Breadth-first search COMS20010 2020, Video 4-3

John Lapinskas, University of Bristol

Shortest path-finding

Last time: Given a graph G and two vertices $x, y \in V(G)$, is there a path from x to y?

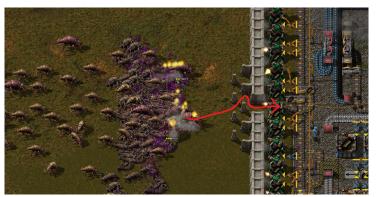
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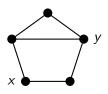
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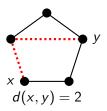
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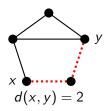
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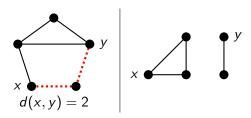
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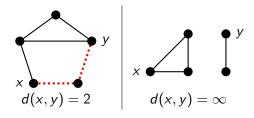
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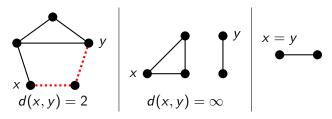
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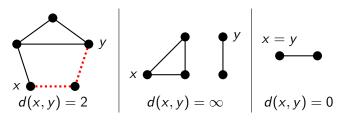
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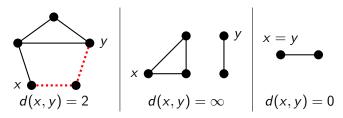
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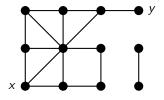
The **distance** between x and y, d(x, y), is the length in edges of a shortest path between x and y, or ∞ if no such path exists.



In directed graphs, it's the same except that the path is **from** x **to** y. So... we might not have d(x,y) = d(y,x)!

Input: A graph G and two vertices x and y.

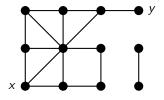
Output: A shortest path from x to y.



Let L_i be the set of vertices at distance i from x.

Input: A graph G and two vertices x and y.

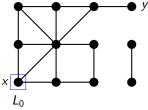
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Let L_i be the set of vertices at distance i from x. So $L_0 = \{x\}$.

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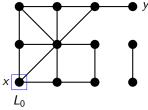
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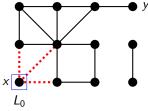


Let L_i be the set of vertices at distance i from x. So $L_0 = \{x\}$.

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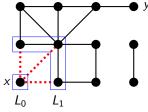


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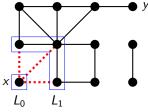


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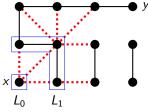
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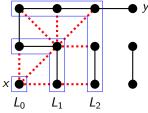
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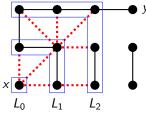
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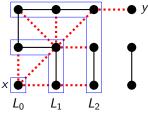
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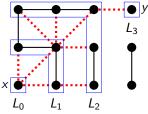
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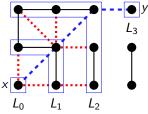
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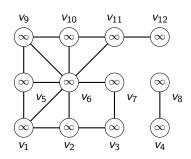


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queue:

Algorithm: BFS

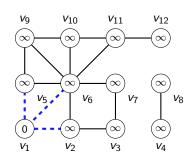
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 (v, v_i) with $\{v, v_i\} \in E$.

5 while queue is not empty do

Remove front tuple (v_i, v_i) from queue. if $L[i] = \infty$ then

Add (v_i, v_k) to queue for all $\{v_j,v_k\}\in E,\ k\neq i.$ Set $L[j] \leftarrow L[i] + 1$, pred[j] = i. 9



queue: (1,5), (1,6), (1,2)

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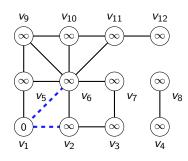
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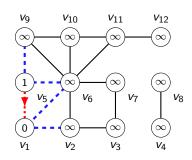
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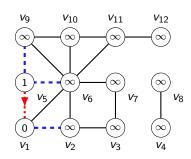
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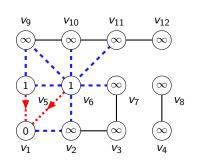
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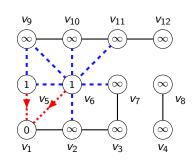
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Return L and pred.



$$\begin{array}{l} \text{queue:} \, (5,9), (5,6), (6,5), (6,9), \\ (6,10), (6,11), (6,7), (6,2), \end{array}$$

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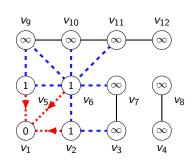
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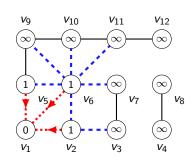


$$\begin{array}{c} \mathtt{queue:} \, (5,9), (5,6), (6,5), (6,9), \\ (6,10), (6,11), (6,7), (6,2), \\ (2,6), (2,3) \end{array}$$

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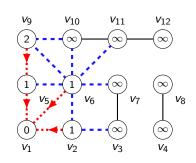
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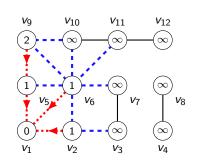
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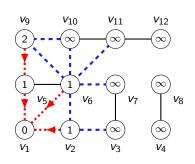
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$$\begin{array}{l} \mathtt{queue:}\,(6,5),(6,9),(6,10),(6,11),\\ (6,7),(6,2),(2,6),(2,3),\\ (9,10),(9,6) \end{array}$$

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queue:
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, $(6,10)$, $(6,11)$, $(6,7)$, $(6,2)$, $(2,6)$, $(2,3)$, $(9,10)$, $(9,6)$

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Output : d(v, y) for all $y \in V$ and "a way of finding shortest paths".

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Let $L[i] \leftarrow \infty$ for all $i \in [n]$.

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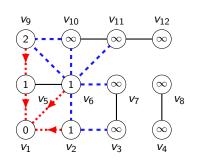
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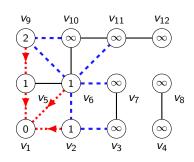
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$$\begin{array}{c} \mathtt{queue:}\,(6,11),(6,7),(6,2),(2,6),\\ (2,3),(9,10),(9,6) \end{array}$$

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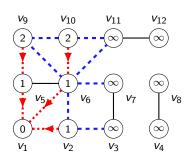
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 4 Let queue be a queue containing all tuples (v, v_i) with {v, v_i} ∈ E.

5 while queue is not empty do

6 Remove front tuple (v_i, v_j) from queue. 7 if $L[j] = \infty$ then 8 Add (v_j, v_k) to queue for all $\{v_j, v_k\} \in E, k \neq i$. 9 Set $L[j] \leftarrow L[j] + 1$, pred[j] = i.

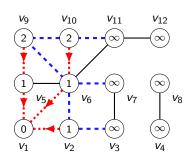


queue:
$$(6,11)$$
, $(6,7)$, $(6,2)$, $(2,6)$, $(2,3)$, $(9,10)$, $(9,6)$, $(10,11)$, $(10,9)$

Algorithm: BFS

: Graph G = (V, E), vertex $v \in V$. Input : d(v, y) for all $y \in V$ and "a way of finding shortest paths". Number the vertices of G as $v = v_1, \dots, v_n$. 2 Let $L[i] \leftarrow \infty$ for all $i \in [n]$. 3 Let $L[1] \leftarrow 0$, pred $[1] \leftarrow None$. 4 Let queue be a queue containing all tuples (v, v_i) with $\{v, v_i\} \in E$. 5 while queue is not empty do Remove front tuple (v_i, v_i) from queue. if $L[j] = \infty$ then

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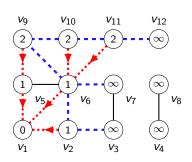
$$\begin{array}{c} \text{queue:} \, (6,7), (6,2), (2,6), (2,3), \\ (9,10), (9,6), (10,11), (10,9) \end{array}$$

Algorithm: BFS

: Graph G = (V, E), vertex $v \in V$. Input Output : d(v, y) for all $y \in V$ and "a way of finding shortest paths". Number the vertices of G as $v = v_1, \dots, v_n$.

- 2 Let $L[i] \leftarrow \infty$ for all $i \in [n]$.
- 3 Let $L[1] \leftarrow 0$, pred $[1] \leftarrow None$.
- 4 Let queue be a queue containing all tuples (v, v_i) with $\{v, v_i\} \in E$.
- 5 while queue is not empty do

Remove front tuple (v_i, v_i) from queue. if $L[j] = \infty$ then Add (v_i, v_k) to queue for all $\{v_i, v_k\} \in E, k \neq i.$ Set $L[i] \leftarrow L[i] + 1$, pred[i] = i.



queue:
$$(6,7)$$
, $(6,2)$, $(2,6)$, $(2,3)$, $(9,10)$, $(9,6)$, $(10,11)$, $(10,9)$, $(11,10)$, $(11,12)$

Algorithm: BFS

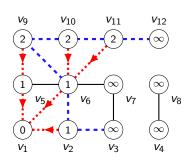
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 $\{v_i, v_k\} \in E, k \neq i.$

Set $L[i] \leftarrow L[i] + 1$, pred[i] = i.

Return L and pred.

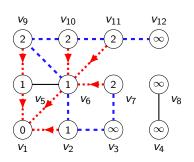
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queue:
$$(6,2)$$
, $(2,6)$, $(2,3)$, $(9,10)$, $(9,6)$, $(10,11)$, $(10,9)$, $(11,10)$, $(11,12)$

Algorithm: BFS

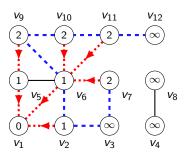
```
: Graph G = (V, E), vertex v \in V.
  Input
              : d(v, y) for all y \in V and "a way of
  Output
                finding shortest paths".
1 Number the vertices of G as v = v_1, \dots, v_n.
2 Let L[i] \leftarrow \infty for all i \in [n].
3 Let L[1] \leftarrow 0, pred[1] \leftarrow None.
4 Let queue be a queue containing all tuples
    (v, v_i) with \{v, v_i\} \in E.
5 while queue is not empty do
        Remove front tuple (v_i, v_i) from queue.
        if L[i] = \infty then
             Add (v_i, v_k) to queue for all
              \{v_i, v_k\} \in E, k \neq i.
             Set L[i] \leftarrow L[i] + 1, pred[i] = i.
  Return L and pred.
```



queue:
$$(6,2), (2,6), (2,3), (9,10),$$
 $_{9}$ $\begin{bmatrix} \{v_j, v_k\} \\ \text{Set L}[j] \leftarrow \\ (9,6), (10,11), (10,9), (11,10), \\ (11,12), (7,3) \end{bmatrix}$ Return L and pred.

Algorithm: BFS

```
: Graph G = (V, E), vertex v \in V.
  Input
              : d(v, y) for all y \in V and "a way of
                finding shortest paths".
 Number the vertices of G as v = v_1, \dots, v_n.
2 Let L[i] \leftarrow \infty for all i \in [n].
3 Let L[1] \leftarrow 0, pred[1] \leftarrow None.
4 Let queue be a queue containing all tuples
    (v, v_i) with \{v, v_i\} \in E.
5 while queue is not empty do
        Remove front tuple (v_i, v_i) from queue.
        if L[i] = \infty then
             Add (v_i, v_k) to queue for all
              \{v_i, v_k\} \in E, k \neq i.
             Set L[j] \leftarrow L[i] + 1, pred[j] = i.
```



queue:
$$(2,6), (2,3), (9,10), (9,6),$$
 g $\begin{bmatrix} \{v_j, v_k\} \\ \text{Set L}[j] \leftarrow \\ (7,3) \end{bmatrix}$ 10 Return L and pred.

Algorithm: BFS

```
Input : Graph G = (V, E), vertex v \in V.

Output : d(v, y) for all y \in V and "a way of finding shortest paths".

1 Number the vertices of G as v = v_1, \ldots, v_n.

2 Let L[i] \leftarrow \infty for all i \in [n].

3 Let L[i] \leftarrow 0, pred[i] \leftarrow None.

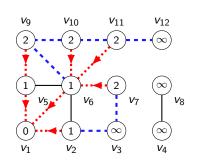
4 Let queue be a queue containing all tuples (v, v_j) with \{v, v_j\} \in E.

5 while queue is not empty do

6 Remove front tuple (v_i, v_j) from queue.

7 If L[i] = \infty then
```

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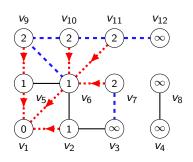


queue:
$$(2,3), (9,10), (9,6), (10,11), _{9}$$

 $(10,9), (11,10), (11,12), (7,3)$

Algorithm: BFS

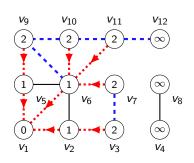
```
: Graph G = (V, E), vertex v \in V.
  Input
              : d(v, y) for all y \in V and "a way of
  Output
                finding shortest paths".
 Number the vertices of G as v = v_1, \dots, v_n.
2 Let L[i] \leftarrow \infty for all i \in [n].
3 Let L[1] \leftarrow 0, pred[1] \leftarrow None.
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             Add (v_i, v_k) to queue for all
              \{v_i, v_k\} \in E, k \neq i.
             Set L[j] \leftarrow L[i] + 1, pred[j] = i.
```



queue:
$$(9,10)$$
, $(9,6)$, $(10,11)$, $(10,9)_9$
 $(11,10)$, $(11,12)$, $(7,3)$

Algorithm: BFS

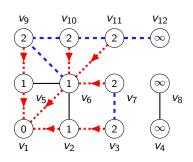
```
: Graph G = (V, E), vertex v \in V.
  Input
              : d(v, y) for all y \in V and "a way of
  Output
                finding shortest paths".
1 Number the vertices of G as v = v_1, \ldots, v_n.
2 Let L[i] \leftarrow \infty for all i \in [n].
3 Let L[1] \leftarrow 0, pred[1] \leftarrow None.
4 Let queue be a queue containing all tuples
    (v, v_i) with \{v, v_i\} \in E.
5 while queue is not empty do
        Remove front tuple (v_i, v_i) from queue.
        if L[i] = \infty then
             Add (v_i, v_k) to queue for all
              \{v_i,v_k\}\in E,\ k\neq i.
             Set L[i] \leftarrow L[i] + 1, pred[i] = i.
```



queue:
$$(9,10)$$
, $(9,6)$, $(10,11)$, $(10,9)_9$
 $(11,10)$, $(11,12)$, $(7,3)$, $(3,7)$

Algorithm: BFS

```
: Graph G = (V, E), vertex v \in V.
  Input
              : d(v, y) for all y \in V and "a way of
  Output
                finding shortest paths".
 Number the vertices of G as v = v_1, \dots, v_n.
2 Let L[i] \leftarrow \infty for all i \in [n].
3 Let L[1] \leftarrow 0, pred[1] \leftarrow None.
4 Let queue be a queue containing all tuples
    (v, v_i) with \{v, v_i\} \in E.
5 while queue is not empty do
        Remove front tuple (v_i, v_i) from queue.
        if L[i] = \infty then
             Add (v_i, v_k) to queue for all
              \{v_i, v_k\} \in E, k \neq i.
             Set L[i] \leftarrow L[i] + 1, pred[i] = i.
```



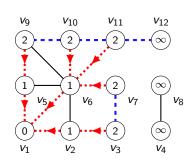
queue:
$$(9,6)$$
, $(10,11)$, $(10,9)$, $(11,10_9)$, $(11,12)$, $(7,3)$, $(3,7)$

Algorithm: BFS

: Graph G = (V, E), vertex $v \in V$. Input : d(v, y) for all $y \in V$ and "a way of Output finding shortest paths". 1 Number the vertices of G as $v = v_1, \dots, v_n$.

- 2 Let $L[i] \leftarrow \infty$ for all $i \in [n]$.
- 3 Let $L[1] \leftarrow 0$, pred $[1] \leftarrow None$.
- 4 Let queue be a queue containing all tuples (v, v_i) with $\{v, v_i\} \in E$.
- 5 while queue is not empty do

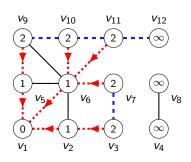
Remove front tuple (v_i, v_i) from queue. if $L[i] = \infty$ then Add (v_i, v_k) to queue for all $\{v_i,v_k\}\in E,\ k\neq i.$ Set $L[i] \leftarrow L[i] + 1$, pred[i] = i.



Algorithm: BFS

: Graph G = (V, E), vertex $v \in V$. Input **Output** : d(v, y) for all $y \in V$ and "a way of finding shortest paths". 1 Number the vertices of G as $v = v_1, \dots, v_n$. 2 Let $L[i] \leftarrow \infty$ for all $i \in [n]$. 3 Let $L[1] \leftarrow 0$, pred $[1] \leftarrow None$. 4 Let queue be a queue containing all tuples (v, v_i) with $\{v, v_i\} \in E$. 5 while queue is not empty do Remove front tuple (v_i, v_i) from queue.

if $L[i] = \infty$ then Add (v_i, v_k) to queue for all $\{v_i, v_k\} \in E, k \neq i.$ Set $L[i] \leftarrow L[i] + 1$, pred[i] = i.

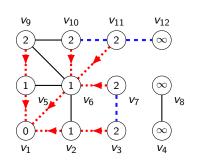


queue:
$$(10,9)$$
, $(11,10)$, $(11,12)$, $(7,3_9)$, $(3,7)$

Algorithm: BFS

: Graph G = (V, E), vertex $v \in V$. Input : d(v, y) for all $y \in V$ and "a way of Output finding shortest paths". 1 Number the vertices of G as $v = v_1, \dots, v_n$. 2 Let $L[i] \leftarrow \infty$ for all $i \in [n]$. 3 Let $L[1] \leftarrow 0$, pred $[1] \leftarrow None$. 4 Let queue be a queue containing all tuples (v, v_i) with $\{v, v_i\} \in E$. 5 while queue is not empty do

Remove front tuple (v_i, v_i) from queue.



queue: $(11, 10), (11, 12), (7, 3), (3, 7)_9$

Algorithm: BFS

```
Input : Graph G = (V, E), vertex v \in V.

Output : d(v, y) for all y \in V and "a way of finding shortest paths".

1 Number the vertices of G as v = v_1, \ldots, v_n.

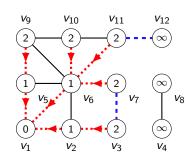
2 Let L[i] \leftarrow \infty for all i \in [n].

3 Let L[i] \leftarrow 0, pred[i] \leftarrow None.

4 Let queue be a queue containing all tuples (v, v_j) with \{v, v_j\} \in E.

5 while queue is not empty do

6 | Remove front tuple (v_i, v_j) from queue.
```



queue: (11, 12), (7, 3), (3, 7)

Algorithm: BFS

```
Input : Graph G = (V, E), vertex v \in V.

Output : d(v, y) for all y \in V and "a way of finding shortest paths".

1 Number the vertices of G as v = v_1, \ldots, v_n.

2 Let L[I] \leftarrow \infty for all i \in [n].

3 Let L[I] \leftarrow 0, pred[I] \leftarrow None.

4 Let queue be a queue containing all tuples (v, v_j) with \{v, v_j\} \in E.
```

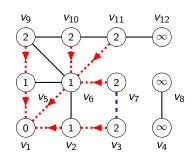
5 while queue is not empty do

Remove front tuple (v_i, v_j) from queue.

If $L[j] = \infty$ then

Add (v_j, v_k) to queue for all $\{v_j, v_k\} \in E, k \neq i$.

 $\begin{cases} \{v_j, v_k\} \in E, \ k \neq i. \\ \text{Set L}[j] \leftarrow \text{L}[i] + 1, \ \text{pred}[j] = i. \end{cases}$



queue: (7,3),(3,7)

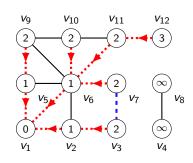
Algorithm: BFS

Input : Graph G = (V, E), vertex $v \in V$. **Output** : d(v, y) for all $y \in V$ and "a way of finding shortest paths". 1 Number the vertices of G as $v = v_1, \dots, v_n$. 2 Let $L[i] \leftarrow \infty$ for all $i \in [n]$. 3 Let $L[1] \leftarrow 0$, pred $[1] \leftarrow None$. 4 Let queue be a queue containing all tuples (v, v_i) with $\{v, v_i\} \in E$.

5 while queue is not empty do

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Add (v_i, v_k) to queue for all $\{v_j,v_k\}\in E,\ k\neq i.$ Set $L[j] \leftarrow L[i] + 1$, pred[j] = i. 9



queue: (7,3),(3,7)

Algorithm: BFS

```
Input : Graph G = (V, E), vertex v \in V.

Output : d(v, y) for all y \in V and "a way of finding shortest paths".

1 Number the vertices of G as v = v_1, \dots, v_n.

2 Let L[i] \leftarrow \infty for all i \in [n].

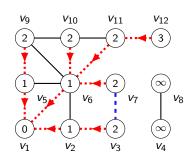
3 Let L[i] \leftarrow 0, pred[i] \leftarrow None.

4 Let queue be a queue containing all tuples (v, v_i) with \{v, v_i\} \in E.
```

5 while queue is not empty do

Remove front tuple (v_i, v_j) from queue. If $L[j] = \infty$ then Add (v_i, v_k) to queue for all

 $\{v_j, v_k\} \in E, \ k \neq i.$ Set $L[j] \leftarrow L[i] + 1$, pred[j] = i.



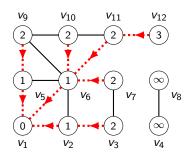
queue: (3,7)

Algorithm: BFS : Graph G = (V, E), vertex $v \in V$. Input **Output** : d(v, y) for all $y \in V$ and "a way of finding shortest paths". 1 Number the vertices of G as $v = v_1, \dots, v_n$. 2 Let $L[i] \leftarrow \infty$ for all $i \in [n]$. 3 Let $L[1] \leftarrow 0$, pred $[1] \leftarrow None$.

4 Let queue be a queue containing all tuples (v, v_i) with $\{v, v_i\} \in E$.

5 while queue is not empty do

Remove front tuple (v_i, v_i) from queue. if $L[i] = \infty$ then Add (v_i, v_k) to queue for all $\{v_j,v_k\}\in E,\ k\neq i.$ Set $L[j] \leftarrow L[i] + 1$, pred[j] = i. 9



Algorithm: BFS

```
Input : Graph G = (V, E), vertex v \in V.

Output : d(v, y) for all y \in V and "a way of finding shortest paths".

1 Number the vertices of G as v = v_1, \dots, v_n.

2 Let L[i] \leftarrow \infty for all i \in [n].

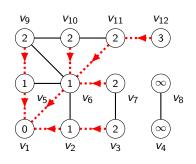
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```

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 $\{v_j, v_k\} \in E, \ k \neq i.$ $\text{Set L[j]} \leftarrow \text{L[j]} + 1, \ \text{pred[j]} = i.$



Algorithm: BFS

```
Input : Graph G = (V, E), vertex v \in V.

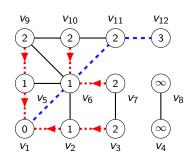
Output : d(v, y) for all y \in V and "a way of finding shortest paths".

1 Number the vertices of G as v = v_1, \ldots, v_n.
```

- 2 Let $L[i] \leftarrow \infty$ for all $i \in [n]$.
- 3 Let $L[1] \leftarrow 0$, pred $[1] \leftarrow None$.
- 4 Let queue be a queue containing all tuples (v, v_j) with $\{v, v_j\} \in E$.
- 5 while queue is not empty do
 6 | Remove front tuple (v_i, v_i) from queue.
- 7 if $L[j] = \infty$ then 8 $Add(v_j, v_k)$ to queue for all $\{v_j, v_k\} \in E, k \neq i$. 9 $Set L[j] \leftarrow L[i] + 1$, pred[j] = i.

10 Return L and pred.

In the output, $L[i] = d(v, v_i)$. By following edges back from v_i via pred, we can also quickly reconstruct a shortest path from v to v_i .



```
Algorithm: BFS
```

```
Input : Graph G = (V, E), vertex v \in V.

Output : d(v, y) for all y \in V and "a way of finding shortest paths".

Number the vertices of G as v = v_1, \ldots, v_n.

Let L[i] \leftarrow \infty for all i \in [n].

Let L[i] \leftarrow 0, Pred[i] \leftarrow None.

Let queue be a queue containing all tuples (v, v_j) with \{v, v_j\} \in E.

while queue is not empty do

Remove front tuple (v_i, v_j) from queue.

If L[j] = \infty then
```

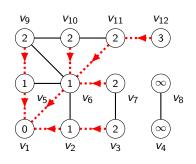
Add (v_j, v_k) to queue for all $\{v_j, v_k\} \in E$, $k \neq i$. Set $L[j] \leftarrow L[i] + 1$, pred[j] = i.

10 Return L and pred.

In the output, $L[i] = d(v, v_i)$. By following edges back from v_i via pred, we can also quickly reconstruct a shortest path from v to v_i .

9

E.g. $v_1v_6v_{11}v_{12}$ is a shortest path from v_1 to v_{12} .



Algorithm: BFS

```
Input : Graph G = (V, E), vertex v \in V.

Output : d(v, y) for all y \in V and "a way of finding shortest paths".

Number the vertices of G as v = v_1, \dots, v_n.

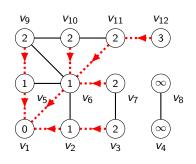
Let L[i] \leftarrow \infty for all i \in [n].

Let L[i] \leftarrow 0, pred[i] \leftarrow 0.
```

- 4 Let queue be a queue containing all tuples (v, v_j) with $\{v, v_j\} \in E$.
- 5 while queue is not empty do 6 Remove front tuple (v_i, v_j) from queue. 7 if $L[j] = \infty$ then 8 Add (v_j, v_k) to queue for all
- $\begin{cases} \{v_j, v_k\} \in E, \ k \neq i. \\ \text{Set L}[j] \leftarrow \text{L}[i] + 1, \ \text{pred}[j] = i. \end{cases}$

10 Return L and pred.

Time analysis: If G is in adjacency list form, each edge is added to queue at most twice, incurring O(1) overhead each time, so the running time is O(|V| + |E|).



```
Algorithm: BFS
```

```
Input : Graph G = (V, E), vertex v \in V.

Output : d(v, y) for all y \in V and "a way of finding shortest paths".

1 Number the vertices of G as v = v_1, \dots, v_n.

2 Let L[I] \leftarrow \infty for all i \in [n].

3 Let L[I] \leftarrow 0, pred[I] \leftarrow None.

4 Let queue be a queue containing all tuples (v, v_j) with \{v, v_j\} \in E.
```

5 while queue is not empty do
6 Remove front tuple (v_i, v_i) from queue.

7 if
$$L[j] = \infty$$
 then
8 Add (v_j, v_k) to queue for all $\{v_j, v_k\} \in E, k \neq i$.
9 Set $L[j] \leftarrow L[i] + 1$, pred $[j] = i$.

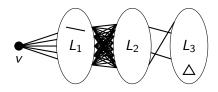
10 Return L and pred.

Important: There is a significant **space** inefficiency in this version of breadth-first search! See example sheet.

BFS trees

Definition: A **BFS** tree T of G is a rooted tree (call its root x) with:

- **1** V(T) is the vertex set of a component of G;
- 2 The *i*'th layer of T is $\{x: d_G(x, v) = i\}$;
- **③** If $\{x,y\} \in E(G)$, then $|d_G(v,x) d_G(v,y)| \le 1$, i.e. x and y must be in the same or adjacent layers of T.

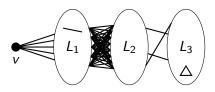


Theorem: The tree of edges from pred is always a BFS tree.

BFS trees

Definition: A **BFS** tree T of G is a rooted tree (call its root x) with:

- **1** V(T) is the vertex set of a component of G;
- ② The *i*'th layer of T is $\{x: d_G(x, v) = i\}$;
- **③** If $\{x,y\} \in E(G)$, then $|d_G(v,x) d_G(v,y)| \le 1$, i.e. x and y must be in the same or adjacent layers of T.



Theorem: The tree of edges from pred is always a BFS tree.

Proof: We already proved (1) and (2), so suppose $\{x,y\} \in E(G)$.

If P is a shortest path from v to x, then Pxy is a path from v to y, so $d(v,y) \le d(v,x) + 1$. Likewise $d(v,x) \le d(v,y) + 1$.

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