Programming Languages and Computation

Week 9: Computable functions and predicates

* 1. Show that the identity function $\mathsf{id}_{\mathbb{N}} : \mathbb{N} \rightharpoonup \mathbb{N}$ is computable.

* 2. Show that the function $\lfloor \sqrt{-} \rfloor : \mathbb{N} \rightharpoonup \mathbb{N}$ which computes the integer square root of a natural number is computable.

* 3. Argue that the variable name does not matter in the definition of computability. That is, if a program `S` computes $f : \mathbb{N} \rightharpoonup \mathbb{N}$ with respect to `x`, then there is a program `S'` that also computes f, but with respect to any variable of our choice.

** 4. Show that if $f : \mathbb{N} \rightharpoonup \mathbb{N}$ and $g : \mathbb{N} \rightharpoonup \mathbb{N}$ are computable, then so is their composition $g \circ f : \mathbb{N} \rightharpoonup \mathbb{N}$.

* 5. Show that the predicate

$$P = \{ n \in \mathbb{N} \mid n \text{ is a prime number} \}$$

is decidable. (NB 1 is *not* a prime number.)

** 6. Show that if the predicates P and Q are decidable, then so is $P \cap Q$. Is the same true if P and Q are semi-decidable?

**** 7. (Trick question.) Is it true that if P and Q are semi-decidable, then so is $P \cup Q$?

**** 8. This exercise is about showing that predicates are the same thing as functions to the set of Boolean values.
Prove that predicates $U \subseteq \mathbb{N}$ and functions $f : \mathbb{N} \to \mathbb{B}$ to the set $\mathbb{B} = \{\top, \bot\}$ of Boolean values are in a perfect correspondence. That is, prove that there is a bijection between the *powerset* $\mathscr{P}(\mathbb{N}) = \{ U \mid U \subseteq \mathbb{N} \}$ and the *function space* $\mathbb{B}^{\mathbb{N}} = \{ f \mid f : \mathbb{N} \to \mathbb{B} \}$.

(a) Given a predicate $U \subseteq \mathbb{N}$, construct a function $f_U : \mathbb{N} \to \mathbb{B}$ that corresponds to it.

(b) Given a function $f : \mathbb{N} \to \mathbb{B}$, construct a predicate $U_f \subseteq \mathbb{N}$ that corresponds to it.

(c) Prove that the constructions $U \mapsto f_U$ and $f \mapsto U_f$ are inverses. That is, for a predicate $S \subseteq \mathbb{N}$ prove that $U_{f_S} = S$; and that for a function $h : \mathbb{N} \to \mathbb{B}$ prove that $f_{U_h} = h : \mathbb{N} \to \mathbb{B}$.

> (What does it mean for two functions to be equal?)