Homework 8

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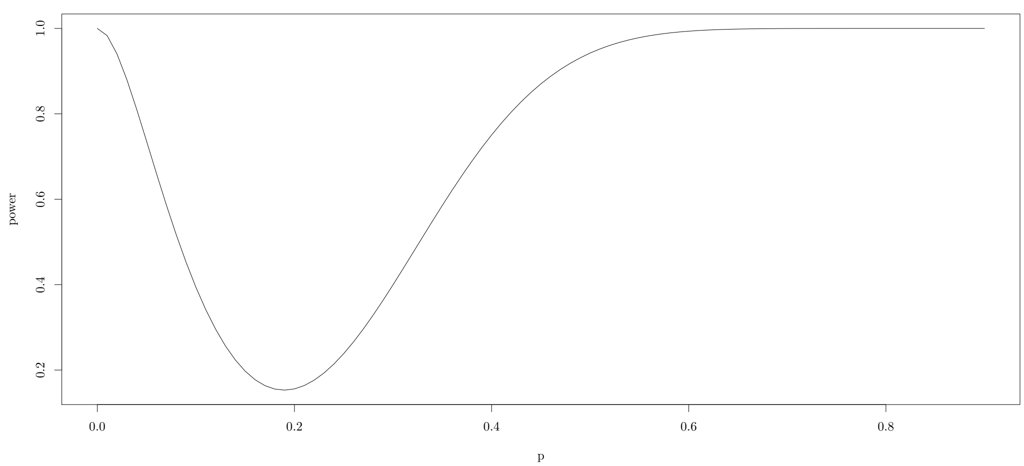
Problem 1

(a)

$$P_{H_1}(\sum_{i=1}^{20} x_i \geq 7 \text{ or } \sum_{i=1}^{20} x_i \leq 1) = (1-p)^{20} + 20p(1-p)^{19} + \sum_{k=7}^{20} \binom{20}{k} p^k (1-p)^{20-k}$$

The power of the decision rule is shown in the following table and graph.

p	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Power	1.00	0.39	0.16	0.40	0.75	0.94	0.99	1.00	1.00	1.00



(b)

$$P_{H_0}(\sum_{i=1}^{20} x_i \geq 7 \text{ or } \sum_{i=1}^{20} x_i \leq 1) = 0.8^{20} + 20 \times 0.2 \times 0.8^{19} + \sum_{k=7}^{20} \binom{20}{k} 0.2^k 0.8^{20-k} = 0.16$$

So type 1 error probability of the rule ϕ is 0.16.

$$P_{p=0.05}(1 < \sum_{i=1}^{20} x_i < 7) = 0.26$$

So type 2 error probability when $p = 0.05$ is 0.26.

Problem 2

Let $Y = \frac{X}{\theta} \sim U(0,1)$, $Y_{(10)} = \frac{X_{(10)}}{\theta}$ has distribution $f_{Y_{(10)}}(y) = 10y^9$ and $F_{Y_{(10)}}(y) = y^{10}$.

Let $c = (1-\alpha)^{0.1}$, $P(Y_{(10)} > c) = \alpha$

Under H_0, $X_{(10)} \le Y_{(10)}$, $P_{H_0}(X_{(10)} > c) \le \alpha$. So, we reject H_0 if $X_{(10)} > c$.

Problem 3

First, we find upper $(1-\frac{\beta}{2}, 1-\frac{\gamma}{2})$ tolerance limit. We guess $\lambda X_{(n)}$ is the tolerance upper limit for some $\lambda > 0$.

$$P(X \le \lambda X_{(n)}) = P(\frac{X}{\theta} \le \lambda \frac{X_{(n)}}{\theta}) = \lambda \frac{X_{(n)}}{\theta}$$

$$P\Big(P(X \le \lambda X_{(n)}) \ge 1-\frac{\beta}{2}\Big) = P\Big(\lambda\frac{X_{(n)}}{\theta} \ge 1-\frac{\beta}{2}\Big) = P\Big(\frac{X_{(n)}}{\theta} \ge \frac{1}{\lambda}(1-\frac{\beta}{2})\Big) = 1 - \Big[\frac{1}{\lambda}(1-\frac{\beta}{2})\Big]^n$$

So, $1 - \Big[\frac{1}{\lambda}(1-\frac{\beta}{2})\Big]^n \ge 1-\frac{\gamma}{2} \Rightarrow \lambda \ge \Big(1-\frac{\beta}{2}\Big)^n \Big(\frac{\gamma}{2}\Big)^{-\frac{1}{n}}$

Similarly, we guess $\psi X_{(n)}$ is the tolerance lower limit for some $\psi > 0$.

$$P(X \ge \psi X_{(n)}) = P(\frac{X}{\theta} \ge \psi \frac{X_{(n)}}{\theta}) = 1 - \psi \frac{X_{(n)}}{\theta}$$

$$P\Big(P(X \ge \psi X_{(n)}) \ge 1-\frac{\beta}{2}\Big) = P\Big(1-\psi\frac{X_{(n)}}{\theta} \ge 1-\frac{\beta}{2}\Big) = P\Big(\frac{X_{(n)}}{\theta} \le \frac{\beta}{2\psi}\Big) = \Big(\frac{\beta}{2\psi}\Big)^n$$

So, $\Big(\frac{\beta}{2\psi}\Big)^n \ge 1-\frac{\gamma}{2} \Rightarrow \psi \le \Big(\frac{\beta}{2}\Big)^n \Big(1-\frac{\gamma}{2}\Big)^{-\frac{1}{n}}$

Therefore, a $(1-\beta, 1-\gamma)$ tolerance interval is $\Big[\Big(1-\frac{\beta}{2}\Big)^n\Big(\frac{\gamma}{2}\Big)^{-\frac{1}{n}} X_{(n)}, \Big(\frac{\beta}{2}\Big)^n\Big(1-\frac{\gamma}{2}\Big)^{-\frac{1}{n}} X_{(n)}\Big]$