2-3-4 trees I: Search and insertion
COMS20010 2020, Video 6-3

John Lapinskas, University of Bristol

The limitations of binary search

From COMS10007: If we have an n-element sorted array, we can search for a given value in $O(\log n)$ time with binary search.

Find 7:

$\downarrow$ $7 < 10$: Check left half subarray

4	5	8	10	13	18	21

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$7 > 5$: Check right quarter subarray

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Find 7:

$7 \neq 8$: Return `Not found`

4	5	8	10	13	18	21

Why can't we use this to implement a dictionary, storing an array of key-value pairs sorted by key?

Because we can't easily insert or remove things from the middle of the array — this takes $\Omega(n)$ time! And if we used a linked list instead... it would take $\Omega(n)$ time to find the halfway point.

Instead, we can use a **binary search tree**.

Binary search trees

Idea: Each node has 0–2 children. If a node's value is x, then **all** its left descendants' values are $< x$, and **all** its right descendants' values are $> x$.

(For simplicity, we assume all values are distinct.)

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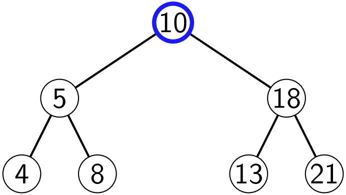

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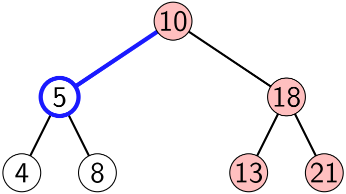

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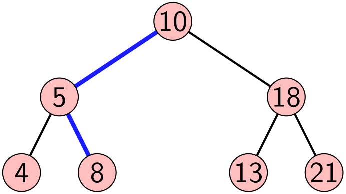

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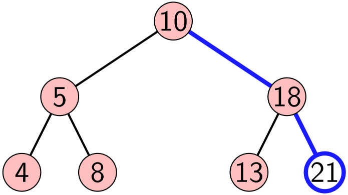

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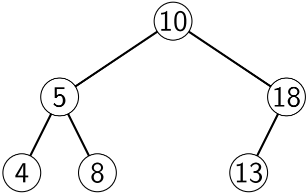

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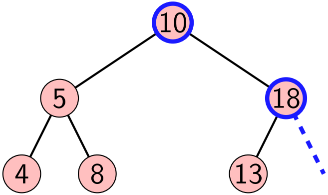

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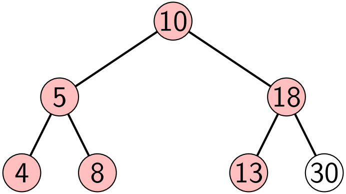

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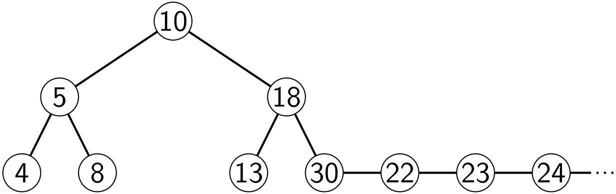
Insert 23;
Insert 24;
⋮



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Ideally, if the tree has n elements, then all but the bottom layer is full — the tree is **balanced**, as above. In that case,

$$n \approx 2^d + 2^{d-1} + \cdots + 1 = 2^{d+1} - 1 \Rightarrow d \in \Theta(\log n).$$

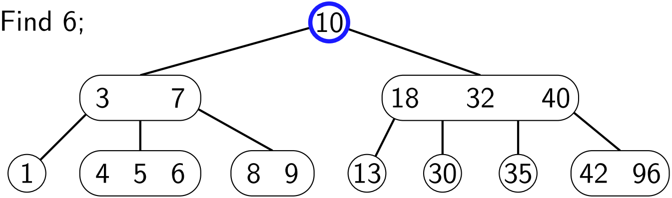
Problem: What if that doesn't happen? We could get $d \in \Omega(n)$...

2-3-4 trees

Idea: Force the tree to be **perfectly** balanced, with all levels full. To make this possible to maintain, allow nodes to contain more than one value.

A ***k*-node** can have up to k children and contain $k-1$ values (so a binary search tree is made entirely of 2-nodes). We will allow $k \in \{2, 3, 4\}$.

Find 6;



Say a 3-node has values $x_1 \leq x_2$, and children c_1, c_2 and c_3.

Then all descendants of c_1 must have values at most x_1...
All descendants of c_2 must have values greater than x_1 and less than x_2...
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4-nodes work the same way. So we can still find a value in $O(d)$ time.

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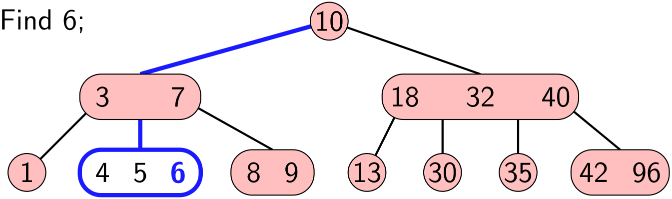

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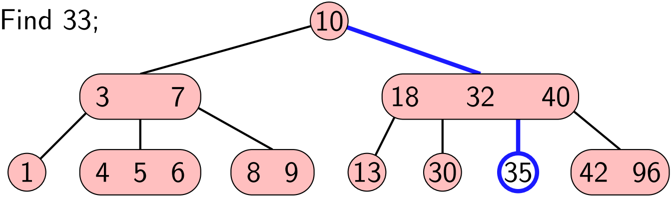

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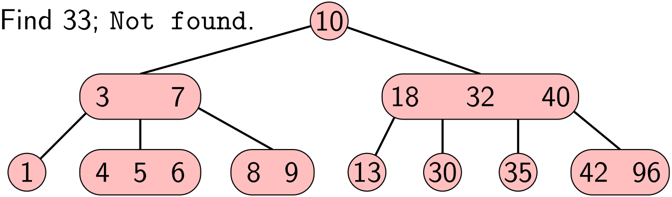
Find 33; `Not found`.



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Inserting new values

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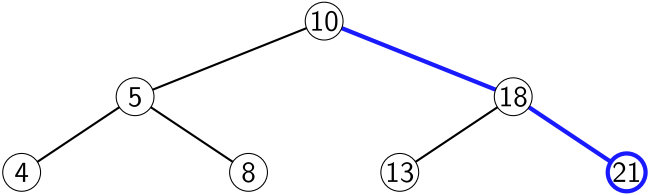

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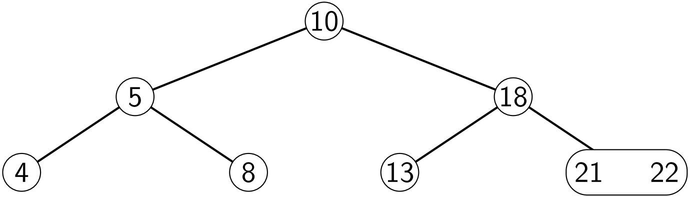

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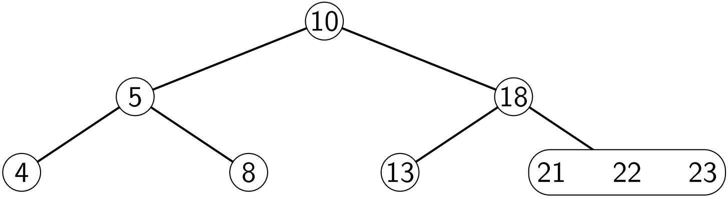

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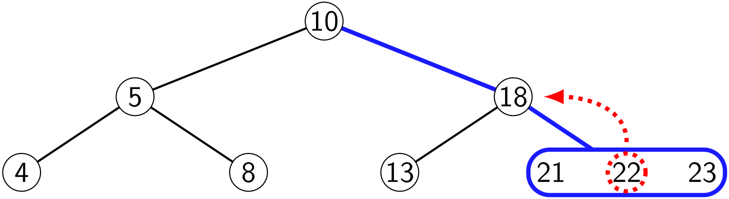

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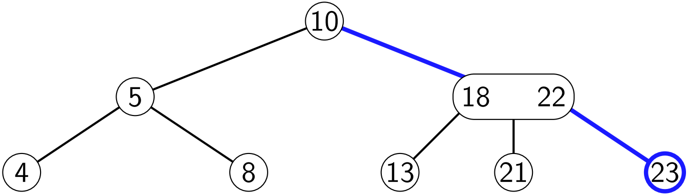

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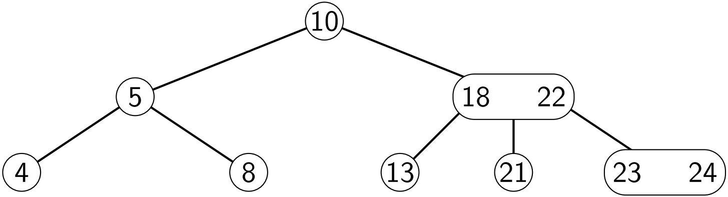

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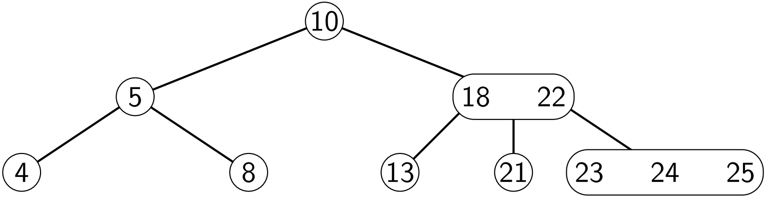

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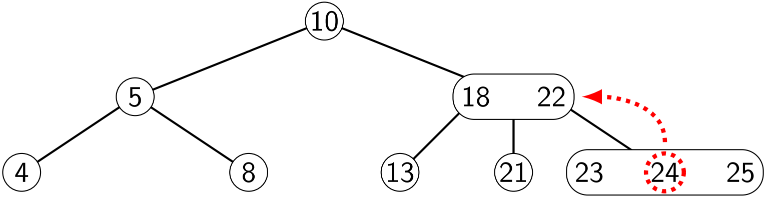

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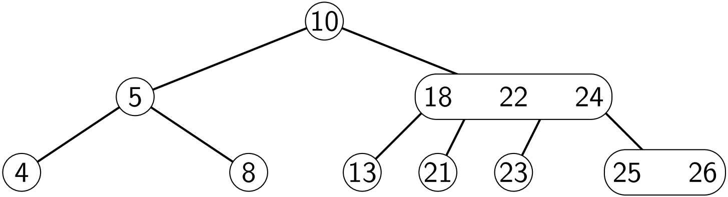

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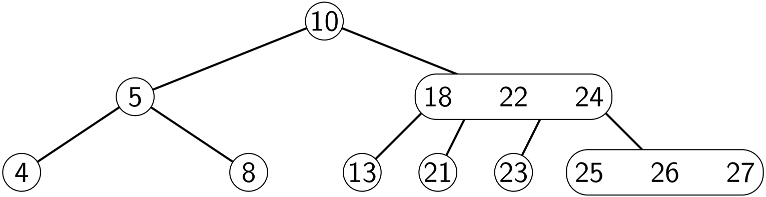

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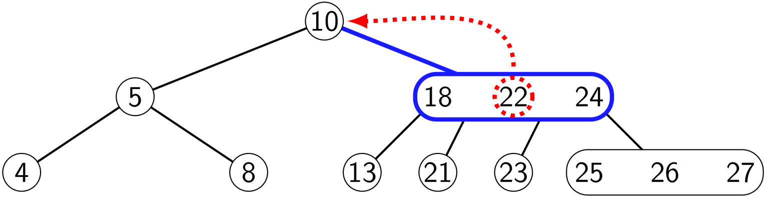

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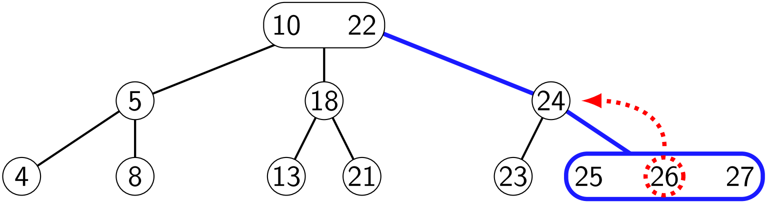

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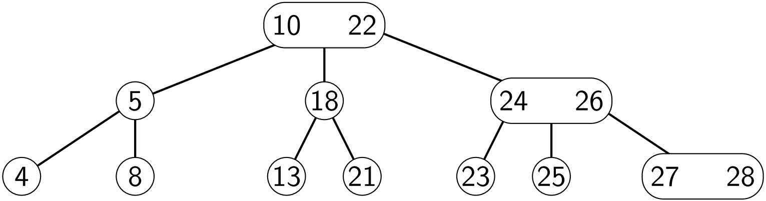

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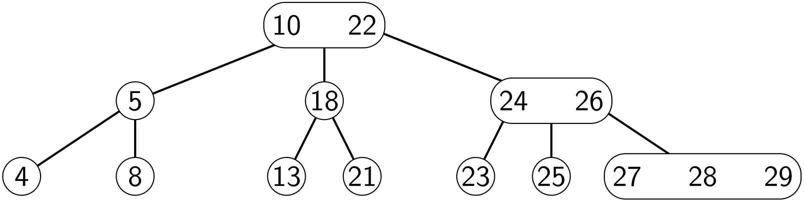

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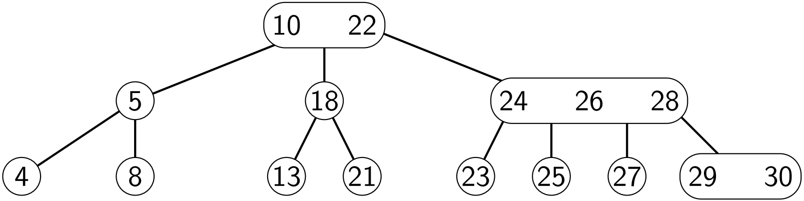

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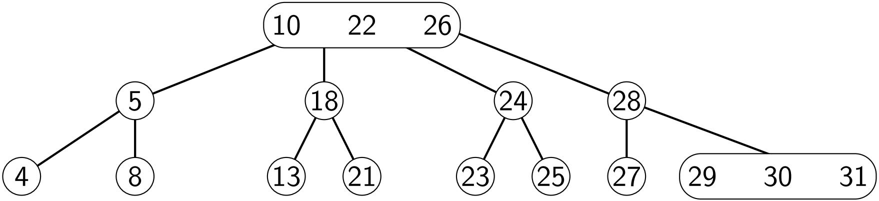

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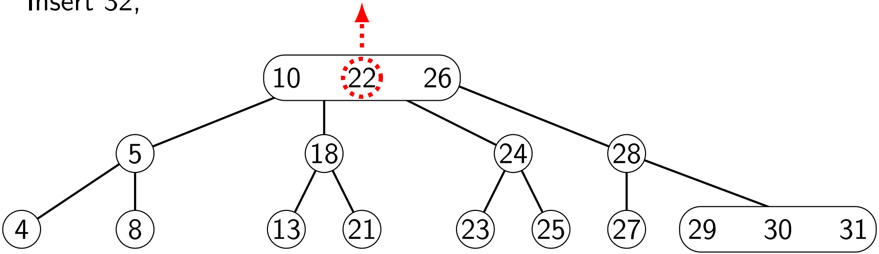

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If we have to split the root, d increases by 1.

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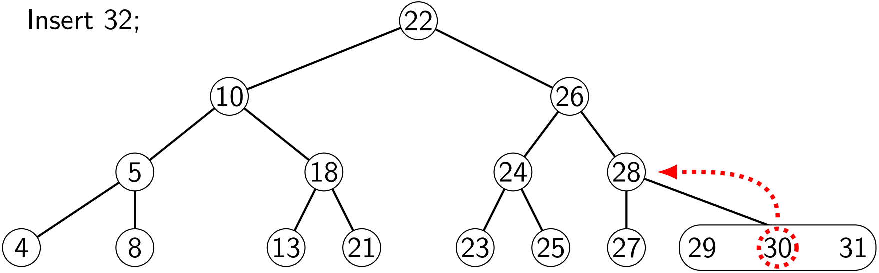

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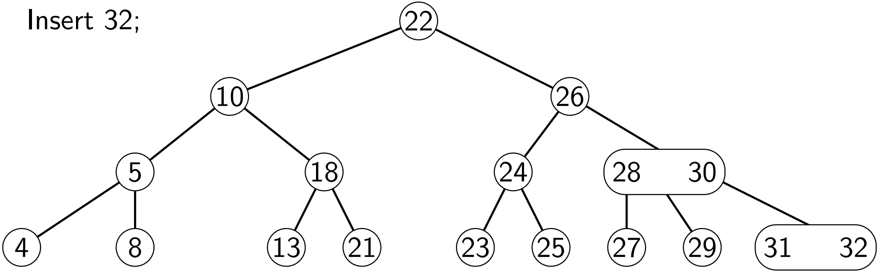

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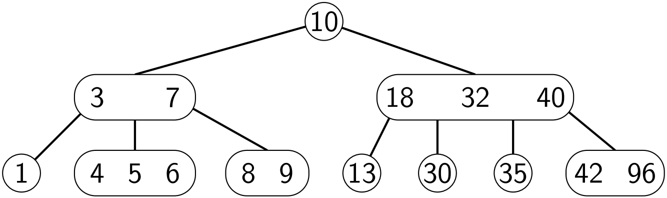
Summary of a 2-3-4 tree with distinct values (so far)



Finding a value v: Let x be the root. If $v \in x$, return a pointer to x. Otherwise, if x is a leaf, return `Not Found`. Otherwise, let k be such that x is a k-node, let $x_1 \leq \cdots \leq x_{k-1}$ be the values in x, let $x_0 = -\infty$, and let $x_k = \infty$; then $x_{i-1} < v < x_i$ for some i. Let c be the i'th child of x. Then repeat the process from the start, taking $x = c$.

Inserting a value v: First attempt to find v as above, **split**ting any 4-nodes encountered (including the root). After reaching a leaf L, and splitting it if it is a 4-node, add v to L.

Deleting a value v: Next time!