NP versus Co-NP
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The asymmetry of NP

Given a decision problem X, we write $\overline{X}$ for its **complement**:

Yes instances of X are No instances of $\overline{X}$ and vice versa.

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Conjecture: In general, this isn't possible. If it were, it wouldn't mean much for algorithms, but it would be a revolution in mathematics.

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Cook reductions are useless here, since every problem in NP reduces to $\overline{\mathrm{SAT}}$ and every problem in Co-NP reduces to SAT.

We need a notion of reduction that can make finer distinctions and tell the two complexity classes apart...

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Karp reductions are **stronger** than Cook reductions; $X \leq_K Y \Rightarrow X \leq_c Y$, since we can apply our oracle to $f(x)$, but the reverse doesn't hold.

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As with Cook reductions, we say a decision problem Y is **NP-hard under Karp reductions** if $X \leq_K Y$ for all $X \in \mathrm{NP}$.

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The proof of Cook-Levin implies SAT is NP-complete under Karp reductions, and $\overline{\mathrm{SAT}}$ is Co-NP-complete under Karp reductions.

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For SAT $\leq_C$ 3-SAT, we built a 3-SAT instance with the same answer as the SAT instance...

$$F = u \wedge (\neg u \vee \neg v) \wedge (v \vee \neg w \vee x \vee \neg y \vee \neg z) \wedge (y \vee z) \wedge (\neg v \vee w \vee z)$$

$$\downarrow$$

$$\begin{aligned} F' = {} & (u \vee f_1 \vee f_2) \wedge (\neg u \vee \neg v \vee f_1) \wedge (e_1 \vee \neg v \vee f_1) \wedge (e_1 \vee w \vee f_1) \wedge (\neg e_1 \vee v \vee \neg w) \\ & \wedge (e_2 \vee \neg e_1 \vee f_1) \wedge (e_2 \vee \neg x) \wedge (\neg e_2 \vee e_1 \vee x) \wedge (e_3 \vee \neg e_2 \vee f_1) \wedge (e_3 \vee y \vee f_1) \vee (\neg e_3 \vee e_2 \vee \neg y) \\ & \wedge (e_3 \vee \neg z \vee f_1) \wedge (y \vee z \vee f_1) \wedge (\neg v \vee w \vee z) \wedge (\neg f_1 \vee a_1 \vee a_2) \wedge (\neg f_1 \vee a_1 \vee \neg a_2) \wedge (\neg f_1 \vee \neg a_1 \vee a_2) \\ & \wedge (\neg f_1 \vee \neg a_1 \vee \neg a_2) \wedge (\neg f_2 \vee a_1 \vee a_2) \wedge (\neg f_2 \vee a_1 \vee \neg a_2) \wedge (\neg f_2 \vee \neg a_1 \vee a_2) \wedge (\neg f_2 \vee \neg a_1 \vee \neg a_2). \end{aligned}$$

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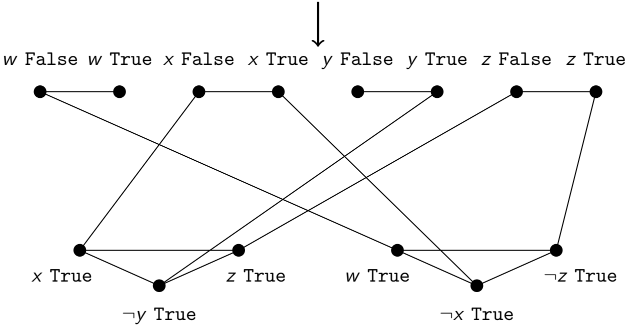
For 3-SAT $\leq_C$ IS, we built an independent set instance with the same answer as our 3-SAT instance...

$$F = (x \vee \neg y \vee z) \wedge (w \vee \neg x \vee \neg z)$$



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For IS $\leq_C$ VC, we built a vertex cover instance with the same answer as our independent set instance...

$$(G, k) \longrightarrow (G, |V| - k).$$

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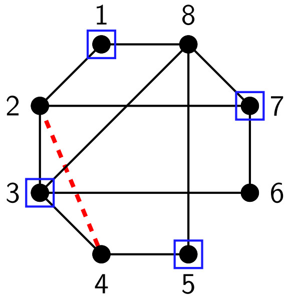
And for VC $\leq_C$ ILP, we built an integer linear programming instance with the same answer as our vertex cover instance.



$$\longrightarrow$$

$$\begin{aligned} &\textstyle\sum_v x_v \to \min \text{ subject to} \\ x_u + x_v &\geq 1 \text{ for all } \{u, v\} \in E; \\ x_v &\leq 1 \text{ for all } v \in V; \\ x_v &\geq 0 \text{ for all } v \in V; \\ x_v &\in \mathbb{N} \text{ for all } v \in V. \end{aligned}$$

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The worse news: Different people use different definitions. Complexity theorists use Karp reductions, programmers use Cook reductions. And both groups usually just say "NP-hard" or "NP-complete".

In this course: If I don't give more detail, "NP-complete" means under Karp reductions, and "NP-hard" means under Cook reductions.

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The slightly better news: *Almost* every NP-complete problem is NP-complete under both Cook and Karp reductions. So thinking only in terms of Karp reductions still saves effort without really sacrificing power.

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And in areas where Karp-unfriendly reduction techniques are more common (e.g. counting problems), everyone just uses Cook reductions, even the pure theorists.