Homework 14

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1. (a)

$$\mathbb{E}(\tilde{\beta}_s) = \mathbb{E}((S^TS)^{-1}S^Ty) = (S^TS)^{-1}S^TX\beta = \beta_s + (S^TS)^{-1}S^TQ\beta_q = \beta_s + A\beta_q$$

(b)

$$\mathrm{Cov}(\tilde{\beta}_s) = \mathrm{Cov}((S^TS)^{-1}S^Ty) = (S^TS)^{-1}S^T\mathrm{Cov}(y)((S^TS)^{-1}S^T)^T = \sigma^2(S^TS)^{-1}$$

(c)

$$\begin{aligned}
\mathrm{Cov}(\hat{\beta}_s) &= \mathrm{Cov}[((S^TS)^{-1}S^T + AD(SA-Q)^T)y] \\
&= \sigma^2((S^TS)^{-1}S^T + AD(SA-Q)^T)((S^TS)^{-1}S^T + AD(SA-Q)^T)^T \\
&= \sigma^2\Big((S^TS)^{-1} + (S^TS)^{-1}S^T(SA-Q)D^TA^T + AD(SA-Q)^TS(S^TS)^{-1} \\
&\qquad + \underbrace{AD(SA-Q)^T(SA-Q)D^TA^T}_{\geq 0}\Big) \\
&\geq \mathrm{Cov}(\tilde{\beta}_s) + \sigma^2\Big(AD^TA^T - (S^TS)^{-1}S^TQD^TA^T + ADA^T - ADQ^TS(S^TS)^{-1}\Big) \\
&= \mathrm{Cov}(\tilde{\beta}_s) + \sigma^2\Big((AD(A^T - Q^TS(S^TS)^{-1}))^T + (AD(A^T - Q^TS(S^TS)^{-1}))\Big) \\
&= \mathrm{Cov}(\tilde{\beta}_s)
\end{aligned}$$

$\square$

2. (a)

$$\mathbb{E}(\tilde{u}) = \mathbb{E}(y_0 - x_s^T\tilde{\beta}_s) = x_0^T\beta - x_s^T(\beta_s + A\beta_Q) = x_Q^T\beta_Q - x_s^TA\beta_Q$$

(b)

$$\mathrm{Var}(\tilde{u}) = \mathrm{Var}(y_0 - x_s^T\tilde{\beta}_s) = \sigma^2 + x_s^T\mathrm{Cov}(\tilde{\beta}_s)x_s = \sigma^2 + \sigma^2x_s^T(S^TS)^{-1}x_s$$

$$\mathrm{Var}(u) = \mathrm{Var}(y_0 - x_0^T\hat{\beta}) = \sigma^2 + x_0^T\mathrm{Cov}(\hat{\beta})x_0 = \sigma^2 + \sigma^2x_0^T(X^TX)^{-1}x_0$$

Note that D is PSD because $\forall\alpha \neq 0$,

$$\begin{aligned}
&S(S^TS)^{-1}S^T = S(S^TS)^{-1}S^TS(S^TS)^{-1}S^T \Rightarrow I - S(S^TS)^{-1}S^T = (I - S(S^TS)^{-1}S^T)^2 \\
&\Rightarrow I - S(S^TS)^{-1}S^T \text{is PSD since its eigenvalue is 0 or 1} \\
&\Rightarrow \alpha^TD^{-1}\alpha = \alpha^T(Q^TQ - Q^TS(S^TS)^{-1}S^TQ)\alpha = (Q\alpha)^T(I - S(S^TS)^{-1}S^T)(Q\alpha) \geq 0
\end{aligned}$$

So, $\mathrm{Var}(u) \geq \mathrm{Var}(\tilde{u})$ since

$$\begin{aligned}
x_0^T(X^TX)^{-1}x_0 &= (x_s^T, x_Q^T)\begin{pmatrix} (S^TS)^{-1} + ADA^T & -AD \\ -(AD)^T & D \end{pmatrix}(x_s^T, x_Q^T)^T \\
&= x_s^T(S^TS)^{-1}x_s + x_s^TADA^Tx_s - x_Q^T(AD)^Tx_s - x_s^TADx_Q + x_Q^TDx_Q \\
&= x_s^T(S^TS)^{-1}x_s + (A^Tx_s - x_Q)^TD(A^Tx_s - x_Q) \\
&\geq x_s^T(S^TS)^{-1}x_s
\end{aligned}$$

(c) Note that $\mathrm{Cov}(\hat{\beta}_Q) = \sigma^2 D \geq \beta_Q \beta_Q^T$

$$\mathbb{E}(u^2) = \mathbb{E}(u)^2 + \mathrm{Var}(u) = \mathrm{Var}(u) = \sigma^2 + \sigma^2 x_0^T (X^T X)^{-1} x_0$$

$$\mathbb{E}(\tilde{u}^2) = \mathbb{E}(\tilde{u})^2 + \mathrm{Var}(\tilde{u}) = (x_Q^T \beta_Q - x_s^T A \beta_Q)^2 + \sigma^2 + \sigma^2 x_s^T (S^T S)^{-1} x_s$$

$$\begin{aligned}
\mathbb{E}(u^2) - \mathbb{E}(\tilde{u}^2) &= \sigma^2 \Big(x_0^T (X^T X)^{-1} x_0 - x_s^T (S^T S)^{-1} x_s \Big) - (x_Q^T \beta_Q - x_s^T A \beta_Q)^2 \\
&= \sigma^2 \Big((A^T x_s - x_Q)^T D (A^T x_s - x_Q) \Big) - (x_Q^T \beta_Q - x_s^T A \beta_Q)^2 \\
&\geq \Big((A^T x_s - x_Q)^T \beta_Q \beta_Q^T (A^T x_s - x_Q) \Big) - (x_Q^T \beta_Q - x_s^T A \beta_Q)^2 \\
&= \Big((x_s^T A \beta_Q - x_Q^T \beta_Q)(x_s^T A \beta_Q - x_Q^T \beta_Q)^T \Big) - (x_Q^T \beta_Q - x_s^T A \beta_Q)^2 \\
&= 0
\end{aligned}$$

So, $\mathbb{E}(u^2) \geq \mathbb{E}(\tilde{u}^2)$