Dijkstra's algorithm
COMS20010 2020, Video 4-4

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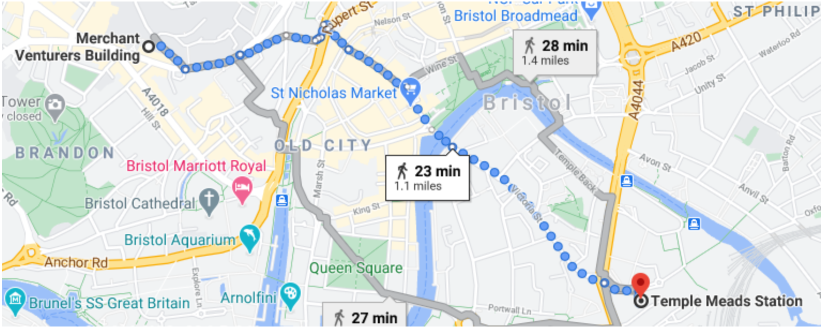
Distances in real networks are weighted!

We often model road networks as graphs: junctions and destinations are vertices, roads are edges, one-way roads are directed edges.

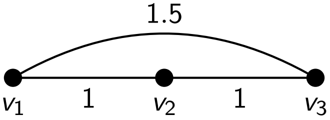


But when we want to find a “shortest path” in this graph, we don’t care about the number of edges, we care about the **physical distance**.

(We may also want to weight by e.g. elevation changes or current traffic.)

Weighted graphs

A **weighted graph** is a pair (G, w), where G is a graph and $w\colon E(G) \to \mathbb{R}$ is a **weight function**. This could represent distances, costs, times, etc.



$w(v_1, v_2) = 1$
$w(v_2, v_3) = 1$
$w(v_1, v_3) = 1.5$

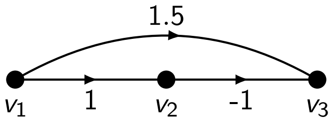
The **length** of a path/walk $P = x_1 \dots x_t$ is the total weight of P's edges:

$$\text{length}(P) = \sum_{i=1}^{t-1} w(x_i, x_{i+1}).$$

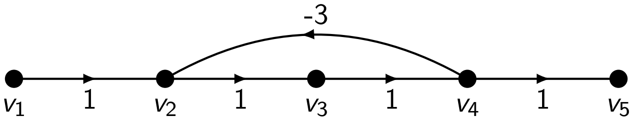
The **distance** from x to y is the shortest length of any path/walk from x to y, or ∞ if they are in different components. E.g. $d(v_1, v_3) = 1.5$.

Negative-weight edges

For some applications, it can make sense to allow edges to have **negative weight**. (E.g. costs versus profits...) This can be counterintuitive!



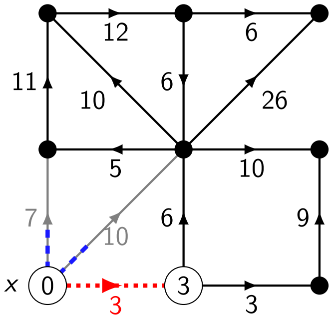
Here, $d(v_1, v_3) = \mathbf{0}$, since $v_1 v_2 v_3$ has cost $w(v_1, v_2) + w(v_2, v_3) = 0$.



Here, "distance" doesn't even make sense — there are walks from v_1 to v_5 with **arbitrarily low** length. E.g. $\mathrm{length}(v_1 v_2 v_3 v_4 v_2 v_3 v_4 \boldsymbol{v_2 v_3 v_4} v_5) = \mathbf{2}$...

This lecture, we ignore negative weights. (This is also faster!)

Dijkstra's algorithm: The idea



Think of breadth-first search as water flooding a set of pipes, starting from x... and now allow the pipes to have **different lengths**.

When the water first reaches a vertex v, you know $d(x, v)$ and a shortest path from x to v.

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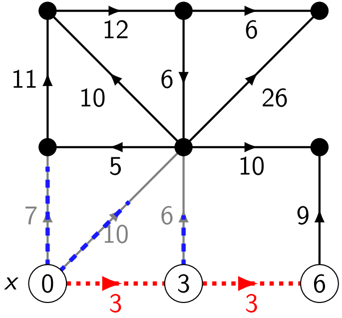

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When the water first reaches a vertex v, you know $d(x, v)$ and a shortest path from x to v.

At each stage, pick an edge (u, v) with $d(x, u)$ known and $d(x, v)$ unknown that minimises $d(x, u) + \mathrm{length}(u, v)$. (Break ties arbitrarily.)

Then set $d(x, v) = d(x, u) + \mathrm{length}(u, v)$.

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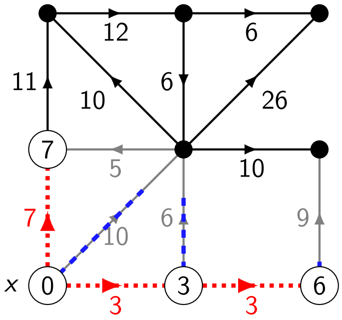

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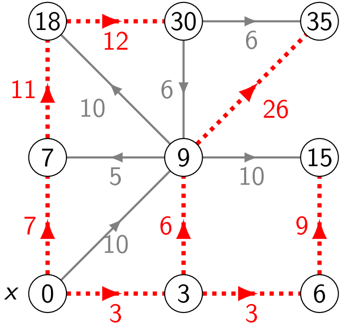

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Dijkstra's algorithm: Correctness

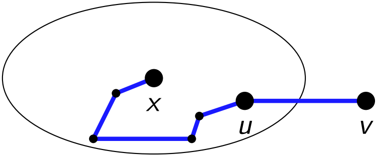
For input graph G, vertex x: At each stage, pick an edge (u, v) with $d(x, u)$ known and $d(x, v)$ unknown that minimises $d(x, u) + \mathrm{length}(u, v)$. Then set $d(x, v) = d(x, u) + \mathrm{length}(u, v)$.

Claim: Dijkstra's algorithm calculates distances correctly.

Proof: By induction on the number of vertices u with $d(x, u)$ set.

Base case: We have $d(x, x) = 0$. ✓

Inductive step: Suppose we know $d(x, u)$ for all $u \in X$, for some set X.
Say Dijkstra's algorithm picks an edge (u, v) with $u \in X$.



We can append (u, v) to any path from x to u, so we have

$$d(x, v) \leq d(x, u) + \mathrm{length}(u, v).$$

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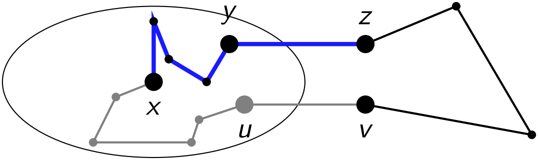


Also, any path P from x to v has to leave X on some edge (y, z). Hence P has length at least $d(x, y) + \mathrm{length}(y, z)$. So from the way we picked (u, v), we have $d(x, v) \geq d(x, u) + \mathrm{length}(u, v)$. □

Priority queues

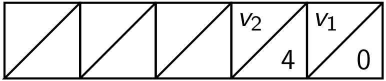
We need a **priority queue** (see COMS10007) to implement this efficiently.

Not like a normal queue: each element has a **priority**, and the "first" element is the one with the **lowest** priority (breaking ties **arbitrarily**).

Relevant operations:

- `StartQueue`(n) returns a new priority queue of maximum length n.
- `Insert`(x, p) inserts a new element x with priority p.
- `Extract`$()$ removes and returns the lowest-priority element.
- `ChangeKey`(x, p) updates the priority of x to p.

`StartQueue` takes $O(n)$ time, all other operations take $O(\log n)$ time.



`StartQueue`(5); `Insert`$(v_1, 0)$; `Insert`$(v_2, 4)$;

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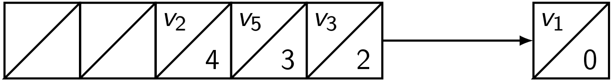

`Insert`$(v_3, 2)$; `Insert`$(v_5, 3)$; `Extract`$()$;

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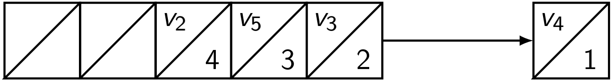

`Insert`($v_4, 20$); `ChangeKey`($v_4, 1$); `Extract`();

Algorithm: DIJKSTRA

Input : Weighted graph $G = ((V,E),w)$, $v \in V$.
Output : $d(v,y)$ for all $y \in V$.

Number the vertices of G as $v = v_1, \dots, v_n$.
queue $\leftarrow$ `StartQueue`(n).
foreach $i = 1$ *to* n **do**
    `dist`$[i] \leftarrow \infty$ and call `queue.Insert`(v_i, ∞).
Call `queue.ChangeKey`$(v_1, 0)$.
do
    $v_i \leftarrow$ `queue.Extract`$()$.
    foreach $(v_i, v_j) \in E$ **do**
        `dist`$[j] \leftarrow \min\{$`dist`$[j]$, `dist`$[i] + w(i,j)\}$.
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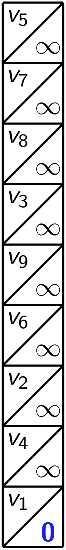
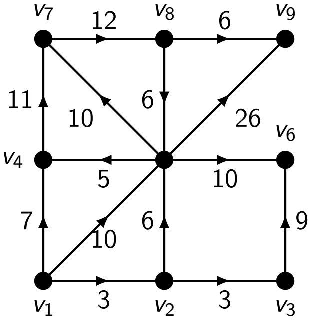
while queue *is not empty*
Return `dist`.





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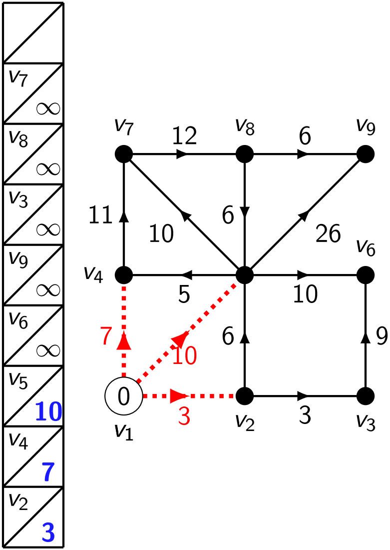

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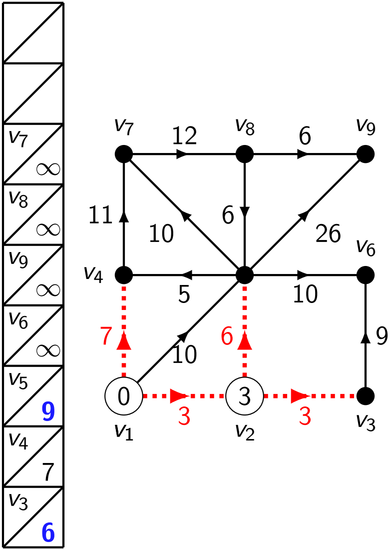

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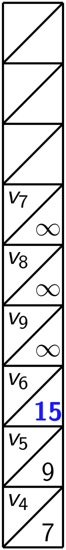

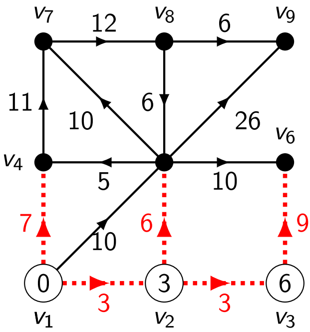

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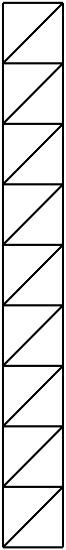
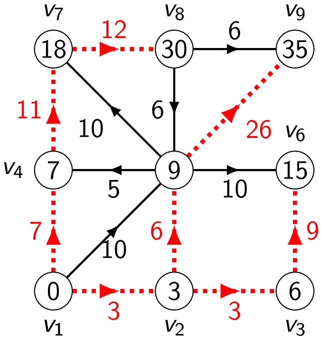
Invariant: `dist`$[j]$ is the minimum value of $d(v_1, v_i) + w(v_i, v_j)$ over all v_i's whose distances are finalised, as in mathematical version. ✓

We can recover shortest paths by storing and returning the dotted red edges.





Dijkstra's algorithm: Time analysis

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Call `queue.ChangeKey`$(v_1, 0)$.
do
    `vert` $\leftarrow$ `queue.Extract`$()$, say `vert` $= v_i$.
    foreach $(v_i, v_j) \in E$ **do**
        `dist`$[j] \leftarrow \min\{$`dist`$[j]$, `dist`$[i] + w(i, j)\}$.
        Call `queue.ChangeKey`$(v_j,$ `dist`$[j])$,
while queue *is not empty*
Return `dist`.

We perform $O(|V|)$ `Insert` operations and `Extract` operations, and $O(|E|)$ `ChangeKey` operations, for a total of $O((|V| + |E|) \log |V|)$ time when G is given in adjacency list form.

We could drop this to $O(|V| \log |V| + |E|)$ time by using a **Fibonacci heap** as a priority queue... But Fibonacci heaps have *awful* constants, and generally $\log |V| \lessapprox 50$, so let's not!