \bar{x}}{n \sum_{i=1}^{n}(x_i - \bar{x})^2} \sigma^2 \\
&= \left\{ \frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \right\} \sigma^2
\end{aligned}$$

5. When $x_i = 0$, it will have no effect on the optimization problem. Therefore, we can suppose that $x_i \neq 0$ for $i = 1, \ldots, n$. Let $u_i = \dfrac{y_i}{x_i}$ and order them, we get $u_{(1)}, \ldots, u_{(n)}$. Define $A_0 = (-\infty, u_{(1)}]$, $A_i = (u_{(i)}, u_{(i+1)}]$ for $i = 1, \ldots, n-1$, $A_n = (u_{(n)}, +\infty)$.

Then for each $b \in A_k, k = 0, \ldots, n$, we can lose the absolute value sign in the optimization problem according to the sign of x_i and y_i and find the local minimal m_k of $\sum_{i=1}^n |y_i - bx_i|$. Let $s = \arg\min_k m_k$, then $\hat{b} = \arg\min_b \sum_{i=1}^n |y_i - bx_i|$ where $b \in A_s$.