The Bellman-Ford algorithm
COMS20010 2020, Video 11-3

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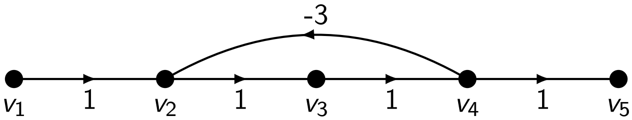
Shortest paths with negative-weight edges

The **length** of a path/walk $P = x_1 \dots x_t$ is the total weight $\sum_{i=1}^{t-1} w(x_i, x_{i+1})$ of P's edges.

The **distance** from x to y is the shortest length of any path/walk from x to y, or ∞ if they are in different components.

We touched on negative-weight edges when we covered Dijkstra's algorithm in week 4, but now we can actually solve the problem.

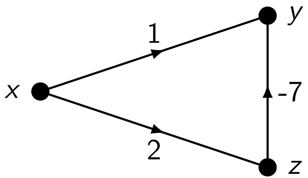
We assume every cycle in the graph has non-negative total weight — this guarantees that a shortest walk from one vertex to another exists, and is a path. Otherwise, it often doesn't exist!



Here there is no shortest walk from v_1 to v_5, since we can keep repeating the cycle $v_2 v_3 v_4$ to send the length of the walk off to $-\infty$...

What goes wrong with Dijkstra?

Dijkstra's algorithm relies on the assumption that the best route out of a set X of vertices is determined by the graph's structure in and near X. With negative weights, this fails.



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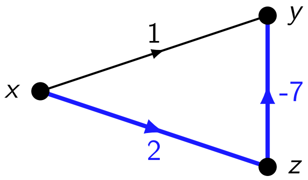

Since (x, y) has lower weight than (x, z), Dijkstra's algorithm run from x finalises $d(x, y) = 1$ as its first step even though $d(x, y) = -5$. It can't "see" the weight-(-7) edge when it's finalising the distance of y.

A dynamic programming approach

Step 1: Find a slow algorithm by reducing the problem to itself.

Original problem: Given a weighted digraph G with no negative-weight cycles and vertices $s, t \in V(G)$, find a shortest path from s to t.

Remember, when a solution is composed of lots of separate choices, a good way of going about this is often to consider the results of each choice.

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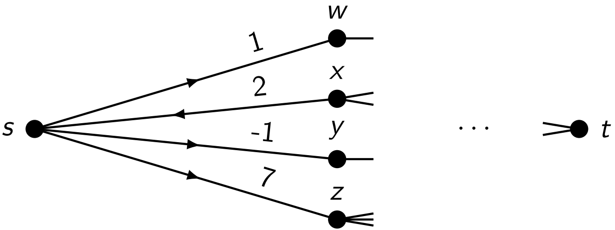

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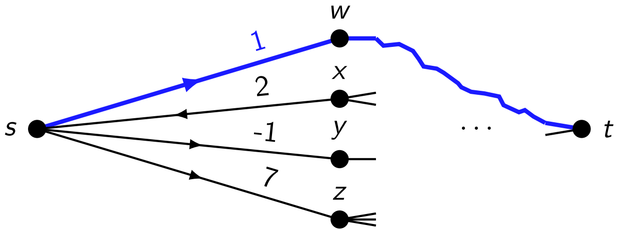

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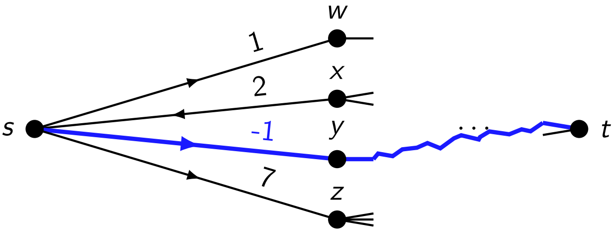

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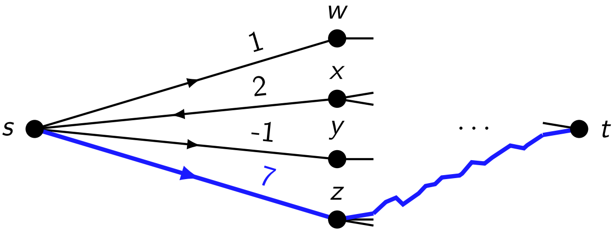

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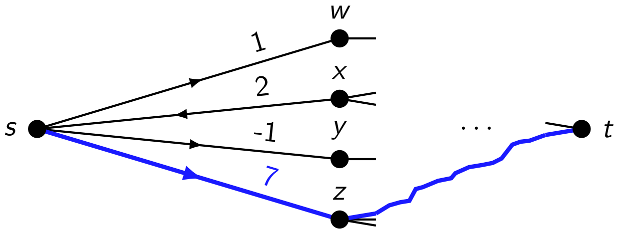

Any shortest path must be an edge from s to some $v \in N^+(s)$, followed by a shortest path from v to t in $G - s$.

The slow recursive algorithm

Algorithm: BadPath

Input : A weighted digraph $G = ((V, E), w)$ with no negative-weight cycles, and two vertices $s, t \in V(G)$.

Output : A shortest path from s to t in G, or `None` if none exists.

begin
  if $s = t$ **then**
    Return the empty path.
  if $d^+(s) = 0$ **then**
    Return `None`.
  Write $N^+(s) = \{v_1, \dots, v_d\}$, where $d \geq 1$.
  Let $P_i \leftarrow \textsc{BadPath}(G - s, v_i, t)$ for all $i \in [d]$.
  if $P_i = \texttt{None}$ *for all* $i \in [d]$ **then**
    Return `None`.
  Return whichever path is shortest in $\{sv_iP_i : i \in [d], P_i \neq \texttt{None}\}$.

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How many possible calls are there to BadPath? If the input graph is a clique, there are $\Theta(|V|2^{|V|})$ — G could be any of the $2^{|V|}$ induced subgraphs, and s could be any of the $|V|$ vertices!

So we can't just memoise this — we need to consolidate the calls.

The hard part: consolidating calls!

We can get around this by using two common tricks in dynamic programming: **reframing the problem** and **adding a parameter**.

Instead of asking for a shortest **path** from s to t in G, we will ask for a shortest **walk** from s to t in G **with at most** $|\boldsymbol{V(G)}| - \mathbf{1}$ **edges**.

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Most of dynamic programming is "cookie-cutter". It's not easy to learn, but once you know how, it's the same method for every problem. This is the part that can be arbitrarily difficult and only comes with practice.

A decent algorithm

Algorithm: GoodPath

Input : A weighted digraph $G = ((V, E), w)$ with no negative-weight cycles, two vertices $s, t \in V(G)$, and an integer $k \geq 0$.

Output : A shortest walk from s to t in G with at most k edges, or `None` if none exists.

begin

    if $k = 0$ **then**

        Return the empty walk if $s = t$, and `None` otherwise.

    Write $N^+(s) = \{v_1, \dots, v_d\}$, where $d \geq 1$.

    Let $P_i \leftarrow \textsc{GoodPath}(G, v_i, t, k-1)$ for all $i \in [d]$.

    if $P_i =$ `None` *for all* $i \in [d]$ **then**

        Return `None`.

    Return whichever walk is shortest in $\{sv_iP_i \colon i \in [d], P_i \neq \texttt{None}\}$.

How many distinct calls are there in $\textsc{GoodPath}(G, s, t, |V| - 1)$?

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```
begin
    if k = 0 then
        Return the empty walk if s = t, and None otherwise.
    Write N⁺(s) = {v_1, ..., v_d}, where d ≥ 1.
    Let P_i ← GoodPath(G, v_i, t, k − 1) for all i ∈ [d].
    if P_i = None for all i ∈ [d] then
        Return None.
    Return whichever walk is shortest in {s v_i P_i : i ∈ [d], P_i ≠ None}.
```

How many distinct calls are there in $\textsc{GoodPath}(G, s, t, |V| - 1)$?

Only $|V|^2$! (One for each possible (k, s) pair, since G and t stay the same between calls.)

Each call takes $O(|V|)$ time, so if we memoise, the algorithm runs in total time $O(|V|^3)$. And as a bonus, we can get $d(v, t)$ for all $v \in V$ for free.