

Breadth-first search

COMS20010 2020, Video 4-3

John Lapinskas, University of Bristol

Shortest path-finding

Last time: Given a graph G and two vertices $x, y \in V(G)$, is there a path from x to y ?

E.g. can an enemy attack the base without breaking down a wall?



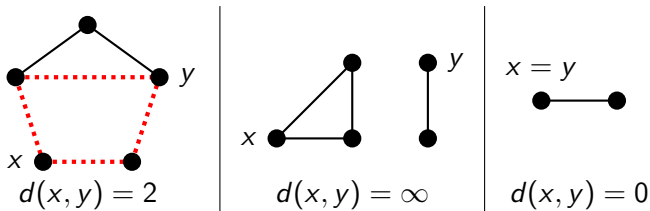
This time: What is the **shortest** path from x to y ?

Graph distance

This time: What is the **shortest** path from x to y ?

What do we mean by “shortest”?

The **distance** between x and y , $d(x, y)$, is the length in edges of a shortest path between x and y , or ∞ if no such path exists.

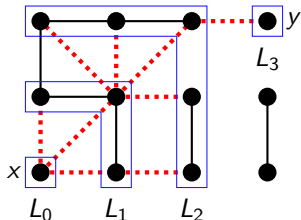


In directed graphs, it's the same except that the path is **from** x **to** y . So... we might not have $d(x, y) = d(y, x)$!

Breadth-first search: The idea

Input: A graph G and two vertices x and y .

Output: A shortest path from x to y .



Let L_i be the set of vertices at distance i from x . So $L_0 = \{x\}$.

L_1 is everything adjacent to x .

L_2 is everything adjacent to L_1 , but **not** in L_0 or L_1 .

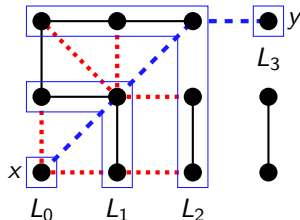
In general, L_{i+1} is everything adjacent to L_i and not in $L_0 \cup \dots \cup L_i$.

By continuing this until we find y , keeping track of which edges we use, we get a shortest path to y .

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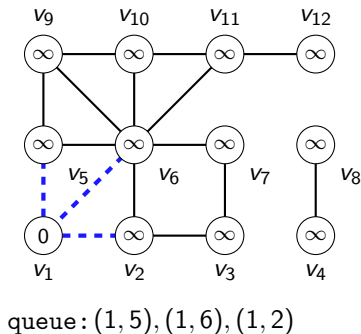
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Breadth-first search: Implementation



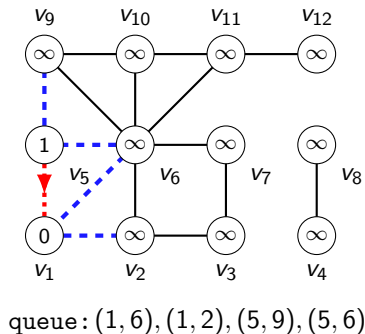
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Breadth-first search: Implementation



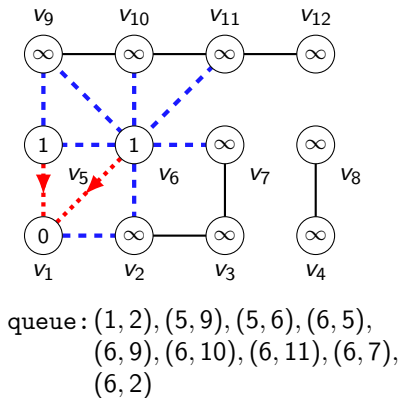
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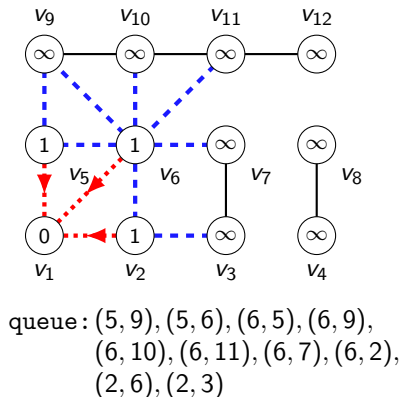
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# Breadth-first search: Implementation



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## Algorithm: BFS

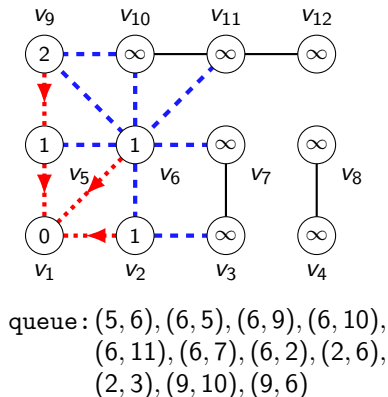
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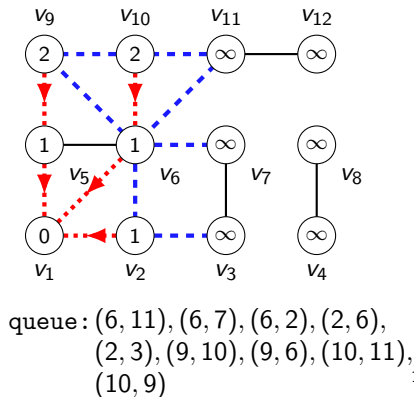
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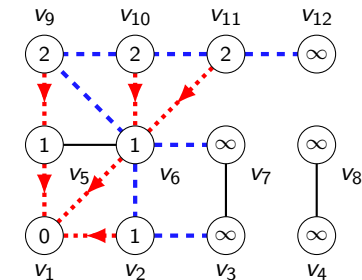
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# Breadth-first search: Implementation



queue: (6, 7), (6, 2), (2, 6), (2, 3),  
 (9, 10), (9, 6), (10, 11), (10, 9),  
 (11, 10), (11, 12)

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## Algorithm: BFS

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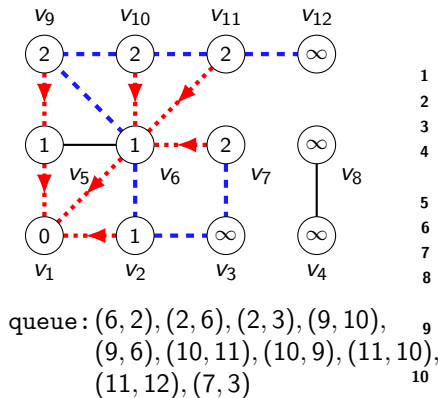
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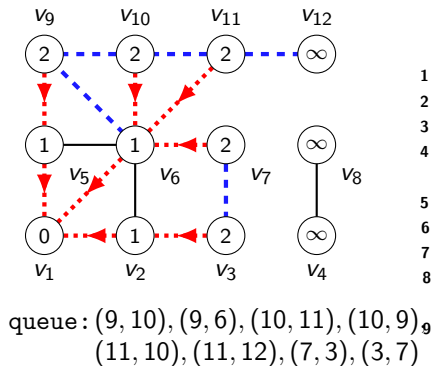
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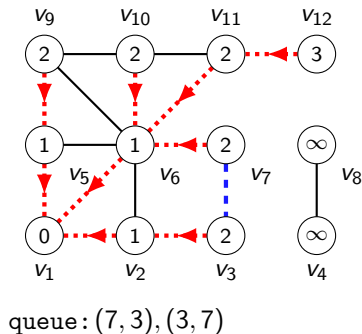
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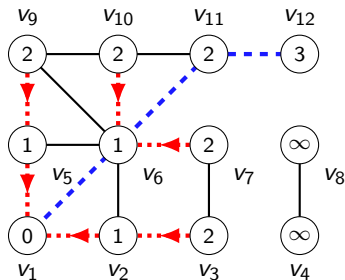
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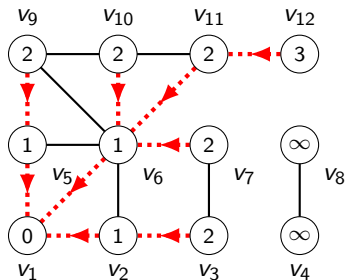
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In the output,  $L[i] = d(v, v_i)$ . By following edges back from  $v_i$  via  $\text{pred}$ , we can also quickly reconstruct a shortest path from  $v$  to  $v_i$ .

E.g.  $v_1 v_6 v_{11} v_{12}$  is a shortest path from  $v_1$  to  $v_{12}$ .



# Breadth-first search: Implementation



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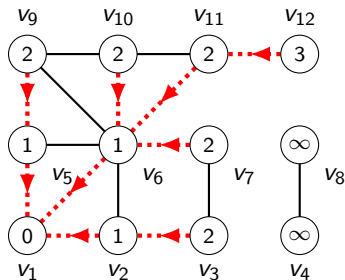
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**Time analysis:** If  $G$  is in adjacency list form, each edge is added to queue at most twice, incurring  $O(1)$  overhead each time, so the running time is  $O(|V| + |E|)$ .

# Breadth-first search: Implementation



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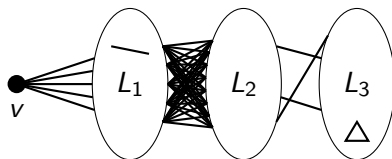
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**Important:** There is a significant **space** inefficiency in this version of breadth-first search! See example sheet.

**Definition:** A **BFS tree**  $T$  of  $G$  is a rooted tree (call its root  $x$ ) with:

- 1  $V(T)$  is the vertex set of a component of  $G$ ;
- 2 The  $i$ 'th layer of  $T$  is  $\{x: d_G(x, v) = i\}$ ;
- 3 If  $\{x, y\} \in E(G)$ , then  $|d_G(v, x) - d_G(v, y)| \leq 1$ , i.e.  $x$  and  $y$  must be in the same or adjacent layers of  $T$ .



**Theorem:** The tree of edges from  $\text{pred}$  is always a BFS tree.

**Proof:** We already proved (1) and (2), so suppose  $\{x, y\} \in E(G)$ .

If  $P$  is a shortest path from  $v$  to  $x$ , then  $P_{xy}$  is a path from  $v$  to  $y$ , so  $d(v, y) \leq d(v, x) + 1$ . Likewise  $d(v, x) \leq d(v, y) + 1$ . ✓