# SAT and the class NP
# COMS20010 2020, Video 9-3

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# Cook reductions

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Suppose we have a polynomial-time algorithm for problem $X$ which calls a polynomial-time algorithm for $Y$ as a subroutine. Then **regardless of whether or not we actually have a polynomial-time algorithm for** $Y$, we call this a **Cook reduction** from $X$ to $Y$ and write $X \leq_c Y$.

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An oracle is explicitly a cheat — we are washing our hands of any responsibility for actually solving problem Y. Maybe a wizard did it.
Or a library function whose code is indistinguishable from wizardry.

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The point of the definition is: Given a Cook reduction from $X$ to $Y$, and a polynomial-time algorithm for Y, we get a polynomial-time algorithm for $X$. We just simulate the oracle using our algorithm for $Y$.

(The "correct" definition is more complicated, involving so-called oracle Turing machines, but the one above is good enough for our purposes.)

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As the notation suggests, if $X \leq_c Y$ and $Y \leq_c Z$ then $X \leq_c Z$, so we can build up chains of reductions. For example:

$$\text{Finding a maximum matching in a bipartite graph} \leq_c \text{Finding a maximum flow} \leq_c \text{Solving a linear program.}$$

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"Polynomial" can hide a multitude of sins...

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To be clear, this was a genuinely good paper! Just not exactly practical.

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Well, the good news is that it's not $2^{1/\varepsilon}$. It's $2^{2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}}$, with $1/\varepsilon^5$ twos.

For comparison, there are about $2^{265}$ atoms in the universe. So it's technically $O(1)$ overhead, but on any conceivable actual input, the algorithm is "solve by brute force"...

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**The point is:** if you're trying to find an algorithm for $X$, then just knowing $X \leq_c Y$ doesn't help you much. So why use the formalism?

# Weakness as a strength: using reductions to prove hardness

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And the really nice thing is: most of the time, from a practical perspective, there's only one problem $X$ that matters.

# Decision problems versus search problems

We focus on **decision problems**, where the desired answer is `Yes` or `No`:

- "Does the input graph contain a matching of size at least $k$?"
- "Does the input flow network contain a flow of value at least $k$?"
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- We are interested in proving problems are hard, not easy — if it's hard to decide whether something exists, then it's certainly hard to find it!
- Decision problems have a simpler theory associated with them.
- It's rare for the decision problem to be easy while the search problem is hard, and often there are easy Cook reductions between them. (See the problem sheet for some examples.)

# The class NP

Within decision problems, we will focus on problems where we can easily verify a `Yes` answer.

Formally, **NP** is the class of all decision problems $X$ with the following property: There is a polynomial-time algorithm `Verify` such that if and only if $x$ is a `Yes` instance of $X$, then there is some bit string $w$ (called a **witness**) with $\texttt{Verify}(x, w) = \texttt{Yes}$.

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- "Is the input a composite number?" is in NP, since given an input $x$ and a pair of integers $y$ and $z$, we can easily verify that $x = yz$ and $y, z > 1$.

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We will reduce the whole of NP to a single problem!

# Key properties of NP

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**Remark 1:** The definition of NP is asymmetric, and does **not** include problems where we can easily verify `No` answers but not `Yes` answers. For example, it is not clear that "Is the input a **prime** number?" is in NP.

For what we're doing now this is just a technical quirk, since we can always just rephrase the question. Later it will be very important!

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Because `Verify` can simply ignore $w$, solve $x$, and return the solution. (So "is the input a prime number?" actually is in NP.)

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A formula in **conjunctive normal form (CNF)** is an AND of OR clauses, such as $x \wedge (y \vee z) \wedge (\neg x \vee \neg z)$. Any formula can be expressed in CNF.

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An **assignment** for a formula is a map from its variables to the set $\{\mathtt{True}, \mathtt{False}\}$, and the formula's **truth value** under that assignment is calculated as you would expect. For example, under the assignment $x \mapsto \mathtt{True}$ and $y \mapsto \mathtt{False}$, the truth value of $x \wedge (\neg x \vee y)$ is

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# SAT

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So if there's a polynomial algorithm for SAT, then there's a polynomial algorithm for **every** problem in NP — that is, P = NP!

# NP-completeness

**P** is the class of all decision problems with a polynomial-time algorithm.

**NP** is the class of all decision problems $X$ with a polynomial-time algorithm `Verify` such that if $x$ is a `Yes` instance of $X$, then there is some bit string $w$ (a **witness**) with $\texttt{Verify}(x, w) = \texttt{Yes}$.

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A proof either way is worth $1,000,000 from the Clay Foundation...

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The good news is: this means you spent a couple of hours writing a hardness proof rather than weeks or months failing to write an algorithm!

NP-hardness can also be a good way of ruling out approaches: "If this worked for problem X, then it would also work for [insert NP-hard problem here], so it's not going to work."

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For example, this problem is NP-complete: "Given a mathematical statement $S$ and $k \geq 1$, is there a proof of $S$ in $k$ lines or fewer?"

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