COMS20010 — Problem sheet 3

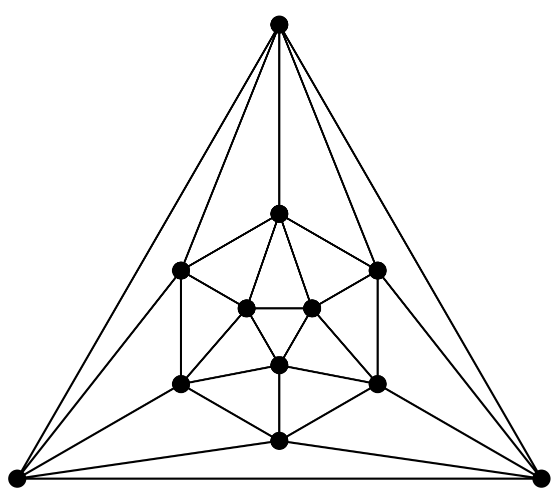
You don't have to finish the problem sheets before the class — focus on understanding the material the problem sheet is based on. If working on the problem sheet is the best way of doing that, for you, then that's what you should do, but don't be afraid to spend the time going back over the quiz and videos instead. (Then come back to the sheet when you need to revise for the exam!) I'll make solutions available shortly after the problem class. As with the Blackboard quizzes, question difficulty is denoted as follows:

- ⋆ You'll need to understand facts from the lecture notes.
- ⋆⋆ You'll need to understand and apply facts and methods from the lecture notes in unfamiliar situations.
- ⋆⋆⋆ You'll need to understand and apply facts and methods from the lecture notes and also move a bit beyond them, e.g. by seeing how to modify an algorithm.
- ⋆⋆⋆⋆ You'll need to understand and apply facts and methods from the lecture notes in a way that requires significant creativity. You should expect to spend at least 5–10 minutes thinking about the question before you see how to answer it, and maybe far longer. Only 20% of marks in the exam will be from questions set at this level.
- ⋆⋆⋆⋆⋆ These questions are harder than anything that will appear on the exam, and are intended for the strongest students to test themselves. It's worth trying them if you're interested in doing an algorithms-based project next year — whether you manage them or not, if you enjoy thinking about them then it would be a good fit.

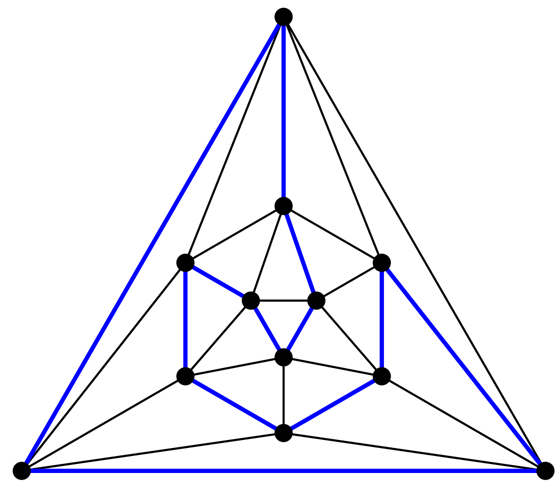
If you think there is an error in the sheet, please post to the unit Team — if you're right then it's much better for everyone to know about the problem, and if you're wrong then there are probably other people confused about the same thing.

This problem sheet covers week 3, focusing on directed graphs, Hamilton cycles, the handshaking lemma, and trees.

1. [⋆⋆] Find a Hamilton cycle in the graph of the icosahedron, drawn below:



Solution: One Hamilton cycle is given by:



2. (a) [⋆] State the handshaking lemma for undirected graphs.

Solution: Let G be a graph; then $\sum_{v \in V(G)} d(v) = 2|E(G)|$.

(b) [⋆⋆] Let G be a graph with six vertices of degree five and two vertices of degree six. How many edges does G have?

Solution: We have $\sum_{v \in V(G)} d(v) = 6 \cdot 5 + 2 \cdot 6 = 42$, so G has 21 edges by the handshaking lemma.

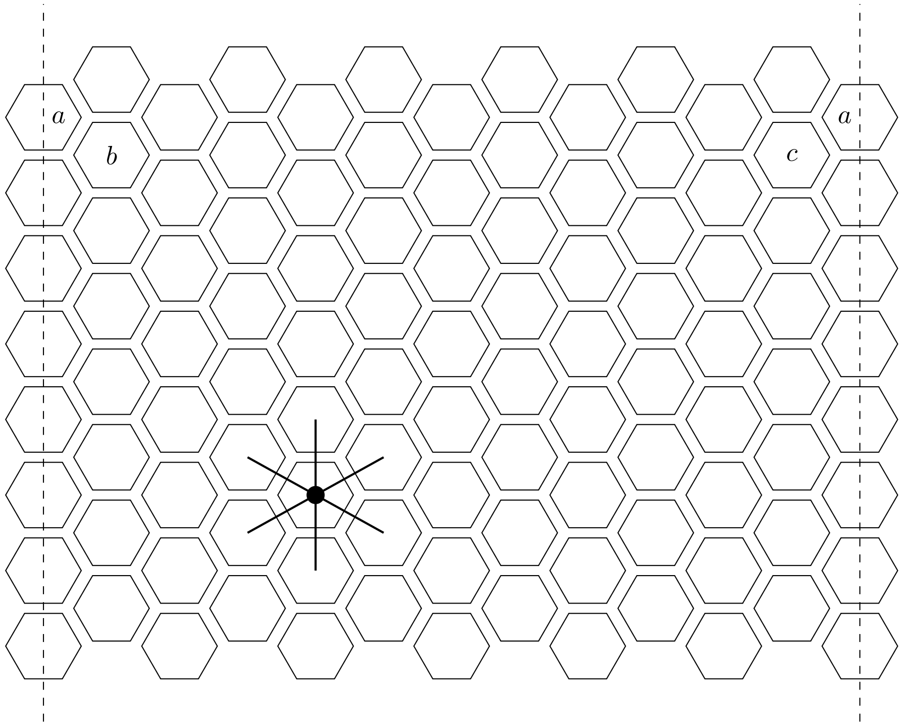
(c) [⋆⋆] In the distant future, the renowned Sol Ballet Corps is putting on a production to emphasise a spirit of love and togetherness between human colonies throughout the solar system. They have 4 dancers from Earth, 5 from Mars, 6 from the moon, and 7 from Ceres. They wish to arrange a procession in which every pair of dancers from different worlds crosses the stage exactly once. For example, one dancer from Earth will cross alongside one dancer from Mars, but not a pair of dancers both from Earth. How many pairs of dancers will need to cross the stage in total?

Solution: Model this as a graph G in which the vertices are dancers, and each pair of dancers from different worlds is an edge — thus the question is asking for the total number of edges in the graph. Dancers from Earth have degree 18, dancers from Mars have degree 17, dancers from the moon have degree 16, and dancers from Ceres have degree 15. The pairs of dancers that cross the stage will be precisely the edges of the graph, and by the handshaking lemma we have

$$|E(G)| = \frac{1}{2} \sum_{v \in V(G)} d(v) = \frac{1}{2}\Big(4 \cdot 18 + 5 \cdot 17 + 6 \cdot 16 + 7 \cdot 15\Big) = 179.$$

This question has appeared in a past exam (with a hint to use the handshaking lemma).

(d) [⋆⋆] The strategy game Civilization VI is played on a hexagonal map as shown below. The map is effectively a cylinder, with the left and right boundaries are joined together, and the top and bottom boundaries are impassable — thus in the picture below, tile a appears twice, and is adjacent to both tile b and tile c. Units can move between any pair of adjacent tiles, and this is stored in memory as a graph in which there is an edge between two tiles if they are adjacent in the map.



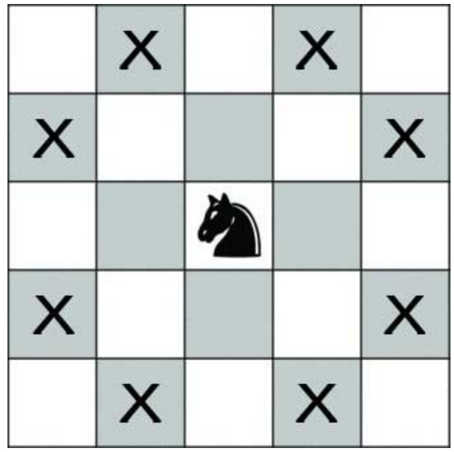
If the map is 90 tiles wide and 56 tiles high, how many edges does the graph have?

Solution: By the handshaking lemma, the total number of edges is half the total degree of the graph. There are 90 tiles of degree 5 and 90 tiles of degree 3, with 45 of each along the top edge and 45 of each along the bottom edge. The remaining $90 \cdot 56 - 90 \cdot 2$ tiles all have degree 6. Thus the total number of edges is

$$\frac{90 \cdot 54 \cdot 6 + 90 \cdot 5 + 90 \cdot 3}{2} = 14940.$$

This question has appeared in a past exam, as a multiple choice question with a hint to use the handshaking lemma.

(e) [$\star\star$ but long] In lectures, we mentioned that Hamilton cycles were first studied in the context of knights' tours. The graph of an $n \times n$ knight's tour is defined as follows: the vertex set is $[n] \times [n]$, and we join (i, j) to (k, ℓ) if either $|i - k| = 1$ and $|j - \ell| = 2$ or vice versa. (In other words, viewing the vertices as squares in $n \times n$ grid, we join two squares if and only if a knight in chess can move from one to the other — see the figure below.) Prove that when $n \geq 2$, an $n \times n$ knight's tour has $4(n-1)(n-2)$ edges.



Solution: This is a prime example of a graph where it's easiest to count edges using the handshaking lemma, since it's easy to find the degree of any particular vertex but hard to visualise the graph as a whole. Let G be an $n \times n$ knight's tour graph. Let's assume $n \geq 5$, since we can just count the edges by hand for $n \leq 4$. For a given vertex (i, j):

(i) if $3 \leq i,\ j \leq n-2$, then (i, j) has degree 8;

(ii) if $i = 2$ and $3 \leq j \leq n-2$, or vice versa, then (i, j) has degree 6;

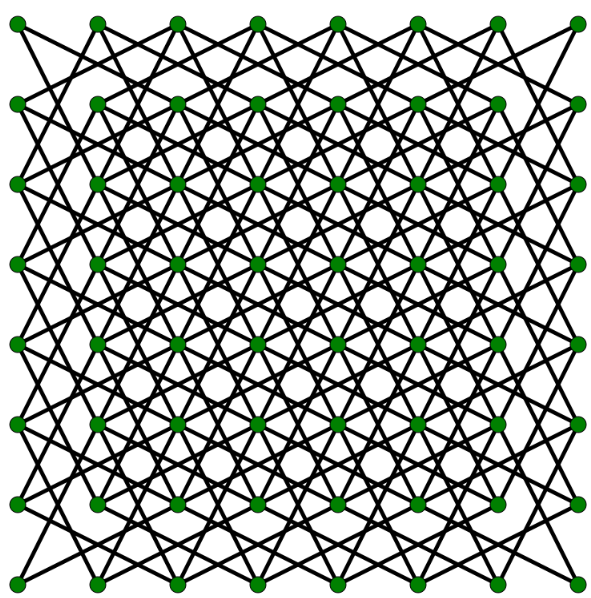
(iii) if $i = 1$ and $3 \leq j \leq n-2$, or vice versa, then (i, j) has degree 4;

(iv) if $i,\ j \in \{2, n-1\}$, then (i, j) has degree 4;

(v) if $i \in \{2, n-1\}$ and $j \in \{1, n\}$, then (i, j) has degree 3;

(vi) if $i,\ j \in \{1, n\}$, then (i, j) has degree 2.

This is illustrated in the figure below for an 8×8 graph.



In total, there are: $(n-4)^2$ vertices of type (i); $4(n-4)$ vertices of type (ii); $4(n-4)$ vertices of type (iii); 4 vertices of type (iv); 8 vertices of type (v); and 4 vertices of type (vi). In total, we therefore have

$$\begin{aligned}\sum_{v \in V} d(v) &= (n-4)^2 \cdot 8 + 4(n-4) \cdot 6 + 4(n-4) \cdot 4 + 4 \cdot 4 + 8 \cdot 3 + 4 \cdot 2\\ &= 8n^2 - 24n + 16 = 8(n-1)(n-2).\end{aligned}$$

It follows by the handshaking lemma that

$$|E(G)| = \frac{1}{2} \sum_{v \in V} d(v) = 4(n-1)(n-2).$$

This question is longer, and with more chances to make a mistake, than anything I'd put on the exam. But conceptually the difficulty is [$\star\star$], since it's "just" an application of the handshaking lemma.

3. [$\star\star$] Let G be a graph.
   (a) Show that G is a tree if and only if G is connected, but removing any edge will disconnect it.
   (b) Show that G is a tree if and only if G has no cycles, but adding any edge will create a cycle.

Solution: There are **many** possible approaches to this question! Here's one.

(a) Suppose G is an n-vertex tree. By the fundamental lemma of trees, G has $n-1$ edges and no cycles. On removing any edge from G, we obtain a new graph G' which has $n-2$ edges and no cycles. Again by the fundamental lemma, all trees have $n-1$ edges, so G' is not a tree; hence G' is disconnected.

Conversely, suppose G is connected, but removing any edge will disconnect it. As proved in lectures, if G contained a cycle, then we could remove any edge from that cycle without disconnecting G; hence G contains no cycles. So G is a tree by definition.

(b) Suppose G is an n-vertex tree, and let G' be the graph formed by adding some edge $\{u, v\}$. By the fundamental lemma of trees, G has a unique path $ux_1 \dots x_k v$ from u to v; then $ux_1 \dots x_k vu$ is a cycle in G', so G' is not a tree.

Conversely, suppose G has no cycles, but that adding any edge will create a cycle. If G were disconnected, then we could add an edge between two of its components without creating a cycle; thus G must be connected, and hence a tree.

4. [$\star\star$] Let T be a tree whose maximum degree is at least $k \geq 2$. Prove that T has at least k leaves. (This extends the result proved in lectures that any tree with at least two vertices has at least 2 leaves.)

Solution: You can directly adapt the proof used in lectures that any tree has at least two leaves. Let x be the number of leaves in T. By the handshaking lemma and the fundamental lemma of trees, writing $n = |V(T)|$,

$$\frac{1}{2} \sum_{v \in V(T)} d(v) = |E(T)| = n - 1.$$

Since every non-leaf vertex of T has degree at least 2, and since at least one non-leaf vertex has degree at least k, we also have

$$\sum_{v \in V(T)} d(v) \geq 2(n-x-1)+k+x.$$

Writing $2(n-1) \geq 2(n-x-1)+k+x$ and solving for x then yields $x \geq k$, as required.

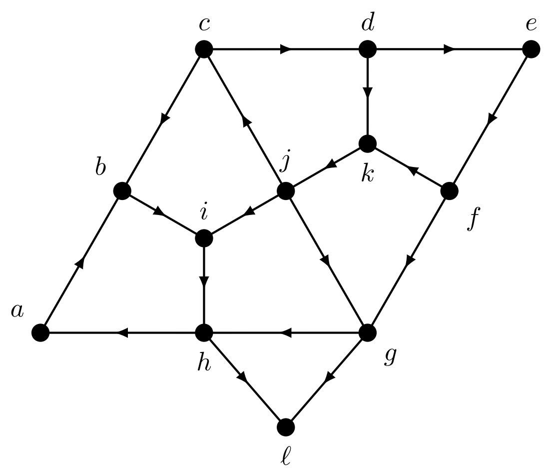
Here's a second way of answering the question. Let $v \in V(T)$ be a vertex of degree at least k, and let $w_1, \dots, w_\Delta$ be the neighbours of v (where $\Delta \geq k$). Since any pair of vertices in a tree is joined by a unique path, the only path between any pair of v's neighbours passes through v itself; thus each vertex w_i lies in a distinct component C_i of $T - v$. Each component C_i is connected (by definition) and acyclic (since T is acyclic), so it is a tree.

If C_i is the single vertex w_i: Then w_i is a leaf of T.

Otherwise: Since C_i is a non-trivial tree it has at least two leaves, so it has at least one leaf not equal to w_i. This is also a leaf of T.

In either case, C_i contains a leaf of T. There are $\Delta \geq k$ components C_i, so the result follows.

5. Let G be the directed graph below.



(a) [⋆⋆] What are $d^+(j)$, $d^-(j)$, $N^+(j)$ and $N^-(j)$?

Solution: We have $N^+(j) = \{c, g, i\}$, $N^-(j) = \{k\}$, $d^+(j) = 3$ and $d^-(j) = 1$.

(b) [⋆⋆] Does G have an Euler walk?

Solution: No. For this to be true, there would need to be exactly zero or two vertices v with $d^+(v) \neq d^-(v)$, and there are far more than that: b, c, d, f, h, i, j, k and ℓ.

(c) [⋆] What does it mean for a graph to be strongly connected?

Solution: A graph G is strongly connected if for any pair of vertices u and v in G, there is a path from u to v. (Equivalently, there is a path from u to v and a path from v to u. Can you tell why this is equivalent?)

(d) [$\star\star$] Is G strongly connected? What are the strong components of G?

Solution: G is not strongly connected. The strong components are $G[\{\ell\}]$, $G[\{a, b, i, h\}]$, $G[\{c, d, e, f, k, j\}]$ and $G[\{g\}]$.

(e) [$\star\star$] List all the cycles in G.

Solution: These are $abiha$, $cdkjc$, $defkd$ and $cdefkjc$. Notice that you can't have a cycle spanning multiple strong components.

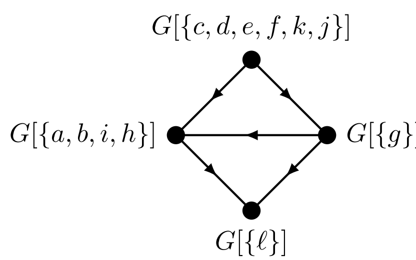
(f) [$\star\star\star$] Show that for **any** graph G, if $X, Y \subseteq V(G)$ such that $G[X]$ is a strong component of G and $G[Y]$ is strongly connected, then either $Y \subseteq X$ or $X \cap Y = \emptyset$.

Solution: For a contradiction, suppose that neither $Y \subseteq X$ nor $X \cap Y = \emptyset$. Then we have $G[X \cup Y] \neq G[X]$, and we claim $G[X \cup Y]$ is strongly connected. If this is true, we have a contradiction, since we have shown that $G[X]$ is not a maximal strongly connected induced subgraph of G (and this is the definition of a strong component).

Let $u, v \in X \cup Y$; we must find a path in $G[X \cup Y]$ from u to v. If $u, v \in X$ or $u, v \in Y$, then this is immediate since $G[X]$ and $G[Y]$ are both strongly connected. Suppose $u \in X$ and $v \in Y$. Since $X \cap Y \neq \emptyset$, there is a vertex $z \in X \cap Y$. Since $G[X]$ and $G[Y]$ are both strongly connected, there is a path in $G[X]$ from u to z and a path in $G[Y]$ from z to v; by joining these paths we find a walk from u to v in $G[X \cup Y]$. Every such walk contains a path from u to v, so we are done. If $u \in Y$ and $v \in X$, then the same argument works, joining a path from u to z in $G[Y]$ to a path from z to v in $G[X]$.

(g) [$\star\star$] Let $\mathrm{c}(G)$ be the following directed graph, sometimes called the *condensation* of G. The vertices of $\mathrm{c}(G)$ are the strong components of G. If C_1 and C_2 are strong components of G, then there is an edge from C_1 to C_2 in $\mathrm{c}(G)$ if and only if there is an edge from some vertex of C_1 to some vertex of C_2 in G. Draw $\mathrm{c}(G)$.

Solution: Here is the resulting graph.



(h) [$\star\star\star$] Show that for **all** directed graphs G, $\mathrm{c}(G)$ contains no cycles.

Solution: For a contradiction, suppose G is a graph such that $\mathrm{c}(G)$ contains a cycle $C_1 \dots C_r C_1$, where $r \geq 2$. We first argue that this corresponds to a cycle in G. By the definition of $\mathrm{c}(G)$, for all $i \in [r-1]$, there must exist vertices $x_i \in V(C_i)$ and $y_i \in V(C_{i+1})$ such that $(x_i, y_i) \in E(G)$; likewise there must exist vertices $x_r \in V(C_r)$ and $y_r \in V(C_1)$ such that $(x_r, y_r) \in E(G)$. Since $C_1, \dots, C_r$ are all strongly connected, for all $i \in [r-1]$ there must be a path P_i from y_i to x_{i+1} in C_{i+1}; likewise there must be a path P_r from y_r to x_1 in C_1. These paths are all vertex-disjoint, since $C_1, \dots, C_r$ are all disjoint by the previous part. Thus $P_1 \dots P_r$ is a cycle in G.

But now we are done by the previous part. Indeed, $P_1 \dots P_r$ is strongly connected (since it's a cycle), it intersects C_1 in P_r, and it intersects C_2 in P_1. Both C_1 and C_2 are strong components of G, and the previous part implies that $V(C_1)$ and $V(C_2)$ are both disjoint, so we have a contradiction.

6. [★★★] Prove that a **directed** graph $G = (V, E)$ with no isolated vertices contains an Euler walk from a vertex v to itself if and only if G is **strongly** connected and every vertex of G has equal in- and out-degrees. **Hint:** Try adapting the proof given in lectures for undirected graphs. (This is actually true for weak connectedness as well — any weakly connected digraph whose vertices have equal in- and out-degrees is strongly connected — but we won't prove this in the unit.)

Solution: If G contains an Euler walk from v to itself, then every vertex of G has equal in- and out-degree as covered in lectures, and G is strongly connected since you can find a walk from u to v for any pair of vertices u and v by following the Euler walk. It remains to prove the "if" direction. Suppose that G is strongly connected, and every vertex of G has equal in- and out-degrees.

Essentially as in the undirected case, we proceed by strong induction on $|E(G)|$. The base case, where $E(G) = \emptyset$, is immediate.

We greedily form a walk $W = w_0 \dots w_k$ by taking $w_0 = v$, taking each w_i to be an arbitrary out-neighbour of w_{i-1} such that (w_{i-1}, w_i) is not already used in W, and stopping when $N^+(w_i) \subseteq \{w_0, \dots, w_{i-1}\}$. Each vertex x has one out-edge in W for each time x appears in $\{w_1, \dots, w_{i-1}\}$, plus one if $x = w_0$, and one in-edge in W for each time x appears in $\{w_1, \dots, w_{i-1}\}$ plus one if $x = w_k$.

Suppose $w_k \neq w_0$, for a contradiction. Since we stopped at w_k, there must be $d^+(w_k)$ out-edges incident to w_k in W, and so w_k appears $d^+(w_k)$ times in $\{w_1, \dots, w_{k-1}\}$. Thus w_k has $d^+(w_k) + 1$ incident in-edges in W. But $d^+(w_k) + 1 = d^-(w_k) + 1 > d^-(w_k)$, so this is impossible. Thus $w_k = w_0 = v$, and we have found a non-trivial directed cycle containing v. (This mirrors the first part of the proof in the undirected setting.)

Now, let the non-trivial strong components of the graph $G - W$ formed by removing W's edges from G be $C_1, \dots, C_r$. Since G is strongly connected, for each i there must be a vertex on W which lies in C_i. Every vertex in $G - W$ has equal in- and out-degrees, and the strong components of $G - W$ are strongly connected by definition, so by induction each C_i contains an Euler walk W_i from v_i to itself. Without loss of generality, v_1 appears on W first, followed by v_2, and so on up to v_r. Then we obtain an Euler walk from v to v in G by following W until reaching v_1, then following W_1 to the end, then following W until reaching v_2, then following W_2, and so on until following W back to v. (This mirrors the second part of the proof in the undirected setting.)

7. [★★★] In lectures we showed that when n is even, Dirac's theorem for n-vertex graphs can't be improved by lowering the degree threshold from $n/2$ to $n/2 - 1$. Prove that when n is odd, Dirac's theorem can't be improved by lowering the degree threshold from $n/2$ to $\lfloor n/2 \rfloor$. (**Hint:** Try considering a graph containing every possible edge between two sets of vertices.)

Solution: Let n be odd, say $n = 2k+1$; then we must find an n-vertex graph with minimum degree at least $\lfloor n/2 \rfloor = k$ and no Hamilton cycle. Let X be a set of k vertices, let Y be a set of $k+1$ vertices, and define a graph G with vertex set $X \cup Y$ and edge set $\{\{x, y\} \colon x \in X,\, y \in Y\}$. Then G has n vertices and minimum degree k. Moreover, suppose for a contradiction that $C = v_1 v_2 \dots v_{2k+1} v_1$ is a Hamilton cycle in G. Without loss of generality, $v_1 \in X$. Then since X and Y don't contain any edges, we must have $v_2 \in Y$, $v_3 \in X$, and so on — in general, we will have $v_i \in X$ if i is odd and $v_i \in Y$ if i is even. In particular, we must have $v_{2k+1} \in X$; but this means v_{2k+1} and v_1 cannot be adjacent, and we have our contradiction.

8. A *tournament* is a directed graph in which every pair of vertices has exactly one edge between them (in one direction) — the name comes from thinking of the graph as a competition, where if the edge between x and y is directed towards y then x beat y in their match.

   (a) [⋆⋆⋆⋆] Prove that any n-vertex tournament contains a vertex of out-degree at least $(n-1)/2$.

   Solution: Any tournament contains exactly $\binom{n}{2}$ edges, so by the handshaking lemma we have

   $$\sum_{v \in V} d^+(v) = \binom{n}{2}.$$

   Suppose every vertex had degree strictly less than $(n-1)/2$. Then we would have

   $$\sum_{v \in V} d^+(v) < n \cdot \frac{n-1}{2} = \binom{n}{2},$$

   which is a contradiction.

   (b) [⋆⋆⋆⋆⋆] An *out-dominating set* in a digraph $G = (V, E)$ is a set $X \subseteq V$ such that for all vertices $v \in V \setminus X$, there is an edge **from** some vertex in X **to** v. Prove that any n-vertex tournament contains an out-dominating set of size $O(\log n)$, and give an algorithm to find it. (**Hint:** Part (a) is helpful for this.)

   Solution: Let $T = (V, E)$ be an n-vertex tournament. We carry out the following algorithm. Let $X_0 = \emptyset$. Given X_i for some $i \geq 0$, let $Y_i \subseteq V$ be the set of vertices that do not receive an edge from X_i; thus X_i is a dominating set in $T[V \setminus Y_i]$. If $Y_i = \emptyset$ then X_i is a dominating set for T, and we terminate; let $I = i$. Otherwise, by part (a), $T[Y_i]$ contains a vertex of out-degree at least $(|Y_i| - 1)/2$; form X_{i+1} by adding this vertex to X_i. This process must terminate with some dominating set X_I, since $|Y_1| > \dots > |Y_i| > 0$ for all $i > 0$.

   It remains to prove that $I = O(\log n)$. Since $|Y_0| = n$ and $|Y_i| \leq |Y_i|/2$ for all i, it follows from an easy induction argument that $|Y_i| \leq n/2^i$ for all i. Moreover, we have $|Y_{I-1}| > 0$, and $|Y_{I-1}|$ is an integer, so $|Y_{I-1}| \geq 1$; thus $1 \leq n/2^{I-1}$. Rearranging, we obtain $I \leq (\log n) + 1 \in O(\log n)$ as required.

   This was originally proved by Erdős, and it is excessively useful as a building block in arguments and algorithms to find larger and more complicated structures in tournaments.

   (c) [⋆⋆⋆] Prove that any n-vertex tournament contains an *in*-dominating set of size $O(\log n)$. (**Hint:** This can be done very quickly without rewriting your answers for (a) and (b).)

   Solution: Let T be an n-vertex tournament. Let T' be the tournament formed by reversing all the edges of T, so that $(a, b) \in E(T')$ if and only if $(b, a) \in E(T)$. By part (b), T' contains an out-dominating set X with size $O(\log n)$, so that X sends an edge to every vertex of $V \setminus X$. But then in T, X **receives** an edge **from** every vertex of $V \setminus X$, so X is an in-dominating set as required.

   It is often much easier to use an existing result or algorithm in a clever way than to come up with a new one — this will be one of the major themes of the course.

9. In lectures, we showed that we can't strengthen Dirac's theorem by lowering the minimum degree bound. This question is about strengthening Dirac's theorem in another direction. Let G be an n-vertex graph with $n \geq 3$. For all pairs $\{u, v\}$, we write $G + \{u, v\}$ for the graph formed from G by adding $\{u, v\}$ as an edge, i.e. $G + \{u, v\} = (V(G), E(G) \cup \{\{u, v\}\})$.

(a) [⋆⋆⋆⋆] Show that if $d(u) + d(v) \geq n$, then G contains a Hamilton cycle if and only if $G + \{u, v\}$ contains a Hamilton cycle. (**Hint:** You will need to use an idea from the proof of Dirac's theorem in lectures.)

Solution: If G contains a Hamilton cycle C, then C is also a Hamilton cycle in $G + \{u, v\}$ — so it remains to prove the other direction. Suppose C is a Hamilton cycle in $G + \{u, v\}$. If C doesn't contain the edge $\{u, v\}$, then C is also present in G and we're done, so suppose C does contain $\{u, v\}$. As in the proof of Dirac's theorem, we will now "rotate" C to find a new Hamilton cycle C' which does not contain $\{u, v\}$, and hence is also present in G.

Write $C = c_1 \dots c_n$, where $c_1 = u$ and $c_n = v$. Let

$$S = \{c_i \colon \{u, c_i\} \in E \text{ and } \{v, c_{i-1}\} \in E\}.$$

Note that if S is non-empty, containing some vertex c_i, then we are done: $c_i c_{i+1} \dots c_n c_{i-1} c_{i-2} \dots c_1$ is a Hamilton cycle not containing $\{u, v\}$, and hence also a Hamilton cycle in G. As in the proof of Dirac's theorem, we prove S is non-empty using the pigeonhole principle. We can write $S = S_1 \cap S_2$, where

$$S_1 = \{c_i \colon \{u, c_i\} \in E\},$$
$$S_2 = \{c_i \colon \{v, c_{i-1}\} \in E\}.$$

We have $|S_1| = d(u)$ and $|S_2| = d(v)$, so by hypothesis we have $|S_1| + |S_2| \geq n$. Both sets are contained in $\{c_2, \dots, c_n\}$, which has only $n - 1$ elements, so by the pigeonhole principle $S = S_1 \cap S_2$ must be non-empty as required.

(b) [⋆⋆⋆] Show that if $d(u) + d(v) \geq n$ for all non-adjacent $u, v \in V(G)$, then G contains a Hamilton cycle;

Solution: We repeatedly apply part (a) to turn G into the *complete graph* (in which every edge is present). Let $G_1 = G$ and let $t = \binom{n}{2} - |E(G)|$. For all $i \in [t-1]$, form G_{i+1} from G_i by adding an edge between two arbitrary non-adjacent vertices u_i and v_i; thus G_t is the complete graph. By hypothesis, $d_{G_i}(u_i) + d_{G_i}(v_i) \geq d_G(u_i) + d_G(v_i) \geq n$, so G_{i+1} contains a Hamilton cycle if and only if G_i does. Thus $G_1 = G$ contains a Hamilton cycle if and only if the complete graph G_t does. And it is easy to find a Hamilton cycle in G_t, since every edge is present — for example, writing $V(G) = [n]$, $12 \dots n$ is a Hamilton cycle in G_t.

Note that this result implies Dirac's theorem, since if G has minimum degree at least $n/2$ then for all non-adjacent u and v we have $d(u) + d(v) \geq n/2 + n/2 \geq n$. It is known as *Ore's theorem*, and the argument of part (a) is the *closure property* and is very helpful in finding Hamilton cycles in general.

(c) [⋆⋆⋆⋆⋆⋆] Show that if writing the degrees of G's vertices as $d_1 \leq \dots \leq d_n$, there is no $i < n/2$ such that $d_i \leq i$ and $d_{n-i} < n - i$, then G contains a Hamilton cycle.

Solution: We will again repeatedly apply part (a) to turn G into the complete graph, but this time the process will be a little harder. We will attempt to do so greedily. Let $G_1 = G$. For each i, given G_i, form a graph G_{i+1} by adding an edge between two arbitrary non-adjacent vertices u_i and v_i with $d_{G_i}(u_i) + d_{G_i}(v_i) \geq n$. If no such pair exists, let $i = t$ and "halt". By part (a), G_i contains a Hamilton cycle if and only if G_{i+1} does. We will show that G_t is the complete graph, so that G must contain a Hamilton cycle and we are done.

Suppose G_t is not the complete graph. Let u and v be two non-adjacent vertices in G_t with $d_{G_t}(u) + d_{G_t}(v)$ as large as possible and (without loss of generality) $d_{G_t}(u) \leq d_{G_t}(v)$. Write

$d_u = d_{G_t}(u)$ and $d_v = d_{G_t}(v)$, for convenience. We must have $d_u + d_v < n$, or we would have added $\{u, v\}$ to G_t — we will use this fact to get a contradiction.

Let S be the set of all vertices **not** adjacent to u in G_t (other than u itself). Since we chose $d_u + d_v$ to be as large as possible, every vertex in S has degree at most d_v in G_t, and $|S| = n - 1 - d_u$. So, counting u since $d_u \leq d_v$, G_t contains at least $n - d_u$ vertices with degree at most d_v. Since $d_u + d_v < n$, it follows that G_t contains at least $n - d_u$ vertices with degree less than $n - d_u$. Since G is a subgraph of G_t, the same must hold for G.

Likewise, let T be the set of all vertices not adjacent to v in G_t (other than v itself); then every vertex in T has degree at most d_u in G_t, and $|T| = n - 1 - d_v$. So G_t has at least $n - 1 - d_v$ vertices with degree at most d_u. Since $d_u + d_v < n$, it follows that G_t has at least d_u vertices with degree at most d_u, and so the same holds for G. But now this contradicts the hypothesis, on taking $i = d_u$. (Note that we must have $d_u < n/2$, since $d_u + d_v < n$ and $d_u \leq d_v$.) So in fact G_t must be a complete graph, so it must contain a Hamilton cycle, so G itself must contain a Hamilton cycle.

This is another generalisation of Dirac's theorem, proved Chvátal in 1972.