

# Dealing with NP-hard problems

## COMS20010 2020, Video 10-4

John Lapinskas, University of Bristol

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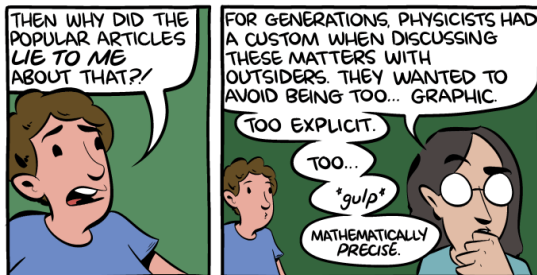
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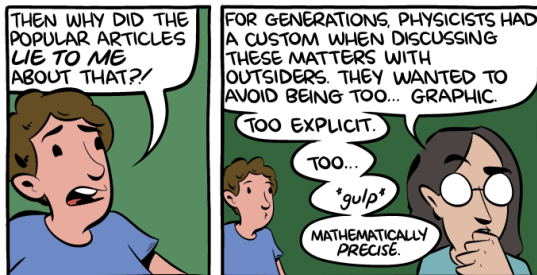
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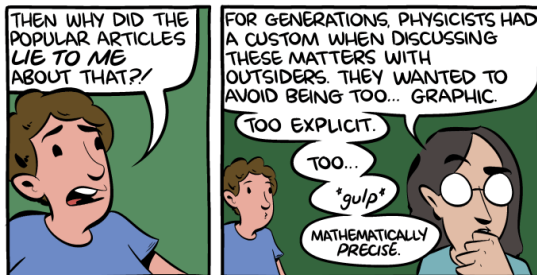
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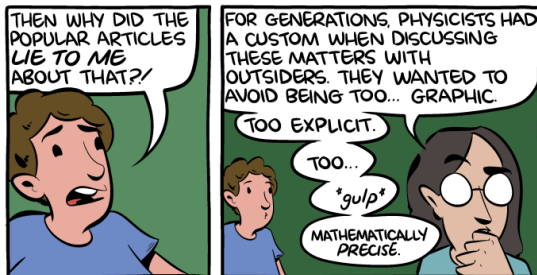


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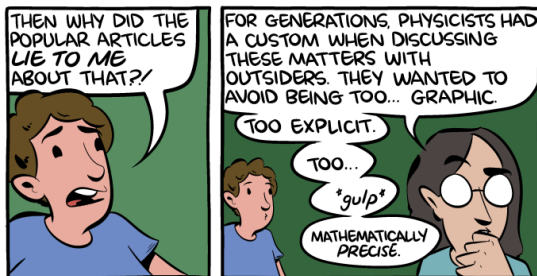


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- Breaking public-key cryptography with e.g. fast prime factorisation;
- Simulating quantum systems.

That's still very useful for e.g. drug discovery! But it's not solving SAT. There's no reason to think your home PC will ever need a "quantum chip".

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Then your output will be a quantum state which, when you measure it, gives No with probability  $1 - 2^{-n}$  and Yes with probability  $2^{-n}$ .

And measuring the state destroys it.

There's **provably** no way to distinguish this state from a pure “No” with any reasonable probability.

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The best evidence for quantum computers being more powerful than classical computers is that they can factor large numbers quickly.

But the evidence this is a hard problem is much weaker than e.g.  $P \neq NP$ . It's in  $NP \cap Co-NP$ , and some people do think this is contained in  $P...$

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For example, it's NP-hard to approximate vertex cover to within a factor of 1.36, and it's “UGC-hard” to approximate it to within a factor of  $2 - \varepsilon$  for any  $\varepsilon > 0$ . (So our algorithm is probably best possible.)

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And it's NP-hard to approximate independent set even to within a factor of  $n^{0.9999}$  on an  $n$ -vertex graph. (You can approximate it to within a factor of  $n$  by outputting a single vertex, so that's not great...)



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One very useful concept that can often be exploited is **low treewidth**.

The definition is too complicated for this course, but if your graph can be split into two by removing only a few vertices, and then you can continue the process recursively, then it probably has low treewidth.

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We call problems with algorithms like this **fixed-parameter tractable (FPT)**. The analogue of NP-hardness in this setting is **W[1]-hardness**.

These can be useful search terms when looking for practical algorithms!

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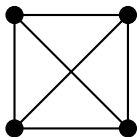
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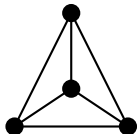
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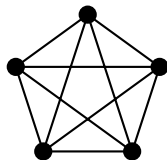
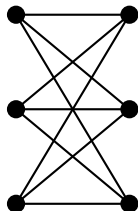
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because it can be drawn  
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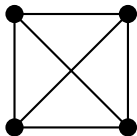


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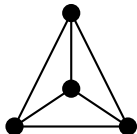
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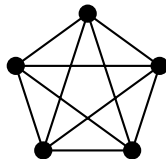
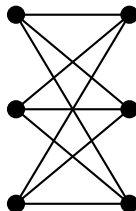
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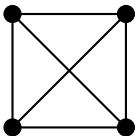


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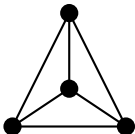
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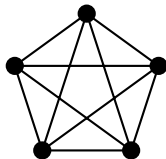
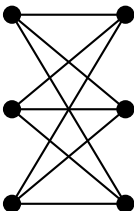
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Of course, if your instances are of size 15–20 you can just use brute force...

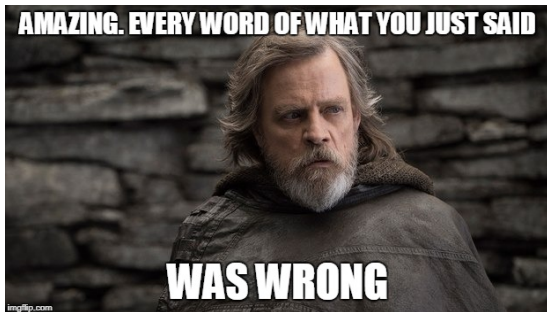
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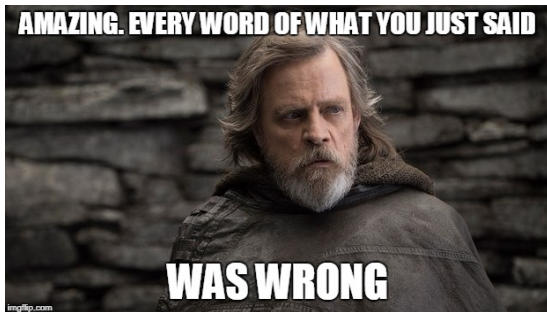
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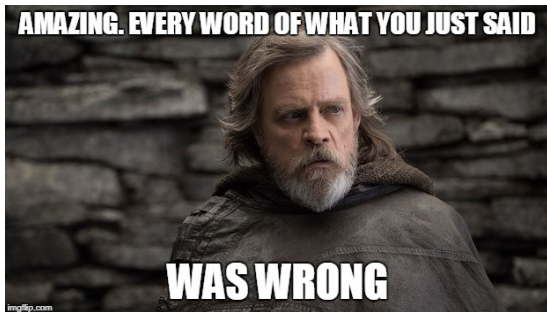
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Clearly “real-world SAT instances” often have some nice properties that solvers are exploiting... but we can’t nail down what those properties are. This is a huge open problem right now.

So just give it a try! It might not work, but it’s worth checking.