Minimum Spanning Trees I: Prim's algorithm
COMS20010 2020, Video 5-3

John Lapinskas, University of Bristol

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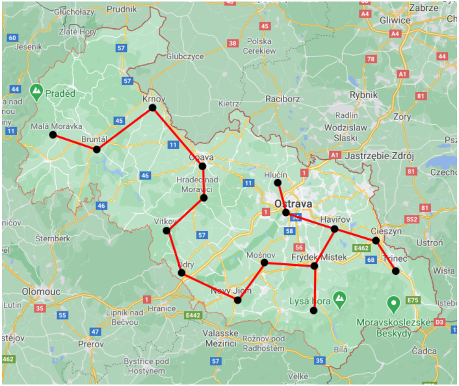

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Motivation

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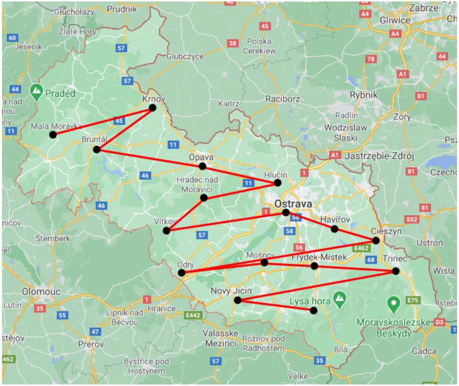
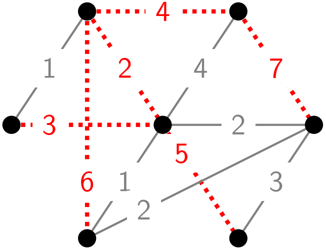


You need every town to be connected to every other town, and you want to spend as little as possible. So you want something like this, not like **this**.

Formal definition

We think of this situation as a connected weighted graph $G = ((V, E), w)$: the vertices are towns, and $w(x, y)$ is the cost of building a connection from x to y. (In this case, E would contain every possible edge.)



Total weight:

$4 + 2 + 7 + 3 + 6 + 5 = 27$

In other words, we seek a subtree T of G with $V(T) = V$ (a **spanning tree**)... whose total weight $\sum_{e \in E(T)} w(e)$ is as small as possible.

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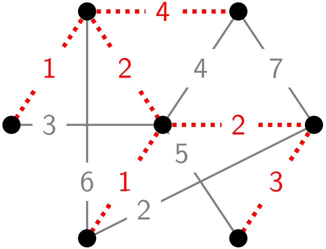


Total weight:

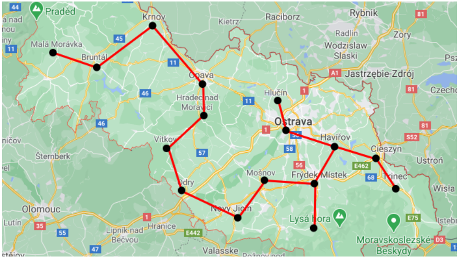
$4 + 1 + 2 + 2 + 1 + 3 = 13$

In other words, we seek a subtree T of G with $V(T) = V$ (a **spanning tree**)... whose total weight $\sum_{e \in E(T)} w(e)$ is as small as possible.

This is called a **minimum spanning tree**.

Wait a second...

Strictly speaking, this might not be the **best** possible solution.



What if we could introduce new vertices?

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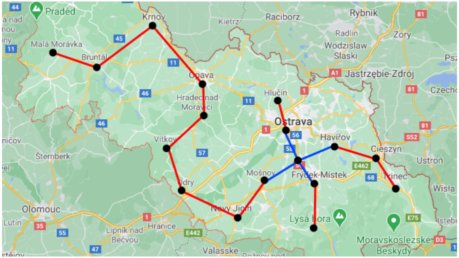


What if we could introduce new vertices?

This version of the problem is called **minimum Steiner tree**. But:

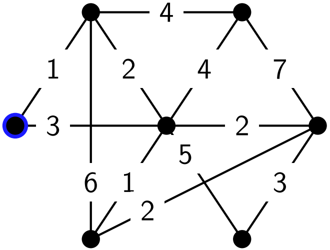
- this is “NP-hard” (read: no polynomial-time algorithm);
- all the approximation algorithms are based on minimum spanning tree;
- using a minimum spanning tree is already “good enough” — at worst twice the weight of a minimum Steiner tree (see problem sheet).

Prim's algorithm: The idea

Input: A connected weighted graph $G = ((V, E), w)$. **Output:** A minimum spanning tree of G.

A **minimum spanning tree** is a subtree T of G covering all of G's vertices, whose total weight $\sum_{e \in E(T)} w(e)$ is as small as possible.

We work **greedily**: pick an arbitrary start vertex, then grow it into a spanning tree by always choosing one of the cheapest available edges.



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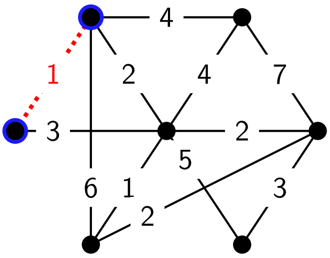

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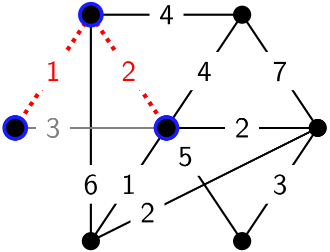

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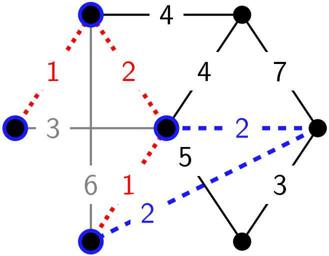

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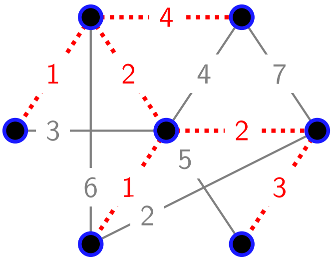

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Prim's algorithm: Formal version and correctness

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A **minimum spanning tree** is a subtree T of G covering all of G's vertices, whose total weight $\sum_{e \in E(T)} w(e)$ is as small as possible.

Formally: Let $T_1 = (\{v\}, \emptyset)$ for some arbitrary $v \in V$.

Let E_i be the set of edges from $V(T_i)$ to $V \setminus V(T_i)$.

Form T_{i+1} by adding a lowest-weight edge $e_i \in E_i$ to T_i, so

$$V(T_{i+1}) = V(T_i) \cup e_i \text{ and } E(T_{i+1}) = E(T_i) \cup \{e_i\}.$$

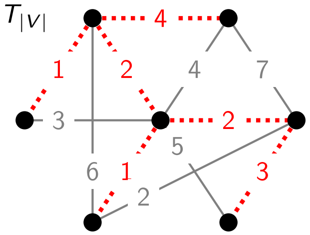
Prim's algorithm is to calculate and return $T_{|V|}$. Why does this work?

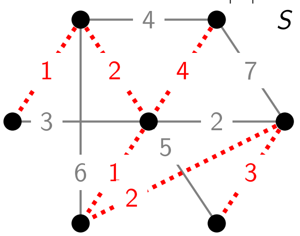
It returns a spanning tree because it's basically breadth-first search!
We just pick a lowest-weight edge at each stage rather than using a queue.

To prove it's a **minimum** spanning tree, we use an exchange argument.

That is, we show we can turn any minimum spanning tree into $T_{|V|}$ without increasing its weight (like with interval scheduling).

$T_{|V|}$ **is minimum:** Let S be a minimum spanning tree with $S \neq T_{|V|}$.





Let $S_i = S[V(T_i)]$, and let $I = \min\{i \colon S_i \neq T_i\}$.

We have $S_1 = T_1$ and $S_{|V|} \neq T_{|V|}$, so $2 \leq I \leq |V|$.

Prim's algorithm: Correctness II

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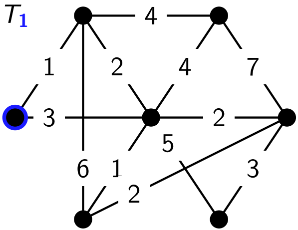

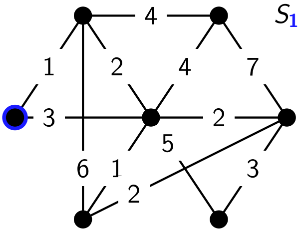

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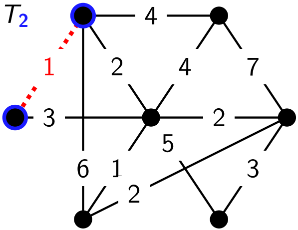

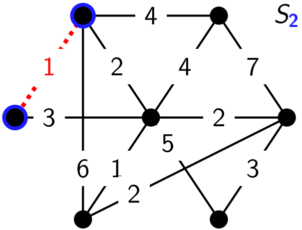

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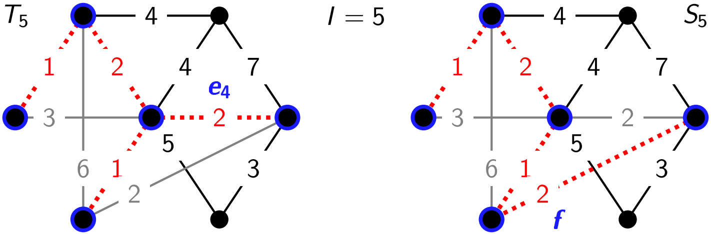

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$S_{I-1} = T_{I-1}$ is a tree, so there is only one edge f from $V(S_{I-1})$ to $V \setminus V(S_{I-1})$ in S_I (or S_I would contain a cycle). Remove it and replace it with e_{I-1}.

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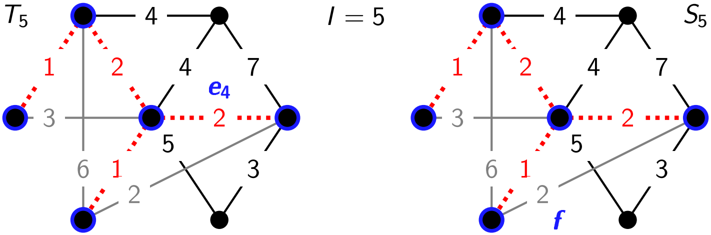

Let $S_i = S[V(T_i)]$, and let $l = \min\{i \colon S_i \neq T_i\}$.

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$S_{l-1} = T_{l-1}$ is a tree, so there is only one edge f from $V(S_{l-1})$ to $V \setminus V(S_{l-1})$ in S_l (or S_l would contain a cycle). Remove it and replace it with e_{l-1}.

Weight doesn't increase: True by Prim's choice of e_{l-1}. ✓

Still a tree: Since there is only one edge f, $S[V \setminus V(S_{l-1})]$ is a tree as well (by the FLoT). Joining two disjoint trees by an edge gives another tree (by the FLoT). ✓

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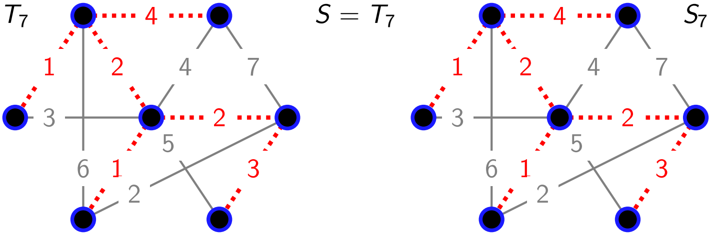

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Weight doesn't increase: ✓ **Still a tree:** ✓

So S is now a spanning tree which is "one edge closer" to $T_{|V|}$.

By repeating the process, we can turn S **into** $T_{|V|}$ without increasing its weight. Hence $w(S) \geq w(T_{|V|})$. Since S was minimum, we're done! $\square$

Prim's algorithm: Implementation

Literally just breadth-first search with a priority queue!

Algorithm: BFS

Input : **Connected weighted** graph $G = ((V, E), w)$.
Output : **A minimum spanning tree for G**.

Number the vertices of G **arbitrarily** as $v_1, \dots, v_n$.
Let $\mathtt{L}[i] \leftarrow \infty$ for all $i \in [n]$.
Let $\mathtt{L}[1] \leftarrow 0$, $\mathtt{pred}[1] \leftarrow \mathtt{None}$.
Let `queue` be a **length-$|E|$ priority** queue containing all tuples (v, v_j) with $\{v, v_j\} \in E$,
  using their edge weights as priorities.
while `queue` *is not empty* **do**
  Remove front tuple (v_i, v_j) from `queue`.
  if $\mathtt{L}[j] = \infty$ **then**
    Add (v_j, v_k) to `queue` for all $\{v_j, v_k\} \in E$, $k \neq i$.
    Set $\mathtt{L}[j] \leftarrow \mathtt{L}[i] + 1$, $\mathtt{pred}[j] = i$.
Return `pred`.

Time analysis: As with breadth-first search, each edge is only processed twice. Processing each edge now takes $\Theta(\log |E|)$ worst-case time, so overall the algorithm runs in $O(|E| \log |E|)$ time. (Note $|E| \geq |V|$.)

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Like with Dijkstra, we could "improve" this to $O(|E| + |V| \log |V|)$ time (with a much worse constant) by using a Fibonacci heap in place of the priority queue.