Shaking hands
COMS20010 2020, Video 3-3

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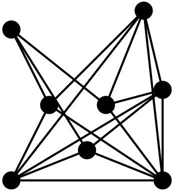
The Handshake Lemma



Counting the edges of this graph seems unpleasant...

but adding up the vertex degrees would be much easier.

Lemma: For any graph $G = (V, E)$, $\sum_{v \in V} d(v) = 2|E|$.

Proof: All edges contain two vertices, and each vertex v is in $d(v)$ edges. Count the number of vertex-edge pairs: Let $X = \{(v, e) \in V \times E : v \in E\}$. Then $|X| = 2|E|$ and $|X| = \sum_{v \in V} d(v)$, so we're done. $\square$.

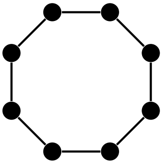
(This proof idea is called **double-counting**.)

Here, $\displaystyle\sum_{v} d(v) = 3 + 5 + 4 + 4 + 5 + 4 + 5 + 6 = 36,$ so 18 edges total.

Example applications

Handshake Lemma: For any graph $G = (V, E)$, $\sum_{v \in V} d(v) = 2|E|$.

Question: How many edges does an n-vertex cycle have?



Answer: Every vertex has degree 2, so

$$\#(\text{edges}) = \frac{1}{2} \sum_{v} d(v) = \frac{1}{2} \cdot n \cdot 2 = n.$$

A graph is ***k*-regular** if every vertex has degree k (so cycles are 2-regular).

Question: Are there 3-regular graphs $G = (V, E)$ with $|V|$ odd?

Answer: No, as then $\sum_{v \in V} d(v)$ would be $3|V|$ (which is odd).

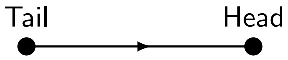
$2|E|$ is even, so this can't happen.

In **directed** graphs, can we express the number of edges in terms of in- and out-degrees? Yes!

Directed Handshake Lemma:

For any digraph $G = (V, E)$, $\sum_{v \in V} d^+(v) = \sum_{v \in V} d^-(v) = |E|$.

Proof: Terminology: we call the first vertex in a directed edge the **tail**, and the second vertex the **head**. (Matching the direction of the arrow!)



Instead of counting vertex-edge pairs, we count tail-edge pairs.

So let $X = \{(v, e) \in V \times E \colon e = (v, w) \text{ for some } w\}$.
Each edge has one tail, so $|X| = |E|$.
And each vertex v is the tail of $d^+(v)$ edges, so $|X| = \sum_{v \in V} d^+(v)$.

Similarly, counting head-edge pairs gives $\sum_{v \in V} d^-(v) = |E|$. $\square$