Hamilton cycles
COMS20010 2020, Video 3-2

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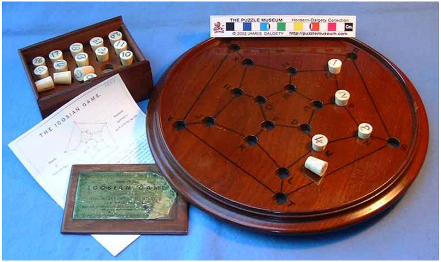
Hamilton cycles

What if instead of using every edge once, we used every vertex once?

A **cycle** is a walk $W = w_0 \dots w_k$ with $w_0 = w_k$ and $k \geq 3$, in which every vertex appears at most once except for w_0 and w_k (which appear twice).

A **Hamilton** cycle is a cycle containing every vertex in the graph. Naturally, they were studied by... Euler, in the context of knights' tours. (But then a century later by William Hamilton...)

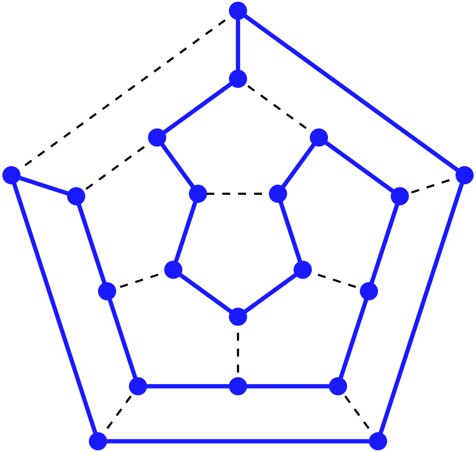


Hamilton actually made and sold a game based on trying to find Hamilton cycles in a dodecahedron!

Perhaps not surprisingly, it didn't sell very well.

Finding Hamilton cycles

In a particular (undirected) graph, Hamilton cycles can be easy to find:



But in general, they can be very hard to find. If you prove an easy-to-check condition like the one for Euler walks, you stand to win a million dollars!

Dirac's theorem

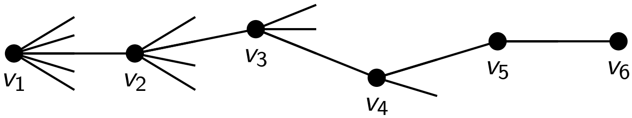
Just because we can't find Hamilton cycles in general doesn't mean we can't find them in special cases...

Dirac's Theorem: Let $n \geq 3$. Then any n-vertex graph G with minimum degree at least $n/2$ has a Hamilton cycle.

Proof: Try to find a long path inductively: start with a trivial one-vertex path and repeatedly extend it.

So suppose G contains a k-vertex path $v_1 \dots v_k$ for some $k \in [n-1]$.

Case 1: $k \leq n/2$. Then being greedy works! E.g. $n = 10$:



In general, $d(v_k) \geq n/2 > |\{v_1, \dots, v_{k-1}\}|$, so there's a vertex v_{k+1} adjacent to v_k other than $v_1, \dots, v_{k-1}$. Then $v_1 \dots v_{k+1}$ is a path of length $k+1$. ✓

Dirac's Theorem: Any n-vertex graph G with minimum degree at least $n/2$ has a Hamilton cycle.

Idea: Repeatedly extend a k-vertex path in G.

Lemma 1: If G contains a k-vertex path with $1 \leq k \leq n/2$, then G contains a $(k+1)$-vertex path. ✓

Case 2: $k > n/2$. Suppose G contains a k-vertex path $v_1 \dots v_k$.
Greedy extension may not work... but try anyway!

Case 2a: There exists a vertex $v_{k+1} \in N(v_k) \setminus \{v_1, \dots, v_{k-1}\}$.
Then $v_1 \dots v_{k+1}$ is a $(k+1)$-vertex path. ✓

Case 2b: There exists a vertex $v_0 \in N(v_1) \setminus \{v_2, \dots, v_k\}$.
Then $v_0 \dots v_k$ is a $(k+1)$-vertex path. ✓

Case 2c: Both $N(v_1) \subseteq \{v_2, \dots, v_k\}$ and $N(v_k) \subseteq \{v_1, \dots, v_{k-1}\}$.
In this case, we *use* the fact that greedy extension fails to extend the path in another way.

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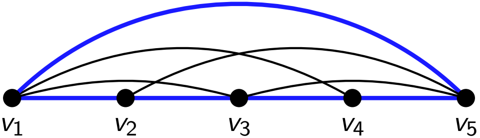
Idea: Repeatedly extend a k-vertex path in G.
Lemma 1: If $k \leq n/2$, then G contains a $(k+1)$-vertex path. ✓

Let $v_1 \ldots v_k$ be a k-vertex path in G.
We are done unless $N(v_1) \subseteq \{v_2, \ldots, v_k\}$ and $N(v_k) \subseteq \{v_1, \ldots, v_{k-1}\}$.

Never think about graphs without a picture. What does this **look like**? Say just for $n = 8$, $k = \frac{1}{2}n + 1 = 5$?



So it looks like we should be able to turn our path into a cycle...

Of course, in general $\{v_1, v_k\}$ might not be an edge!
But there are lots of other cycles available.

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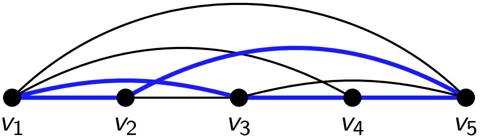

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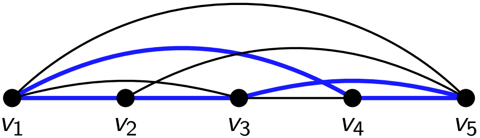

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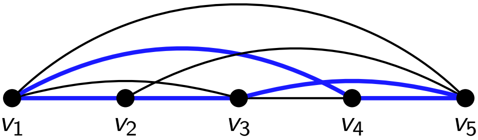
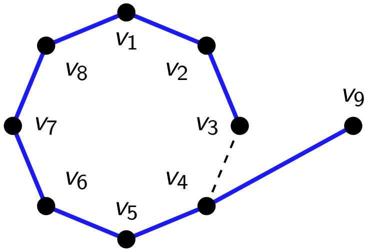


In general: We seek $v_i \in N(v_1)$ such that $v_{i-1} \in N(v_k)$.
Then $v_1 v_i \dots v_k v_{i-1} v_{i-2} \dots v_1$ will be a cycle on $\{v_1, \dots, v_k\}$.

There are at least $n/2$ vertices $v_i \in N(v_1)$, hence at least $n/2$ vertices in $\{v_{i-1} \colon v_i \in N(v_1)\}$. There are also at least $n/2$ vertices in $N(v_k)$.
Both sets are contained in $\{v_1, \dots, v_{k-1}\}$, which has size at most $n-1$.
So they must intersect. **This works even when $k = n$.** ✓

Dirac's Theorem: Any n-vertex graph G with minimum degree at least $n/2$ has a Hamilton cycle.

Idea: Repeatedly extend a k-vertex path in G.
Lemma 1: If $k \leq n/2$, then G contains a $(k+1)$-vertex path. ✓
Lemma 2: If $k > n/2$, then G contains **either** a $(k+1)$-vertex path
or a k-vertex cycle. ✓



So suppose G has a cycle $v_1 \dots v_k$, with $n/2 < k < n$, and let v_{k+1} be an arbitrary vertex not in the cycle.

We have $d(v_{k+1}) \geq n/2$, and $|\{v_1, \dots, v_k\}| > n/2$, and the graph has n vertices. So v_{k+1} must be adjacent to some v_i on the cycle.

Then $v_{k+1} v_i \dots v_k v_1 \dots v_{i-1}$ is a $(k+1)$-vertex path. ✓

Dirac's Theorem: Any n-vertex graph G with minimum degree at least $n/2$ has a Hamilton cycle.

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Lemma 1: If $k \leq n/2$, then G contains a $(k+1)$-vertex path. ✓
Lemma 2: If $k > n/2$, then G contains **either** a $(k+1)$-vertex path
or a k-vertex cycle. ✓
Lemma 3: If $n/2 < k < n$ and G contains a k-vertex cycle, then
G contains a $(k+1)$-vertex path. ✓

So now by starting with a single-vertex path and repeatedly applying our three Lemmas, we reach an n-vertex path.

Then Lemma 2 turns this into a Hamilton cycle and we're done! □

Note this proof gives us a (fairly fast) algorithm for finding a Hamilton cycle when Dirac's theorem applies. This often happens in graph theory!

How good is Dirac's Theorem?

Minimum degree $n/2$ seems like quite a big thing to ask — can Dirac's theorem be improved on? In one sense, no. For example:

This graph G certainly has no Hamilton cycle, and has minimum degree $3 = \frac{1}{2}|V(G)| - 1$. So Dirac's theorem is false for minimum degree $\frac{1}{2}n - 1$.

But there are other ways to improve it. For example, when we do have minimum degree $n/2$, there's more than just one Hamilton cycle.

In fact, for large graphs, we can find $(n-2)/8$ **disjoint** Hamilton cycles, decomposing almost half the graph!

(Proved in 2013–4 by Csaba, Kühn, Lapinskas, Lo, Osthus and Treglown.)