Dynamic programming
COMS20010 2020, Video 11-2

John Lapinskas, University of Bristol

Reminder from COMS10007: The Fibonacci sequence

The **Fibonacci sequence** is given by

$$F(0) = 0; \qquad F(1) = 1; \qquad F(n) = F(n-1) + F(n-2) \text{ for } n \geq 2.$$

Trying to use this recurrence to calculate it directly takes $\Theta(\phi^n)$ time:

$$\boxed{F(5)}$$

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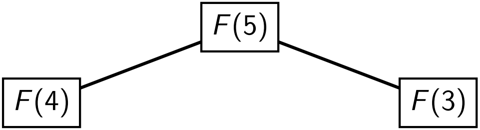

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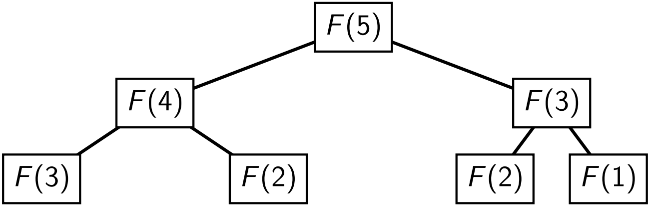

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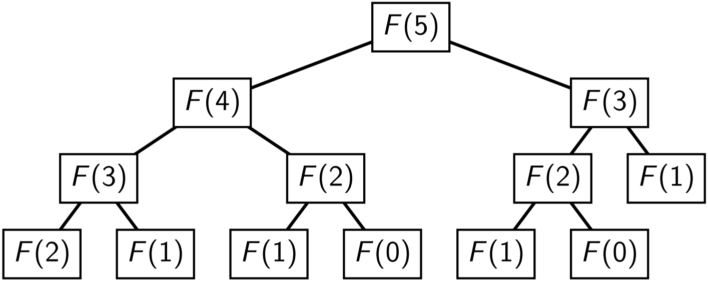

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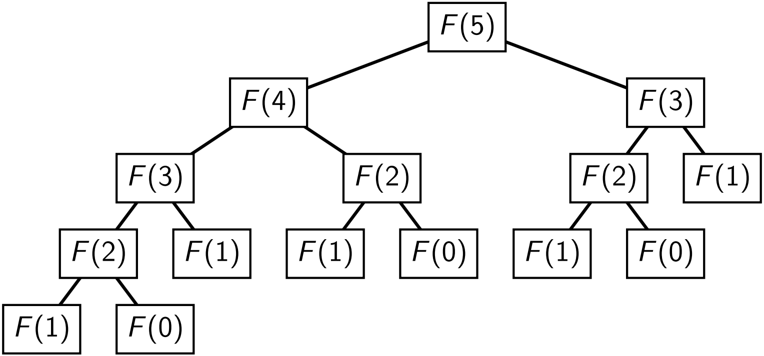

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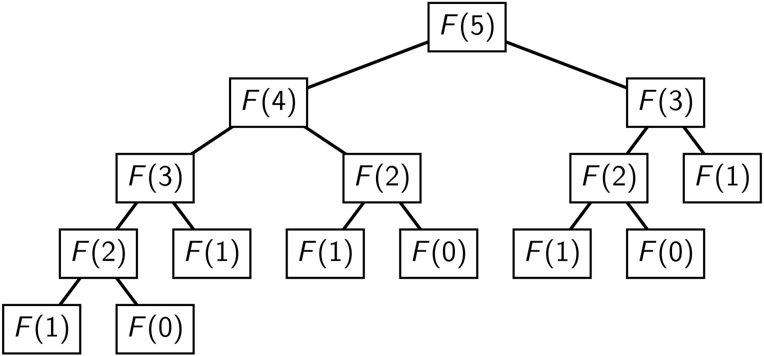

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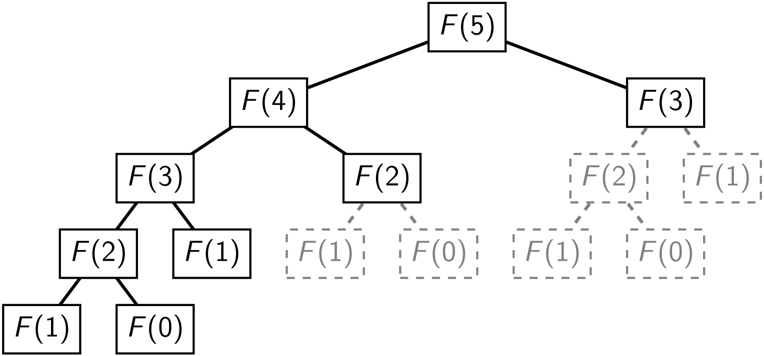

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We can do this literally, e.g. via static variables or a cache argument. This is called **memoization**, and any language can do it. E.g. for Python:

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$\longleftarrow$ Return `cache[5]`.

Either way, we turn a $\Theta(\phi^n)$-time algorithm for calculating F_n into a $\Theta(n)$-time algorithm. This technique is called **dynamic programming**.

Dynamic programming for weighted interval scheduling

In weighted interval scheduling, we have a slow recursive algorithm:

- Pick an arbitrary interval I;
- Recursively find the best schedule containing I;
- Recursively find the best schedule not containing I;
- Return whichever is better.

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Here, if we take I to be the interval with the latest finish time, rather than choosing it arbitrarily, things will work out nicely!

Why “fastest-finishing” works fast

Key point: Say our intervals are $\mathcal{R} = \{(s_1, f_1), \dots, (s_n, f_n)\}$, where $f_1 \le \dots \le f_n$. Then the slowest-finishing interval (s_n, f_n) only overlaps with intervals finishing later than s_n.

So our recursive calls always take $\mathcal{R} = \{(s_1, f_1), \dots, (s_i, f_i)\}$ for some i!

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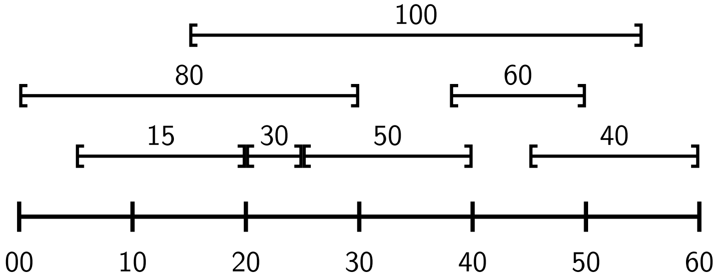

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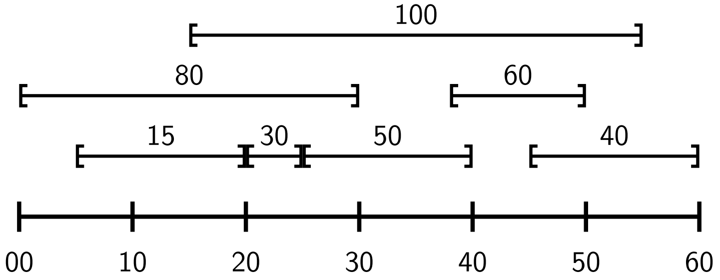

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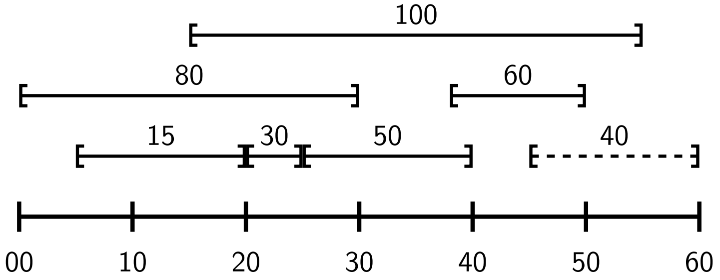

(45, 60) not in schedule: Recurse on $(5, 20), (20, 25), \dots, (15, 55)$.

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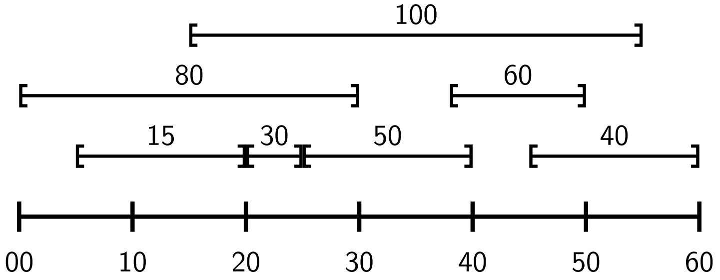

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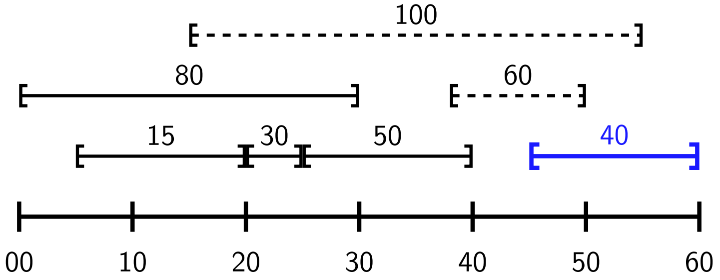

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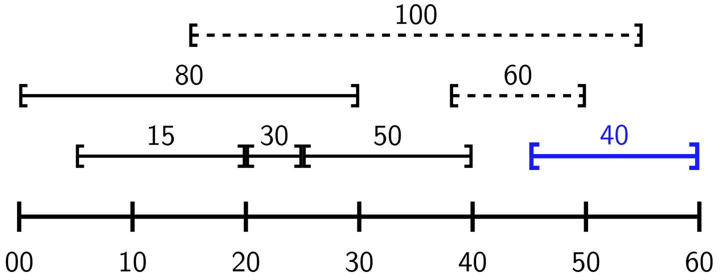

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Choosing I to be the fastest-starting interval works too — see quiz!

The recursive (memoised) version

Algorithm: WIS

Input : A **sorted** array $\mathcal{R}$ of n requests and a weight function w.
Output : A maximum-weight compatible subset of $\mathcal{R}$.

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  Write $\mathcal{R} = (s_1, f_1), \dots, (s_n, f_n)$ with $f_1 \leq \cdots \leq f_n$.
  if $\mathcal{R} = \emptyset$ **then**
    Return $\emptyset$.
  else if $\mathcal{R}$ *is in* `cache` **then**
    Return `cache`$[\mathcal{R}]$.
  else
    Let $X \leftarrow \{(s_i, f_i) \colon f_i > s_n\}$ be the set of intervals in $\mathcal{R}$ incompatible with (s_n, f_n).
    $S_{\text{out}} \leftarrow \mathrm{WIS}(\mathcal{R} \setminus \{(s_n, f_n)\}, w)$.
    $S_{\text{in}} \leftarrow \{I\} \cup \mathrm{WIS}(\mathcal{R} \setminus (\{(s_n, f_n)\} \cup X), w)$.
    if $w(S_{\text{out}}) > w(S_{\text{in}})$ **then** `output` $\leftarrow S_{\text{out}}$, **else** `output` $\leftarrow S_{\text{in}}$.
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Here `cache` is a static dictionary. Any sensible implementation (e.g. a hash table) will take $O(\log n)$ time or $O(1)$ time per access. We can find X in $O(\log n)$ time with binary search. So each call takes $O(\log n)$ time.

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Each call takes $O(\log n)$ time, and there are $O(n)$ total calls, for a total of $O(n \log n)$ time. We also need to sort $\mathcal{R}$ before calling WIS for the first time, which takes $O(n \log n)$ time.

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So overall, the running time is $O(n \log n)$!

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  $\texttt{cache} \leftarrow [\texttt{Null}] \times (n+1)$.
  $\texttt{cache}[0] \leftarrow \emptyset$.
  for $i = 1$ *to* n **do**
    Let $p(i) \leftarrow \max\{\{0\} \cup \{1 \leq j \leq i-1 \colon f_j \leq s_i\}\}$.
    $S_{\mathrm{out}} \leftarrow \texttt{cache}[i-1]$.
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    if $w(S_{\mathrm{out}}) > w(S_{\mathrm{in}})$ **then** $\texttt{cache}[i] \leftarrow S_{\mathrm{out}}$, **else** $\texttt{cache}[i] \leftarrow S_{\mathrm{in}}$.
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begin
  Sort $\mathcal{R} \leftarrow (s_1, f_1), \dots, (s_n, f_n)$ with $f_1 \leq \dots \leq f_n$.
  $\texttt{cache} \leftarrow [\texttt{Null}] \times (n+1)$.
  $\texttt{cache}[0] \leftarrow \emptyset$.
  for $i = 1$ *to* n **do**
    Let $p(i) \leftarrow \max\{\{0\} \cup \{1 \leq j \leq i-1 \colon f_j \leq s_i\}\}$.
    $S_{\mathrm{out}} \leftarrow \texttt{cache}[i-1]$.
    $S_{\mathrm{in}} \leftarrow \texttt{cache}[p(i)] \cup \{(s_i, f_i)\}$.
    if $w(S_{\mathrm{out}}) > w(S_{\mathrm{in}})$ **then** $\texttt{cache}[i] \leftarrow S_{\mathrm{out}}$, **else** $\texttt{cache}[i] \leftarrow S_{\mathrm{in}}$.
  Return $\texttt{cache}[n]$.

This algorithm is doing the same thing as the recursive algorithm, working from the base case $\mathcal{R} = \emptyset$ (corresponding to $\texttt{cache}[0]$) upwards.

Again, we can find $p(i)$ in $O(\log n)$ time with binary search, so the overall running time is $O(n \log n)$ — the same as the recursive version!

The iterative version

Algorithm: WIS

Input : An **unsorted** array $\mathcal{R}$ of n requests and a weight function w.
Output : A maximum-weight compatible subset of $\mathcal{R}$.

begin
  Sort $\mathcal{R} \leftarrow (s_1, f_1), \dots, (s_n, f_n)$ with $f_1 \leq \cdots \leq f_n$.
  $\texttt{cache} \leftarrow [\texttt{Null}] \times (n+1)$.
  $\texttt{cache}[0] \leftarrow \emptyset$.
  for $i = 1$ *to* n **do**
    Let $p(i) \leftarrow \max\{\{0\} \cup \{1 \leq j \leq i-1 \colon f_j \leq s_i\}\}$.
    $S_{\mathrm{out}} \leftarrow \texttt{cache}[i-1]$.
    $S_{\mathrm{in}} \leftarrow \texttt{cache}[p(i)] \cup \{(s_i, f_i)\}$.
    if $w(S_{\mathrm{out}}) > w(S_{\mathrm{in}})$ **then** $\texttt{cache}[i] \leftarrow S_{\mathrm{out}}$, **else** $\texttt{cache}[i] \leftarrow S_{\mathrm{in}}$.
  Return $\texttt{cache}[n]$.

It's generally good practice to make your dynamic programming algorithms iterative, since it often has lower constant overhead, and it can help you identify more significant savings. (See video 11-4!) But it is **not** necessary.

Unless you already know it's a performance bottleneck, do whichever you find easiest — premature optimisation creates bugs!