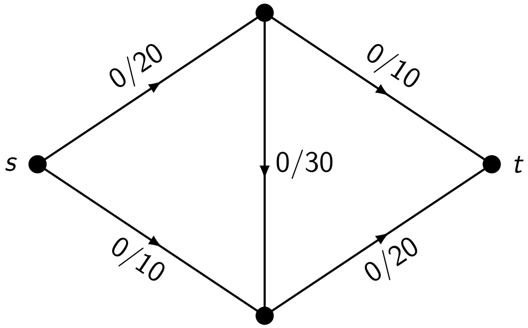
The Ford-Fulkerson algorithm
COMS20010 2020, Video 8-4

John Lapinskas, University of Bristol

The problem: Find a **maximum flow**: a flow f maximising $v(f)$.

Now the definition of value is sorted out, how do we solve the problem?

How about a greedy approach? Repeatedly find paths from s to t with unused capacity and "push" more flow down them.



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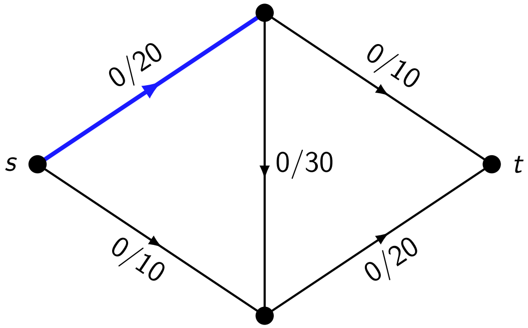

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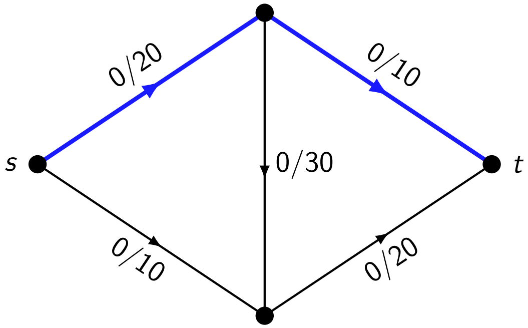

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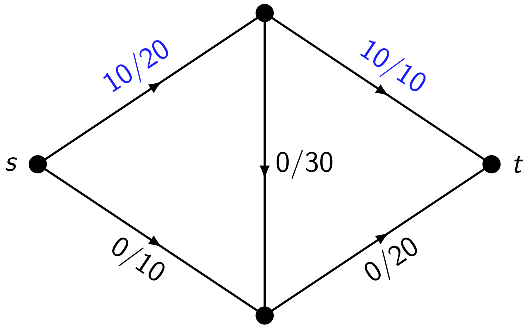

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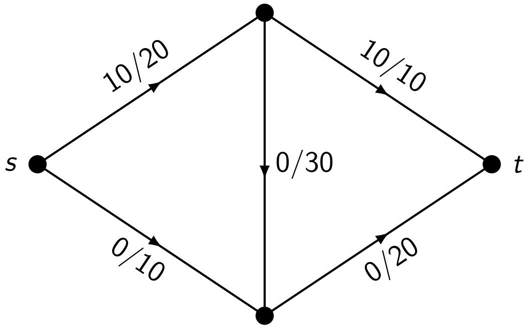

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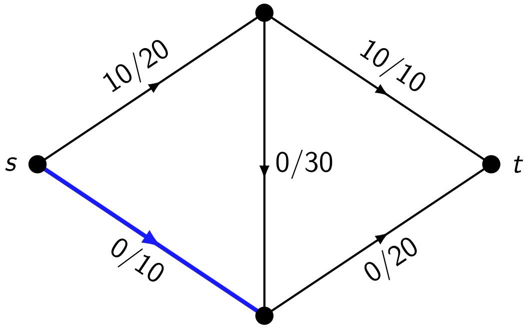

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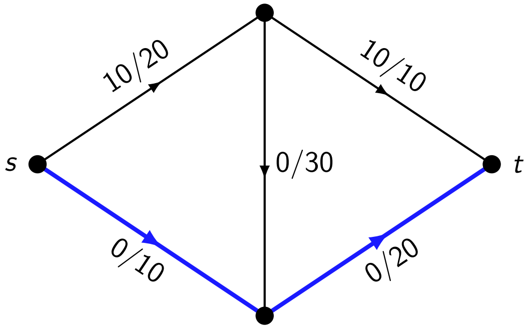

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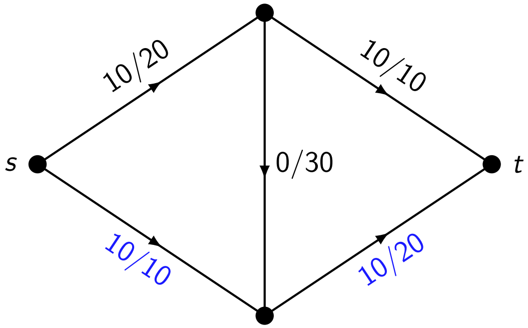

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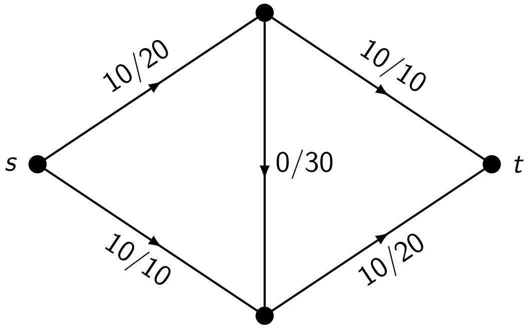

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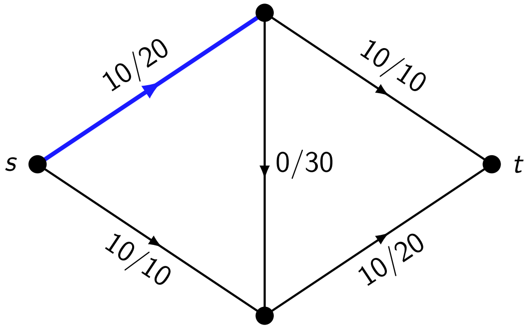

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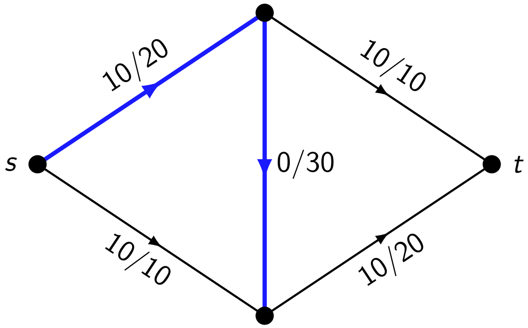

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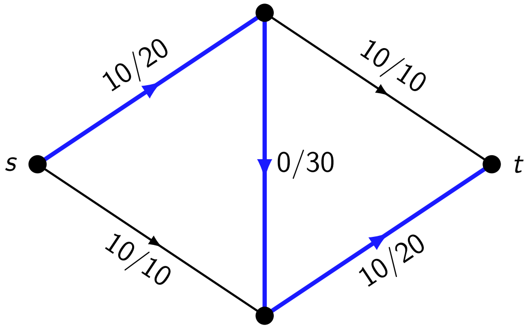

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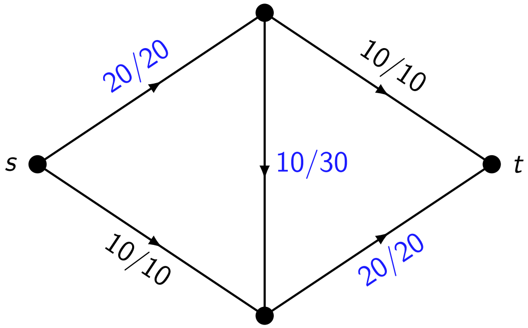

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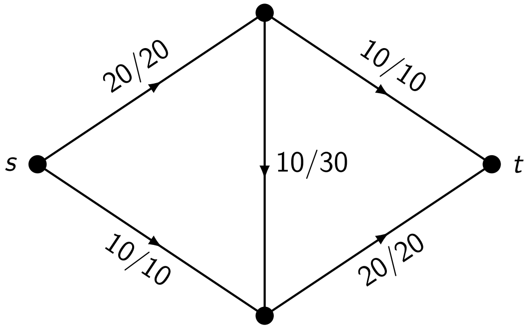

This flow has value $20 + 10 = 30$, which is best possible.
So a greedy approach can work...

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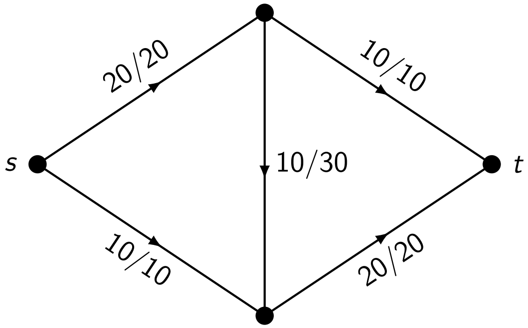

This flow has value $20 + 10 = 30$, which is best possible.
So a greedy approach can work... but it can also fail.

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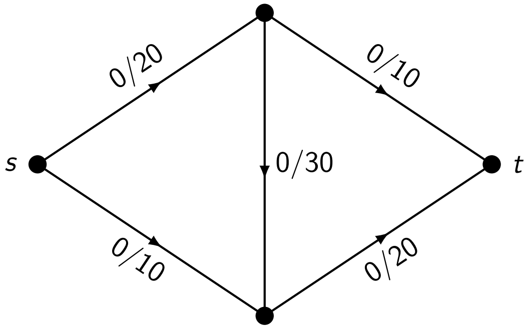
What if we'd chosen this path first instead?



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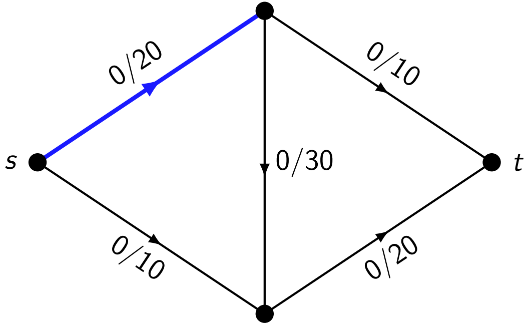

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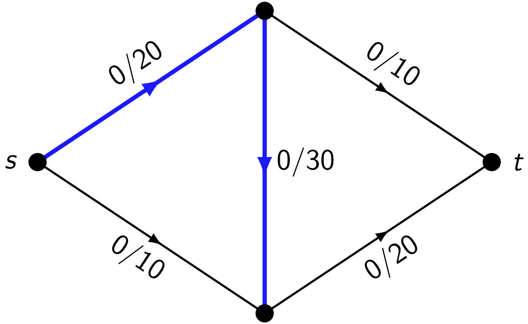

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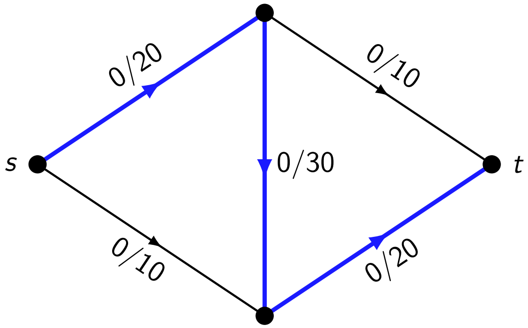

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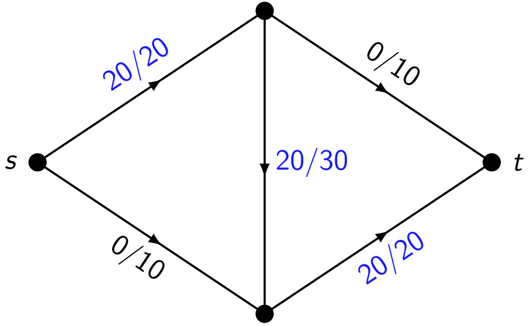

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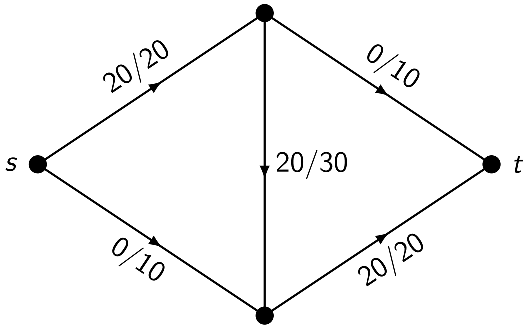

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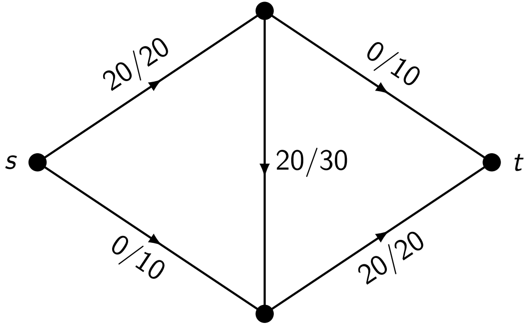

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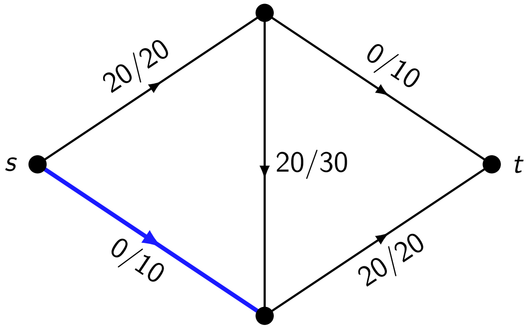

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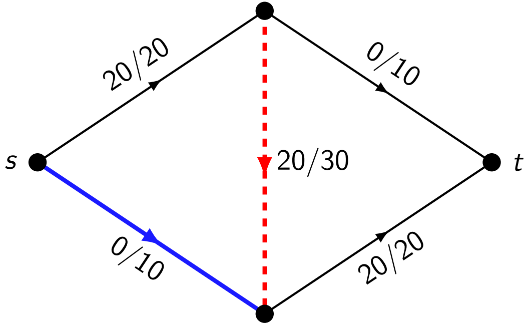

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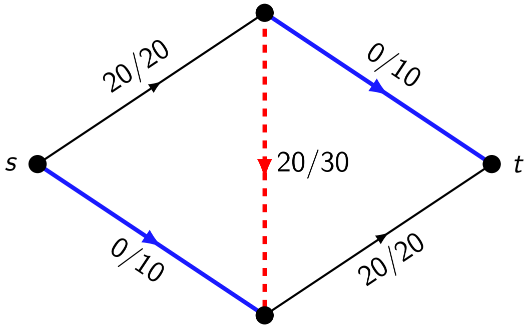

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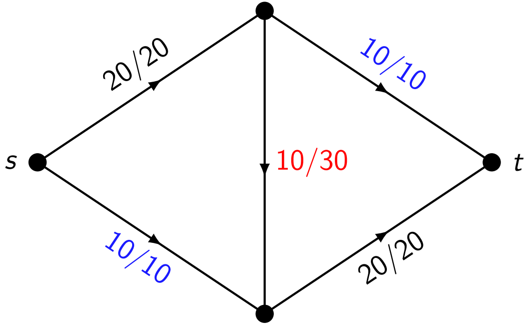

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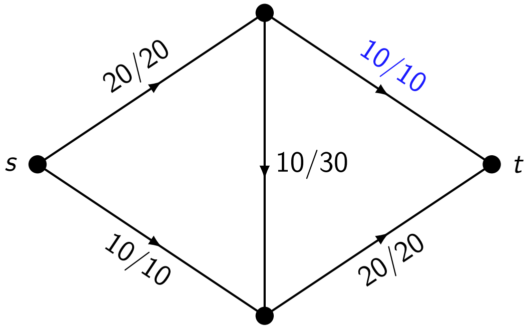

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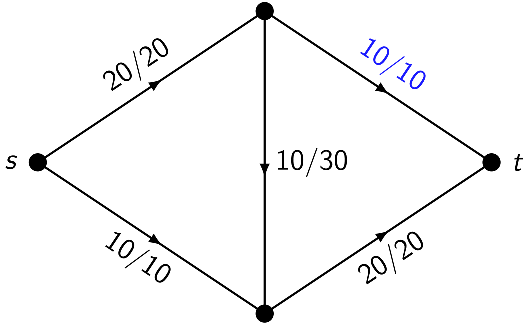

Now there are no more paths from s to t with spare capacity, but our flow only has value 20...

What if we allow ourselves to push flow *backwards* along a path?
We get a maximum flow! Now to generalise this...

A **flow** is a function $f\colon E \to \mathbb{R}$ such that for all $e \in E$ and $v \in V \setminus \{s,t\}$:

- $0 \le f(e) \le c(e)$;
- $f^+(v) := \sum_{u \in N^-(v)} f(u,v) = \sum_{w \in N^+(v)} f(v,w) =: f^-(v)$.

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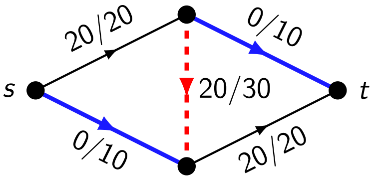

We want to say: an augmenting path for a flow f is an **undirected** path from s to t which we can push flow along. So forward edges e have $f(e) < c(e)$, and backward edges e have $f(e) > 0$.

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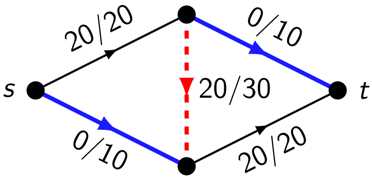

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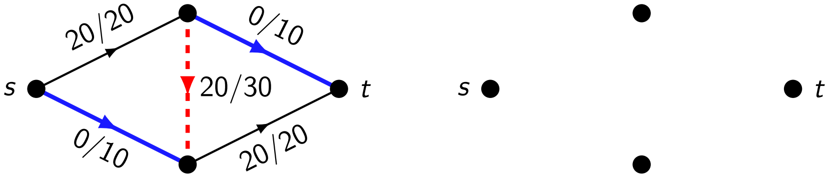
But there's an annoying technicality with bidirected edges...



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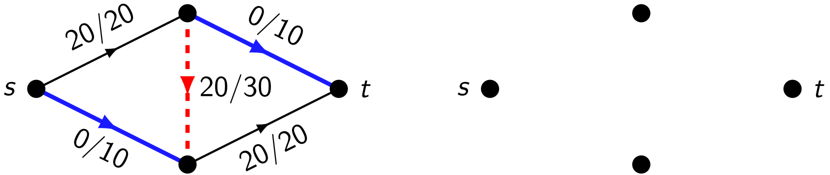

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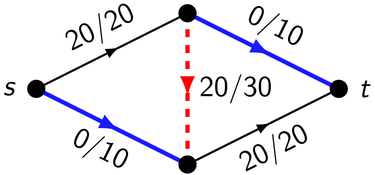

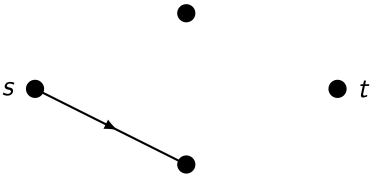

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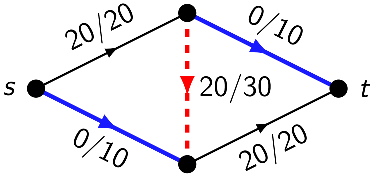

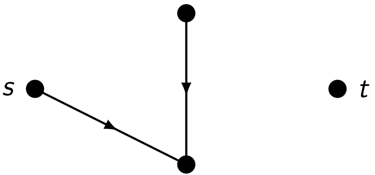

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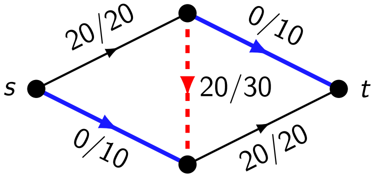

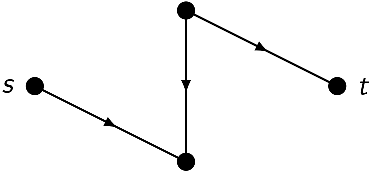

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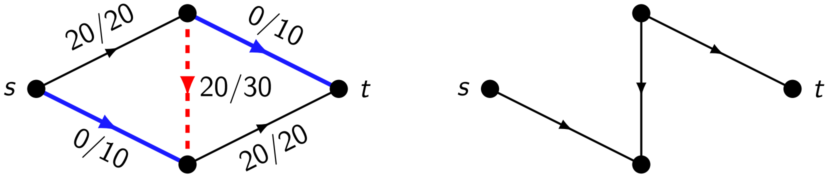

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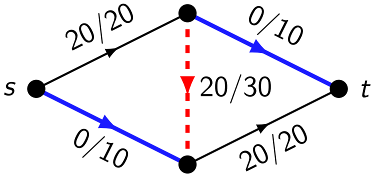

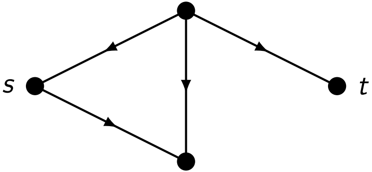

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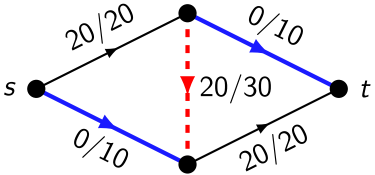

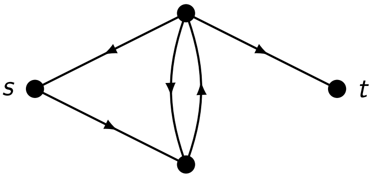

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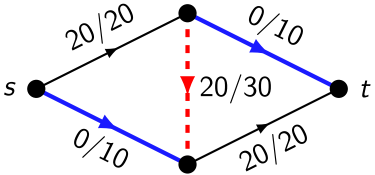

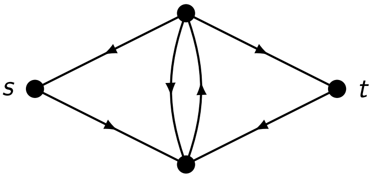

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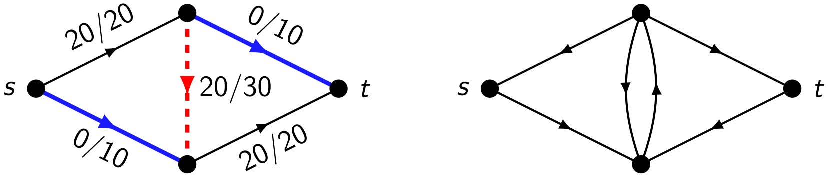

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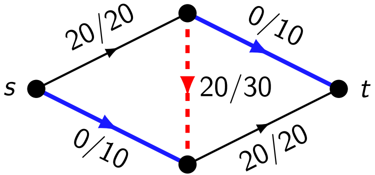

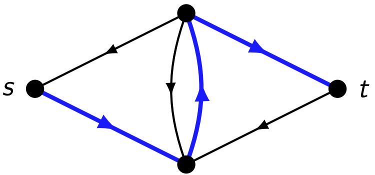

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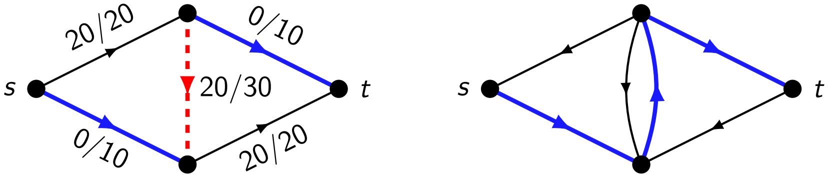

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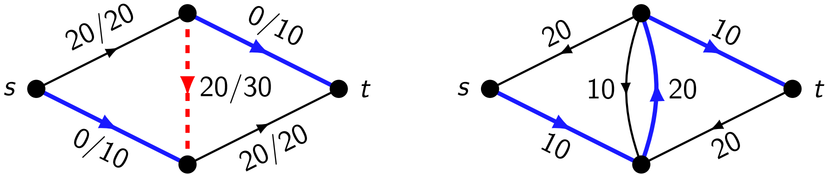

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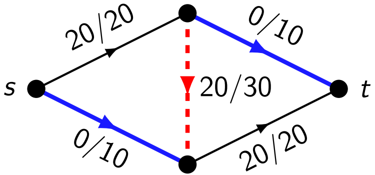

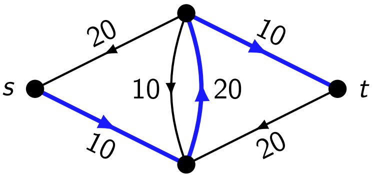

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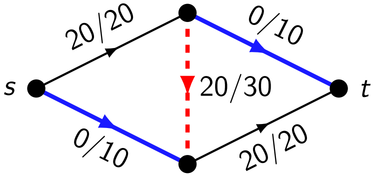

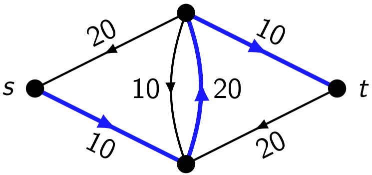

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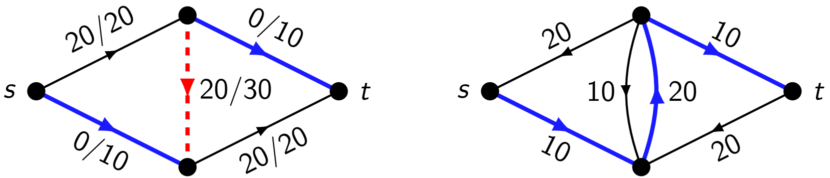

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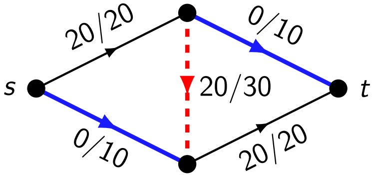

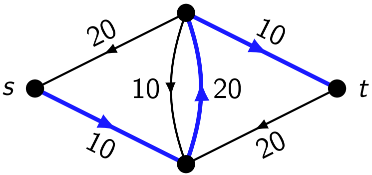

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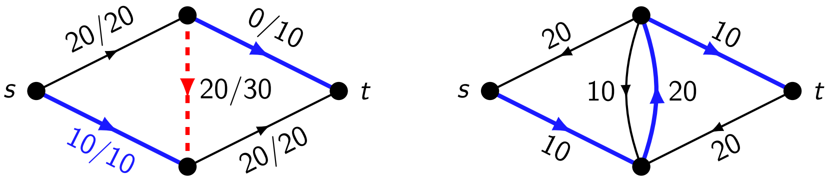

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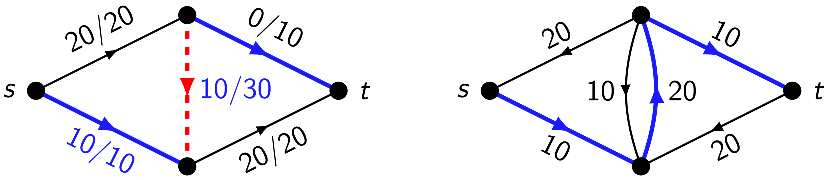

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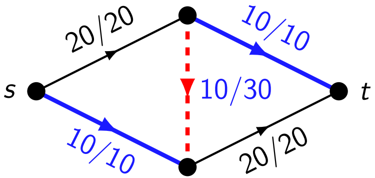

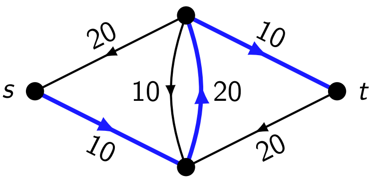

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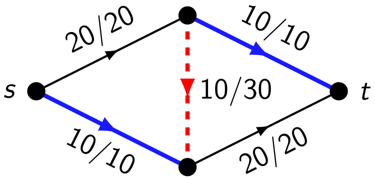

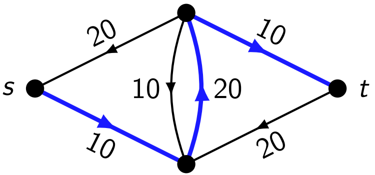

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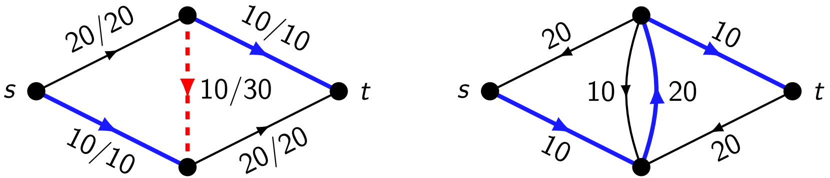

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Lemma 3: $\texttt{Push}(G, c, s, t, f, P)$ returns a new flow f', with value $v(f') = v(f) + C$, in $O(|V(G)|)$ time. $\square$

The Ford-Fulkerson Algorithm

Algorithm: FordFulkerson

Input : A (weakly connected) flow network (G, c, s, t).
Output : A flow f with no augmenting paths.

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begin
    Construct the flow f with f(e) = 0 for all e ∈ E(G).
    Construct the residual graph G_f.
    while G_f contains a path P from s to t do
        Find P using depth-first (or breadth-first) search.
        Update f ← Push(G, c, s, t, f, P).
        Update G_f on the edges of P.
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By Lemma 3, every iteration of 4–7 increases $v(f)$ by at least 1. So if f^* is a maximum flow, there are at most $v(f^*)$ iterations in total.

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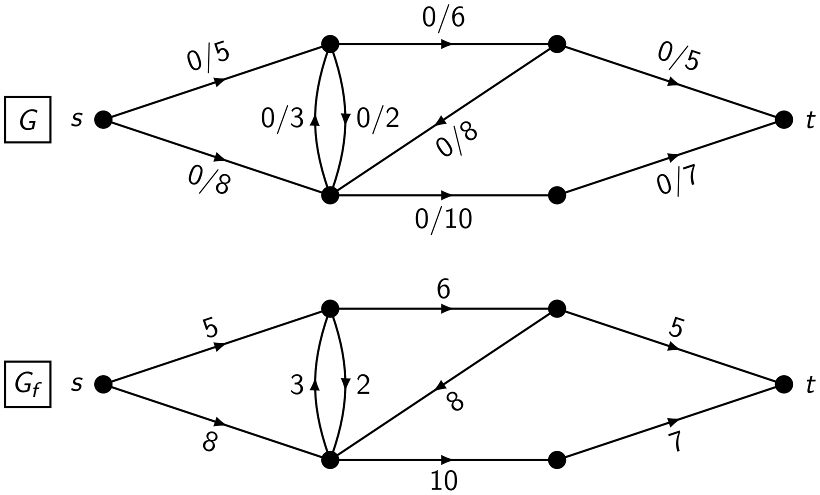
Every step takes $O(|E|)$ time or $O(|V|)$ time, and since G is weakly connected we have $|V| = O(|E|)$. So the running time is $O(v(f^*)|E|)$.

Worked example



Initialise flow and construct G_f.

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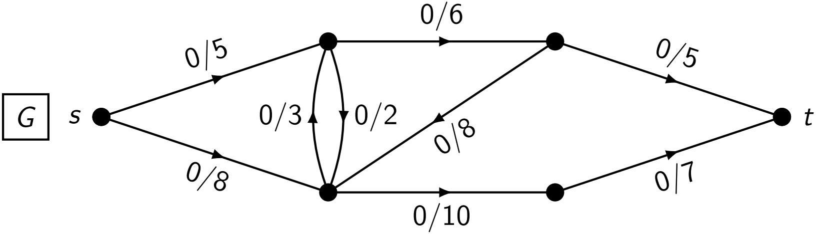

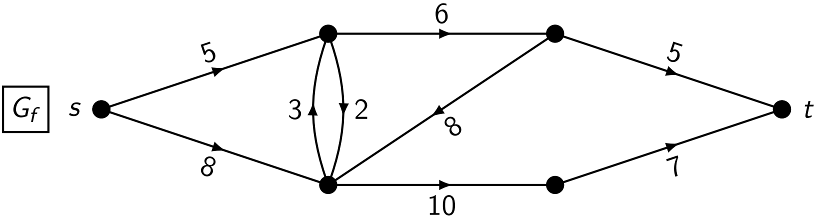

Apply depth-first search to find an augmenting path in G_f.

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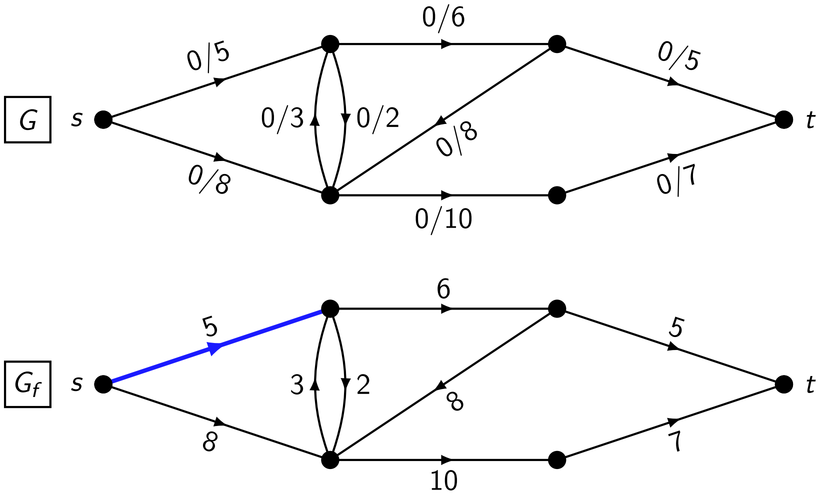

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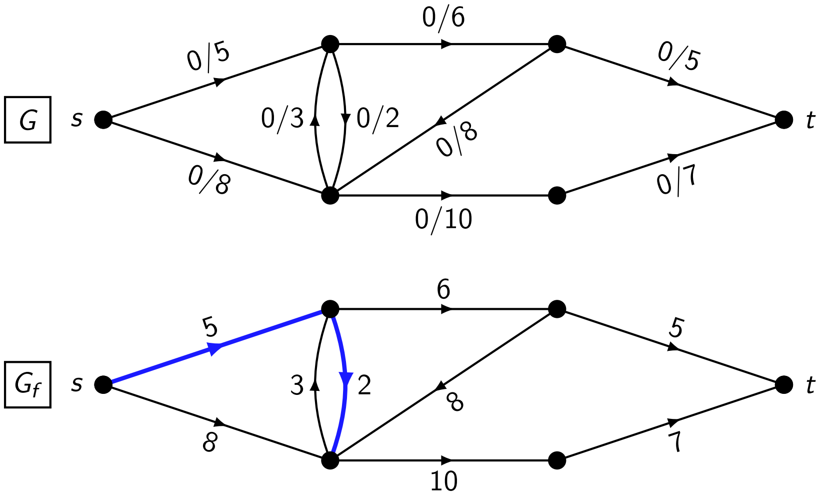

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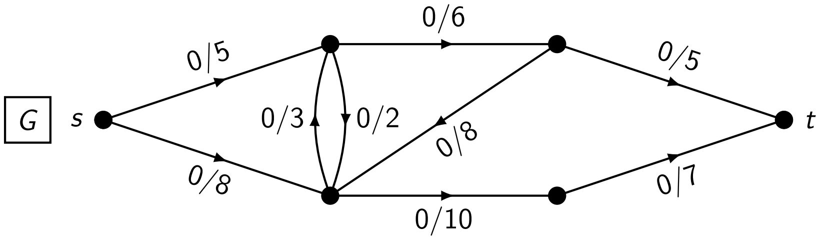

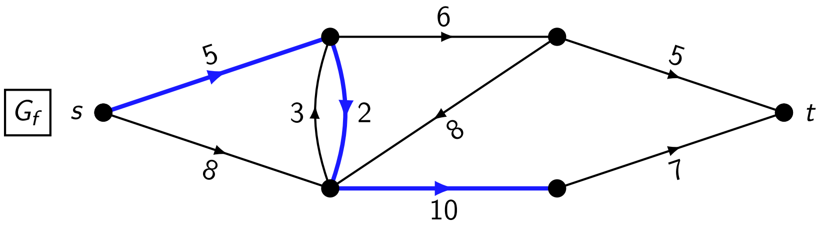

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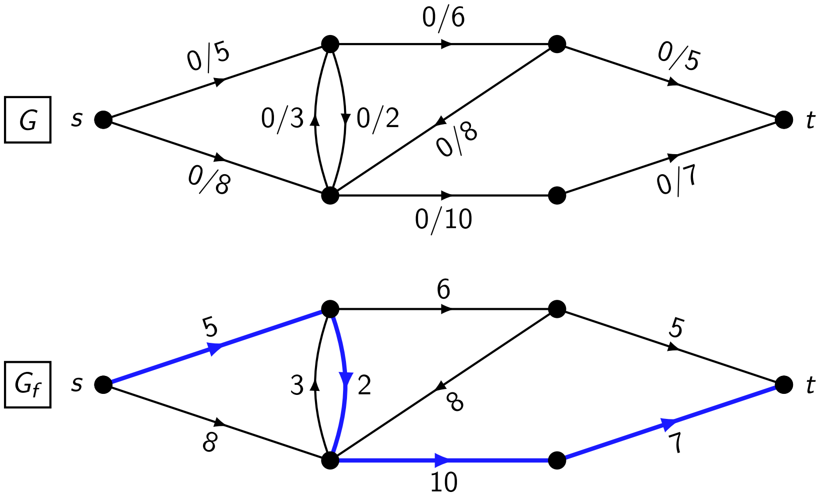

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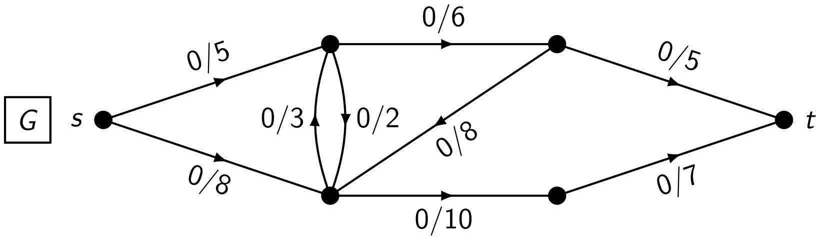
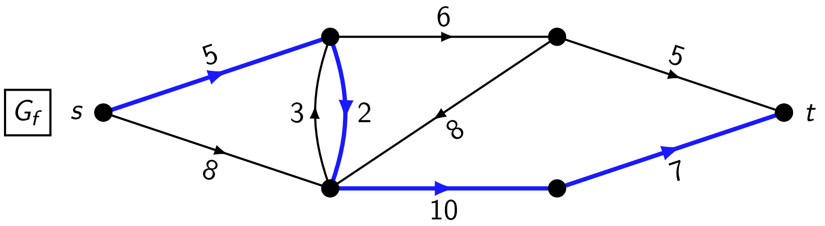
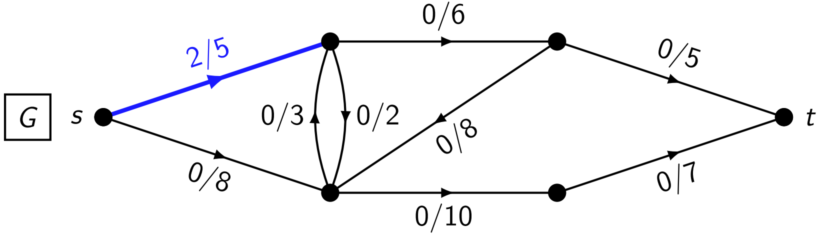
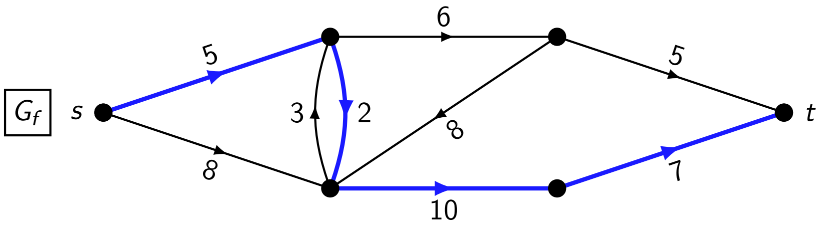




Push flow along the path. (This path has residual capacity 2.)

Worked example





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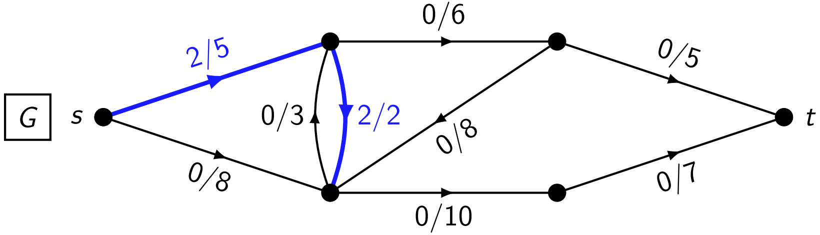

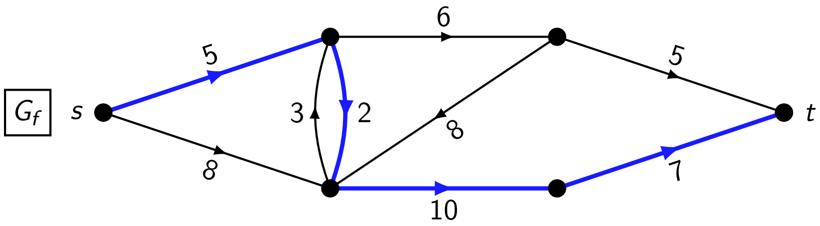

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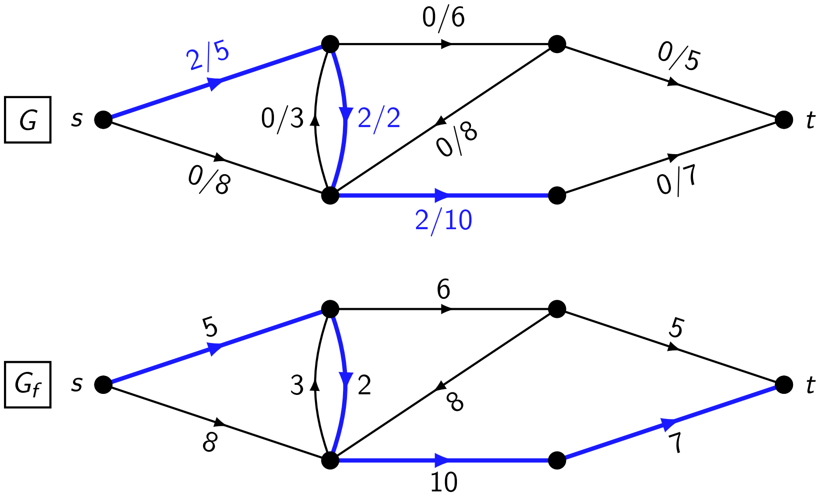

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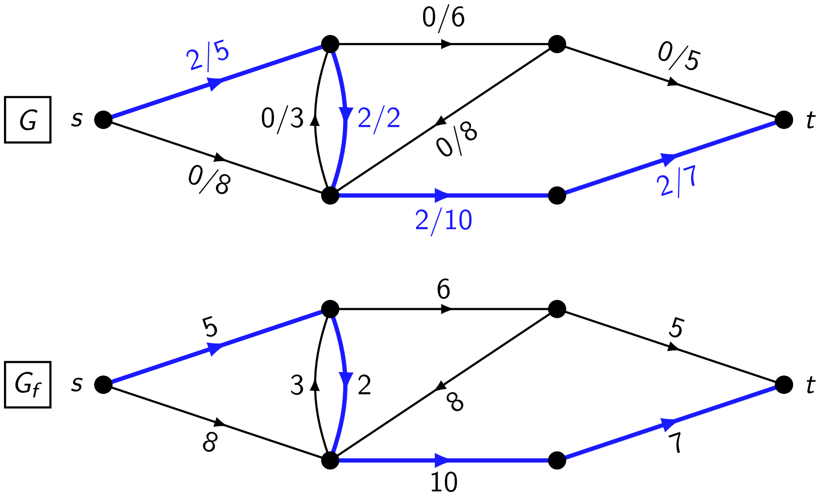

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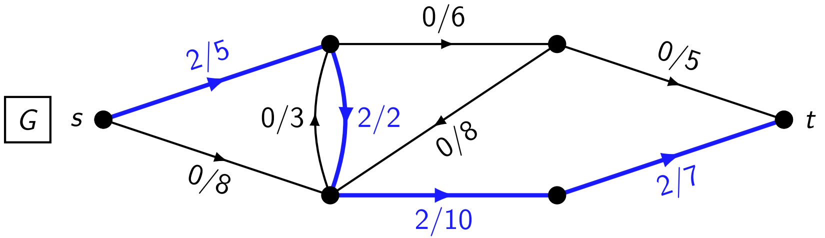

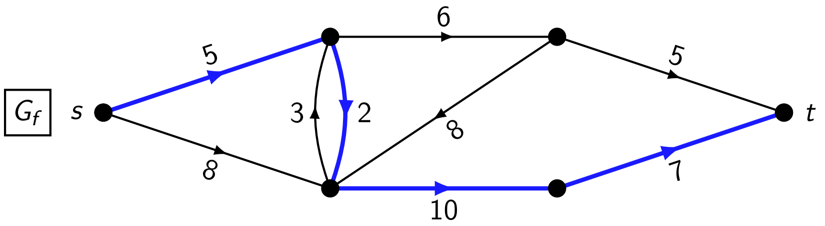

Update G_f along the path.

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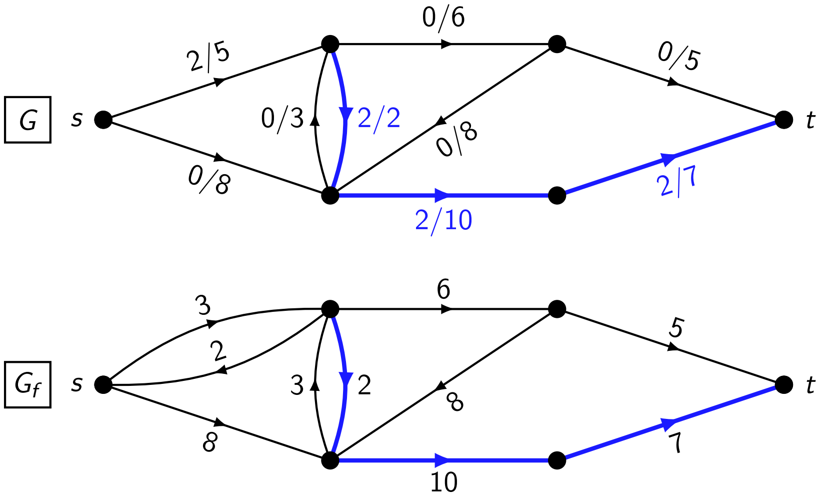

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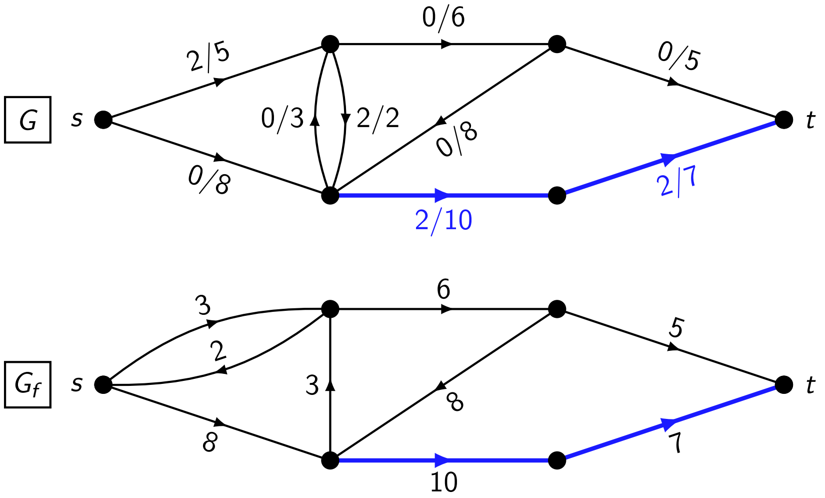

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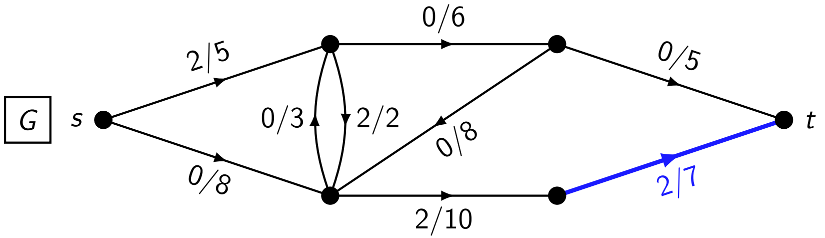

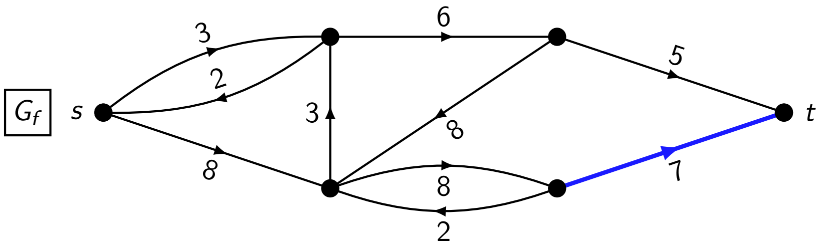

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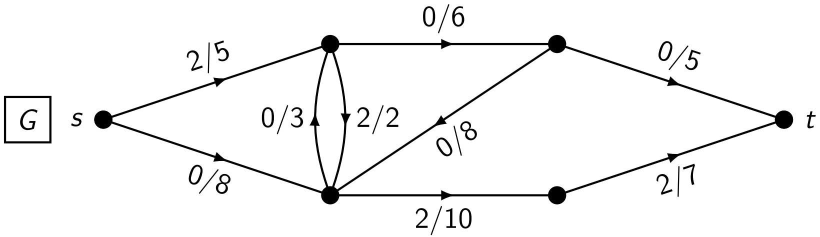

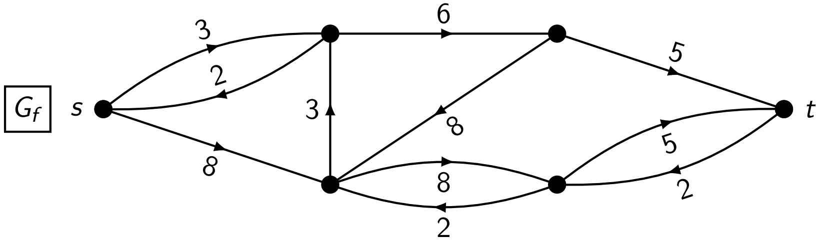

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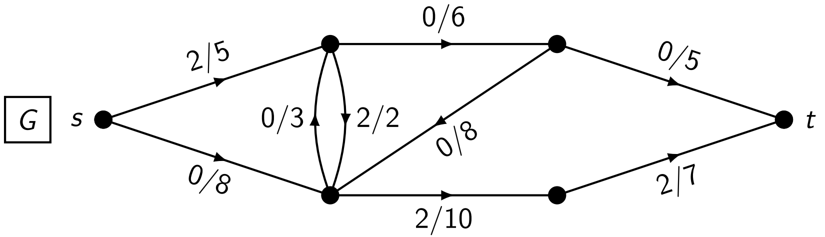
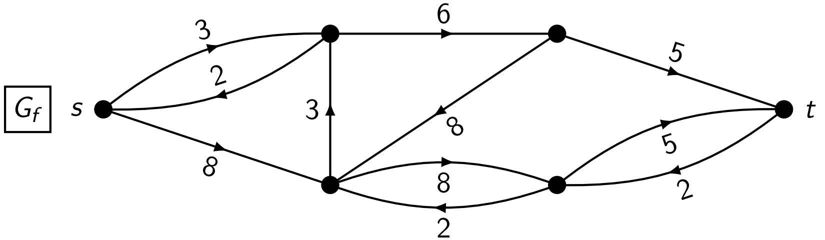
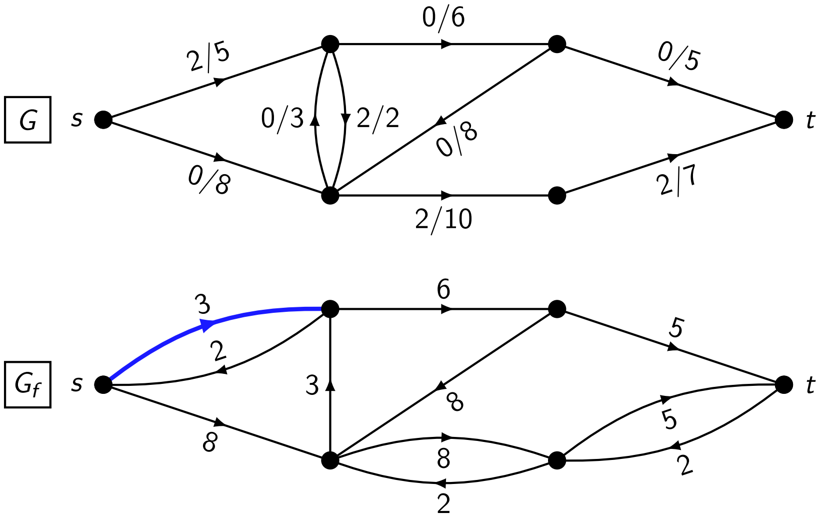




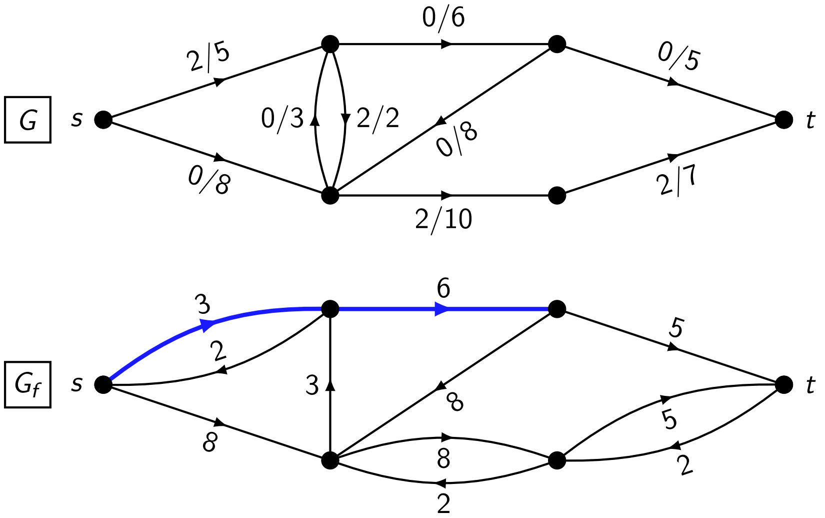
Apply depth-first search to find an augmenting path in G_f.

Worked example



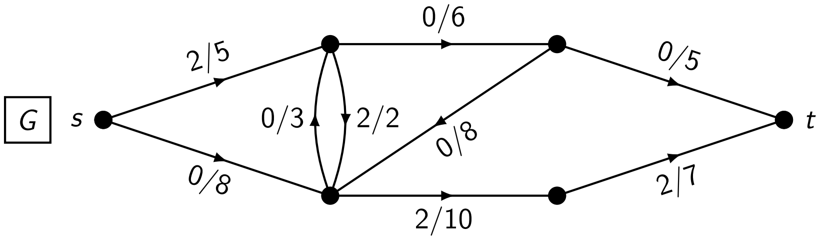
Apply depth-first search to find an augmenting path in G_f.

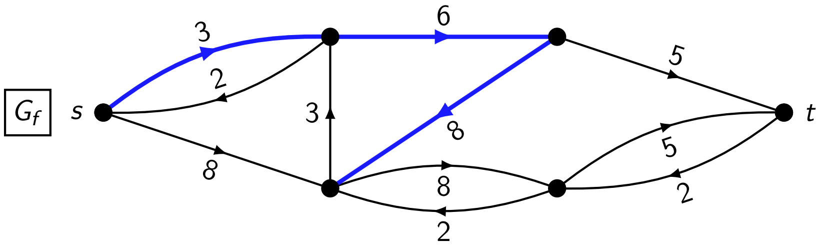
Worked example



Apply depth-first search to find an augmenting path in G_f.

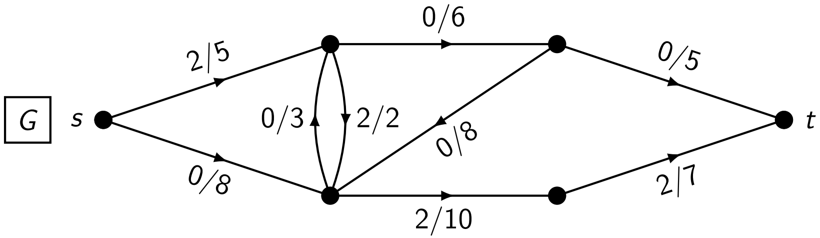
Worked example

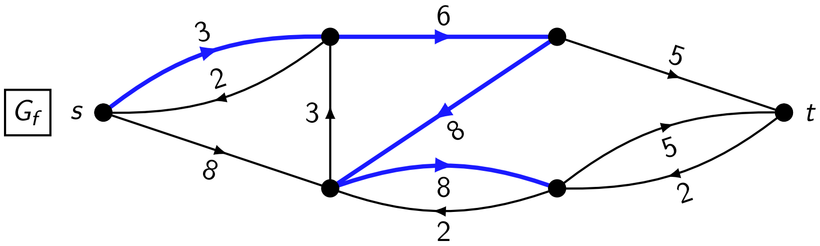




Apply depth-first search to find an augmenting path in G_f.

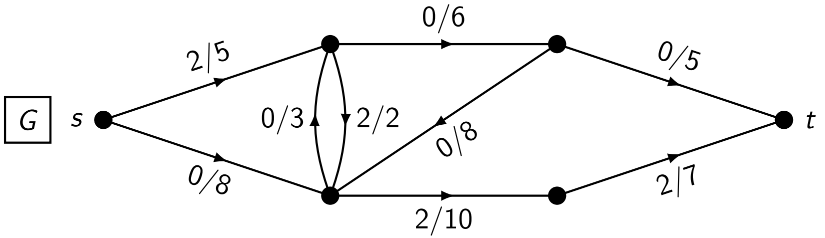
Worked example

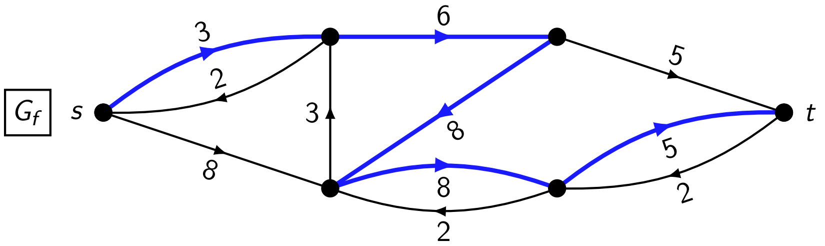




Apply depth-first search to find an augmenting path in G_f.

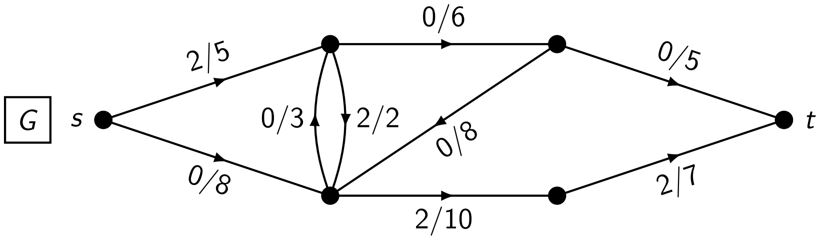
Worked example

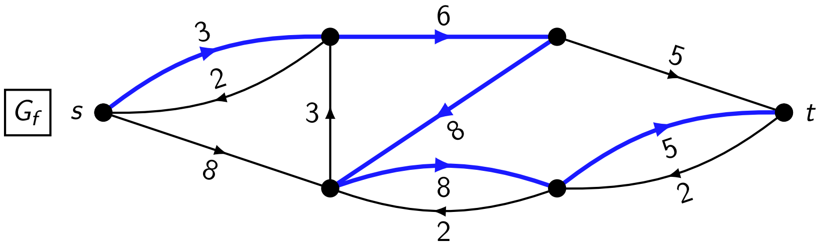




Apply depth-first search to find an augmenting path in G_f.

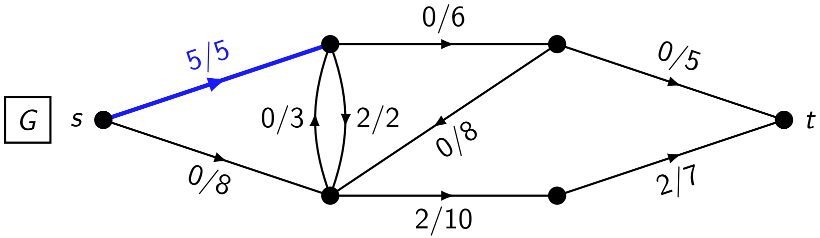
Worked example

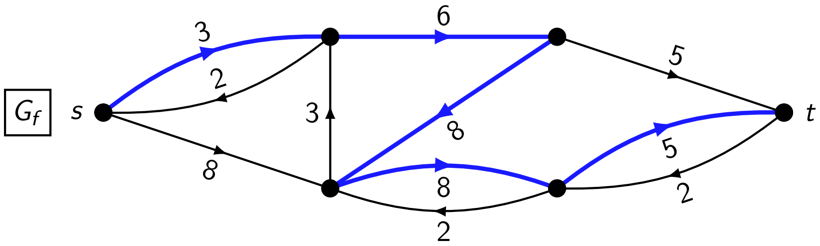




Push flow along the path. (This path has residual capacity 3.)

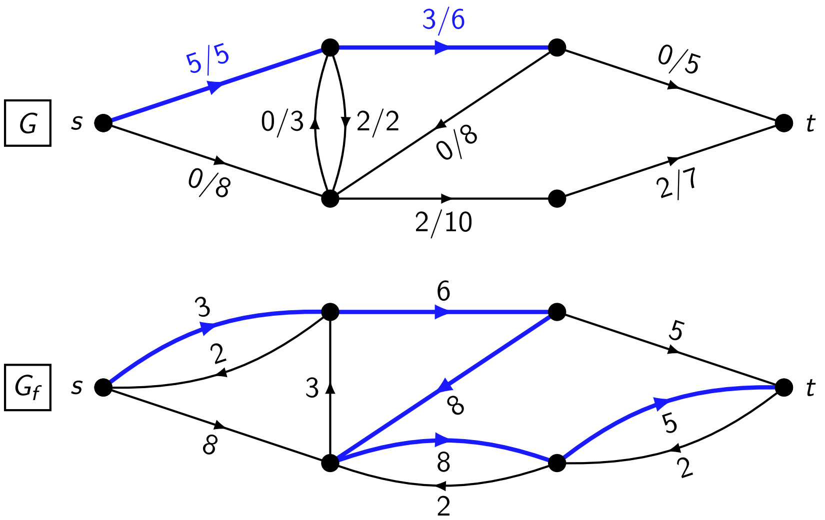
Worked example





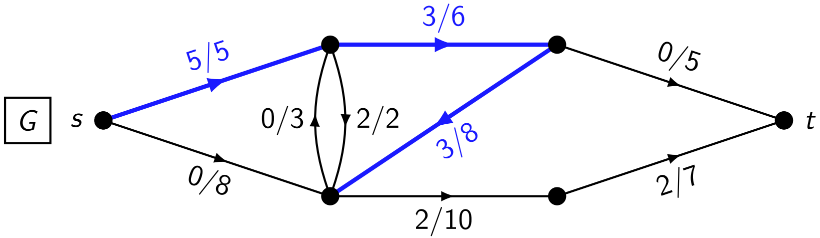
Push flow along the path. (This path has residual capacity 3.)

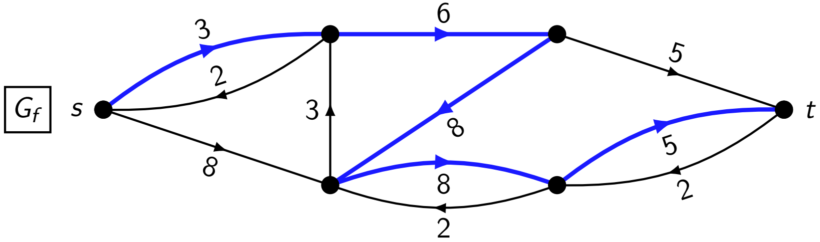
Worked example



Push flow along the path. (This path has residual capacity 3.)

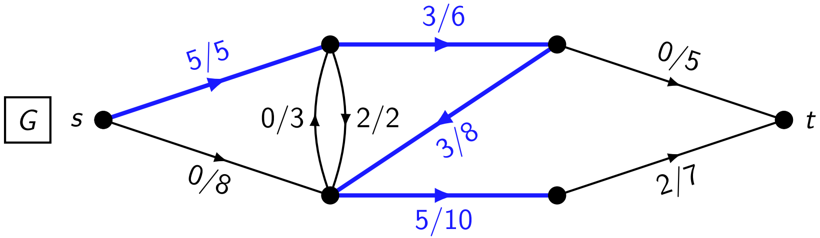
Worked example

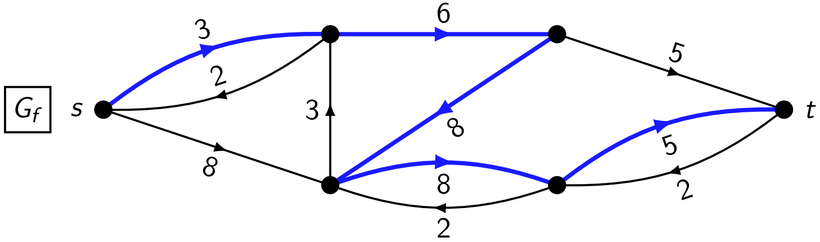




Push flow along the path. (This path has residual capacity 3.)

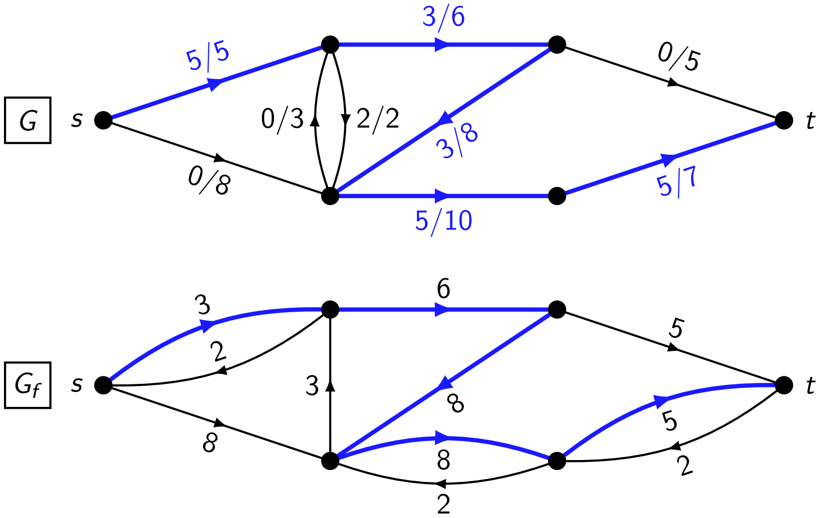
Worked example





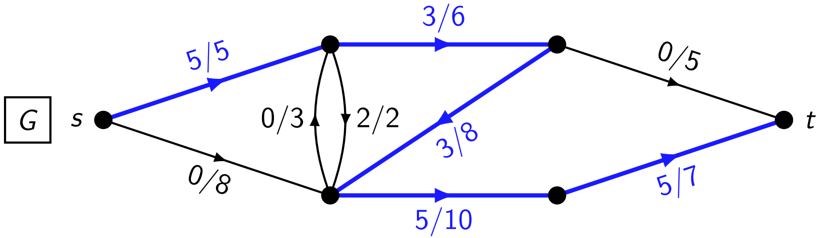
Push flow along the path. (This path has residual capacity 3.)

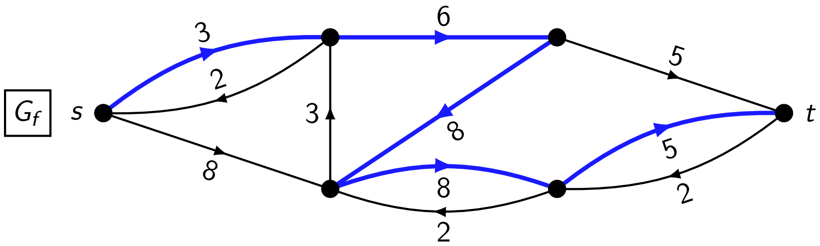
Worked example



Push flow along the path. (This path has residual capacity 3.)

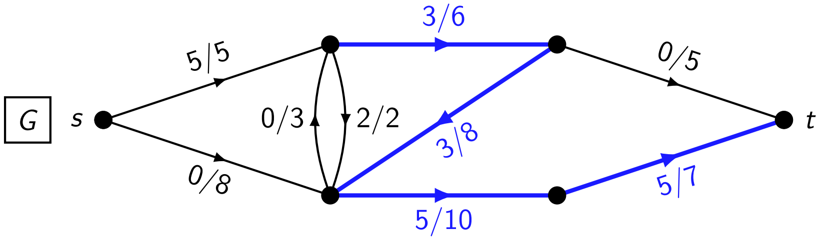
Worked example

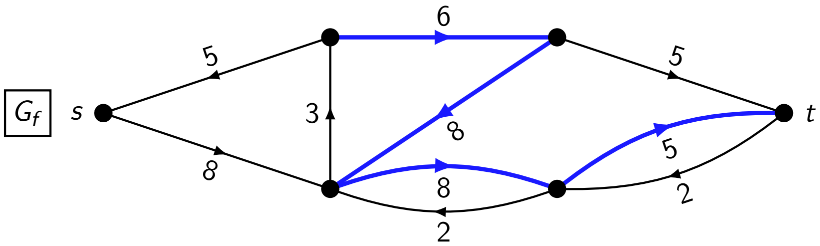




Update G_f along the path.

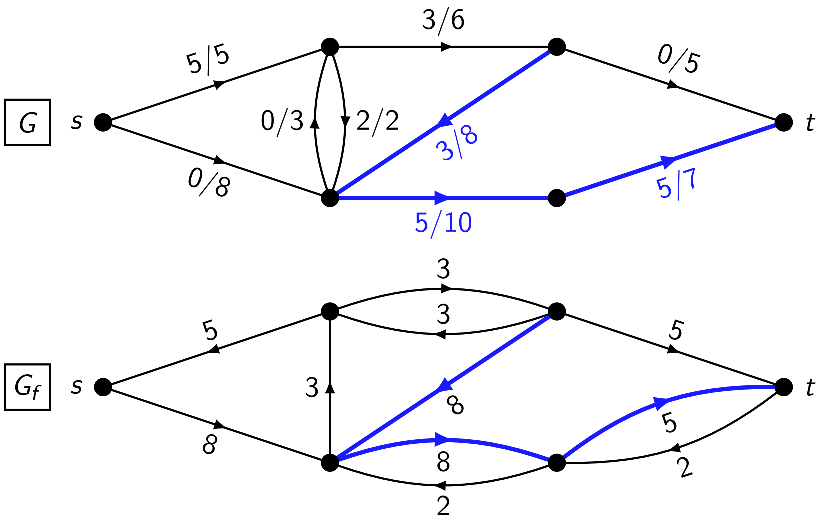
Worked example





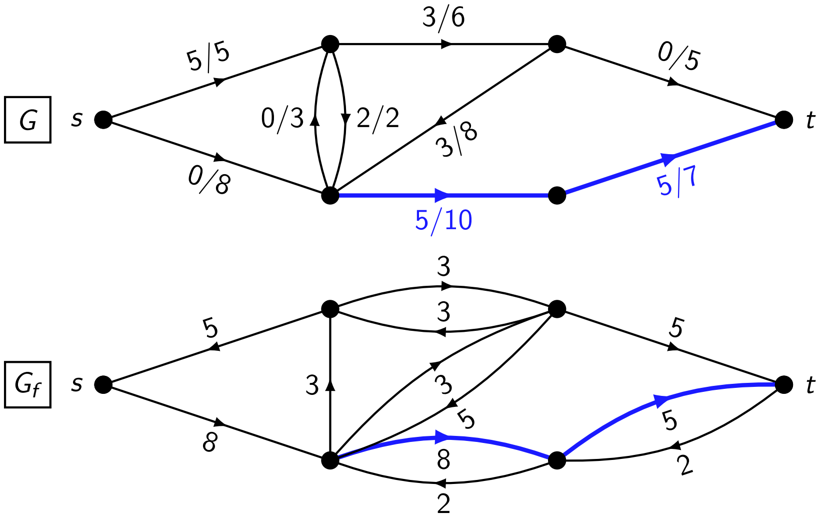
Update G_f along the path.

Worked example



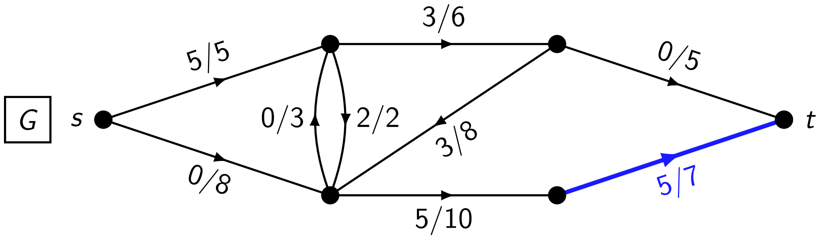
Update G_f along the path.

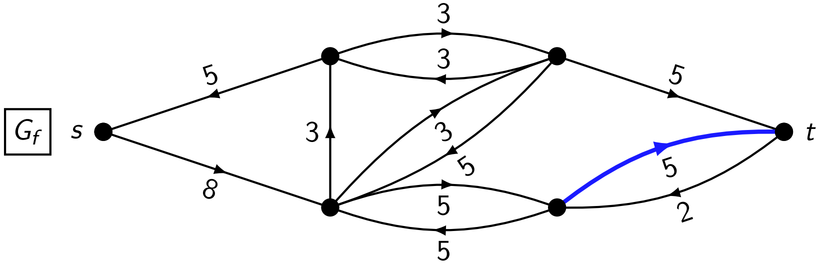
Worked example



Update G_f along the path.

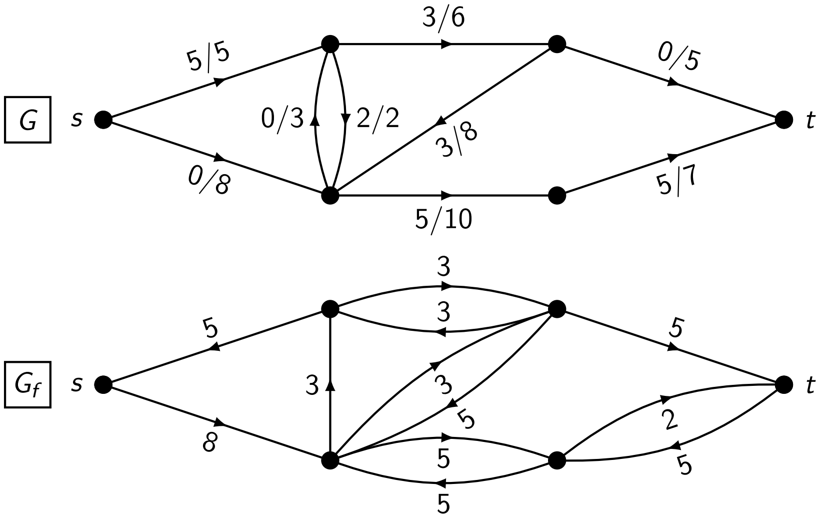
Worked example





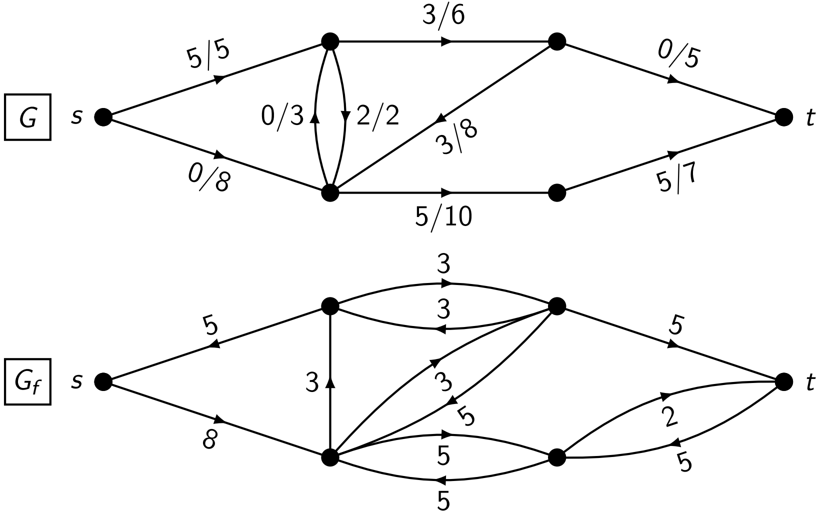
Update G_f along the path.

Worked example



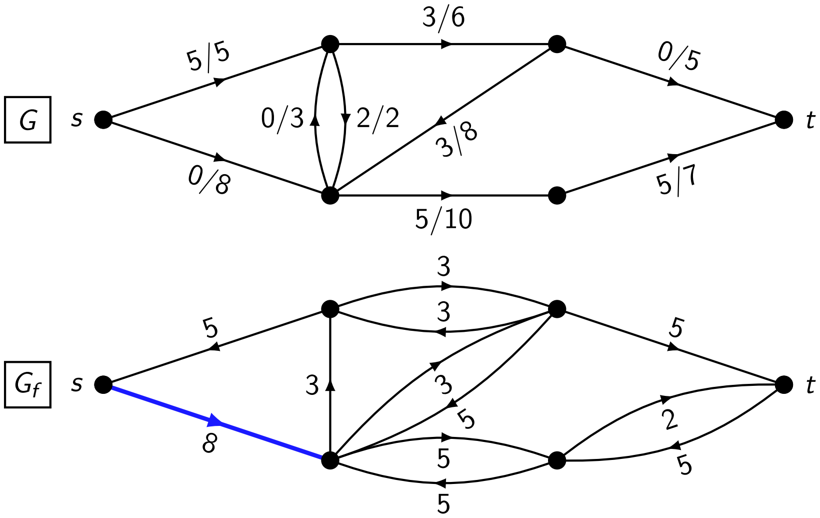
Update G_f along the path.

Worked example



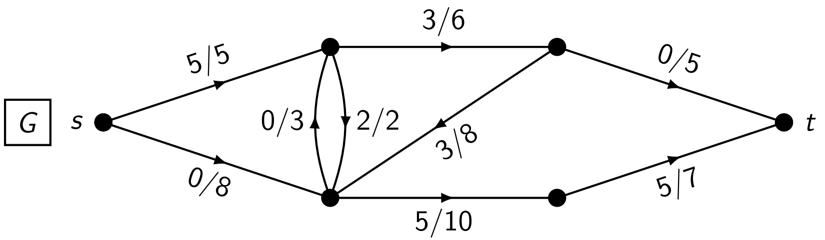
Apply depth-first search to find an augmenting path in G_f.

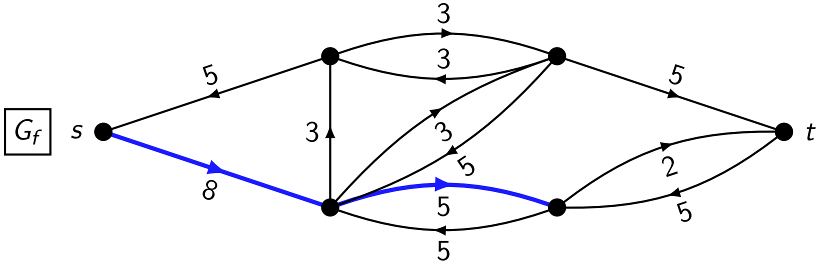
Worked example



Apply depth-first search to find an augmenting path in G_f.

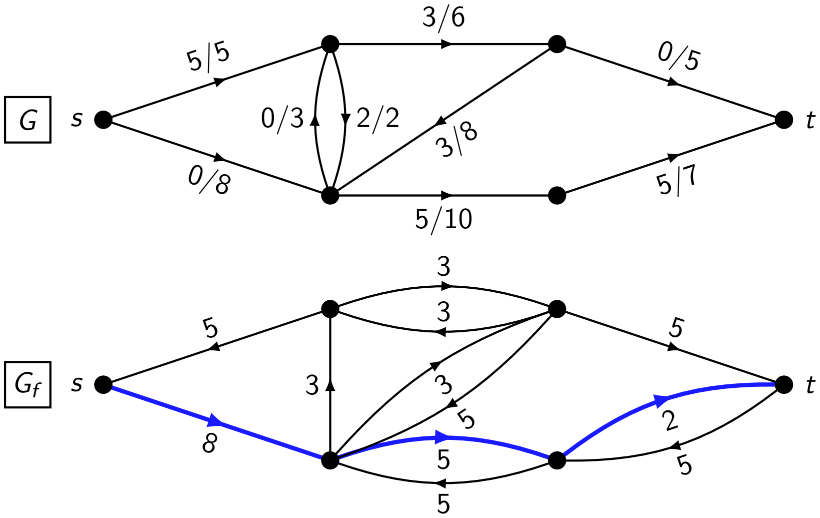
Worked example





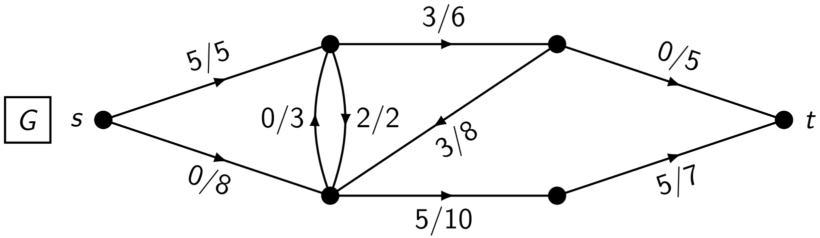
Apply depth-first search to find an augmenting path in G_f.

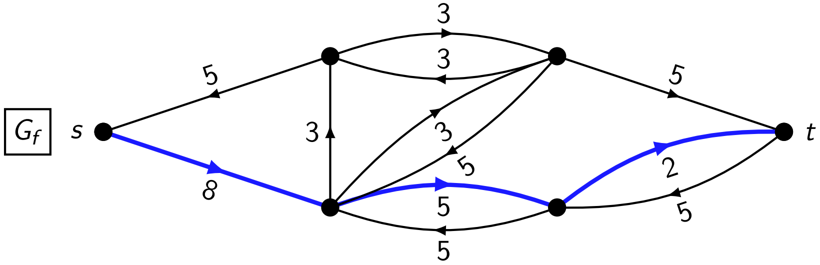
Worked example



Apply depth-first search to find an augmenting path in G_f.

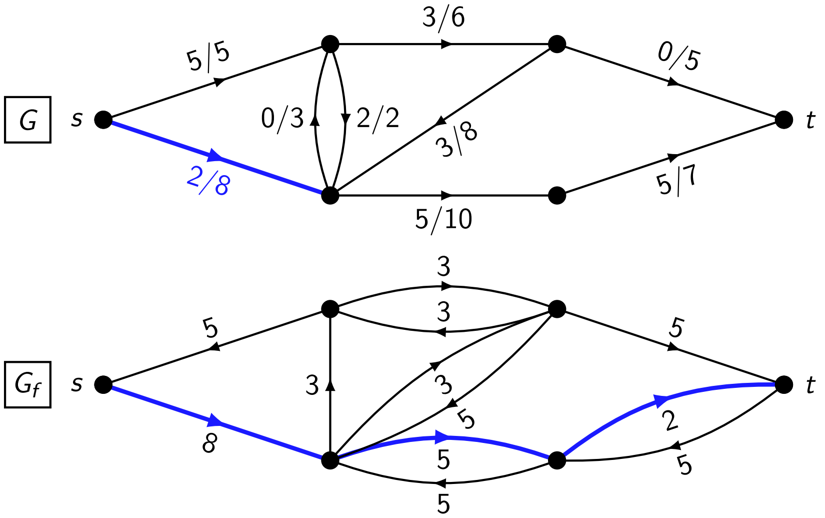
Worked example



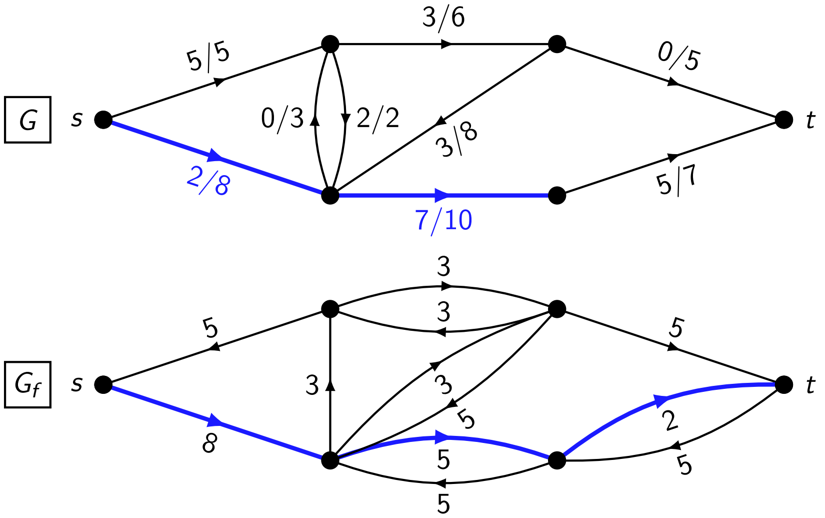


Push flow along the path. (This path has residual capacity 2.)

Worked example

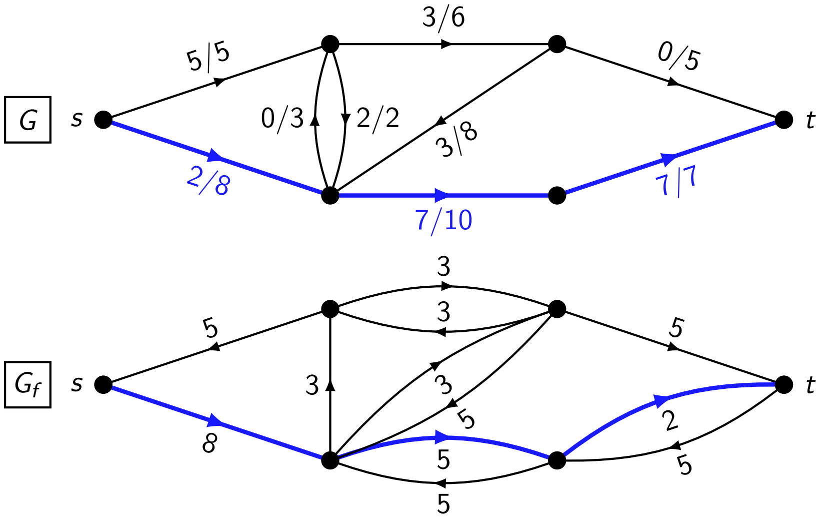


Push flow along the path. (This path has residual capacity 2.)



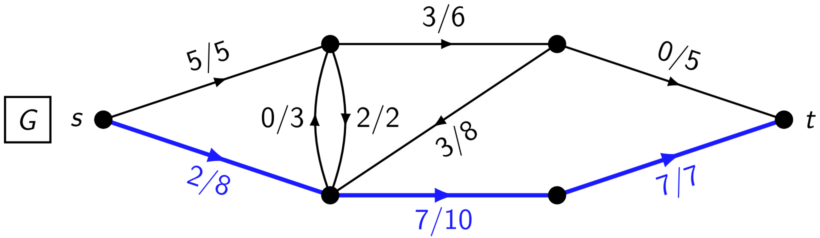
Push flow along the path. (This path has residual capacity 2.)

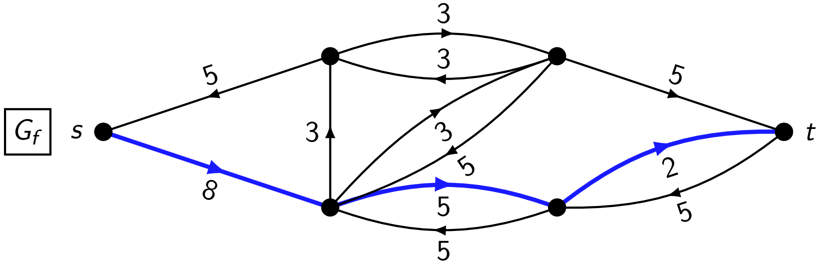
Worked example



Push flow along the path. (This path has residual capacity 2.)

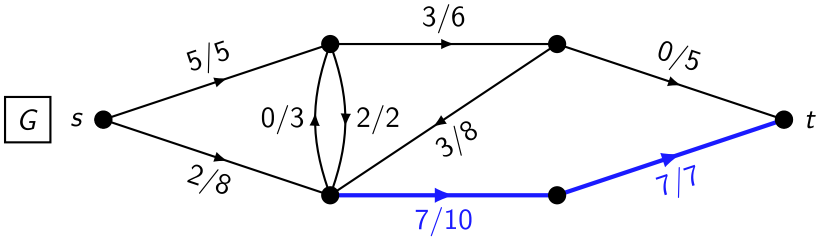
Worked example

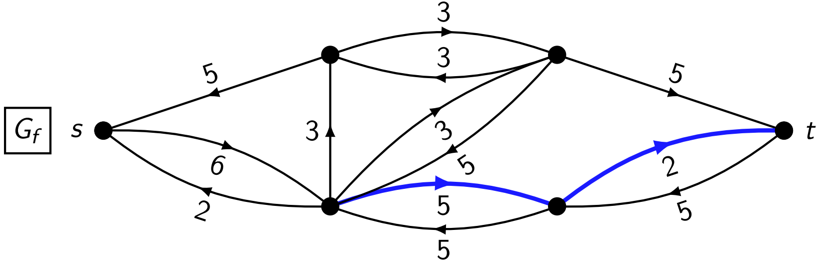




Update G_f along the path.

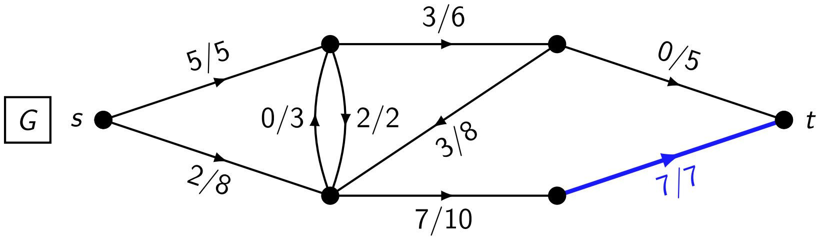
Worked example

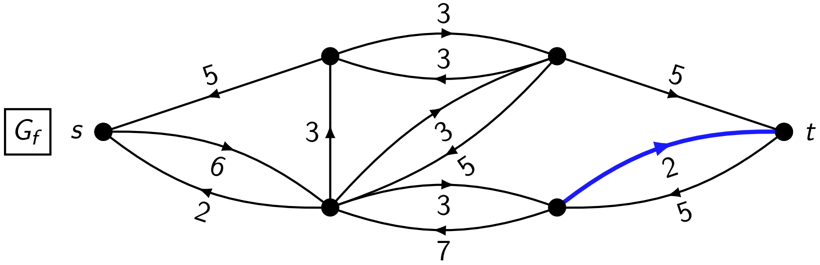




Update G_f along the path.

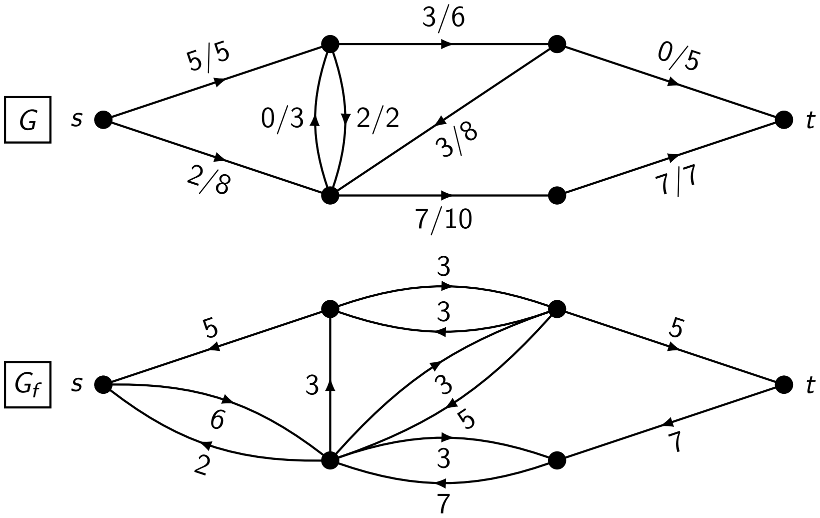
Worked example





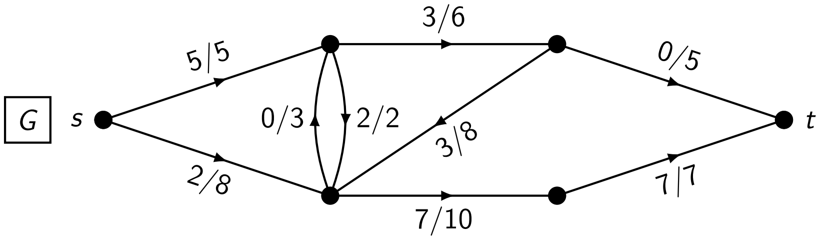
Update G_f along the path.

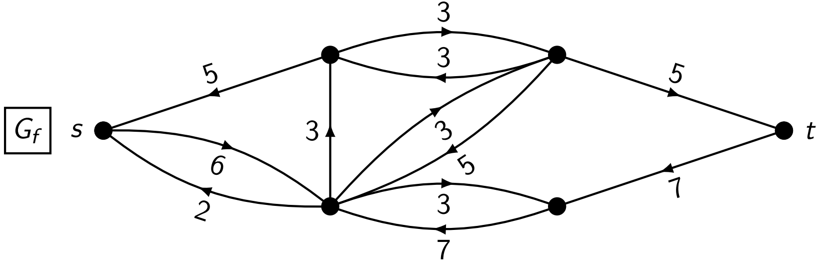
Worked example



Update G_f along the path.

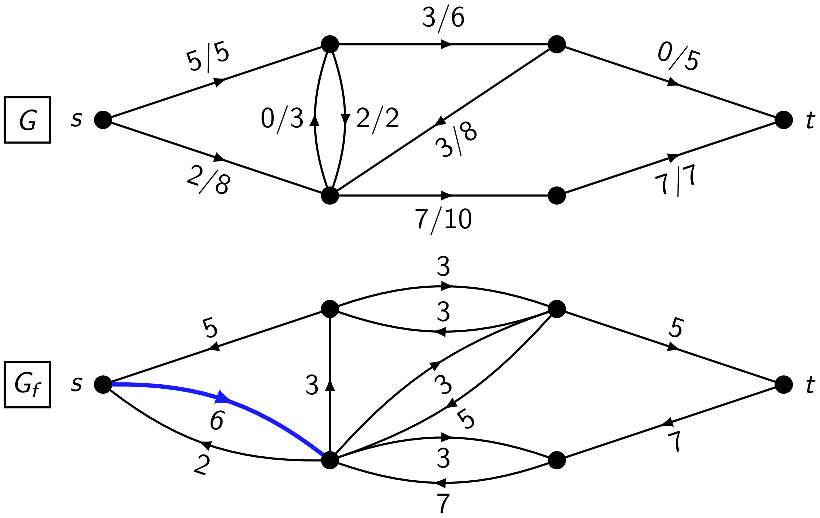
Worked example





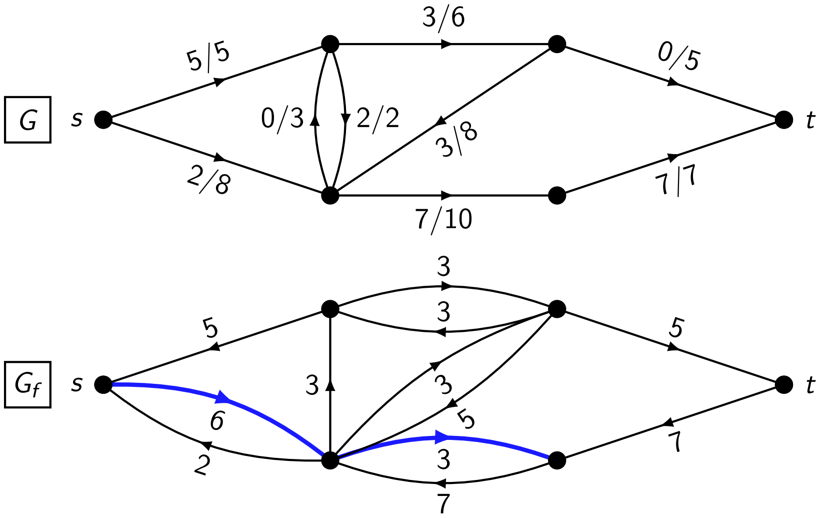
Apply depth-first search to find an augmenting path in G_f.

Worked example



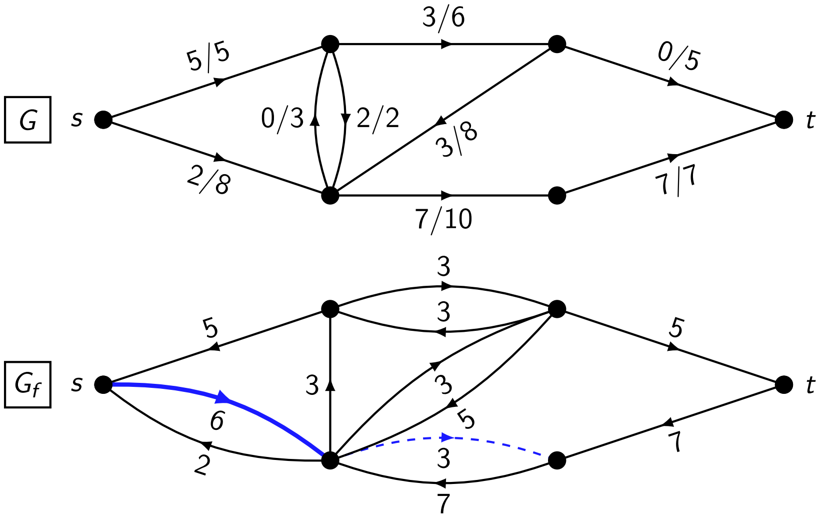
Apply depth-first search to find an augmenting path in G_f.

Worked example



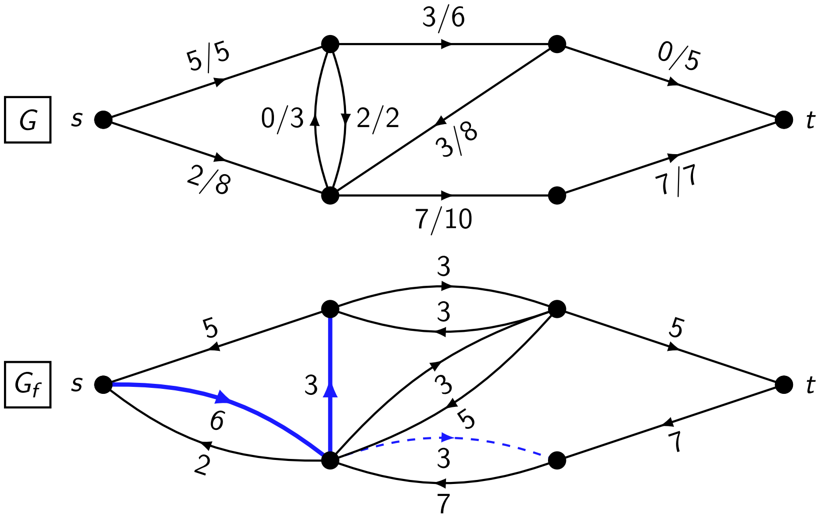
Apply depth-first search to find an augmenting path in G_f.

Worked example



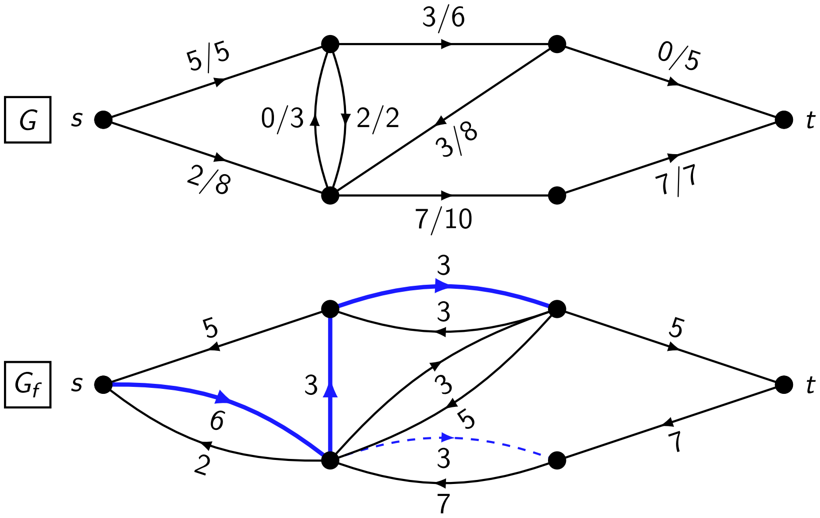
Apply depth-first search to find an augmenting path in G_f.

Worked example



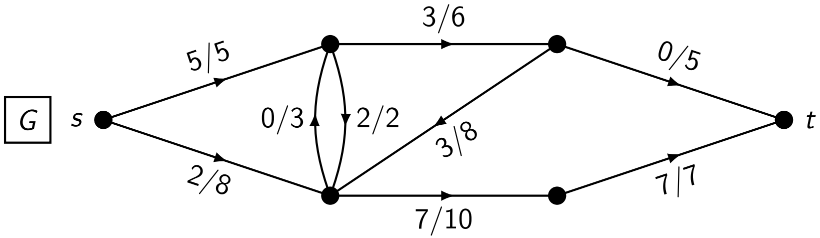
Apply depth-first search to find an augmenting path in G_f.

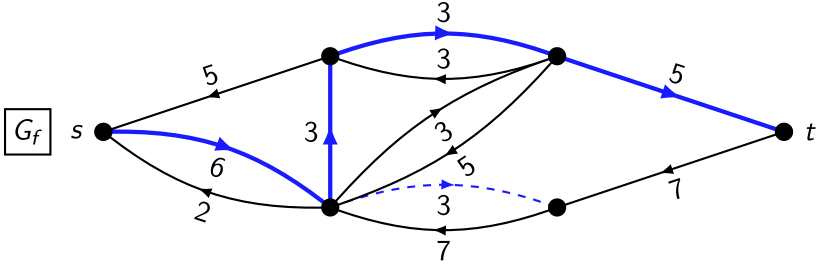
Worked example



Apply depth-first search to find an augmenting path in G_f.

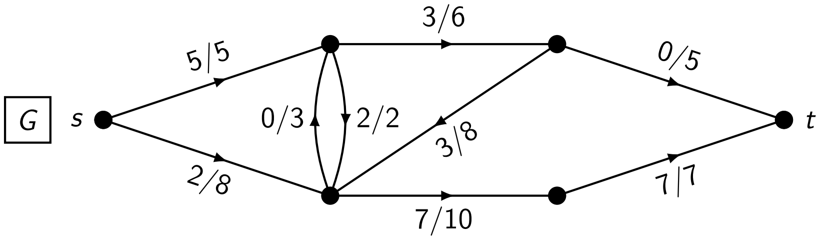
Worked example

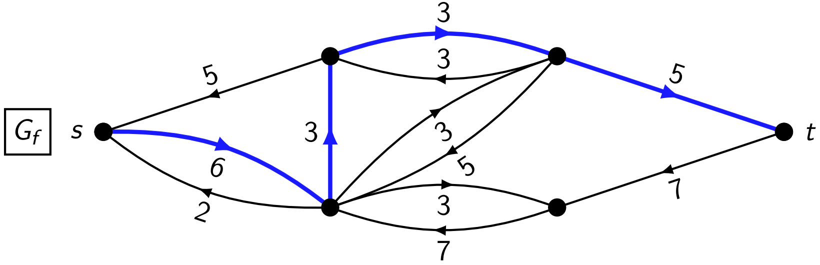




Apply depth-first search to find an augmenting path in G_f.

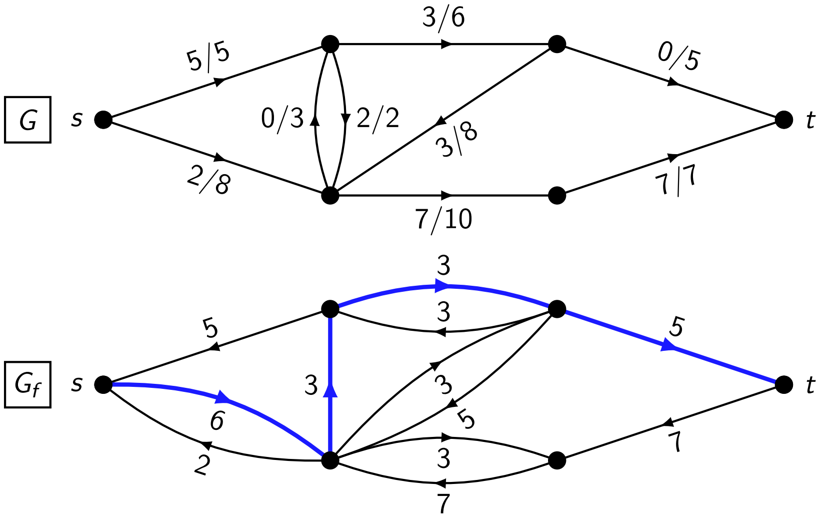
Worked example





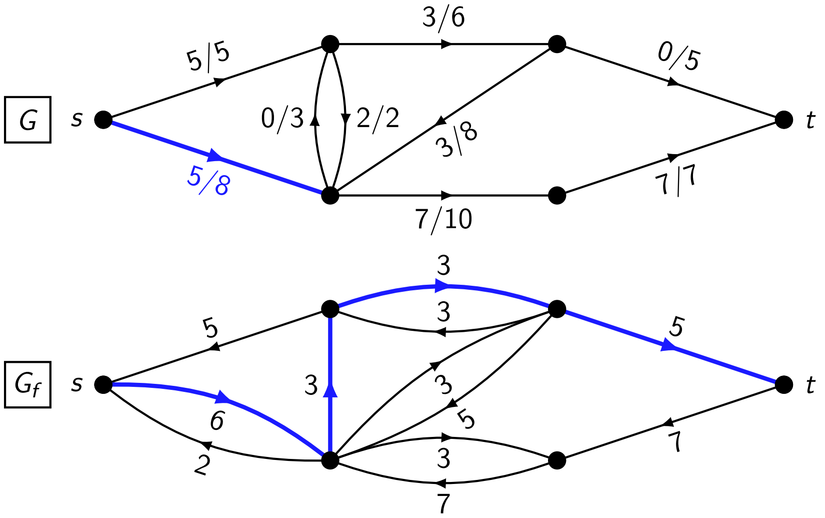
Apply depth-first search to find an augmenting path in G_f.

Worked example



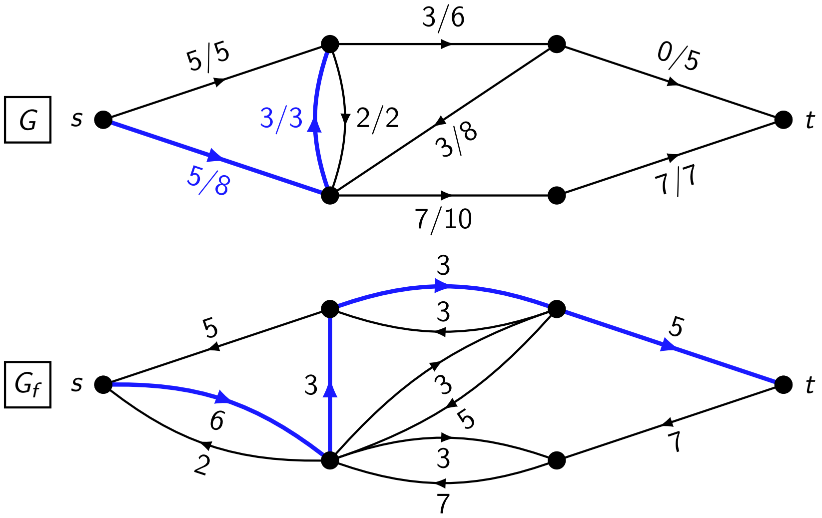
Push flow along the path. (This path has residual capacity 3.)

Worked example



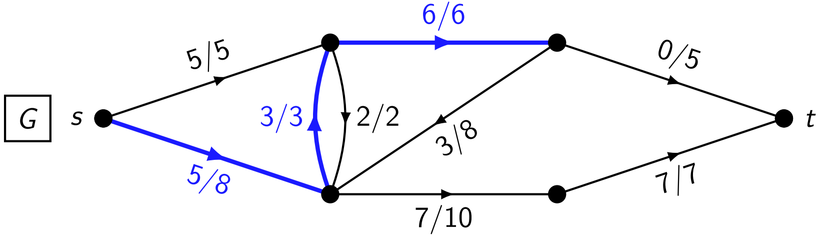
Push flow along the path. (This path has residual capacity 3.)

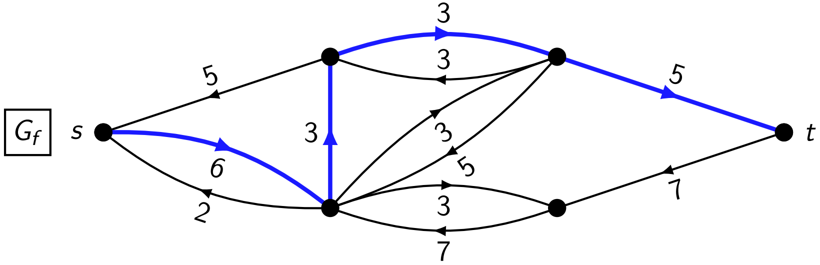
Worked example



Push flow along the path. (This path has residual capacity 3.)

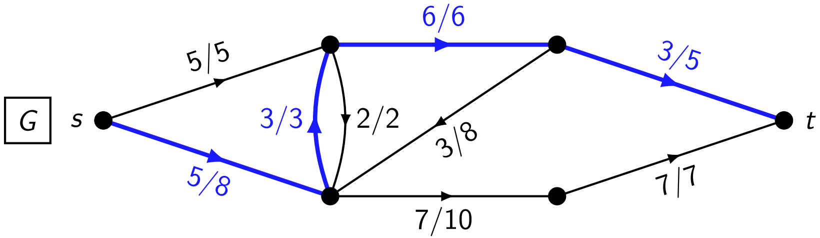
Worked example

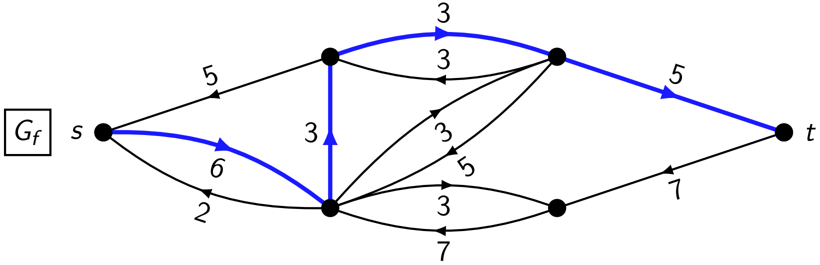




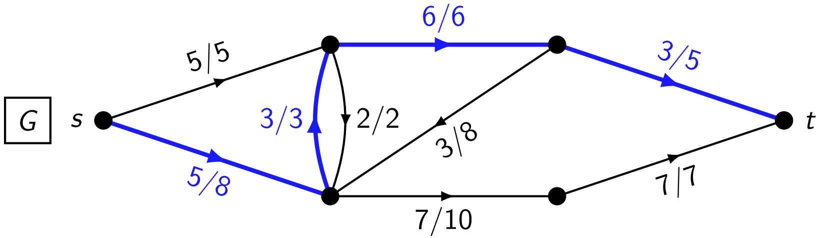
Push flow along the path. (This path has residual capacity 3.)

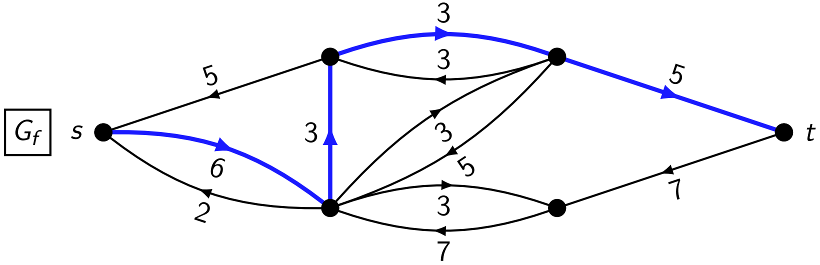
Worked example





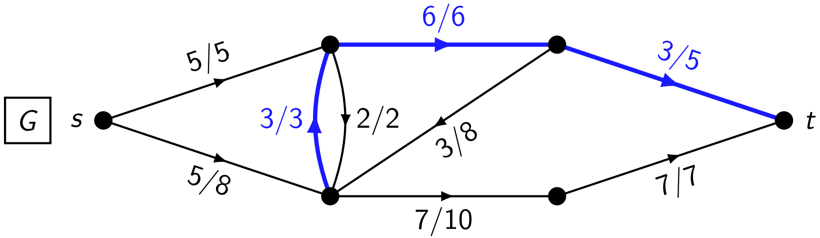
Push flow along the path. (This path has residual capacity 3.)

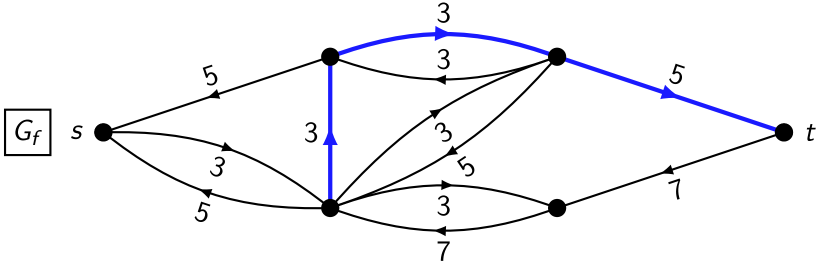




Update G_f along the path.

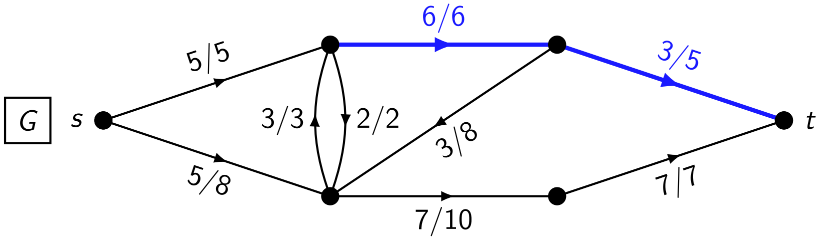
Worked example

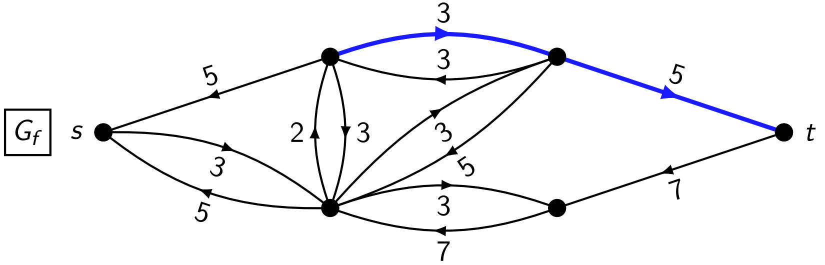




Update G_f along the path.

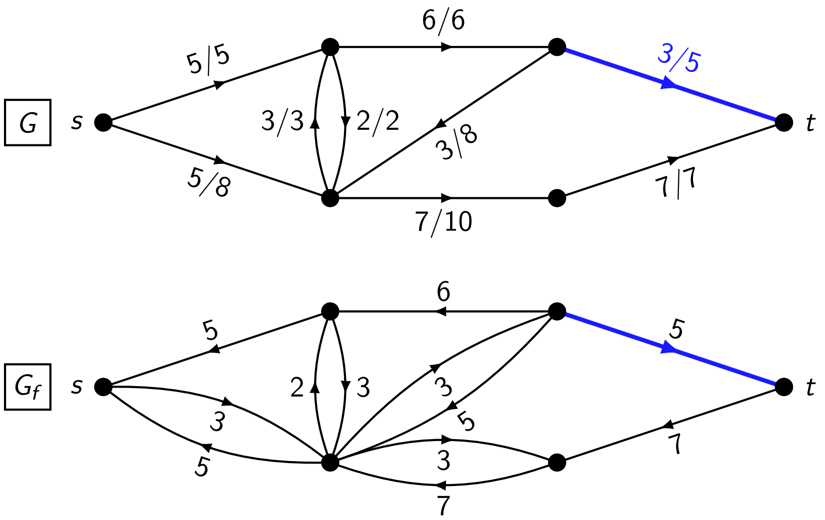
Worked example





Update G_f along the path.

Worked example



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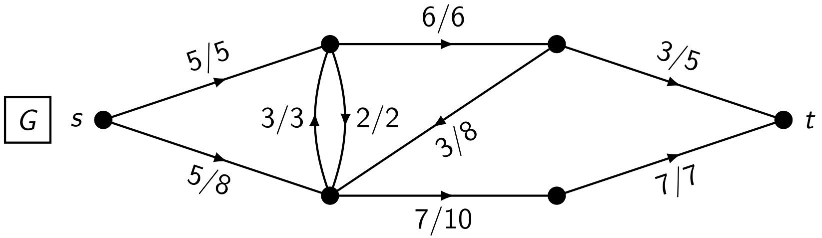

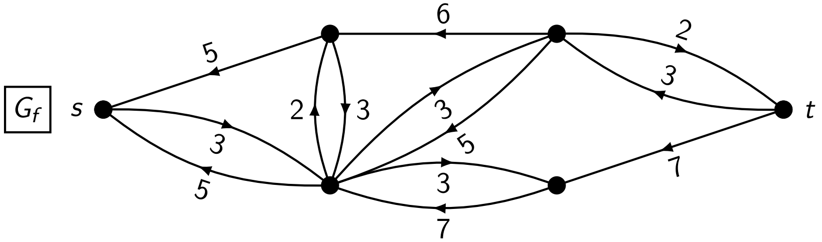

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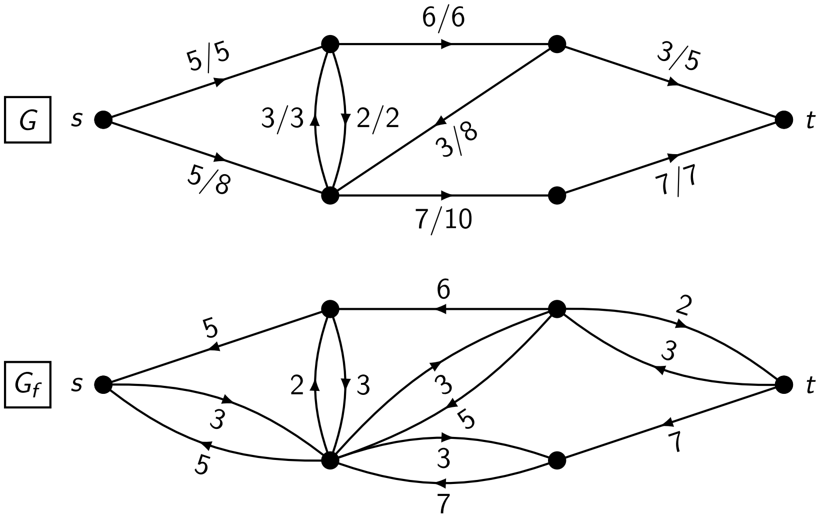
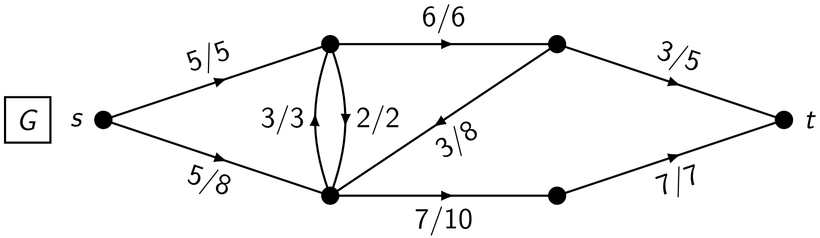
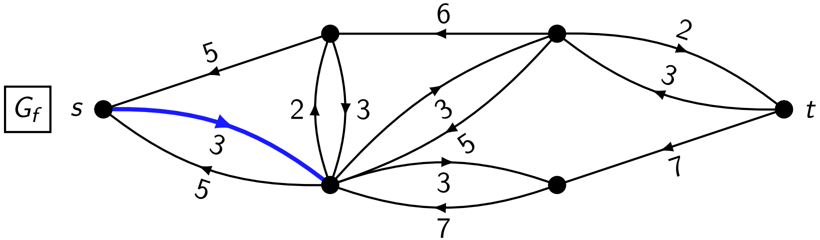


Apply depth-first search to find an augmenting path in G_f.

Worked example





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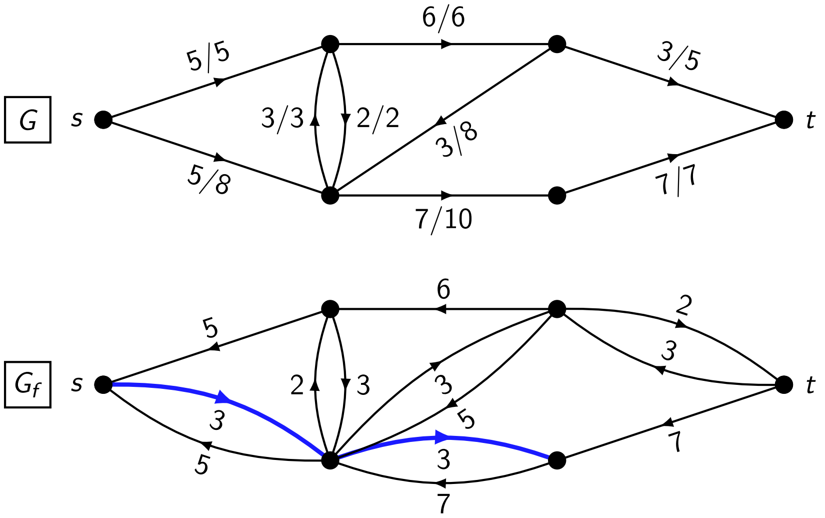

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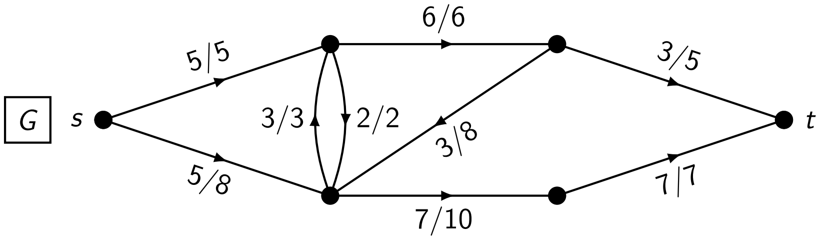

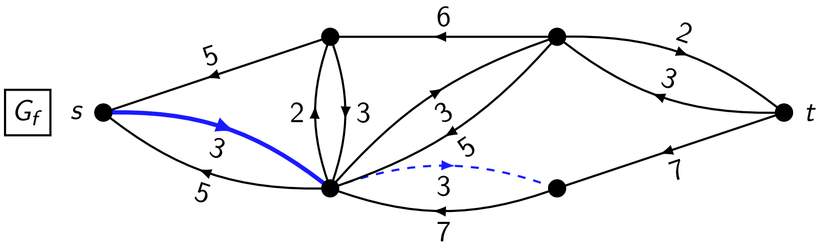

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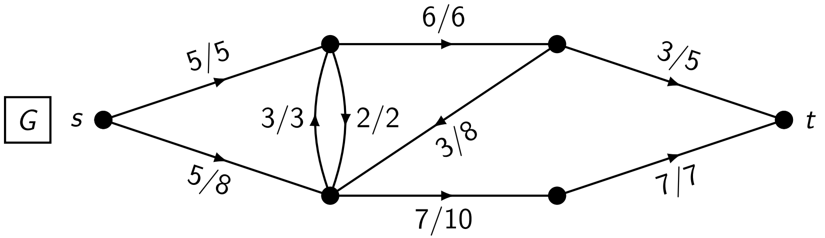

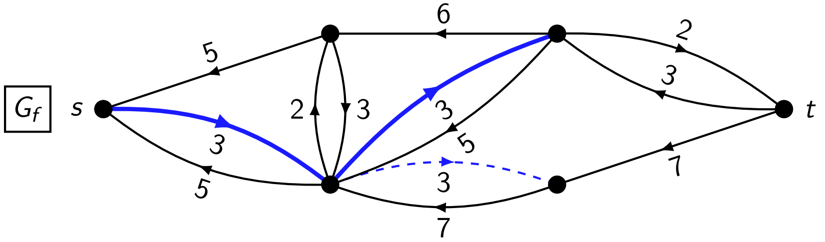

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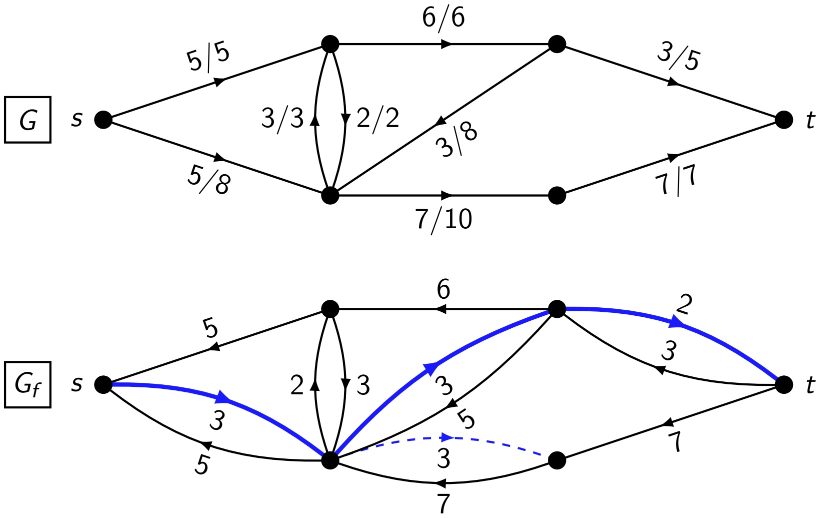

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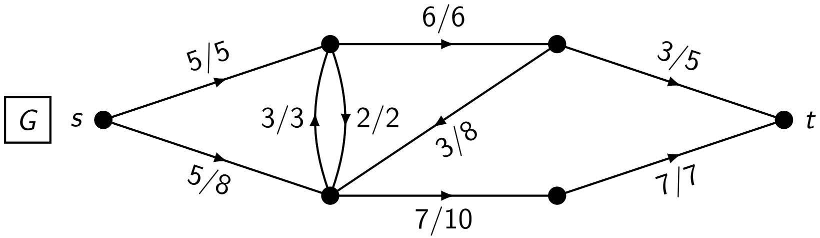

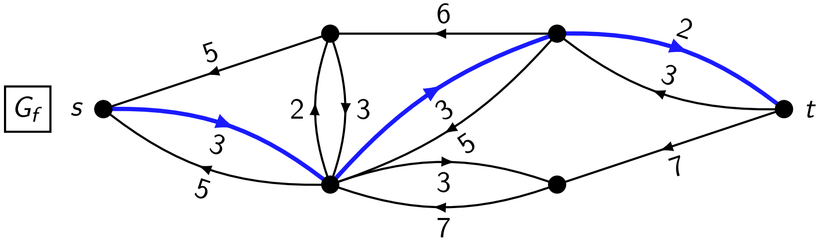

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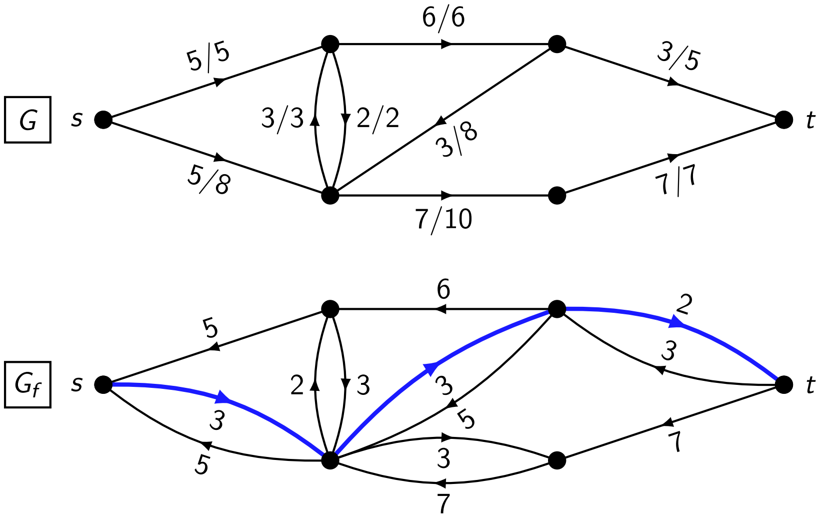
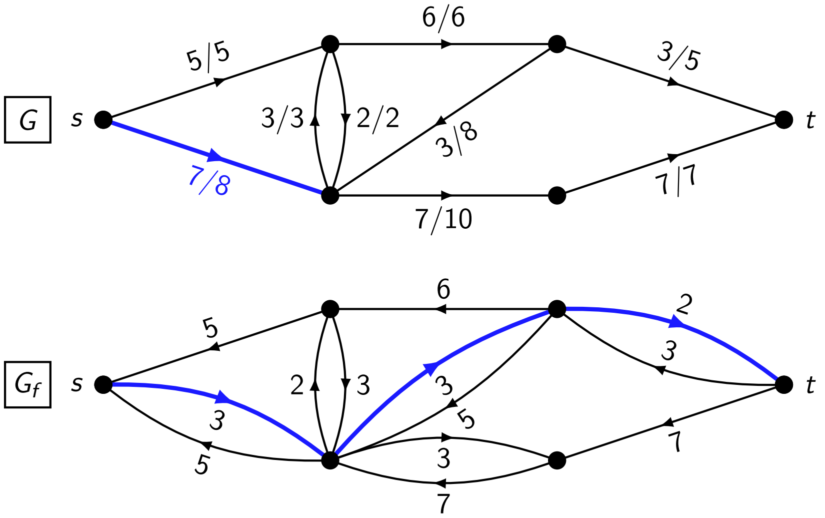


Push flow along the path. (This path has residual capacity 2.)

Worked example



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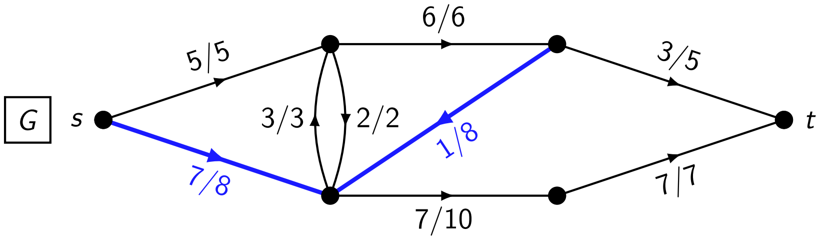

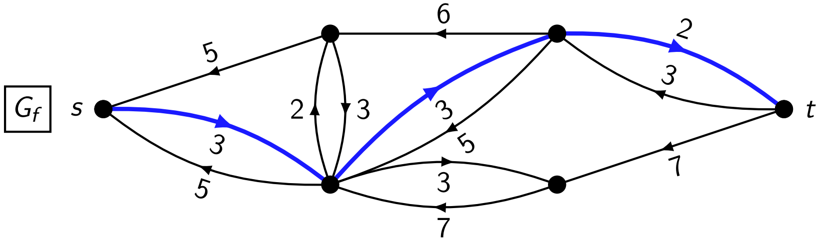

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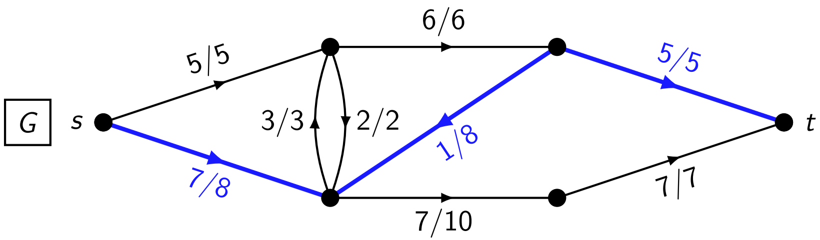

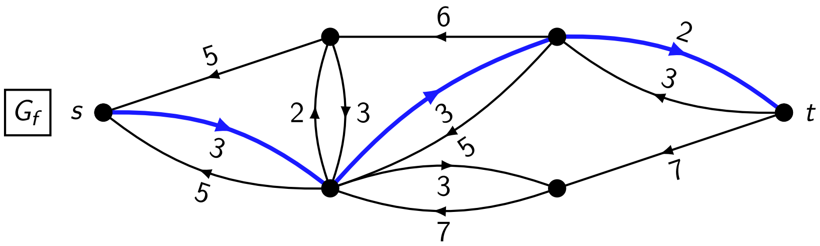

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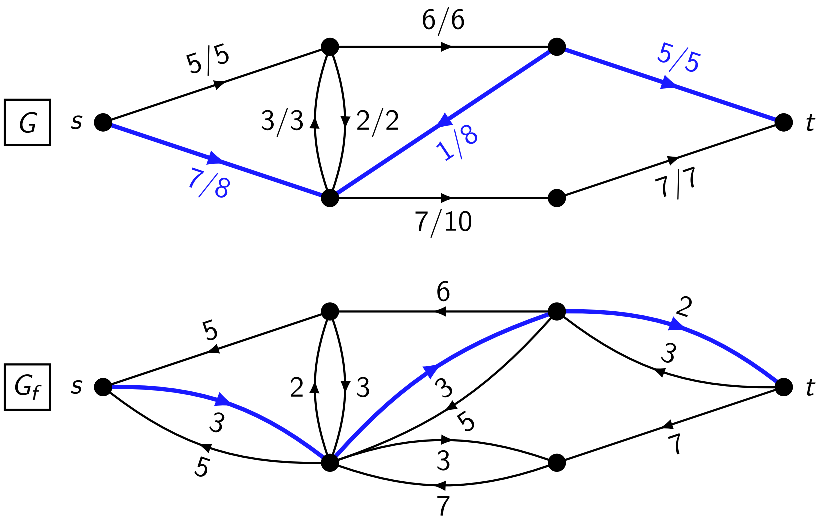

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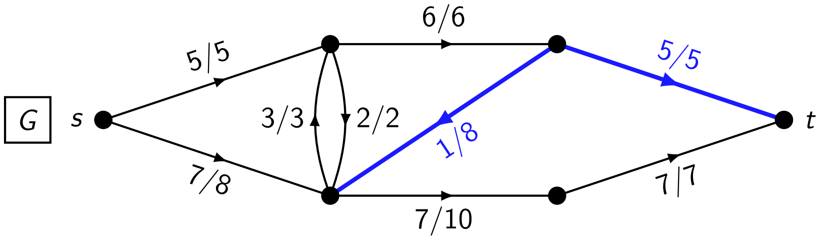

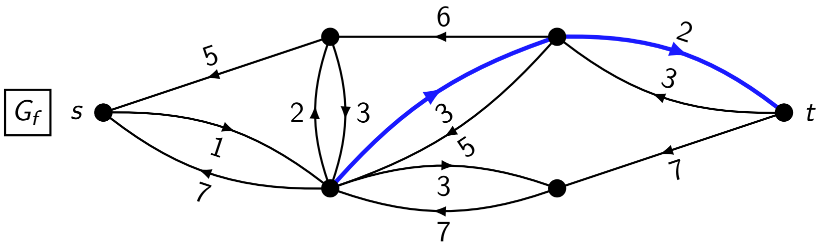

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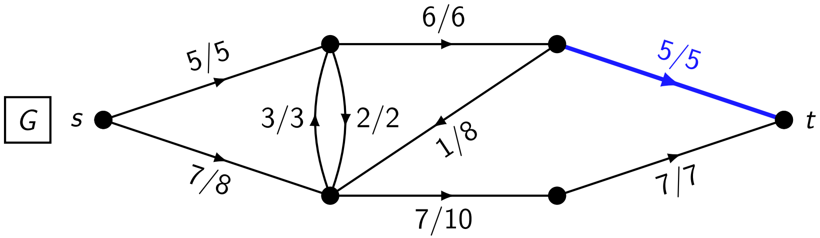

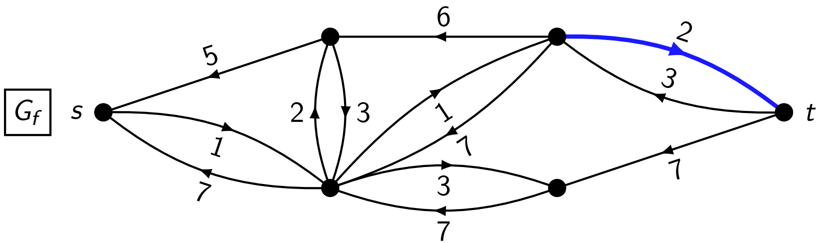

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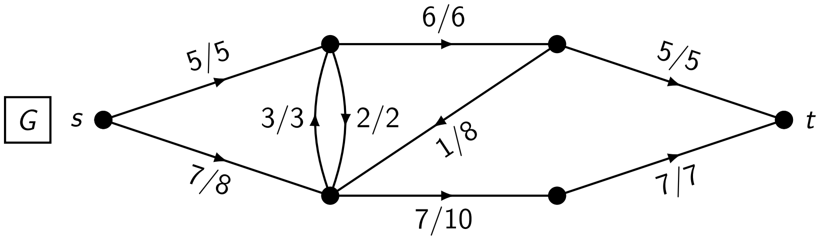

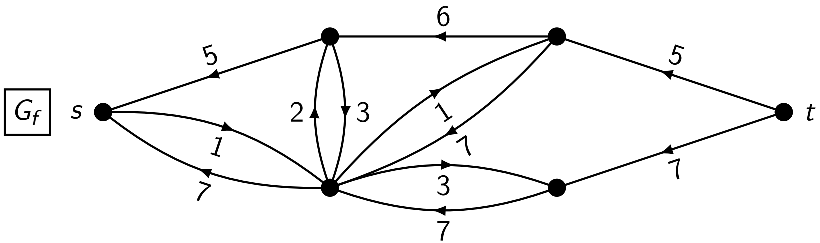

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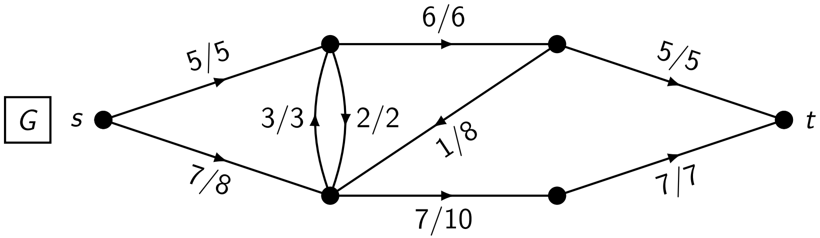

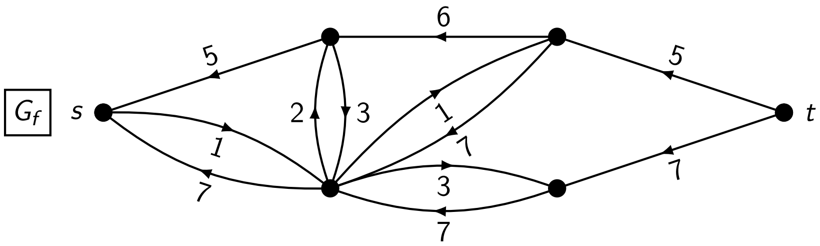

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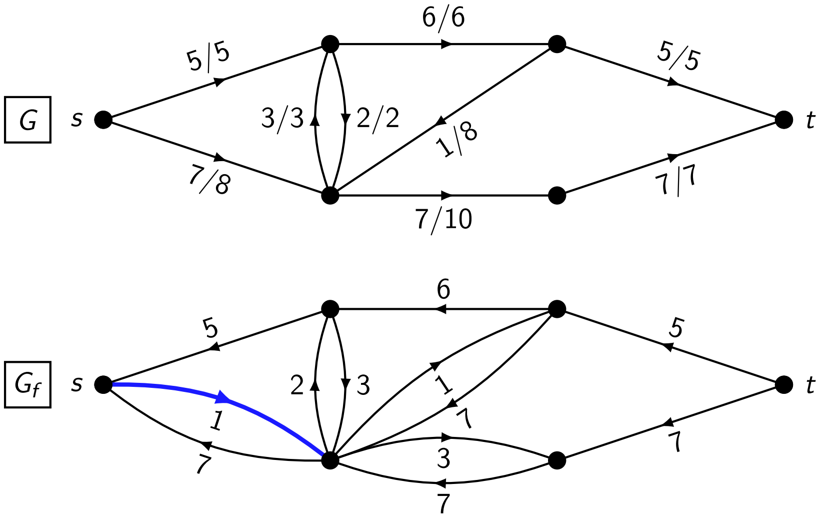

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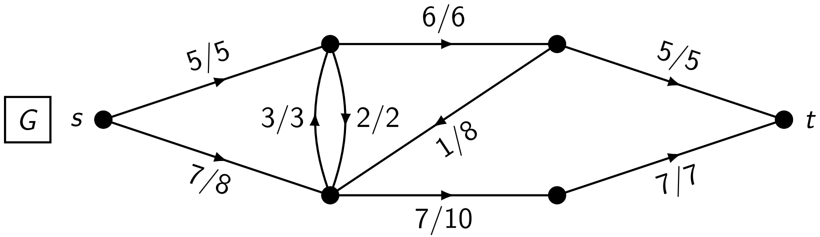

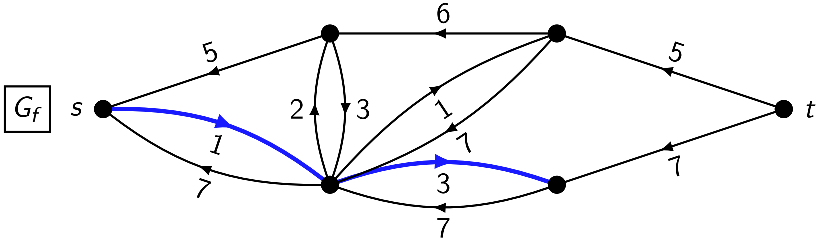

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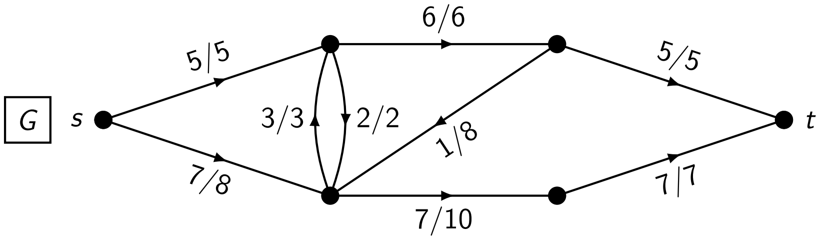

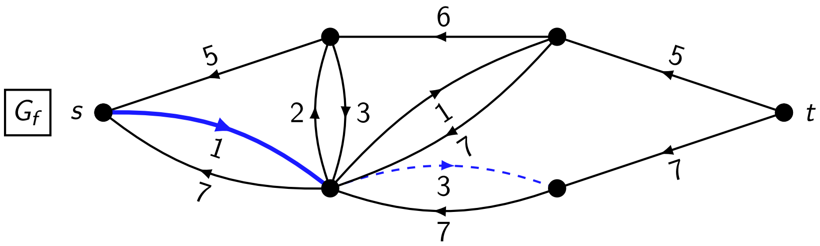

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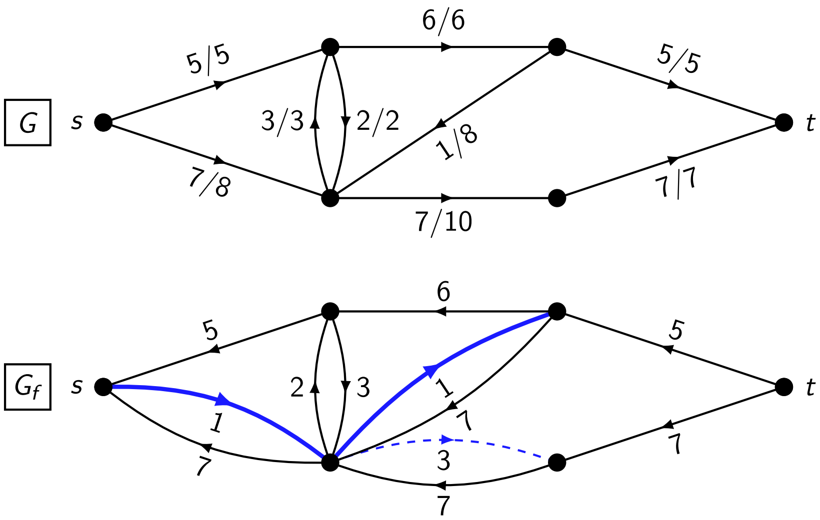

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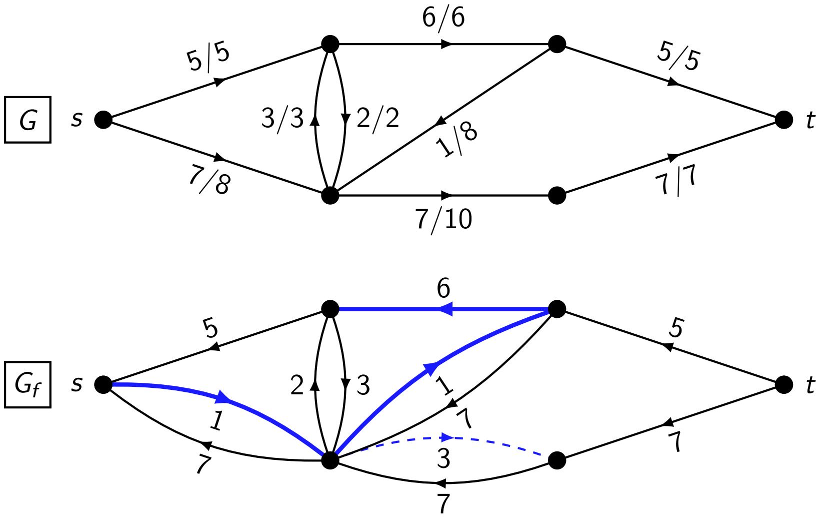

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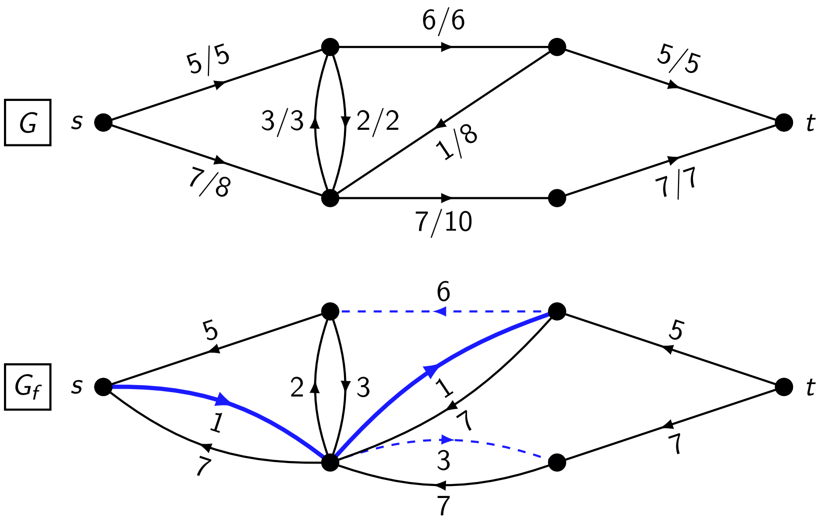

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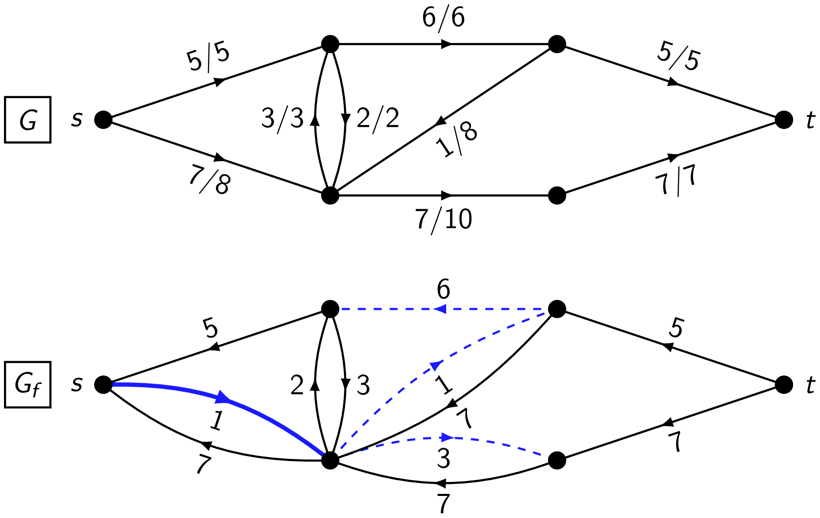

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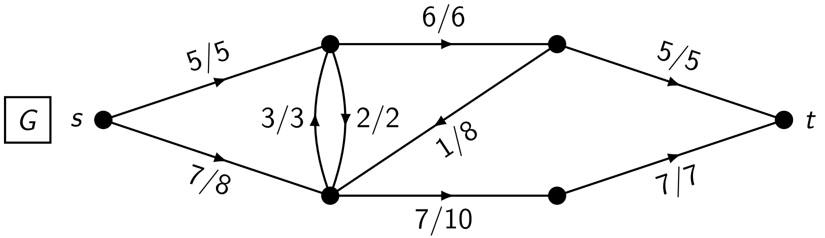

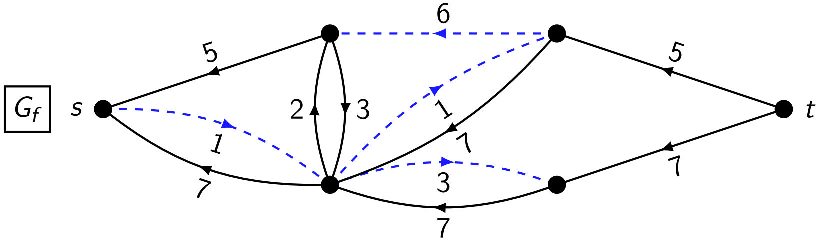

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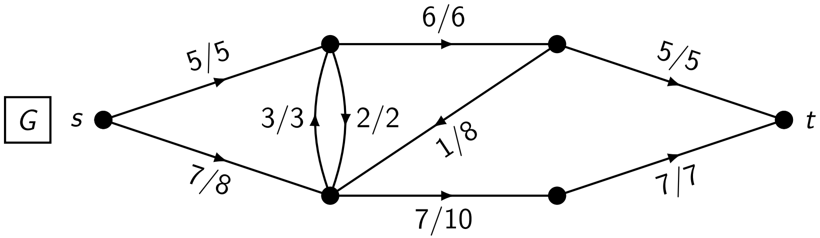
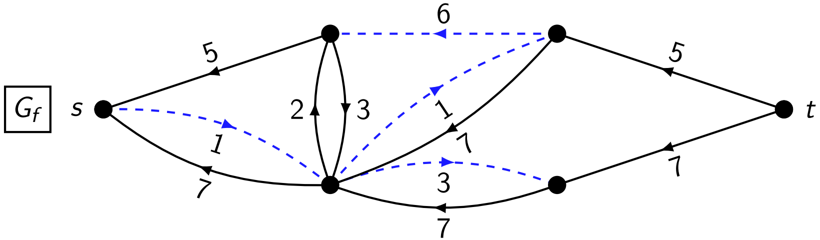


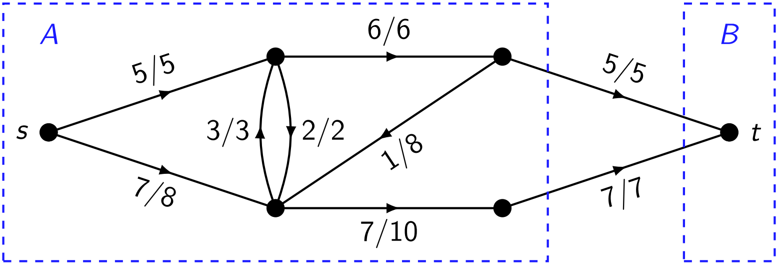


No such path exists, so we're done! This flow has value $5 + 7 = 12$.

Why does this work?

A **cut** is any pair of disjoint sets $A, B \subseteq V$ with $A \cup B = V$, $s \in A$ and $t \in B$. (So A and B partition V, the source is in A and the sink is in B.)

Lemma 2: For all cuts (A, B), $v(f) = f^+(A) - f^-(A) = f^-(B) - f^+(B)$.



Write $c^+(A) = \sum_{e \text{ out of } A} c(e)$. By Lemma 2, **any** flow g has value $v(g) = g^+(A) - g^-(A) \leq c^+(A) = 12$, and our output flow has value 12. So it must be maximum.

We can use the same argument to prove Ford-Fulkerson always works.

Lemma 2: For all cuts (A, B), $v(f) = f^+(A) - f^-(A) = f^-(B) - f^+(B)$.

To prove the flow f returned by Ford-Fulkerson is **always** maximum by this argument, we will show there is **always** a cut (A, B) with $v(f) = c^+(A)$, i.e. with $f^+(A) = c^+(A)$ and $f^-(A) = 0$.

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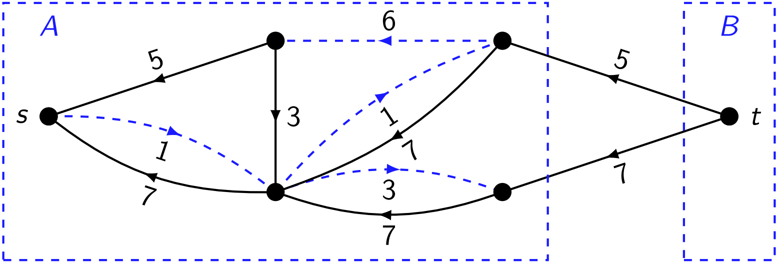

We take $A = \{v \in V(G)\colon v \text{ reachable from } s \text{ in } G_f\}$, and $B = V(G) \setminus A$.

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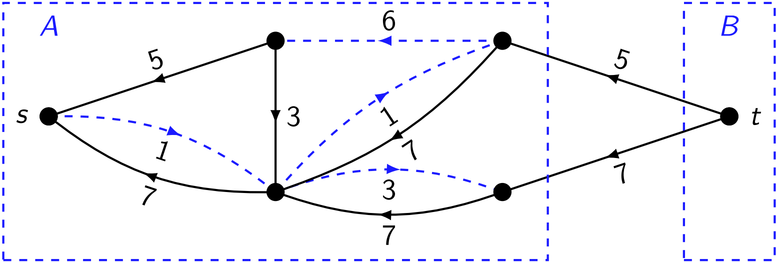

We take $A = \{v \in V(G) \colon v \text{ reachable from } s \text{ in } G_f\}$, and $B = V(G) \setminus A$.

(A, B) is a cut: $s \in A$, and $t \notin A$ since f has no augmenting paths. $\checkmark$

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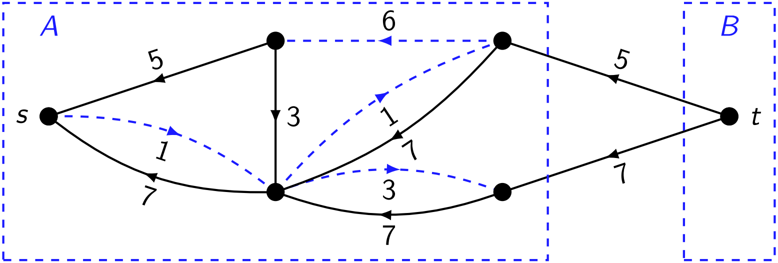

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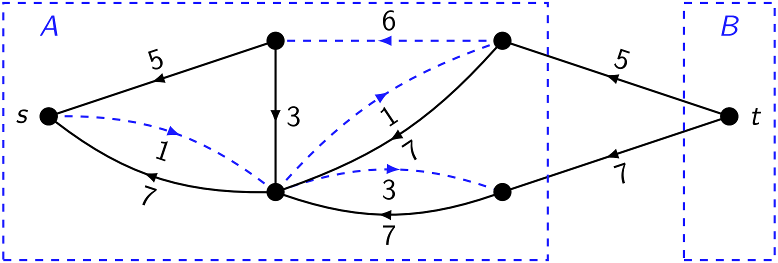

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$f^+(A) = c^+(A)$: No $A \to B$ forward edges in G_f $\Rightarrow$ every $A \to B$ edge in G is filled to capacity $\Rightarrow f^+(A) = c^+(A)$. ✓

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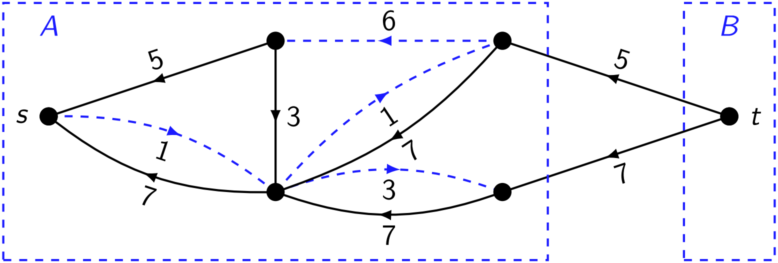

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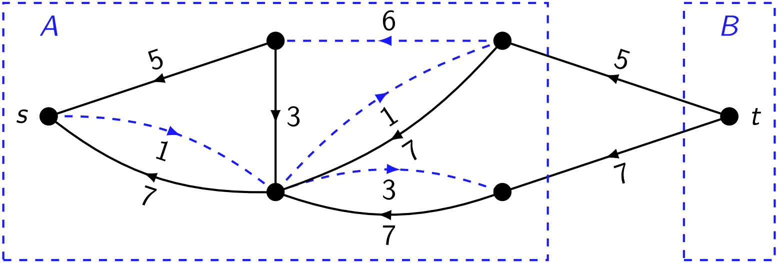

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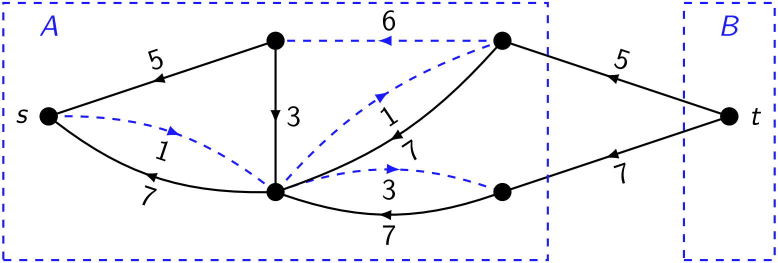

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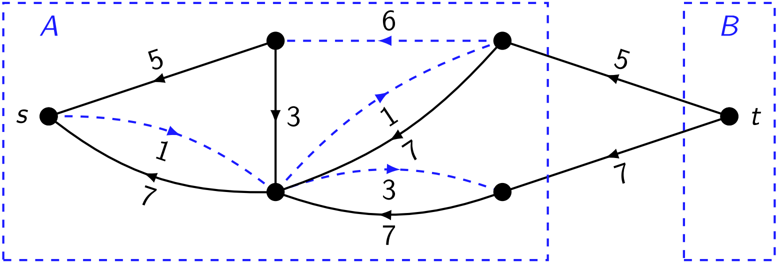

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So by Lemma 2, every other flow g has value $g^+(A) - g^-(A) \leq c^+(A) = v(f)$. Thus f is a maximum flow and Ford-Fulkerson is correct. □

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Max-flow min-cut theorem: The value of a maximum flow is equal to the minimum capacity of a cut, i.e. the minimum value of $c^+(A)$ over all cuts (A, B).

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Theorem: Ford-Fulkerson always returns a maximum flow. $\square$

Theorem: There is always a maximum flow with integer values.

Proof: The maximum flow returned by Ford-Fulkerson has this property. (We can prove this easily with a loop invariant: f starts with value zero, and each iteration of the main loop adds an integer to f's value.) $\square$

Max-flow min-cut theorem: The value of a maximum flow is equal to the minimum capacity of a cut, i.e. the minimum value of $c^+(A)$ over all cuts (A, B).

Proof: Let f be a maximum flow, and let (A, B) be a cut minimising $c^+(A)$. We already proved $v(f) \leq c^+(A)$. Moreover, there is no augmenting path for f, so exactly as before, there is a cut (A', B') with $c^+(A') = v(f)$; thus $v(f) \geq c^+(A)$. The result follows. $\square$