Inductive Invariants

A set P of configurations is an **inductive invariant** just if:

- $I \subseteq P$ – base
- and, for all c, $d \in C$: if $c \in P$ and $c \Rightarrow d$ then $d \in P$ too. – step

The **chameleons example** transition system consists of the following data:

- The configurations consist of all possible triples (r, b, g) of natural numbers.
- The transition relation has $(r, b, g) \Rightarrow (r', b', g')$ just if any one of the following is true:
  - $r' = r + 2$ and $b' = b - 1$ and $g' = g - 1$
  - $r' = r - 1$ and $b' = b + 2$ and $g' = g - 1$
  - $r' = r - 1$ and $b' = b - 1$ and $g' = g + 2$

$$P := \left\{ (r,b,g) \;\middle|\; \begin{array}{l} r-b \% 3 = 2 \\ b-g \% 3 = 2 \\ r-g \% 3 = 1 \end{array} \right\}$$

Base

In our example $I = \{(2,3,4)\}$ so we need to show $(2,3,4) \in P$:

$$2-3 \% 3 = -1 \% 3 = 2$$

$$3-4 \% 3 = -1 \% 3 = 2$$

$$2-4 \% 3 = -2 \% 3 = 1$$

Step We have to show if $c \in P$ and $c \Rightarrow d$ then $d \in P$.

Assume (i) $(r,b,g) \in P$

(ii) $(r,b,g) \Rightarrow (r',b',g')$

We need to show $(r',b',g') \in P$.

From (i) we can deduce:

(iii)
$$r-b \% 3 = 2$$
$$b-g \% 3 = 2$$
$$r-g \% 3 = 1$$

From (ii) and the def^n of $\Rightarrow$ we know that one of the following must be true:

(1) $r' = r+2, \quad b' = b-1, \quad g' = g-1$

or (2) $r' = r-1, \quad b' = b-1, \quad g' = g+2$

or (3) $r' = r-1, \quad b' = b+2, \quad g' = g-1$

We will have to show that $(r', b', g') \in P$ in all cases.

- In case the step is by (1)

$$\begin{aligned} r'-b' \% 3 &= (r+2)-(b-1) \% 3 \\ &= r-b+3 \% 3 \\ &= r-b \% 3 \\ \text{by (iii)} \quad &= 2 \end{aligned}$$

$$
\begin{aligned}
b' - g' \,\%\, 3 &= (b-1) - (g-1) \,\%\, 3 \\
&= b - g \,\%\, 3 \\
\text{by (iii)} \quad &= 2
\end{aligned}
$$

$$
\begin{aligned}
r' - g' \,\%\, 3 &= (r+2) - (g-1) \,\%\, 3 \\
&= r - g + 3 \,\%\, 3 \\
&= r - g \,\%\, 3 \\
\text{by (iii)} \quad &= 1
\end{aligned}
$$