Hamilton cycles
COMS20010 2020, Video 3-2

John Lapinskas, University of Bristol

Hamilton cycles

What if instead of using every edge once, we used every vertex once?

A **cycle** is a walk $W = w_0 \dots w_k$ with $w_0 = w_k$ and $k \geq 3$, in which every vertex appears at most once except for w_0 and w_k (which appear twice).

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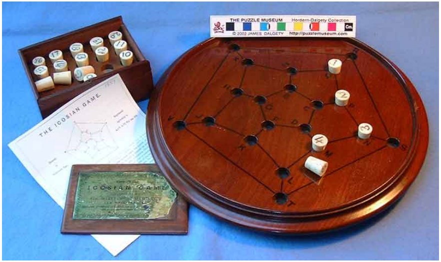
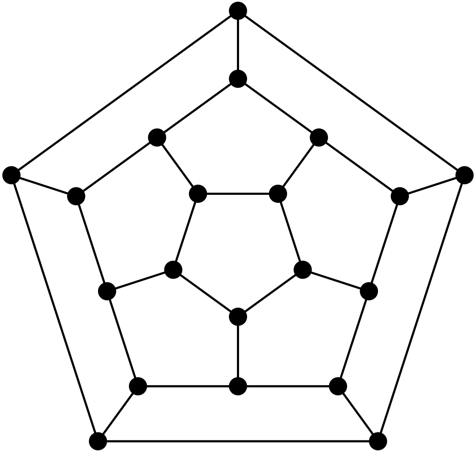


Hamilton actually made and sold a game based on trying to find Hamilton cycles in a dodecahedron!

Perhaps not surprisingly, it didn't sell very well.

Finding Hamilton cycles

In a particular (undirected) graph, Hamilton cycles can be easy to find:



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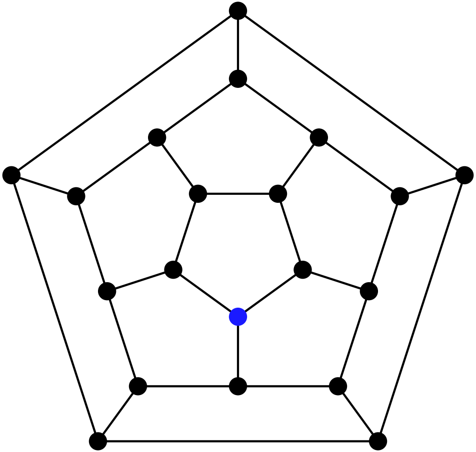

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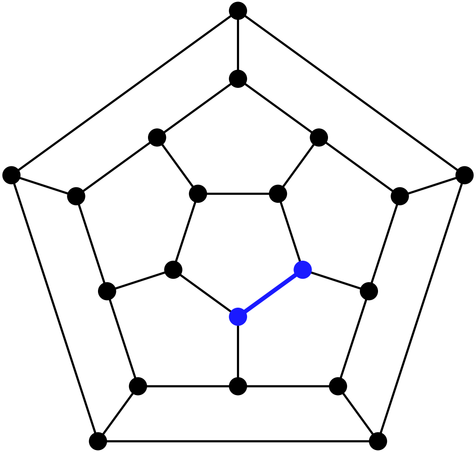

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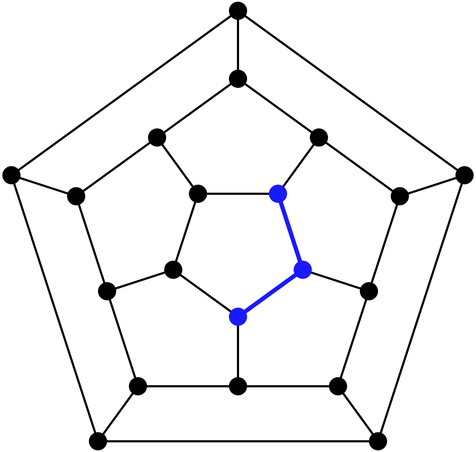

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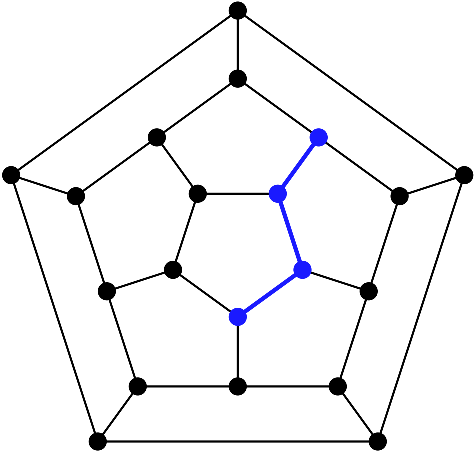

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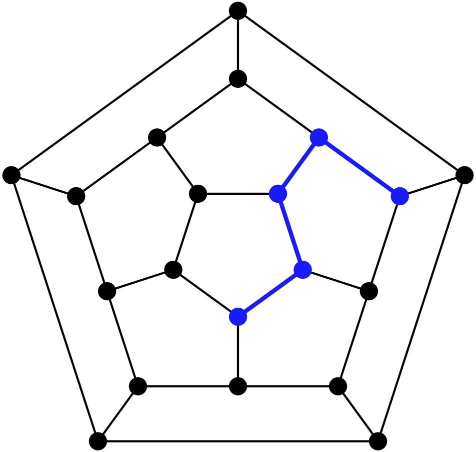

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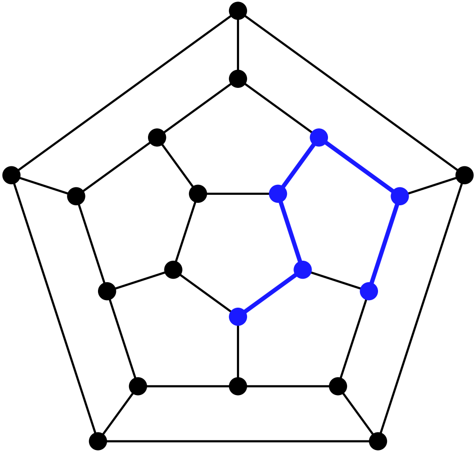

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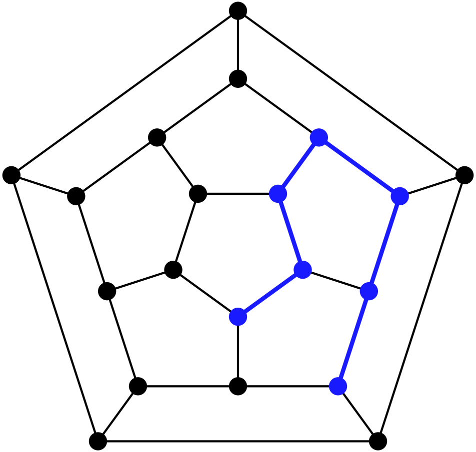

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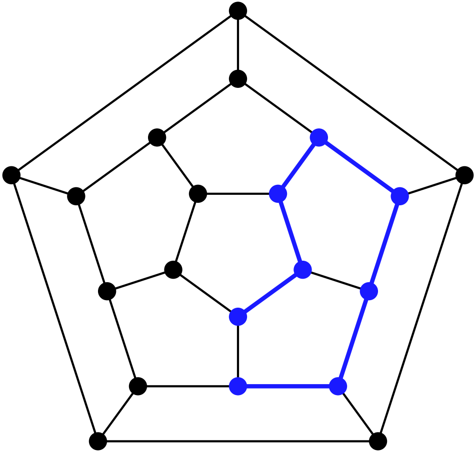

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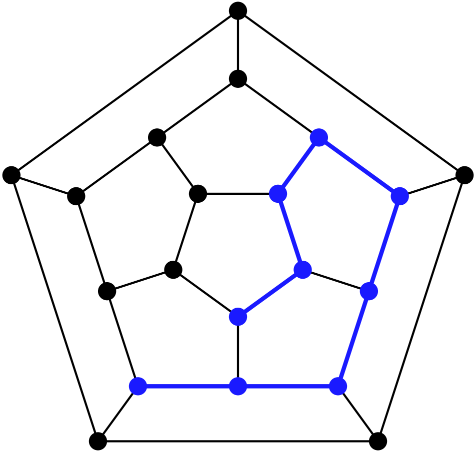

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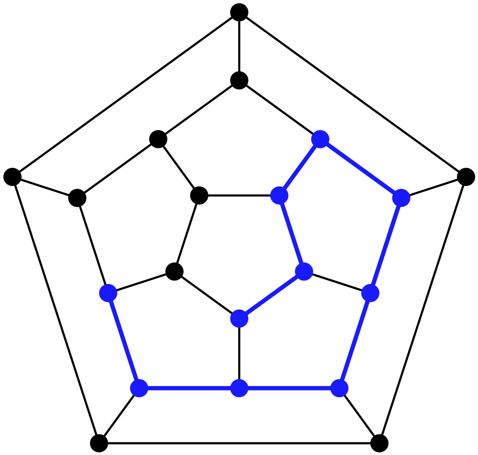

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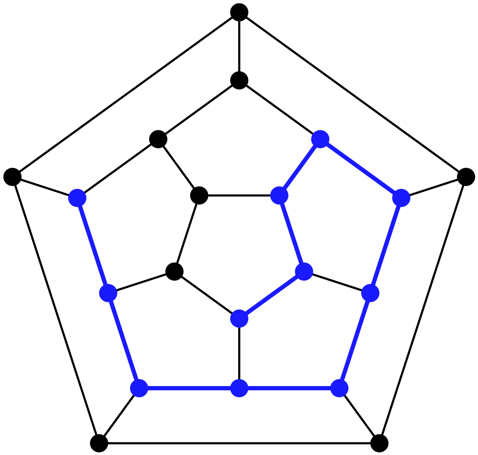

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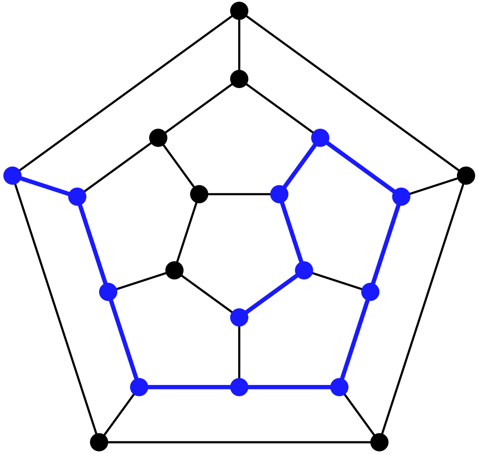

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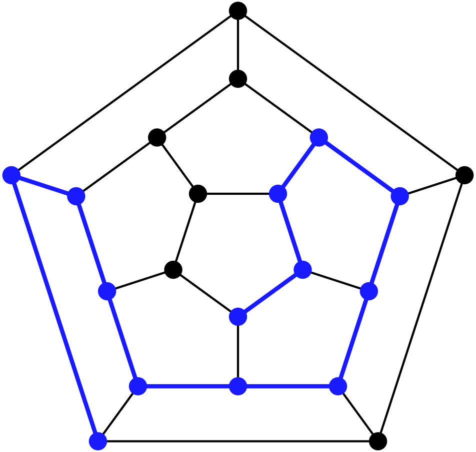

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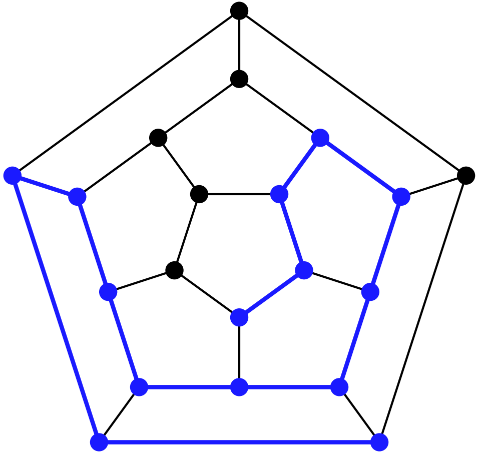

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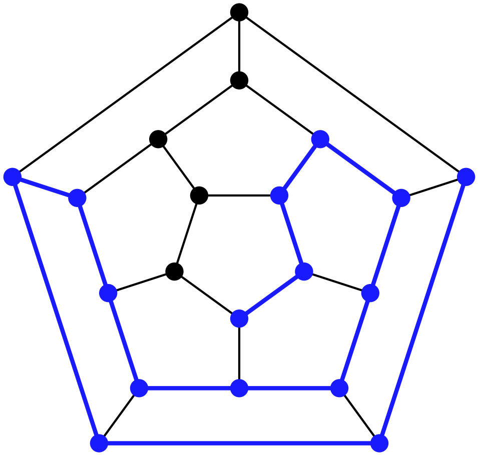

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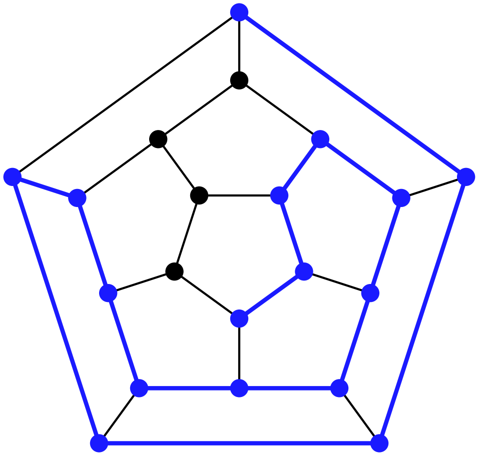

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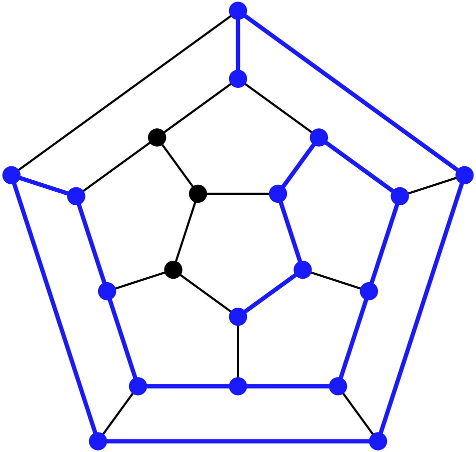

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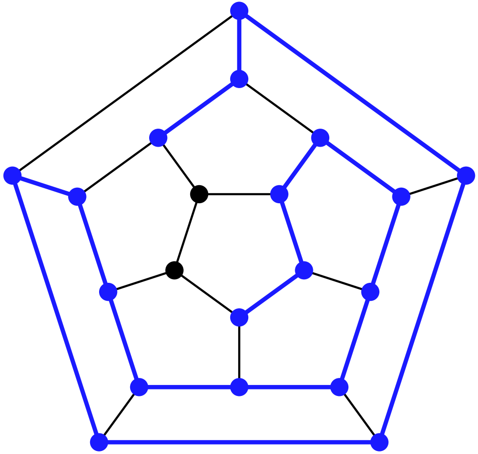

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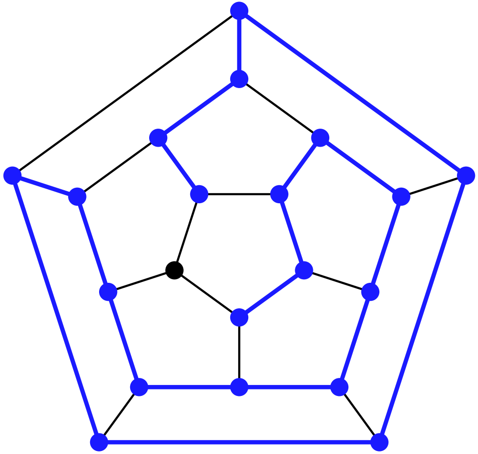

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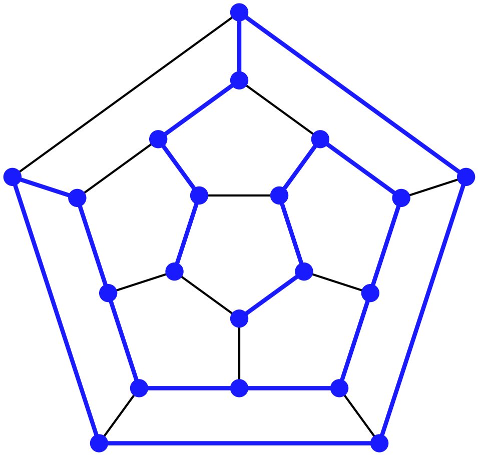

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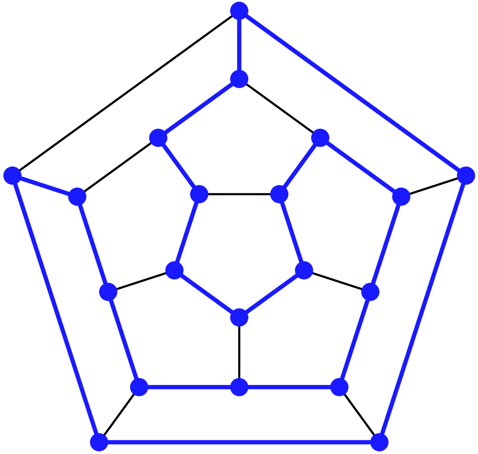

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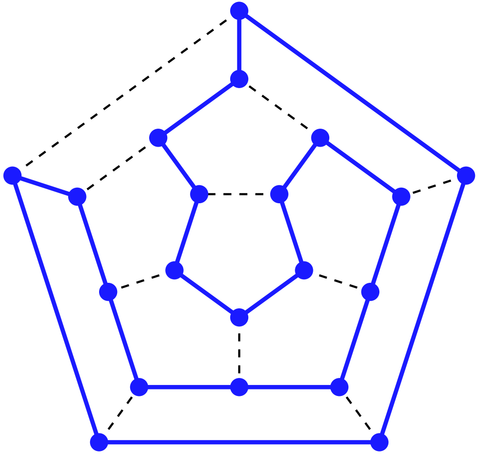

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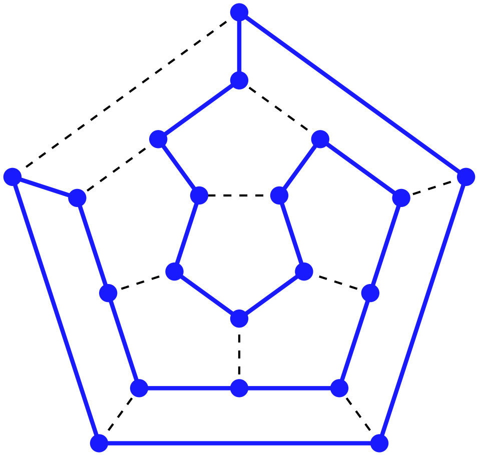

But in general, they can be very hard to find. If you prove an easy-to-check condition like the one for Euler walks, you stand to win a million dollars!

Dirac's theorem

Just because we can't find Hamilton cycles in general doesn't mean we can't find them in special cases...

Dirac's Theorem: Let $n \geq 3$. Then any n-vertex graph G with minimum degree at least $n/2$ has a Hamilton cycle.

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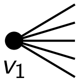
Case 1: $\boldsymbol{k \leq n/2}$. Then being greedy works! E.g. $n = 10$:



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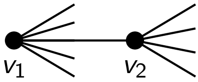

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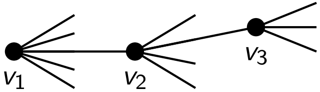

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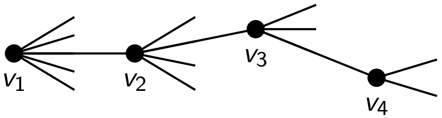

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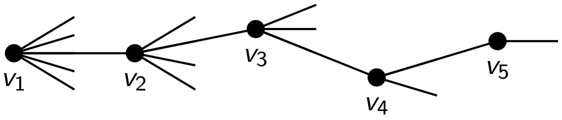

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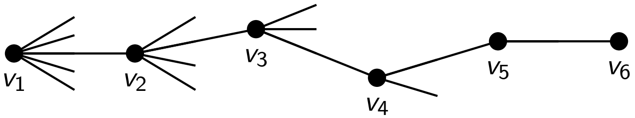

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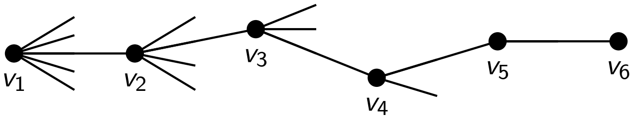

In general, $d(v_k) \geq n/2 > |\{v_1, \dots, v_{k-1}\}|$, so there's a vertex v_{k+1} adjacent to v_k other than $v_1, \dots, v_{k-1}$. Then $v_1 \dots v_{k+1}$ is a path of length $k+1$. ✓

Dirac's Theorem: Any n-vertex graph G with minimum degree at least $n/2$ has a Hamilton cycle.

Idea: Repeatedly extend a k-vertex path in G.

Lemma 1: If G contains a k-vertex path with $1 \leq k \leq n/2$, then G contains a $(k+1)$-vertex path. ✓

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Case 2c: Both $N(v_1) \subseteq \{v_2, \dots, v_k\}$ and $N(v_k) \subseteq \{v_1, \dots, v_{k-1}\}$.
In this case, we *use* the fact that greedy extension fails to extend the path in another way.

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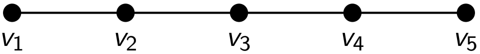
Let $v_1 \dots v_k$ be a k-vertex path in G.
We are done unless $N(v_1) \subseteq \{v_2, \dots, v_k\}$ and $N(v_k) \subseteq \{v_1, \dots, v_{k-1}\}$.

Never think about graphs without a picture. What does this **look like**? Say just for $n = 8$, $k = \frac{1}{2}n + 1 = 5$?



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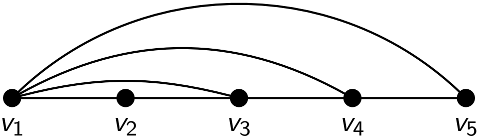

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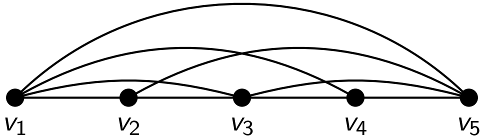

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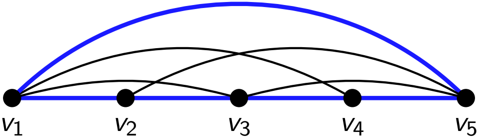

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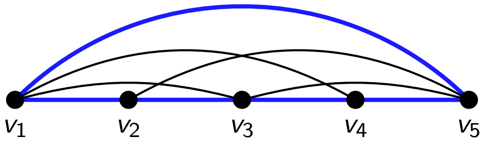

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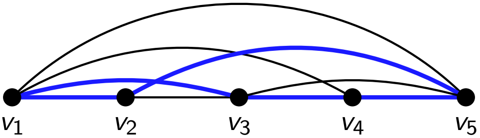

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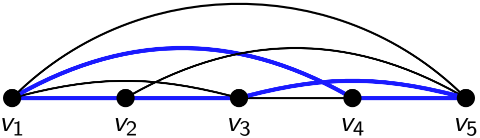

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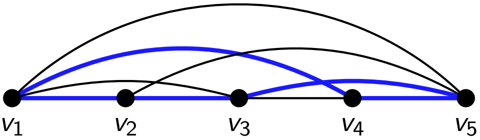

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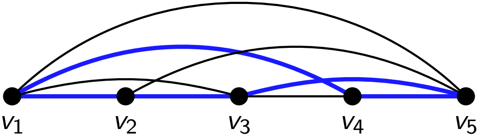

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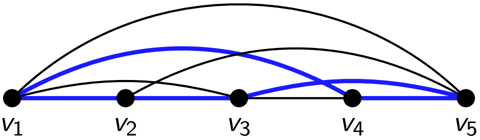

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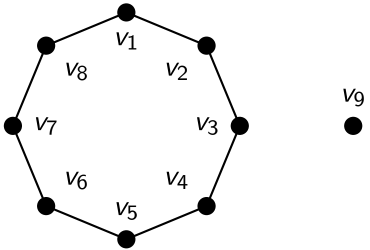

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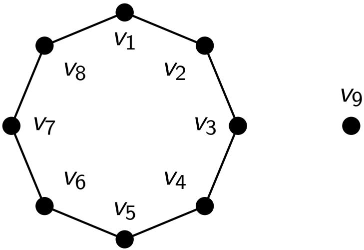

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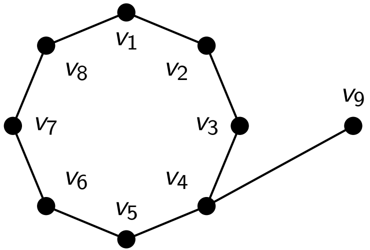

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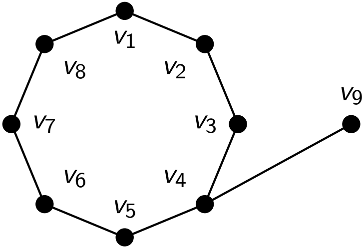

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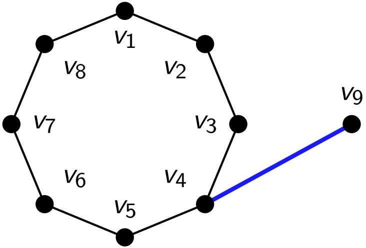

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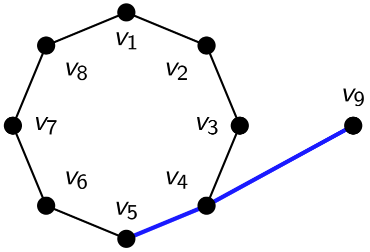

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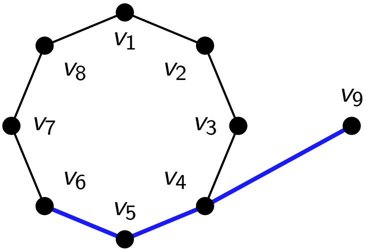

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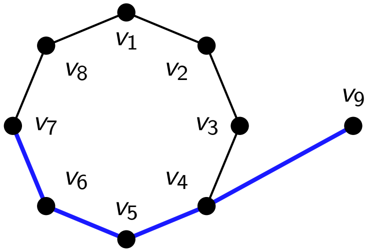

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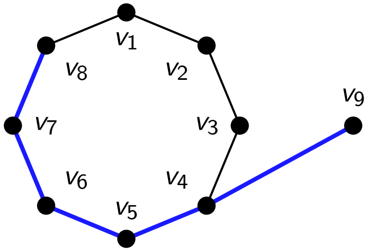

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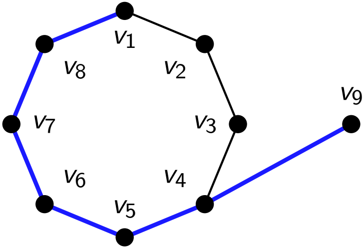

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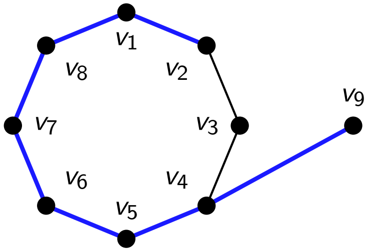

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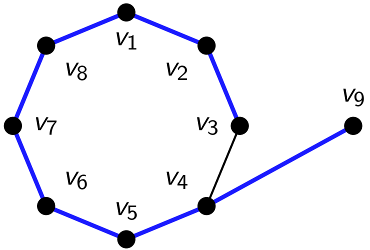

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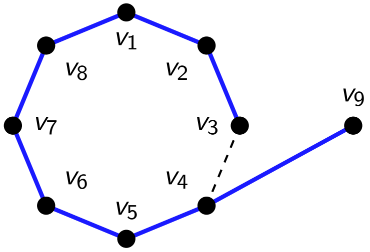

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So now by starting with a single-vertex path and repeatedly applying our three Lemmas, we reach an n-vertex path.

Then Lemma 2 turns this into a Hamilton cycle and we're done! □

Note this proof gives us a (fairly fast) algorithm for finding a Hamilton cycle when Dirac's theorem applies. This often happens in graph theory!

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But there are other ways to improve it. For example, when we do have minimum degree $n/2$, there's more than just one Hamilton cycle.

In fact, for large graphs, we can find $(n-2)/8$ **disjoint** Hamilton cycles, decomposing almost half the graph!

(Proved in 2013–4 by Csaba, Kühn, Lapinskas, Lo, Osthus and Treglown.)