Homework 10

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Problem 1

Consider the test $H_0 : \theta = \theta_1 \le \theta_0, H_1 : \theta = \theta_2 > \theta_0$ Note that

$$\frac{1}{\lambda(\mathbf{X})} = \frac{\theta_2^{-n}\mathbb{I}(X_{(n)} < \theta_2)}{\theta_1^{-n}\mathbb{I}(X_{(n)} < \theta_1)} \begin{cases} \left(\frac{\theta_1}{\theta_2}\right)^n, & X_{(n)} < \theta_1 \\ \infty, & X_{(n)} \ge \theta_1 \end{cases}$$

This is monotone in $X_{(n)}$. So, the UMPT for the test $H_0 : \theta \le \theta_0, H_1 : \theta > \theta_0$ is to reject when $X_{(n)} > c$, Density for $X_{(n)}$ is $g_\theta(t) = \frac{nt^{n-1}}{\theta^n}\mathbb{I}(0 < t < \theta)$, $\mathrm{E}_{\theta_0}(\phi(\mathbf{X})) = \int_c^{\theta_0} \frac{nt^{n-1}}{\theta_0^n}dt = 1 - \frac{c^n}{\theta_0^n} = \alpha \Rightarrow c = \theta_0\sqrt[n]{1-\alpha}$.

The power function is $P_\theta(X_{(n)} > c) = \int_c^\theta \frac{nt^{n-1}}{\theta^n}dt = 1 - (1-\alpha)\left(\frac{\theta_0}{\theta}\right)^n$

Problem 2

There exists θ_1, θ_2, $g(\cdot)$, $h(\cdot)$ such that

$$\lambda(\mathbf{X}) = \frac{\sup_{\theta\in\Theta_0} f(\mathbf{X}|\theta)}{\sup_{\theta\in\Theta} f(\mathbf{X}|\theta)} = \frac{f(\mathbf{X}|\theta_1)}{f(\mathbf{X}|\theta_2)} = \frac{g(T(\mathbf{X})|\theta_1)h(\mathbf{X})}{g(T(\mathbf{X})|\theta_2)h(\mathbf{X})} = \frac{g(T(\mathbf{X})|\theta_1)}{g(T(\mathbf{X})|\theta_2)}$$

$\lambda(\mathbf{X})$ is a function of $T(\mathbf{X})$. Note that $\phi(\mathbf{X})$ is a function of $\lambda(\mathbf{X})$, so $\phi(\mathbf{X})$ is a function of T.

Problem 3

We have

$$f(\mathbf{X},\mathbf{Y}|\mu_1,\mu_2,\sigma_1^2,\sigma_2^2) = (2\pi\sigma_1^2)^{-m}(2\pi\sigma_2^2)^{-n}\exp\Big\{-\frac{\sum_{i=1}^m (X_i-\mu_1)^2}{2\sigma_1^2} - \frac{\sum_{j=1}^n (Y_j-\mu_2)^2}{2\sigma_2^2}\Big\}$$

Consider the test, $H_0 : \frac{\sigma_1^2}{\sigma_2^2} = c, H_1 : \frac{\sigma_1^2}{\sigma_2^2} \ne c$. Under H_0, let $S_1^2 = \frac{1}{m-1}\sum_{i=1}^m (X_i - \bar{X})^2$, $S_2^2 = \frac{1}{n-1}\sum_{j=1}^n (Y_j - \bar{Y})^2$, we have $\frac{c(n-1)S_2^2}{(m-1)S_1^2} = \frac{(n-1)S_2^2/\sigma_2^2}{(m-1)S_1^2/\sigma_1^2} \sim F_{n-1,m-1}$

A level α test can be $\phi(\mathbf{X},\mathbf{Y}) = \begin{cases} 0, & \frac{c(n-1)S_2^2}{(m-1)S_1^2} \in (F_{n-1,m-1,\alpha/2}, F_{n-1,m-1,1-\alpha/2}) \\ 1, & \text{otherwise} \end{cases}$, so a $1-\alpha$ confidence interval for $\frac{\sigma_1^2}{\sigma_2^2}$ is $\left(\frac{(m-1)S_1^2}{(n-1)S_2^2}F_{n-1,m-1,\alpha/2}, \frac{(m-1)S_1^2}{(n-1)S_2^2}F_{n-1,m-1,1-\alpha/2}\right)$.

Problem 4

Suppose there exists a level α UMPT $\phi^\star(\mathbf{X})$ for the test $H_0 : \theta = \theta_0, H_1 : \theta \ne \theta_0$, then it is also the UMPT for the test $H_0 : \theta = \theta_0, H_1 : \theta > \theta_0$ and $H_0 : \theta = \theta_0, H_1 : \theta < \theta_0$.

However, for $H_0 : \theta = \theta_0, H_1 : \theta > \theta_0$, a level α UMPT is reject H_0 when $\frac{\sqrt{n}(\bar{X}-\theta_0)}{\sigma} > z_{1-\alpha}$; for $H_0 : \theta = \theta_0, H_1 : \theta < \theta_0$, a level α UMPT is reject H_0 when $\frac{\sqrt{n}(\bar{X}-\theta_0)}{\sigma} < -z_\alpha$. When we observe $\mathbf{X}$ such that $\frac{\sqrt{n}(\bar{X}-\theta_0)}{\sigma} > z_{1-\alpha}$, we cannot reject H_0 for the second test, which contradicts the assumption that $\phi^\star(\mathbf{X})$ is a level α UMPT for the test $H_0 : \theta = \theta_0, H_1 : \theta \ne \theta_0$.