Flow networks
COMS20010 2020, Video 8-3

John Lapinskas, University of Bristol

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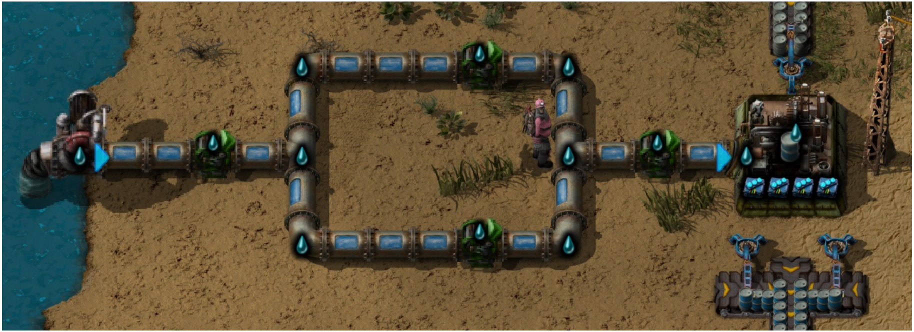
- Survey design;
- Professional baseball. (See KT 7.12!)

For now, let's just consider a toy problem. One pump supplies water for one factory, passing through a network of pipes of different capacities.



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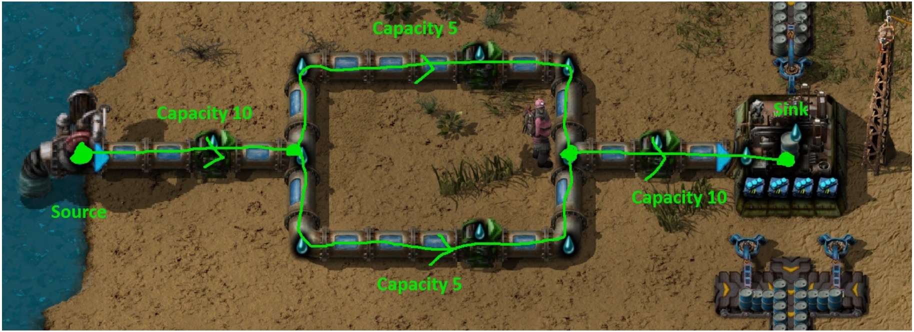

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More generally: A **flow network** (G, c, s, t) consists of a directed graph $G = (V, E)$, a **capacity** function $c\colon E \to \mathbb{N}$, a **source** vertex $s \in V$ with $N^-(s) = \emptyset$, and a **sink** vertex $t \in V$ with $N^+(t) = \emptyset$.

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The problem: Find a **maximum flow**: a flow f maximising $v(f)$.

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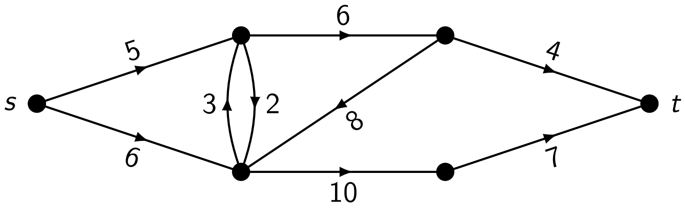

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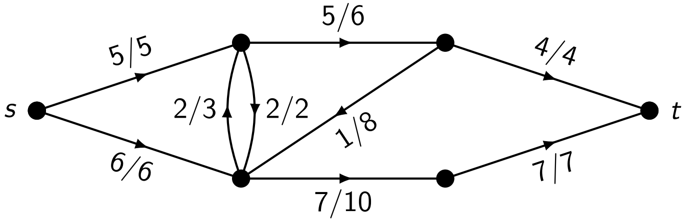

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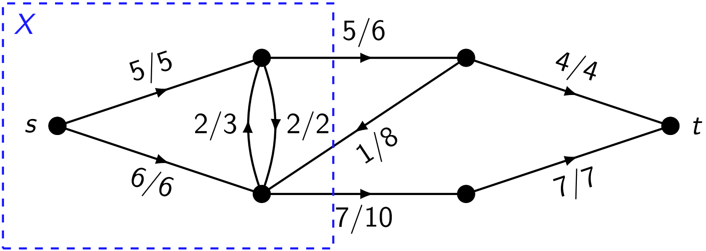

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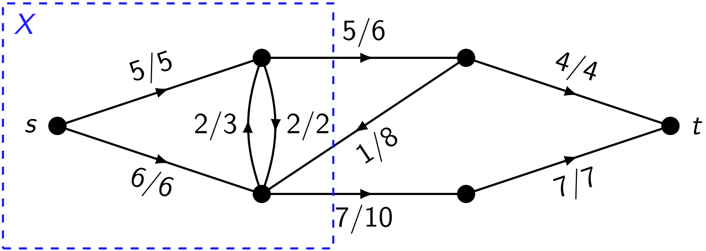

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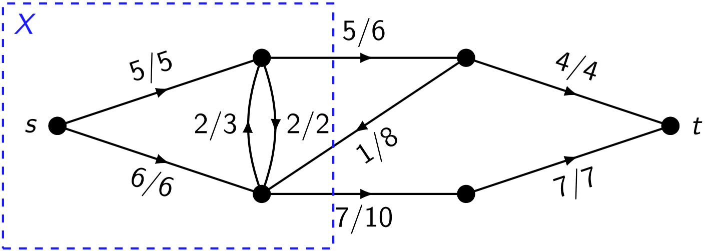

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For example, here $f^+(X) = 5 + 7 = 12$ and $f^-(X) = 1$.

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For all $e \subseteq X$, $f(e)$ appears once on each side; after cancelling those terms we're left with $f^+(X) = f^-(X)$. $\square$

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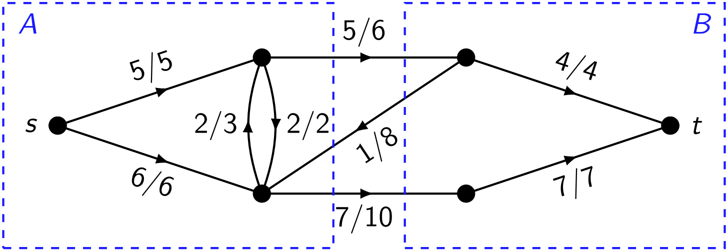

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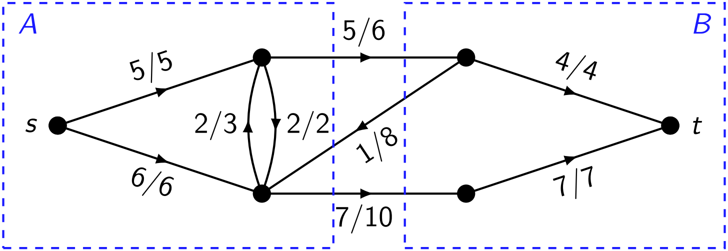

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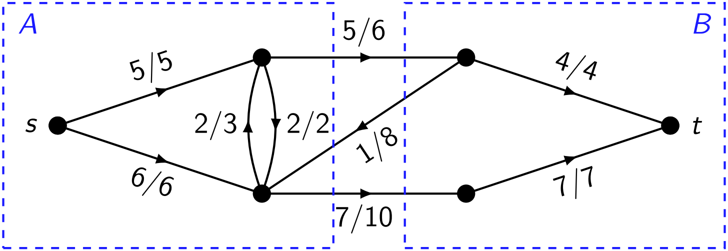

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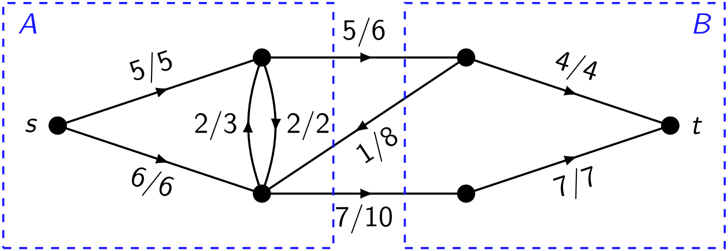

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But $f^+(A \setminus \{s\}) = f^+(A) - f(s,B)$ and $f^-(A \setminus \{s\}) = f^-(A) + f(s,A)$...

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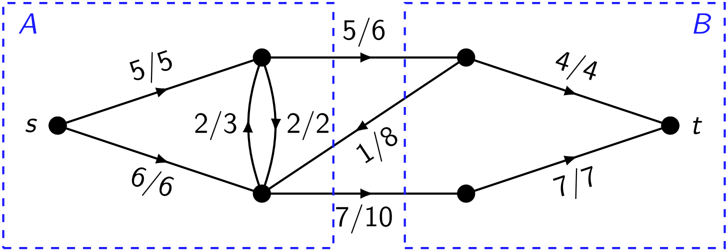

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So $f^+(A) - f(s,B) = f^-(A) + f(s,A)$.

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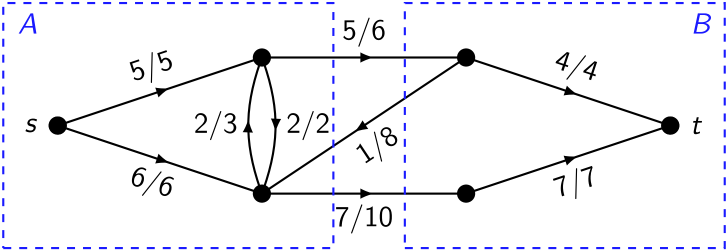

Lemma 2: For all cuts (A, B), $f^+(A) - f^-(A) = f^-(B) - f^+(B) = v(f)$.

Proof: By Lemma 1, we have $f^+(A \setminus \{s\}) = f^-(A \setminus \{s\})$.

Rearranging $f^+(A) - f(s, B) = f^-(A) + f(s, A)$:

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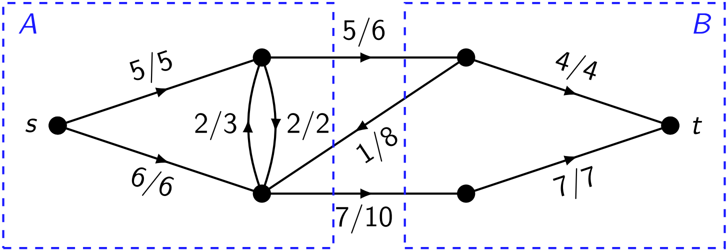

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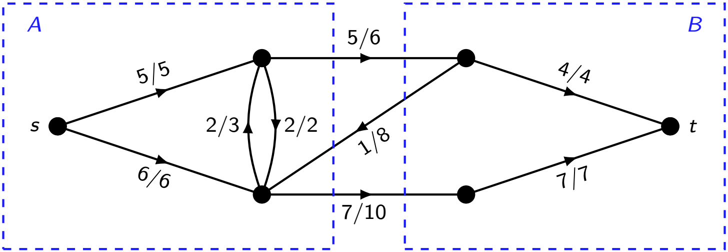

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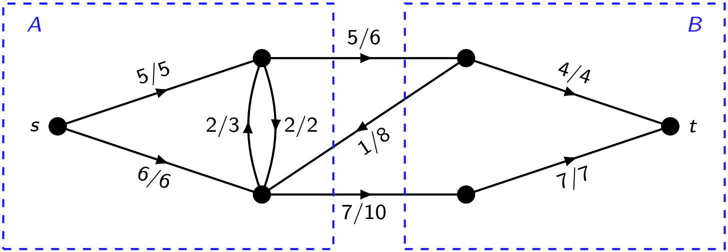

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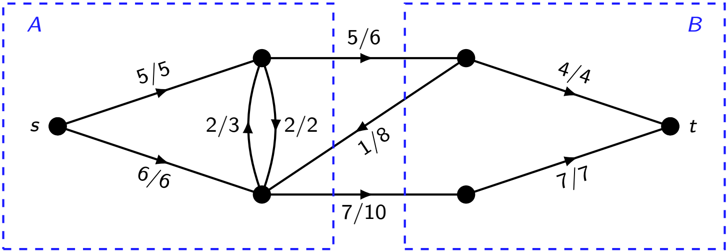

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Lemma 2 implies we could have defined $v(f)$ via **any** cut in the network. In particular, $f^+(s) = f^-(t)$.