Flow networks
COMS20010 2020, Video 8-3

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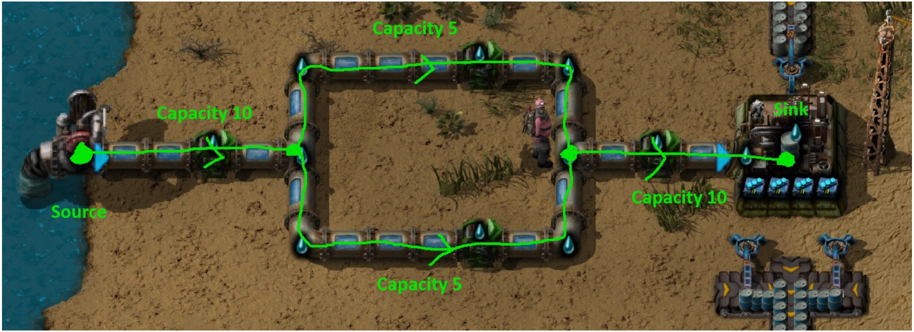
In these two lectures, we'll talk about "flow networks", where something is travelling from place to place inside a graph. Examples include:

- Water networks;
- Power networks;
- Road systems;
- Internet infrastructure;
- People moving through stalls at Freshers' Fair.

They're also useful in a wide variety of other settings, including:

- Airline scheduling;
- Image segmentation;
- Proving graph theory results;
- Survey design;
- Professional baseball. (See KT 7.12!)

For now, let's just consider a toy problem. One pump supplies water for one factory, passing through a network of pipes of different capacities.



The problem: How much water can get to the factory?

(The reason we're considering such a basic problem is that it will turn out most of the more interesting problems **reduce** to this one...!)

More generally: A **flow network** (G, c, s, t) consists of a directed graph $G = (V, E)$, a **capacity** function $c\colon E \to \mathbb{N}$, a **source** vertex $s \in V$ with $N^-(s) = \emptyset$, and a **sink** vertex $t \in V$ with $N^+(t) = \emptyset$.

A **flow** in (G, c, s, t) is a function $f\colon E \to \mathbb{R}$ with the following properties:

- No edge has more flow than capacity; for all $e \in E$, $0 \le f(e) \le c(e)$.
- Flow is conserved at vertices; for all $v \in V \setminus \{s, t\}$,

$$\sum_{u \in N^-(v)} f(u, v) = \sum_{w \in N^+(v)} f(v, w).$$

For brevity, we write $f^-(v) = \sum_{u \in N^-(v)} f(u, v)$ for the total flow into v, and $f^+(v) = \sum_{w \in N^+(v)} f(v, w)$ for the total flow out of v.

The **value** of f, denoted $v(f)$, is $f^+(s)$.

The problem: Find a **maximum flow**: a flow f maximising $v(f)$.

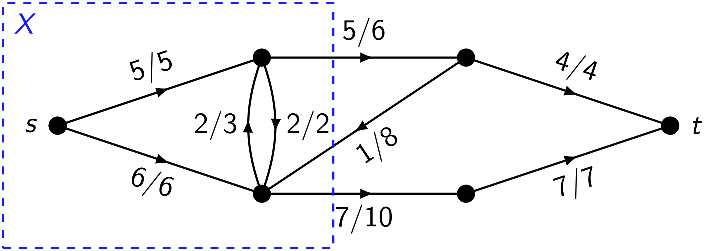
A **flow network** (G, c, s, t) is a directed graph $G = (V, E)$, a **capacity** $c \colon E \to \mathbb{N}$, a **source** $s \in V$, and a **sink** $t \in V$, with $N^-(s) = N^+(t) = \emptyset$.

A **flow** is a function $f \colon E \to \mathbb{R}$ such that for all $e \in E$ and $v \in V \setminus \{s, t\}$:

- $0 \le f(e) \le c(e)$;
- $f^-(v) := \sum_{u \in N^-(v)} f(u, v) = \sum_{w \in N^+(v)} f(v, w) =: f^+(v)$.

Why do we define the value of f by $v(f) = f^+(s)$ rather than e.g. $f^-(t)$?

Because we get the same answer either way! Let's make that formal.



We write $f^+(X) := \sum_{e \text{ out of } X} f(e)$ and $f^-(X) := \sum_{e \text{ into } X} f(e)$.

For example, here $f^+(X) = 5 + 7 = 12$ and $f^-(X) = 1$.

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The **value** of f, denoted $v(f)$, is $f^+(s)$.
We write $f^+(A) := \sum_{e \text{ out of } A} f(e)$ and $f^-(A) := \sum_{e \text{ into } A} f(e)$.

Lemma 1: For all sets $X \subseteq V \setminus \{s, t\}$, we have $f^+(X) = f^-(X)$.
(So flow is conserved in sets as well as at individual vertices.)

Proof: By summing conservation of flow over all $v \in X$:

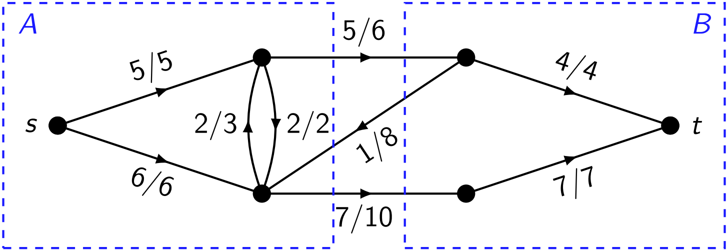
$$\sum_{v \in X} \sum_{u \in N^-(v)} f(u, v) = \sum_{v \in X} \sum_{w \in N^+(v)} f(v, w).$$

For all $e \subseteq X$, $f(e)$ appears once on each side; after cancelling those terms we're left with $f^+(X) = f^-(X)$. $\square$

The **value** of a flow f, denoted $v(f)$, is $f^+(s)$.
We write $f^+(A) := \sum_{e \text{ out of } A} f(e)$ and $f^-(A) := \sum_{e \text{ into } A} f(e)$.

Lemma 1: For all sets $X \subseteq V \setminus \{s,t\}$, we have $f^+(X) = f^-(X)$.

A **cut** is any pair of disjoint sets $A, B \subseteq V$ with $A \cup B = V$, $s \in A$ and $t \in B$. (So A and B partition V, the source is in A and the sink is in B.)



Lemma 2: For all cuts (A,B), $f^+(A) - f^-(A) = f^-(B) - f^+(B) = v(f)$.

Proof: By Lemma 1, we have $f^+(A \setminus \{s\}) = f^-(A \setminus \{s\})$.

But $f^+(A \setminus \{s\}) = f^+(A) - f(s,B)$ and $f^-(A \setminus \{s\}) = f^-(A) + f(s,A)$...
So $f^+(A) - f(s,B) = f^-(A) + f(s,A)$.

The **value** of a flow f, denoted $v(f)$, is $f^+(s)$.
We write $f^+(A) := \sum_{e \text{ out of } A} f(e)$ and $f^-(A) := \sum_{e \text{ into } A} f(e)$.

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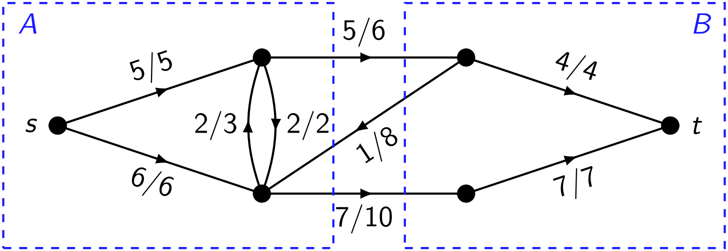


Lemma 2: For all cuts (A, B), $f^+(A) - f^-(A) = f^-(B) - f^+(B) = v(f)$.

Proof: By Lemma 1, we have $f^+(A \setminus \{s\}) = f^-(A \setminus \{s\})$.

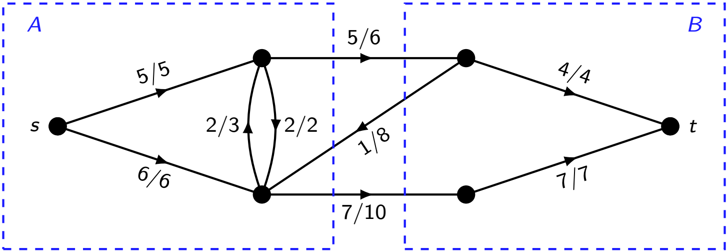
Rearranging $f^+(A) - f(s, B) = f^-(A) + f(s, A)$:
$f^+(A) - f^-(A) = f(s, B) + f(s, A) = f^+(s) = v(f)$.

The **value** of a flow f, denoted $v(f)$, is $f^+(s)$.
We write $f^+(A) := \sum_{e \text{ out of } A} f(e)$ and $f^-(A) := \sum_{e \text{ into } A} f(e)$.

Lemma 1: For all sets $X \subseteq V \setminus \{s,t\}$, we have $f^+(X) = f^-(X)$.

A **cut** is any partition (A,B) of V with $s \in A$ and $t \in B$.

Lemma 2: For all cuts (A,B), $f^+(A) - f^-(A) = f^-(B) - f^+(B) = v(f)$. **Proof:** We have shown $v(f) = f^+(A) - f^-(A)$.



Since A and B are disjoint and $A \cup B = V$, the edges out of A are the edges into B, so $f^+(A) = f^-(B)$. Likewise $f^-(A) = f^+(B)$. □

Lemma 2 implies we could have defined $v(f)$ via **any** cut in the network. In particular, $f^+(s) = f^-(t)$.