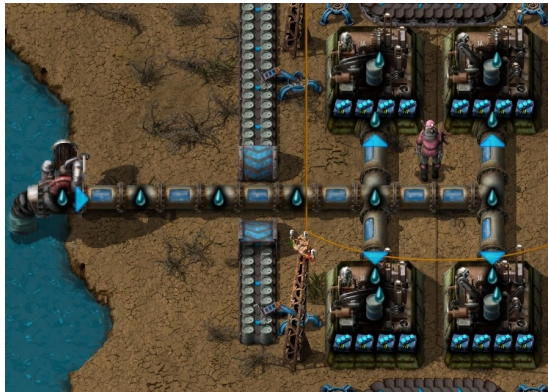# Applications of Ford-Fulkerson
COMS20010 2020, Video 9-2

John Lapinskas, University of Bristol

Circulations: Multiple sources and sinks

One of our simplifying assumptions was that a flow network only has one source and one sink. But in e.g. water networks, this is often not true.

We can model situations like these as **circulations**, in which we allow **every** node to be a source or sink.
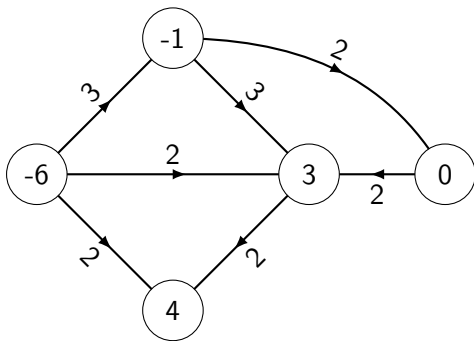
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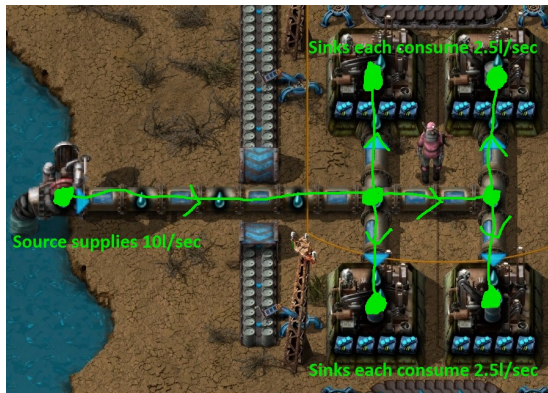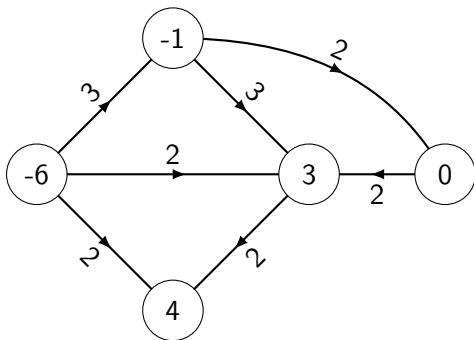

Instead of maximising flow, we specify how much each source supplies and how much each sink consumes, and try to satisfy these requirements.

A **circulation network** (G, c, D) is a directed graph $G = (V, E)$, a capacity function $c: E \to \mathbb{N}$, and a **demand** function $D: V \to \mathbb{Z}$.

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A vertex v with demand $D(v) > 0$ is a **sink**.
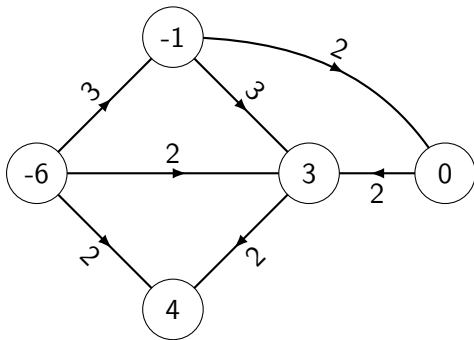A vertex v with demand $D(v) < 0$ is a **source**.

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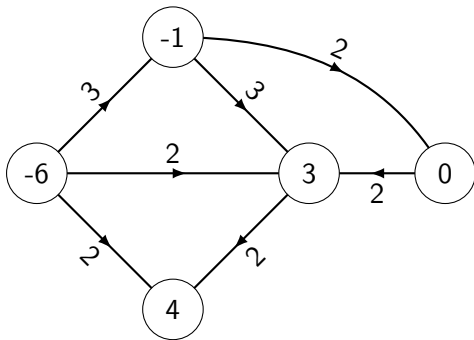

A vertex v with demand $D(v) > 0$ is a **sink**.

A vertex v with demand $D(v) < 0$ is a **source**.

A **circulation** is a function $f\colon E \to \mathbb{R}$ with $0 \leq f(e) \leq c(e)$ for all $e \in E$, and $f^-(v) - f^+(v) = D(v)$ (not zero!) for all $v \in V$. Note flow is conserved except at sources and sinks.

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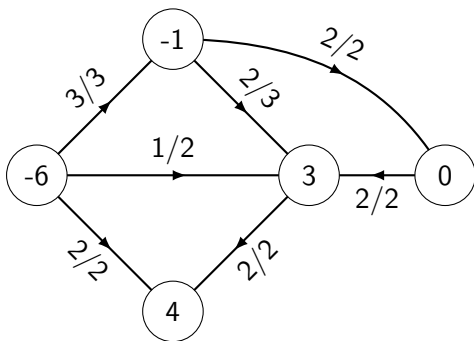

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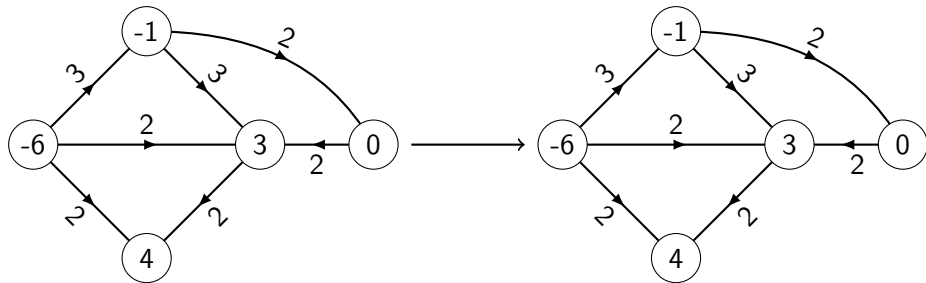A vertex v with demand $D(v) < 0$ is a **source**.

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Our problem: Does a circulation exist? Here: yes!

Finding circulations

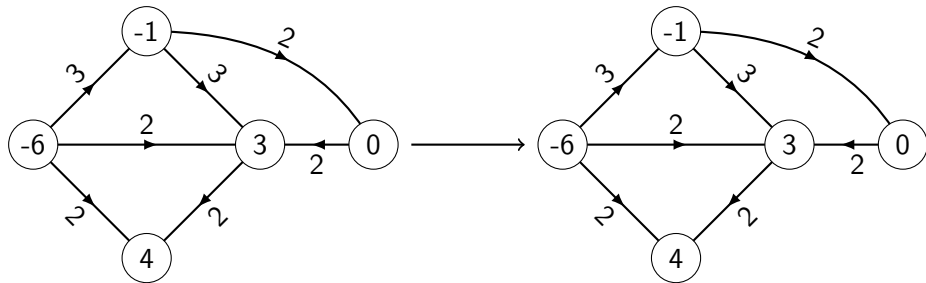Again, we can reduce to finding a maximum flow in a flow network.

Write $S = \{\text{sources}\}$, $T = \{\text{sinks}\}$.
Then transform our input circulation network into a flow network:

Finding circulations

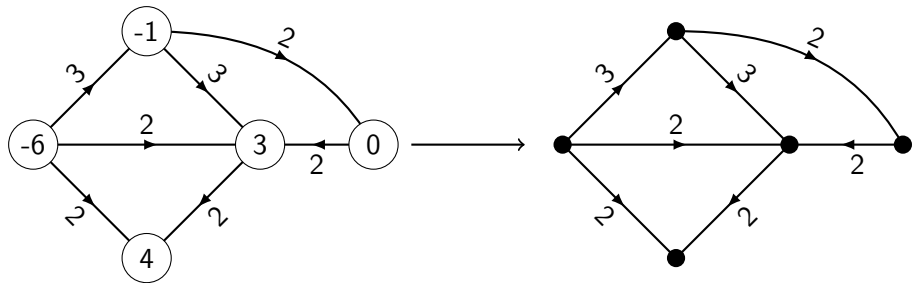Again, we can reduce to finding a maximum flow in a flow network.

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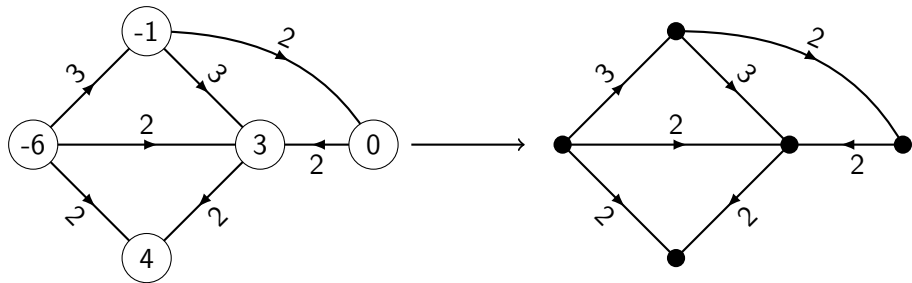Again, we can reduce to finding a maximum flow in a flow network.

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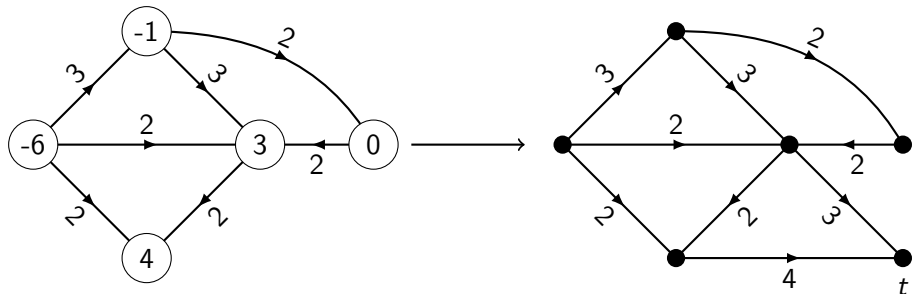Again, we can reduce to finding a maximum flow in a flow network.

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- Add a new vertex t and add every possible edge $T \to t$.
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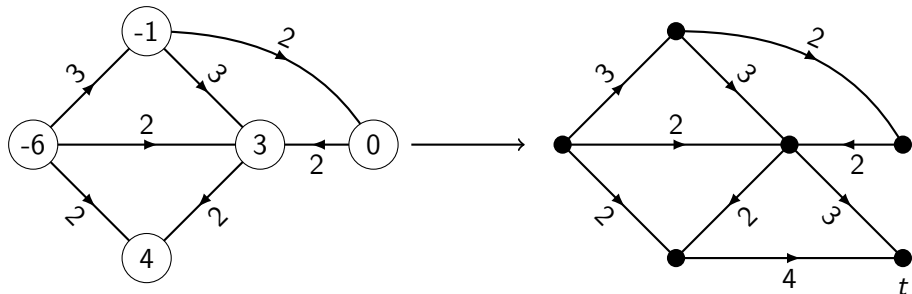Again, we can reduce to finding a maximum flow in a flow network.

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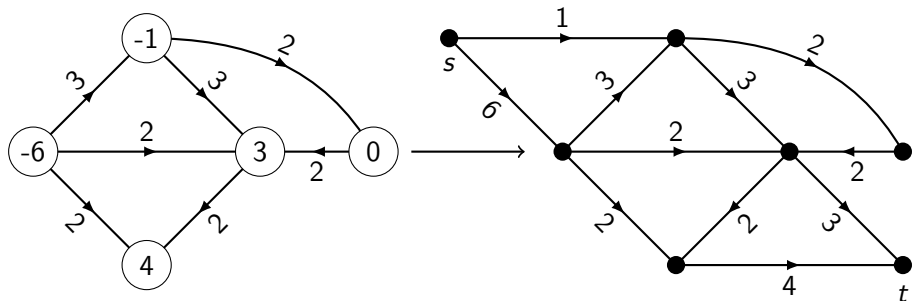Again, we can reduce to finding a maximum flow in a flow network.

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Finding circulations

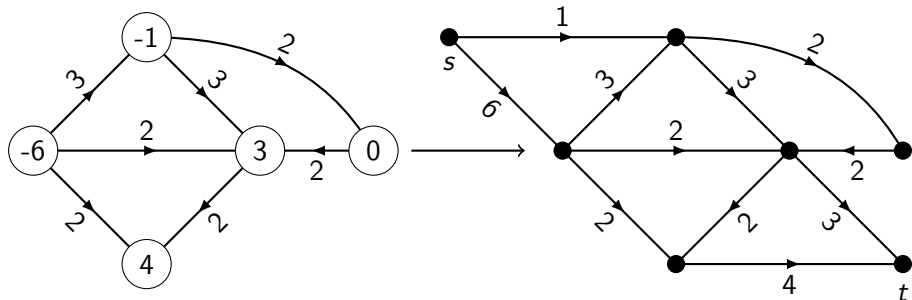Again, we can reduce to finding a maximum flow in a flow network.

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We claim every circulation on the left corresponds to a flow with value $\sum_{v \in T} D(v) = -\sum_{v \in S} D(v)$ on the right, and vice versa:
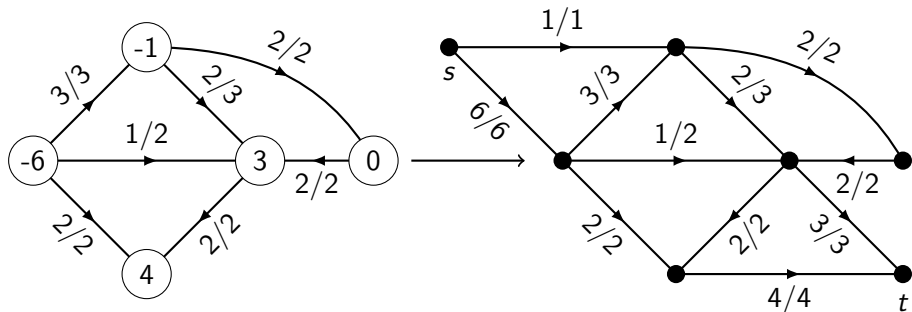
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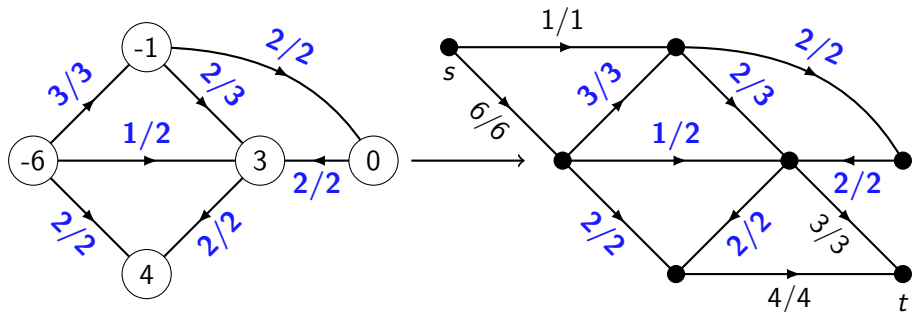
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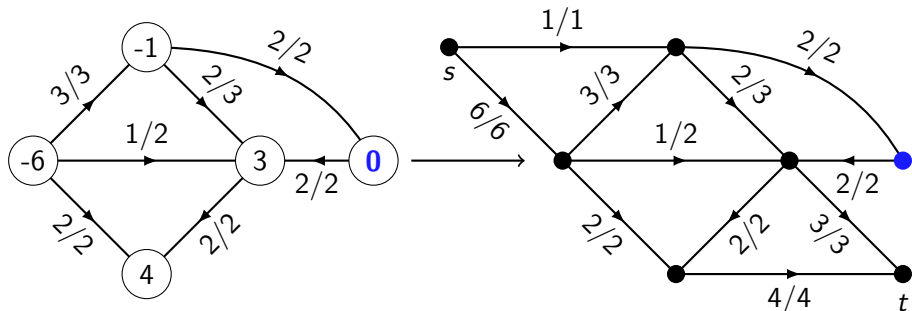Again, we can reduce to finding a maximum flow in a flow network.

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Capacity constraints on left $\Leftrightarrow$ Corresponding constraints on right

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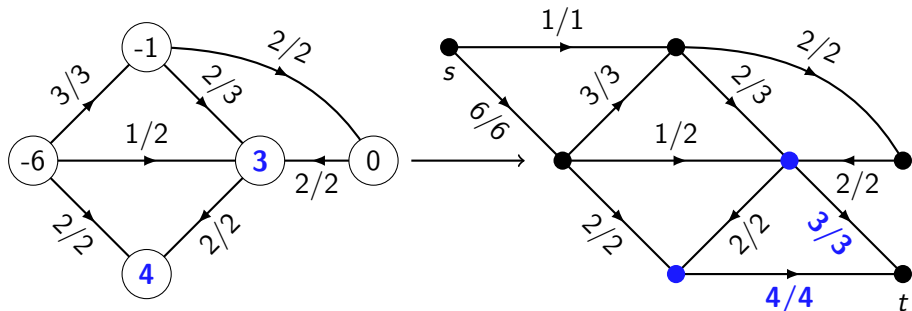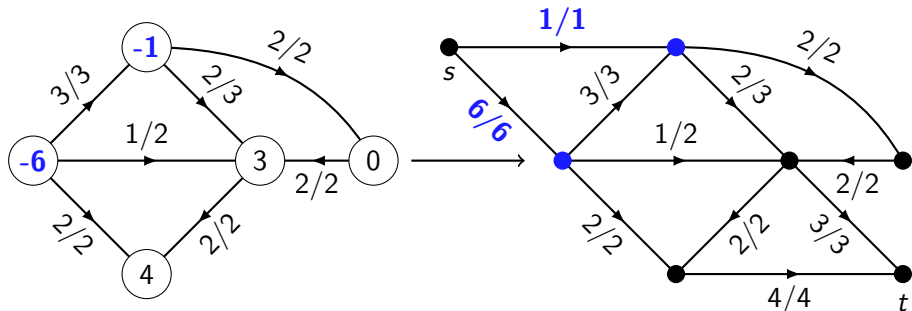

We claim every circulation on the left corresponds to a flow with value $\sum_{v \in T} D(v) = -\sum_{v \in S} D(v)$ on the right, and vice versa:

$$\text{Demand satisfied in } T \Leftrightarrow \begin{array}{l}\text{Flow conserved in } T \text{ and} \\ \text{flow value is } \sum_{v \in T} D(v)\end{array}$$

(Remember flow value can be taken at any cut; see Lemma 2 last lecture.)

Finding circulations

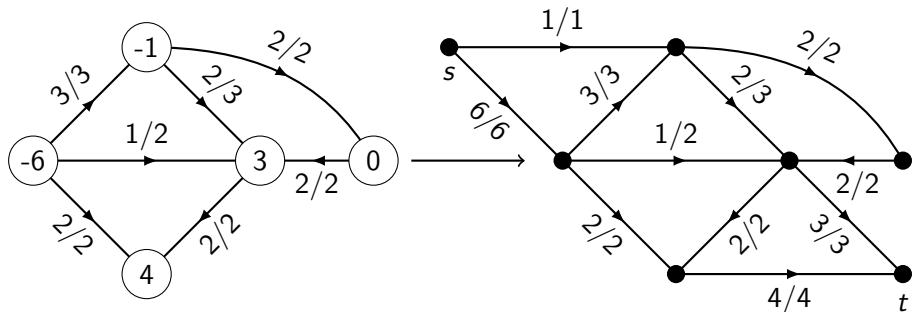Again, we can reduce to finding a maximum flow in a flow network.

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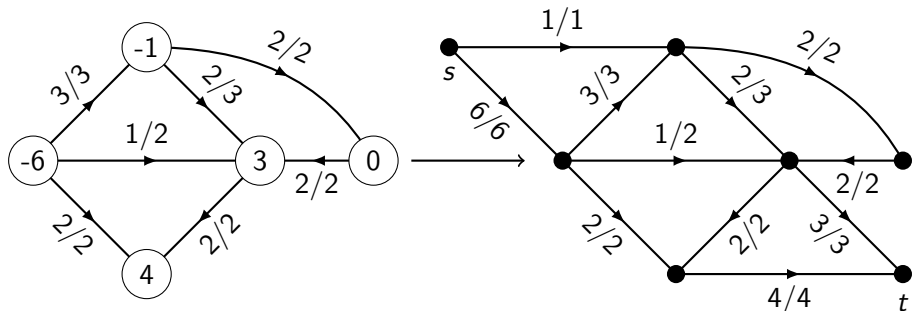
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The algorithm: Construct a flow network as above and run Edmonds-Karp. If it returns a flow with value $\sum_{v \in T} D(v) = -\sum_{v \in S} D(v)$, compute and return the corresponding circulation; otherwise, no circulation exists.
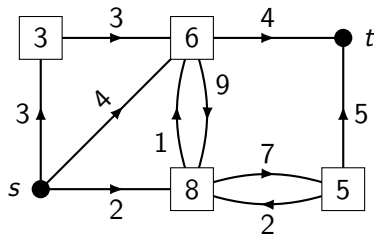
Adding vertex capacities

We've already put limits on the amount of flow that can pass through each edge of the network. What if we want to do the same for vertices? E.g. a pumping station might only be able to handle a limited amount of water.
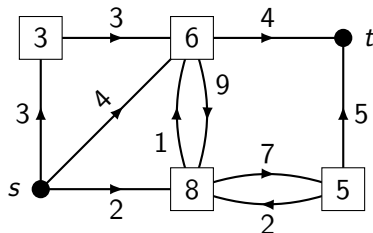
A **vertex flow network** (G, c_E, c_V, s, t) is a flow network (G, c_E, s, t) combined with a **vertex capacity** function $c_V \colon V \setminus \{s, t\} \to \mathbb{N}$.

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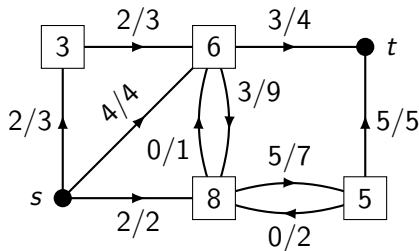

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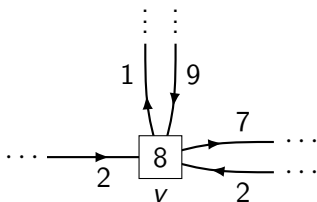

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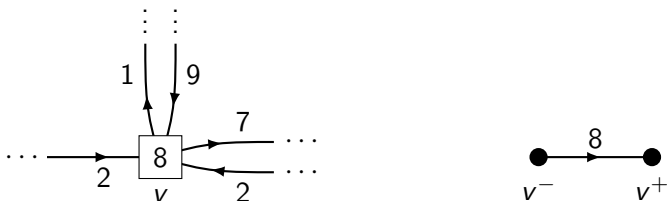
Again, we can reduce this to a flow network!

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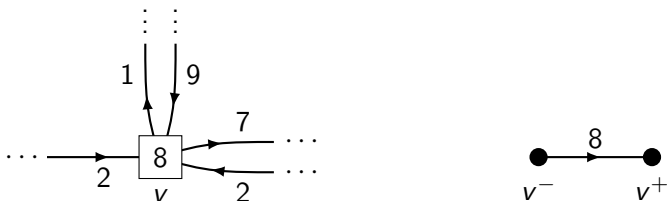

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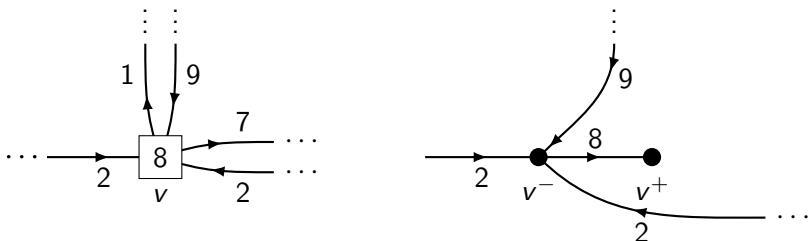

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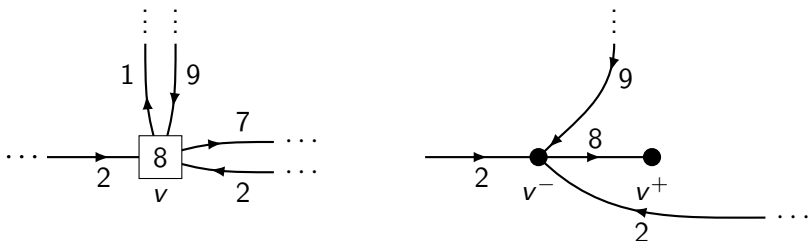

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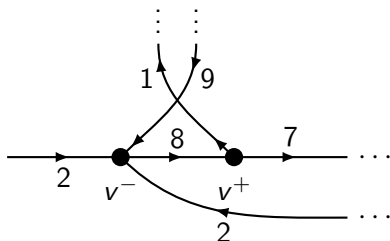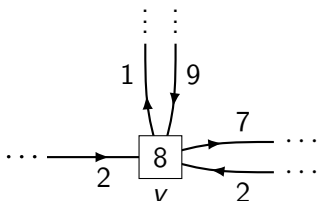

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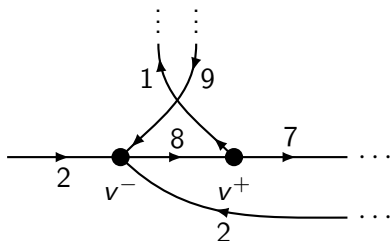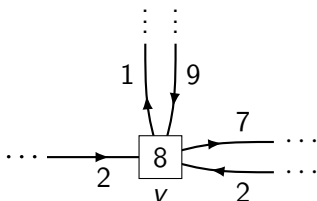

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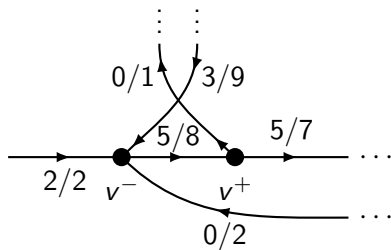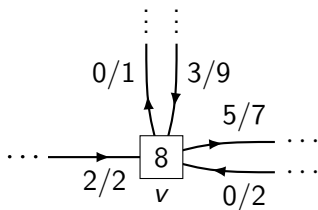Again, we can reduce this to a flow network!

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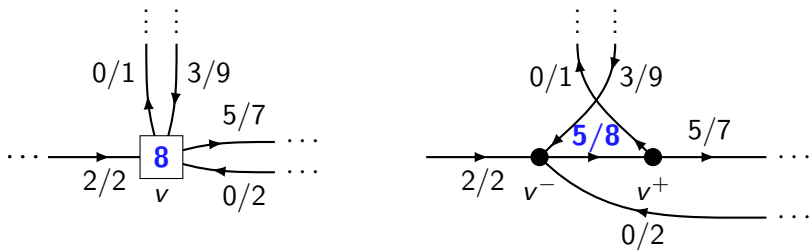
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Vertex capacity constraint for v $\Leftrightarrow$ Edge capacity constraint for (v^-, v^+)

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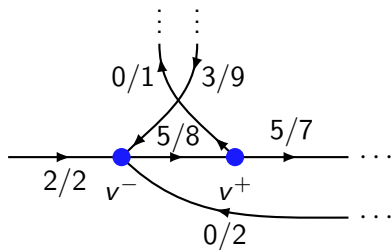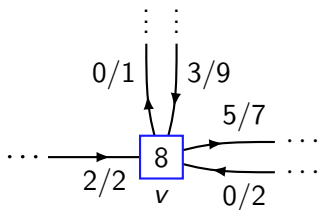

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Flow conservation at v $\Leftrightarrow$ Flow conservation at v^-, v^+

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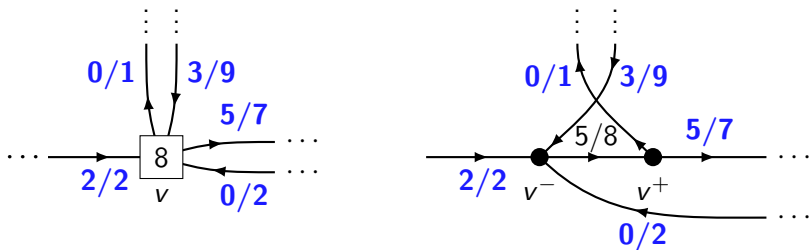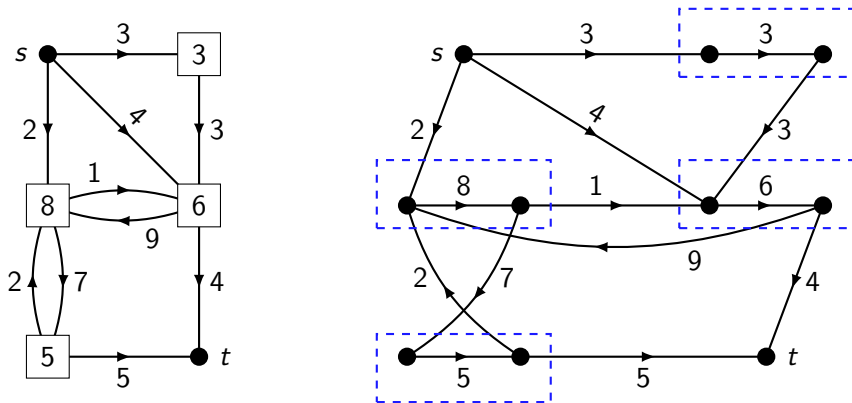

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Edge capacity constraints $\Leftrightarrow$ Edge capacity constraints other than (v^-, v^+).

The full construction looks like this (with boxes to point out the gadgets):

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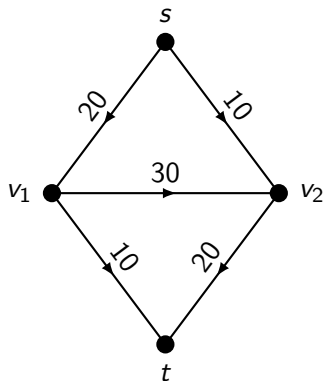
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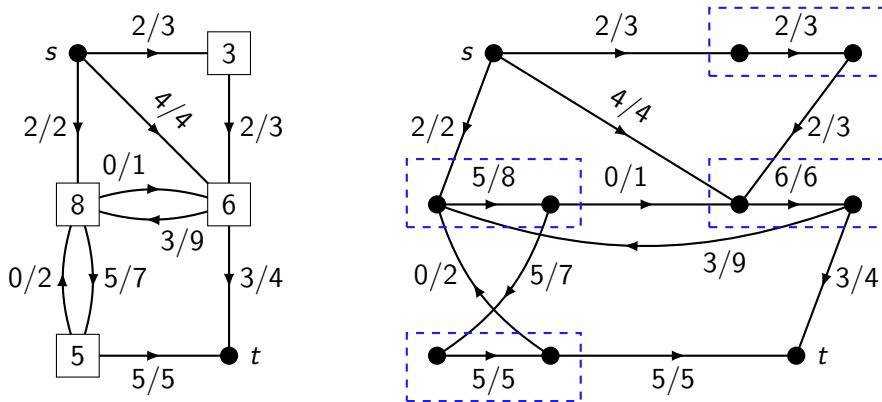
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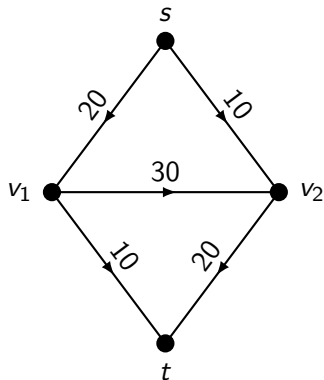
Connection to linear programming

We can also look at flow problems as linear programming problems whose variables are the flow at each edge:

$$f(s, v_1) + f(s, v_2) \to \text{max subject to}$$
$$f(s, v_1) = f(v_1, v_2) + f(v_1, t);$$
$$f(s, v_2) + f(v_1, v_2) = f(v_2, t);$$
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$$0 \le f(v_1, t) \le 10;$$
$$0 \le f(v_2, t) \le 20.$$

The objective function is the value of the flow.

Connection to linear programming

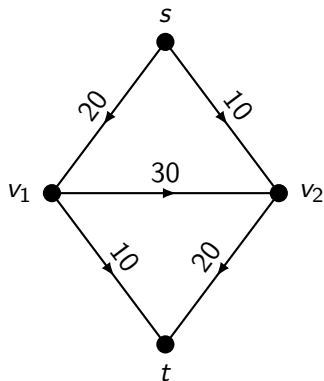We can also look at flow problems as linear programming problems whose variables are the flow at each edge:

$$f(s, v_1) + f(s, v_2) \to \text{max subject to}$$
$$f(s, v_1) = f(v_1, v_2) + f(v_1, t);$$
$$f(s, v_2) + f(v_1, v_2) = f(v_2, t);$$
$$0 \leq f(s, v_1) \leq 20;$$
$$0 \leq f(s, v_2) \leq 10;$$
$$0 \leq f(v_1, v_2) \leq 30;$$
$$0 \leq f(v_1, t) \leq 10;$$
$$0 \leq f(v_2, t) \leq 20.$$

These constraints are satisfied precisely when $0 \leq f(e) \leq c(e)$ for all edges e.

Connection to linear programming

We can also look at flow problems as linear programming problems whose variables are the flow at each edge:

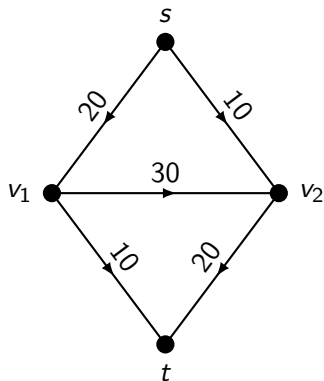$$f(s, v_1) + f(s, v_2) \rightarrow \max \text{ subject to}$$
$$f(s, v_1) = f(v_1, v_2) + f(v_1, t);$$
$$f(s, v_2) + f(v_1, v_2) = f(v_2, t);$$
$$0 \leq f(s, v_1) \leq 20;$$
$$0 \leq f(s, v_2) \leq 10;$$
$$0 \leq f(v_1, v_2) \leq 30;$$
$$0 \leq f(v_1, t) \leq 10;$$
$$0 \leq f(v_2, t) \leq 20.$$

These constraints are satisfied precisely when flow is conserved at v_1 and v_2.

Connection to linear programming

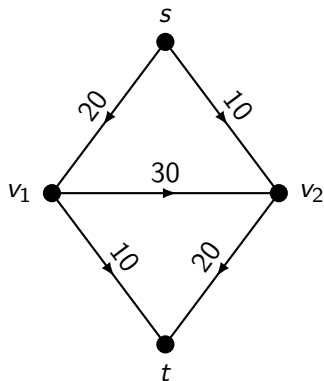We can also look at flow problems as linear programming problems whose variables are the flow at each edge:

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$$0 \leq f(v_2, t) \leq 20.$$

So maximum flows on the left correspond exactly to optimal LP solutions on the right, and we could use LP algorithms to solve flow problems!

Connection to linear programming

We can also look at flow problems as linear programming problems whose variables are the flow at each edge:

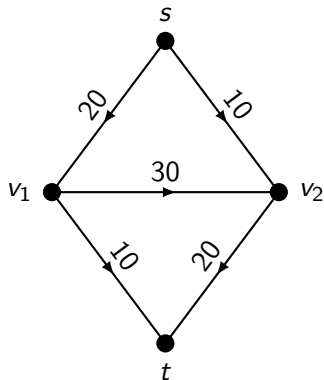$$f(s, v_1) + f(s, v_2) \to \max \text{ subject to}$$
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$$0 \le f(v_1, t) \le 10;$$
$$0 \le f(v_2, t) \le 20.$$

Sadly, this is a nice connection but not a good idea in practice — usually flow-specific algorithms are much faster...