Homework 4

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1. $\text{rank}(A) = 1 \Rightarrow$ we can rewrite A: $A = \alpha\beta^T$ for some $\alpha, \beta \neq 0$. We have $\text{tr}(A) = \beta^T\alpha = a \neq 0$.

$\text{rank}(A) = 1 \Rightarrow |A| = 0 \Rightarrow \lambda = 0$ is an eigenvalue of A. $\lambda = a$ is also an eigenvalue since $A\alpha = \alpha\beta^T\alpha = a\alpha$.

Let $\mathcal{V} = \left\{x : \beta^T x = 0\right\}$, $\dim(\mathcal{V}) = n-1$; let $\phi_1, \ldots, \phi_{n-1}$ be a basis of $\mathcal{V}$, we have $A\phi_i = \alpha\beta^T\phi_i = 0$. That is, $\lambda = 0$'s geometric and algebraic multiplicity is at least $n-1$. We have another eigenvalue $\lambda = a$, its geometric and algebraic multiplicity are at least 1. We know that the sum of algebraic multiplicity is n, so we must take the extreme. That is, there are only two eigenvalues, $\lambda = 0$ and $\lambda = a$, and their geometric and algebraic multiplicity are $n-1$ and 1 respectively.

The sum of geometric multiplicity is $n - 1 + 1 = n$, so A is diagonalizable.

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2. It has already been proven in question 1 that $A = \alpha\beta^T$ is diagonalizable.

$\square$

3. Assume A is diagonalizable, $A = PDP^{-1}$, $D^k = P^{-1}A^kP = 0 \Rightarrow D = 0 \Rightarrow A = 0$, which is a contradiction. So A is not diagonalizable.

$\square$

4. $x^TAx = x^TPDP^Tx = (P^Tx)^TD(P^Tx)$. Suppose $y = P^Tx$, so $\sum_{i=1}^n y_i^2 = ||y|| = y^Ty = x^TPP^Tx = x^Tx = 1$. Then, $x^TAx = y^TDy = \sum_{i=1}^n \lambda_i y_i^2$.

$x^TAx \leq \lambda_{\max}\sum_{i=1}^n y_i^2 = \lambda_{\max}$, and $x^TAx \geq \lambda_{\min}\sum_{i=1}^n y_i^2 = \lambda_{\min}$.

$\square$

5. From question 4 we know that $\lambda_{\min} \leq x^TAx \leq \lambda_{\max}$ for any x with $||x|| = 1$. Let $y = cx, \forall c \neq 0$, we have $y \in \mathbb{R}^n$. Then, $y^TAy = c^2x^TAx \in c^2[\lambda_{\min}, \lambda_{\max}]$.

We can see that A is PD, i.e., $y^TAy > 0$ for any nonzero $y \in \mathbb{R}^n$ iff $\lambda_{\min} > 0$, i.e., eigenvalues are all positive; Similarly, A is PSD iff $\lambda_{\min} \geq 0$, i.e., eigenvalues are all non-negative.

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