# Homework 5

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1. (a) Let $\mathbb{E}(X_i) = \mu$, $X = (X_1, \dots, X_n)^T$, $\boldsymbol{\mu} = (\mu, \dots, \mu)^T$, $\Sigma = \sigma^2 I$, $\sum_{i=1}^n (X_i - \hat{\mu})^2 = (AX)^T(AX)$ where $A = \begin{pmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \dots & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & \dots & -\frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \dots & 1-\frac{1}{n} \end{pmatrix}$, $A^T A = \begin{pmatrix} 1-\frac{1}{n} & 1-\frac{2}{n} & \dots & 1-\frac{2}{n} \\ 1-\frac{2}{n} & 1-\frac{1}{n} & \dots & 1-\frac{2}{n} \\ \vdots & \vdots & \ddots & \vdots \\ 1-\frac{2}{n} & 1-\frac{2}{n} & \dots & 1-\frac{1}{n} \end{pmatrix}$, we have $A\boldsymbol{\mu} = 0$ and

$$\begin{aligned}
\mathbb{E}(\hat{\sigma}^2) &= \frac{1}{n-1}\mathbb{E}\Big(\sum_{i=1}^n (X_i - \hat{\mu})^2\Big) = \frac{1}{n-1}\mathbb{E}\Big((AX)^T(AX)\Big) = \frac{1}{n-1}\mathbb{E}\Big(X^T A^T A X\Big) \\
&= \frac{1}{n-1}(\boldsymbol{\mu}^T A^T A \boldsymbol{\mu} + \mathrm{tr}(A^T A \Sigma)) \\
&= \frac{1}{n-1}\sigma^2 \mathrm{tr}(A^T A) \\
&= \frac{1}{n-1}\sigma^2 n(1 - \frac{1}{n}) \\
&= \sigma^2
\end{aligned}$$

□

(b) Let $B = (\frac{1}{n}, \dots, \frac{1}{n})$, $\hat{\mu} = BX$, We can see that $BA = 0 \Rightarrow BA^2 = 0$. So $\hat{\mu} \perp \hat{\sigma}^2$

□

2. Since $AB = 0$, $X^T AB = 0$, $X^T A \in \mathbb{R}^{p \times n}$, from Theorem 6 we know $X^T AX \perp X^T BX$

□

3. From Lemma 2 we know the eigenvalues of $A$ must be 0 or 1. Since $A$ is symmetric, we can decompose it: $A = PDP^T$, where $P$ is an orthogonal matrix and $D$ is a diagonal matrix whose diagonal elements are either 0 or 1. We also know the algebraic multiplicity of the eigenvalue 0 is $n - \mathrm{rank}(-A) = n - r$, so there are $n - r$ 0's and $r$ 1's in $D$.

Let $p_{ij}$ be the $(i,j)$-th element of $P$ and $d_{ii}$ be the $(i,i)$-th element of $D$, $\mathcal{S} = \Big\{i : d_{ii} = 1\Big\}$, we have $\mu^T A \mu = \sum_{i=1}^n d_{ii}(\sum_{j=1}^n p_{ij}\mu_j)^2 = 0 \Rightarrow \forall i \in \mathcal{S}, \sum_{j=1}^n p_{ij}\mu_j = 0$

$X^T AX = (PX)^T DPX = \sum_{i=1}^n d_{ii}(\sum_{j=1}^n p_{ij}x_j)^2 = \sum_{i\in\mathcal{S}}(\sum_{j=1}^n p_{ij}x_j)^2$

$\forall i \in \mathcal{S}$, $\mathbb{E}(\sum_{j=1}^n p_{ij}x_j) = \sum_{j=1}^n p_{ij}\mathbb{E}(x_j) = \sum_{j=1}^n p_{ij}\mu_j = 0$

$\forall i \in \mathcal{S}$, $\mathrm{Var}(\sum_{j=1}^n p_{ij}x_j) = \sum_{j=1}^n p_{ij}^2 \mathrm{Var}(x_j) = \sum_{j=1}^n p_{ij}^2 = 1$. So $\sum_{j=1}^n p_{ij}x_j \sim N(0,1)$

$\forall i \neq k \in \mathcal{S}$, $\mathrm{Cov}(\sum_{j=1}^n p_{ij}x_j, \sum_{j=1}^n p_{kj}x_j) = \sum_{j=1}^n p_{ij}p_{kj}\mathrm{Var}(x_j) = \sum_{j=1}^n p_{ij}p_{kj} = 0$, so $\sum_{j=1}^n p_{ij}x_j$ are independent among each other.

Therefore, $X^T AX \sim \chi^2(r)$

□