The real Bellman-Ford algorithm
COMS20010 2020, Video 11-4

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Bellman-Ford: A reminder

Algorithm: GoodPath

Input : A weighted digraph $G = ((V, E), w)$ with no negative-weight cycles, two vertices $s, t \in V(G)$, and an integer $k \geq 0$.
Output : A shortest walk from s to t in G with at most k edges, or `None` if none exists.

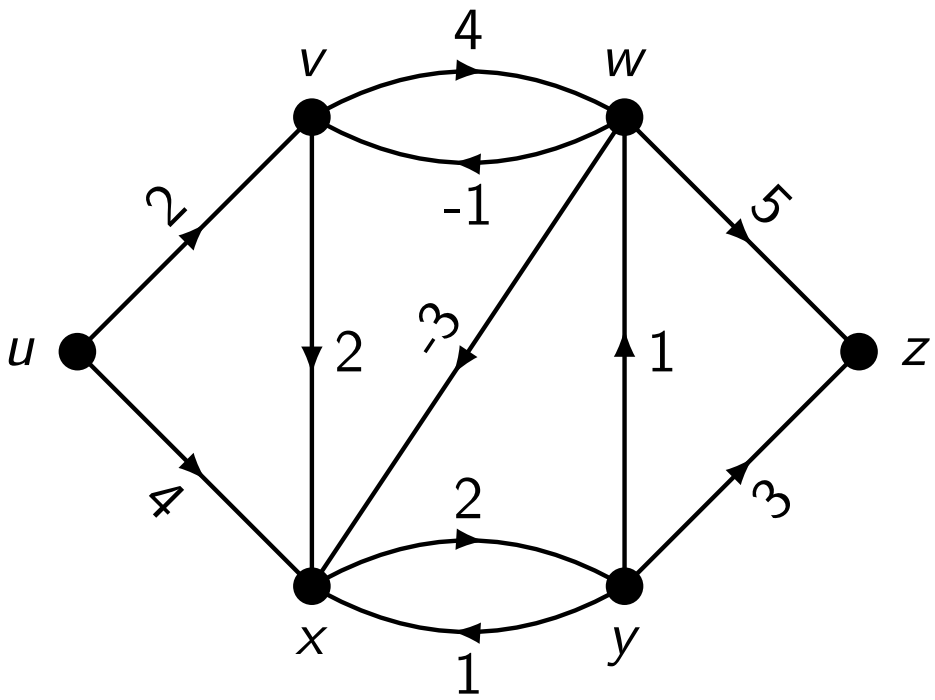
```
1 begin
2     if k = 0 then
3         Return the empty walk if s = t, and None otherwise.
4     Write N⁺(s) = {v_1, ..., v_d}, where d ≥ 1.
5     Let P_i ← GoodPath(G, v_i, t, k − 1) for all i ∈ [k].
6     if P_i = None for all i ∈ [k] then
7         Return None.
8     Return whichever walk is shortest in {sv_i P_i : i ∈ [k], P_i ≠ None}.
```

Memoised, this takes $O(|V|^3)$ time and space, since we need to store the result of $\Omega(|V|^2)$ function calls.

By making the algorithm iterative and being a little smarter, we can drop this to $O(|V||E|)$ time and $O(|V|)$ space. This is why it's often a good idea to de-memoise!

Iterative Bellman-Ford: An example

Say we are trying to find shortest paths from every vertex to z.

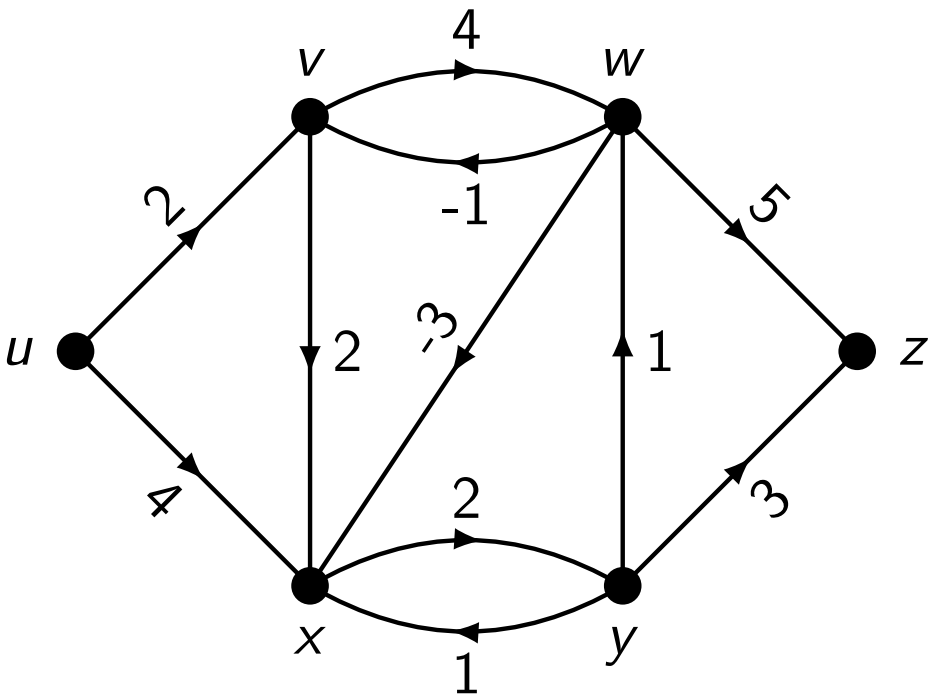


k \ s	u	v	w	x	y	z
5						
4						
3	$uxyz$	$vxyz$	$wxyz$	xyz	yz	z
2	$\emptyset$	vwz	wz	xyz	yz	z
1	$\emptyset$	$\emptyset$	wz	$\emptyset$	yz	z
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	z

This is getting ugly. These paths are taking up a lot of space, and we need to recalculate their lengths each time.

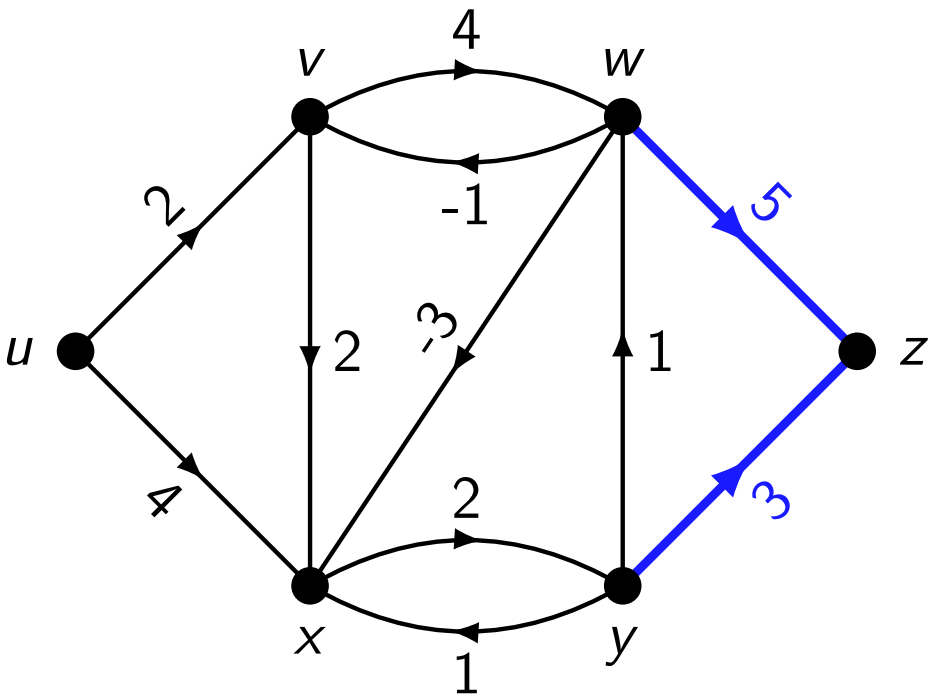
Why not just store the **first edge** of each path, along with its length?

Iterative Bellman-Ford: A better approach



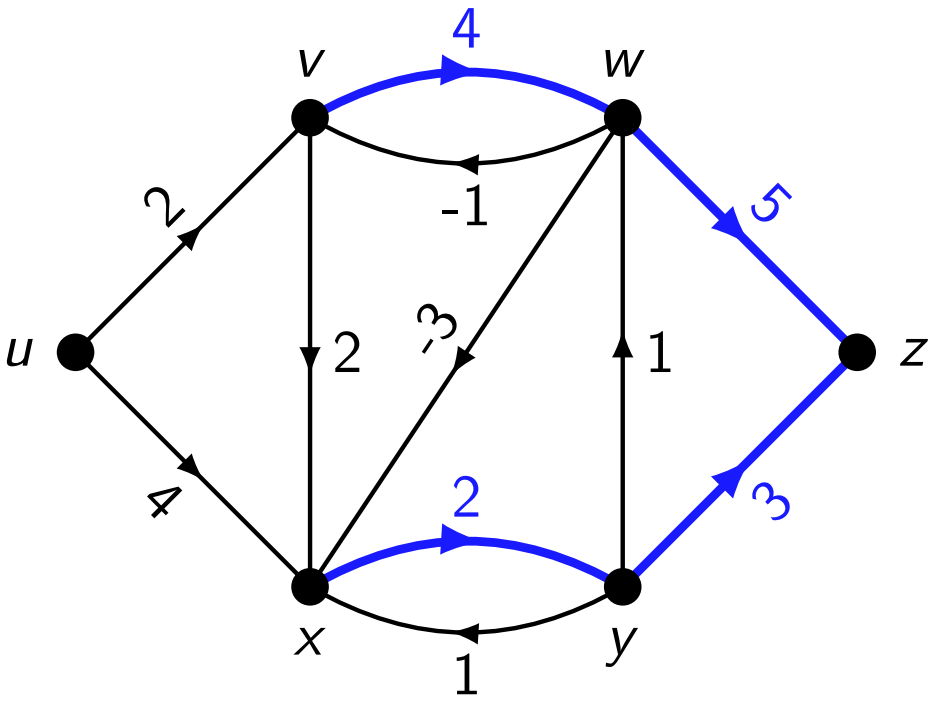
k \ s	u	v	w	x	y	z
5						
4						
3						
2						
1						
0	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(z, 0)$

Iterative Bellman-Ford: A better approach



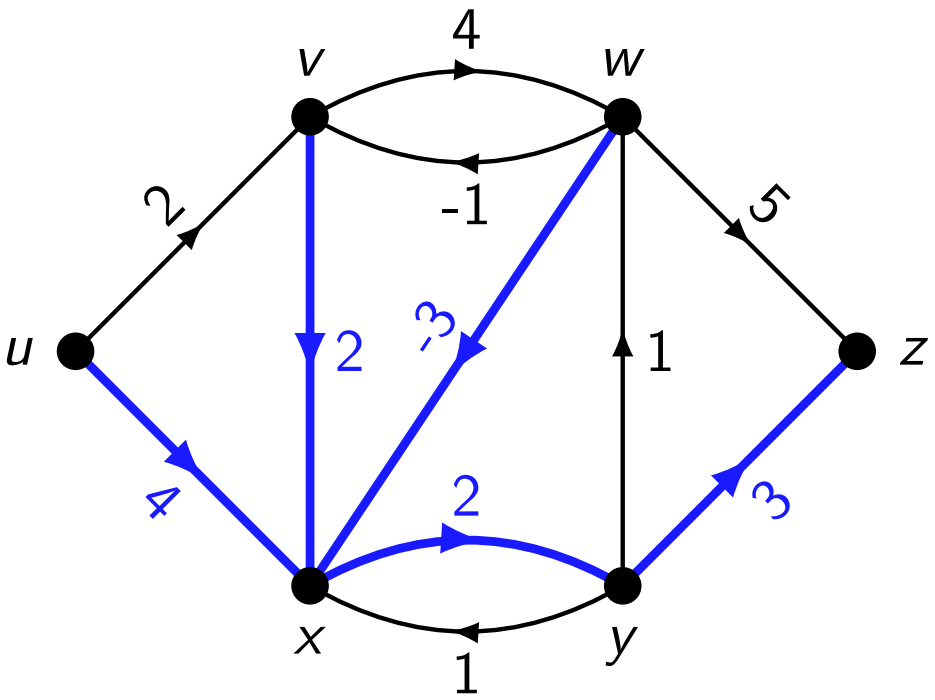
k \ s	u	v	w	x	y	z
5						
4						
3						
2						
1	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(wz, 5)$	$(\emptyset, \infty)$	$(yz, 3)$	$(z, 0)$
0	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(z, 0)$

Iterative Bellman-Ford: A better approach



k \ s	u	v	w	x	y	z
5						
4						
3						
2	$(\emptyset, \infty)$	$(vw, 9)$	$(wz, 5)$	$(xy, 5)$	$(yz, 3)$	$(z, 0)$
1	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(wz, 5)$	$(\emptyset, \infty)$	$(yz, 3)$	$(z, 0)$
0	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(z, 0)$

Iterative Bellman-Ford: A better approach



k \ s	u	v	w	x	y	z
5						
4						
3	$(ux, 9)$	$(vx, 7)$	$(wx, 2)$	$(xy, 5)$	$(yz, 3)$	$(z, 0)$
2	$(\emptyset, \infty)$	$(vw, 9)$	$(wz, 5)$	$(xy, 5)$	$(yz, 3)$	$(z, 0)$
1	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(wz, 5)$	$(\emptyset, \infty)$	$(yz, 3)$	$(z, 0)$
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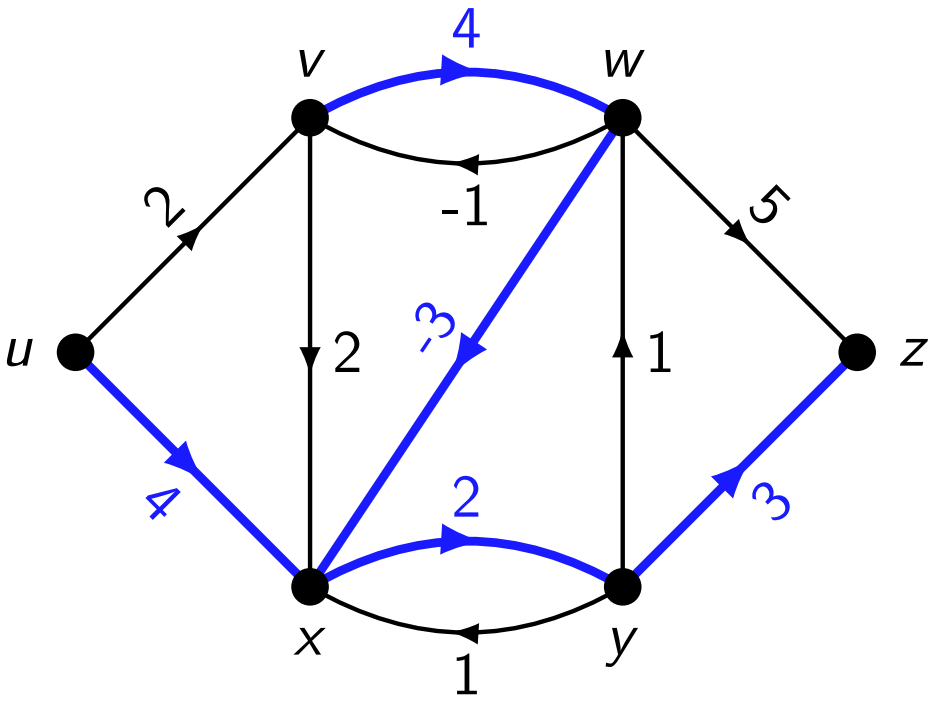
Iterative Bellman-Ford: A better approach



k \ s	u	v	w	x	y	z
5						
4	$(ux, 9)$	$(vw, 6)$	$(wx, 2)$	$(xy, 5)$	$(yz, 3)$	$(z, 0)$
3	$(ux, 9)$	$(vx, 7)$	$(wx, 2)$	$(xy, 5)$	$(yz, 3)$	$(z, 0)$
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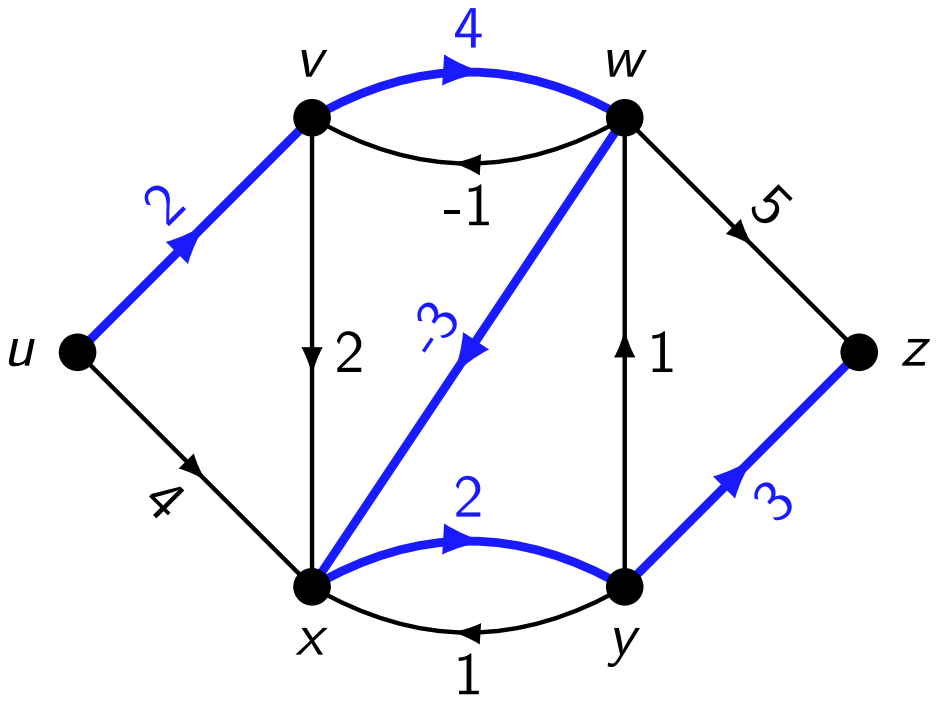
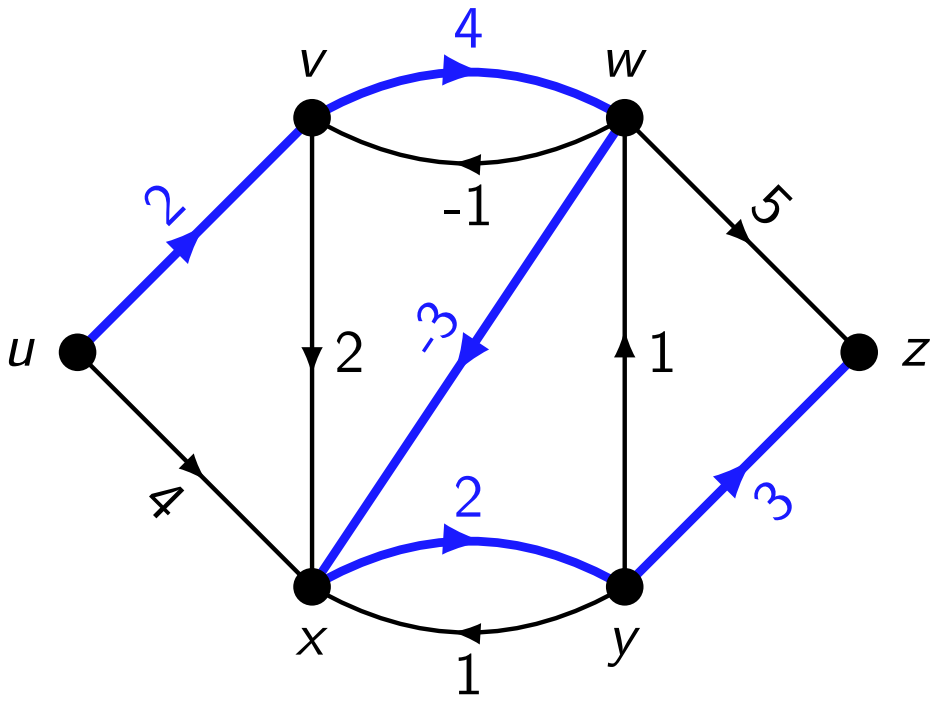


k \ s	u	v	w	x	y	z
5	$(uv, 8)$	$(vw, 6)$	$(wx, 2)$	$(xy, 5)$	$(yz, 3)$	$(z, 0)$
4	$(ux, 9)$	$(vw, 6)$	$(wx, 2)$	$(xy, 5)$	$(yz, 3)$	$(z, 0)$
3	$(ux, 9)$	$(vx, 7)$	$(wx, 2)$	$(xy, 5)$	$(yz, 3)$	$(z, 0)$
2	$(\emptyset, \infty)$	$(vw, 9)$	$(wz, 5)$	$(xy, 5)$	$(yz, 3)$	$(z, 0)$
1	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(wz, 5)$	$(\emptyset, \infty)$	$(yz, 3)$	$(z, 0)$
0	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(z, 0)$

So e.g. $d(u, z) = 8$, via the path $uvwxyz$.

Iterative Bellman-Ford: A better approach



k \ s	u	v	w	x	y	z
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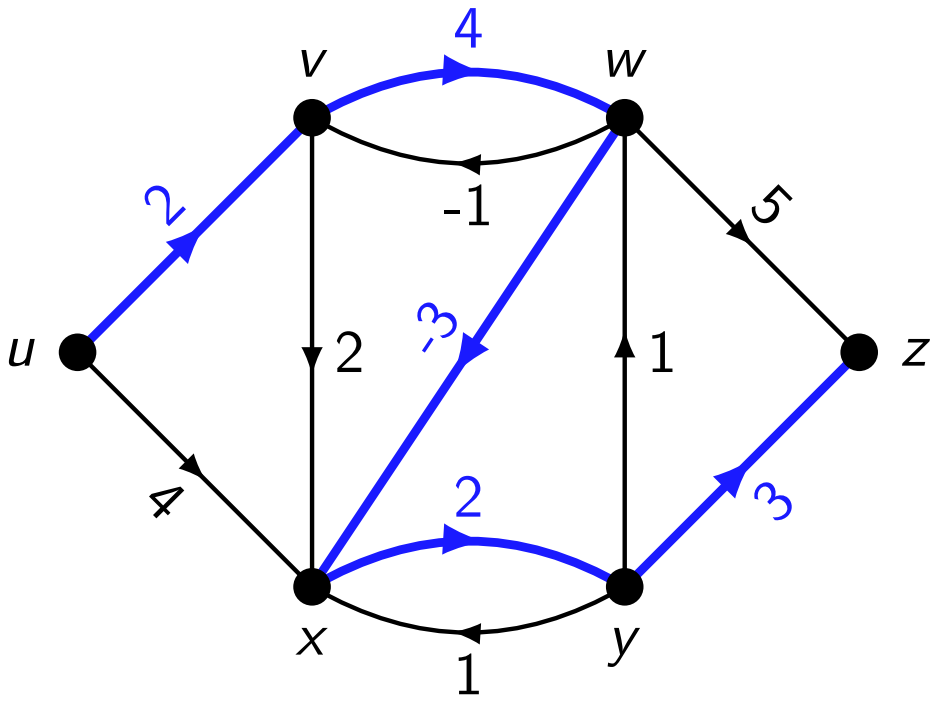
0	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(z, 0)$

So e.g. $d(u, z) = 8$, via the path $uvwxyz$.

We only actually use the $k = 0$ row of the table in calculating the $k = 1$ row, which we only use in calculating the $k = 2$ row...

Let's free the memory after we're done with it. Now we use $O(|V|)$ space!

Iterative Bellman-Ford: A better approach



k \ s	u	v	w	x	y	z
5	$(uv, 8)$	$(vw, 6)$	$(wx, 2)$	$(xy, 5)$	$(yz, 3)$	$(z, 0)$
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0	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(\emptyset, \infty)$	$(z, 0)$

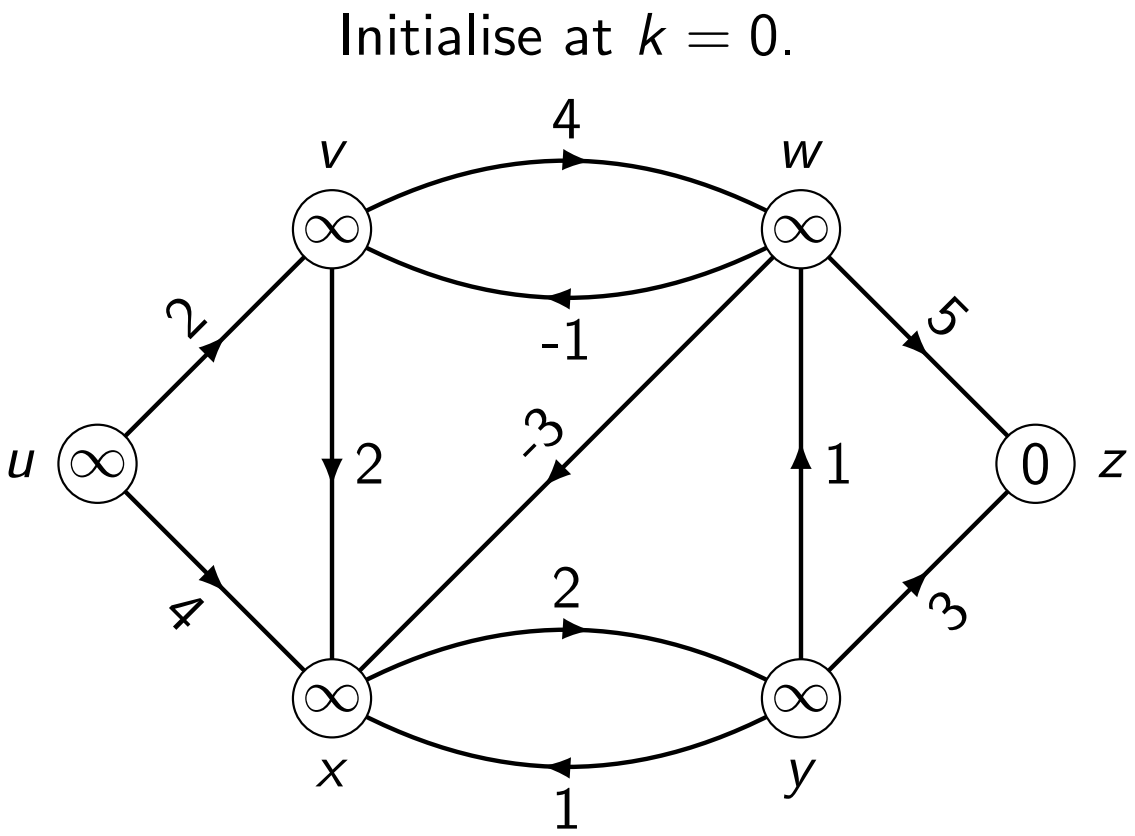
So e.g. $d(u, z) = 8$, via the path $uvwxyz$.

We can be even more cunning, and store **only the current row**.

So when we try and retrieve the value from our table for $s = v$, $k = 3$ (say), we might get the value for $s = v$, $k = 4$ instead if we already updated it. But this is OK — in fact, it means we sometimes find shorter paths faster!

Iterative Bellman-Ford: An even better approach

Here's what this looks like in practice:



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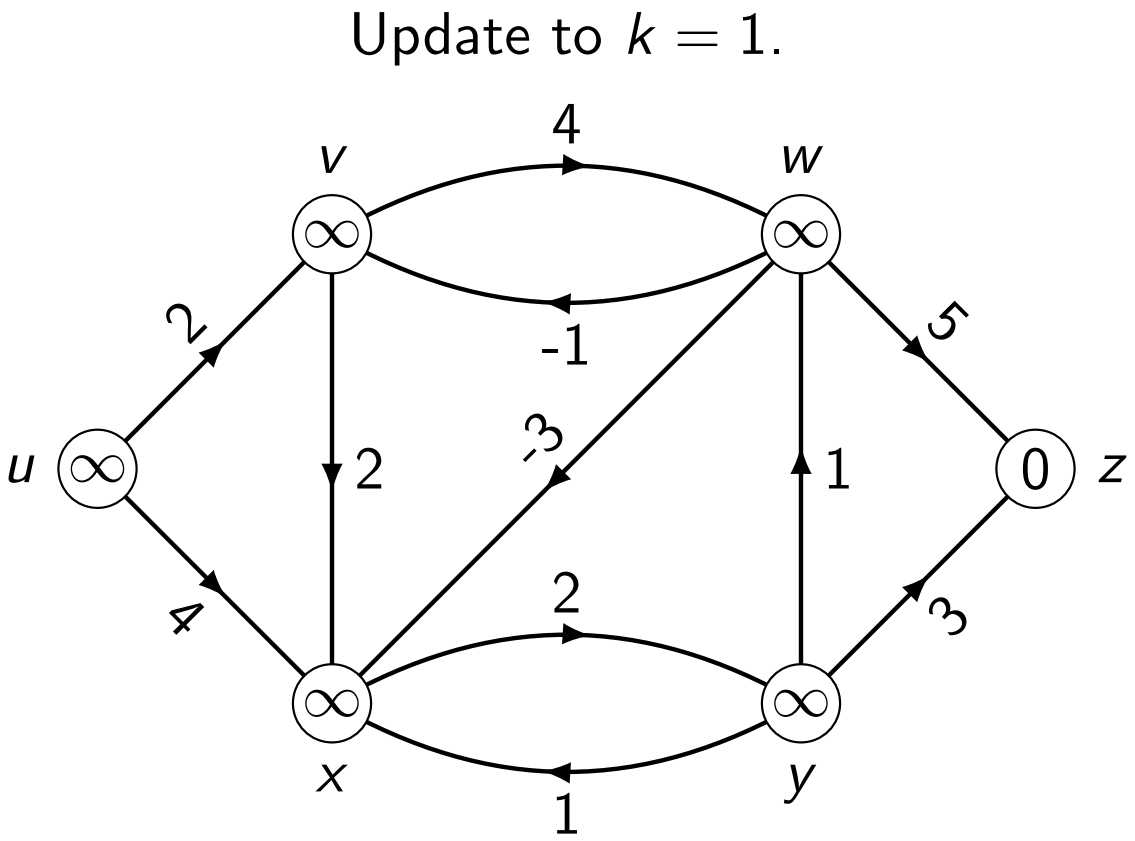

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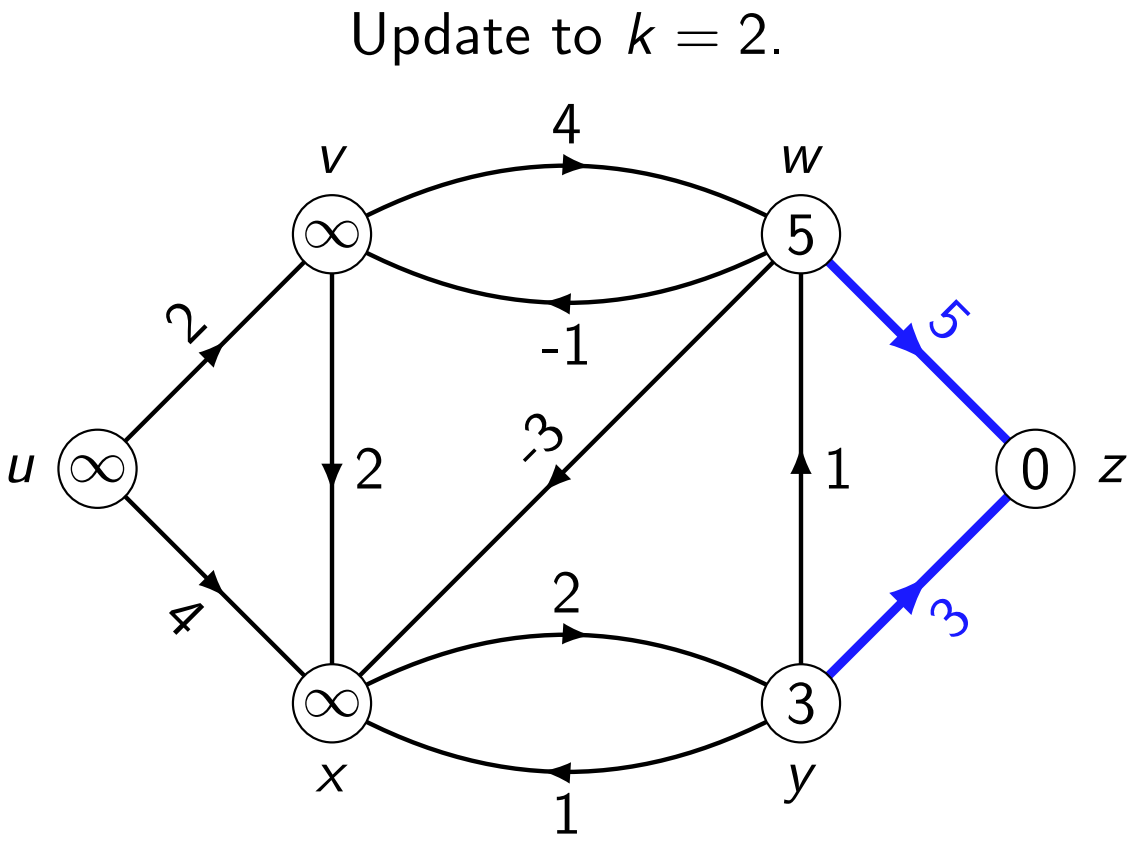

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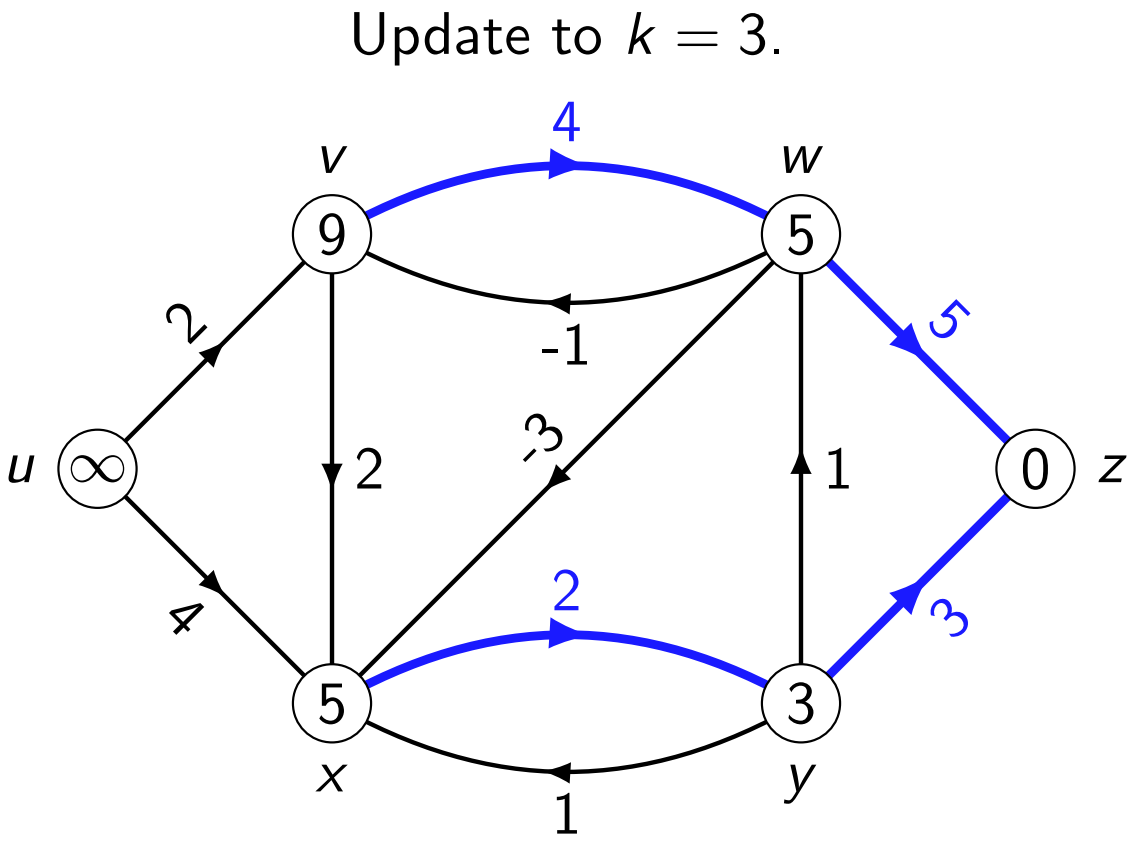

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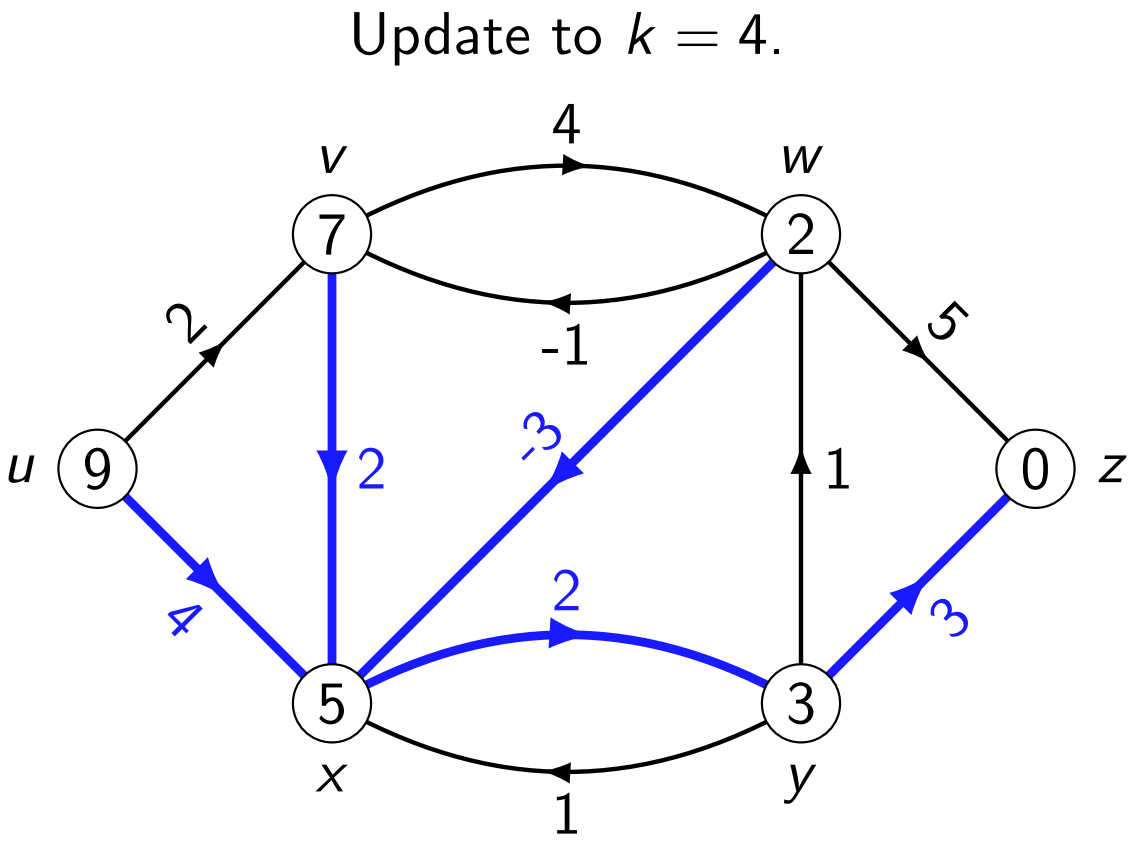
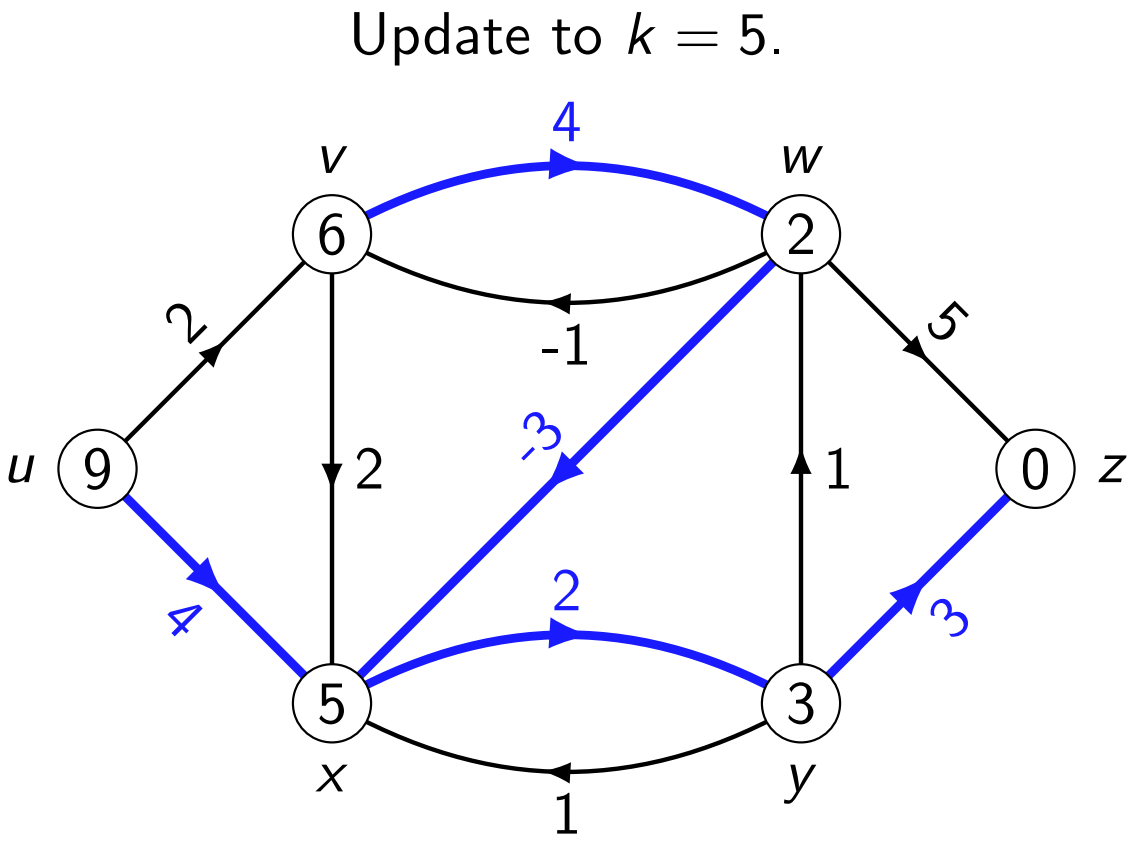


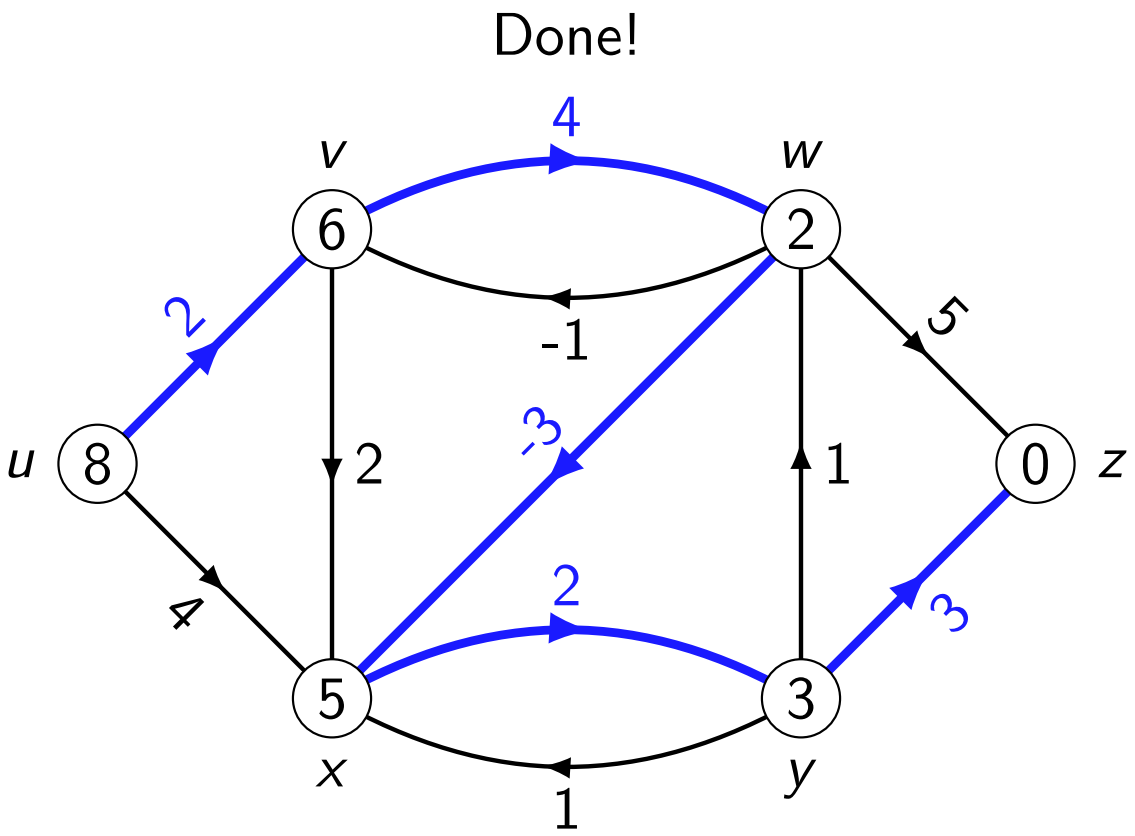
Iterative Bellman-Ford: An even better approach

Here's what this looks like in practice:



Iterative Bellman-Ford: An even better approach

Here's what this looks like in practice:



Now let's put this in pseudocode...

The real Bellman-Ford algorithm

Algorithm: BellmanFord

Input : A weighted digraph $G = ((V, E), w)$ with no negative-weight cycles and vertices $s, t \in V(G)$.

Output : A shortest path from s to t, or `None` if none exists.

```
1  begin
2      Let dist[v] ← ∞ for all v ∈ V \ t, dist[t] ← 0.
3      Let edge[v] ← None for all v ∈ V.
4      for i = 1 to |V| − 1 do
5          for u in V do
6              for v in N⁺(u) do
7                  if dist[u] > w(u, v) + dist[v] then
8                      dist[u] ← w(u, v) + dist[v] and edge[u] ← (u, v).
9      v ← s. while v ≠ t do
10         If edge[v] = None, return None.
11         Else writing edge[v] = (v, w), output (v, w) and set v ← w.
```

This now takes $O(|V| \sum_{u \in V} d^+(u)) = O(|V||E|)$ time, by the handshaking lemma, and $O(|V|)$ space. Using edge, you can also output every *other* shortest path to t in $O(|V|^2)$ time.

Other useful pathfinding algorithms

Bellman-Ford as described gives you all shortest paths to a sink.

You can also adapt it to give you all shortest paths from a source, like Dijkstra. (See problem sheet.)

What if you want shortest paths from **all** sources to **all** sinks, though? Repeatedly applying Dijkstra gives you $O(|V||E| \log |V|)$ time for non-negative edge weights.

You can match this running time even with negative edge weights using **Johnson's algorithm**.

Also, a lot of the time you're not working blind — you have some idea of "which direction is best", e.g. if you're pathfinding in a video game. In this case you should use a heuristic-guided algorithm like **A* search**, which often runs much faster than Dijkstra or Bellman-Ford.