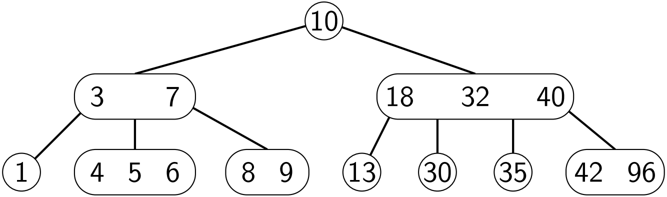
2-3-4 trees II: Deletion and alternative forms
COMS20010 2020, Video 6-4

John Lapinskas, University of Bristol

Summary of a 2-3-4 tree with distinct values (so far)



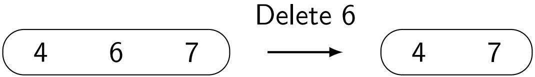
Finding a value *v*: Let x be the root. If $v \in x$, return a pointer to x. Otherwise, if x is a leaf, return `Not Found`. Otherwise, let k be such that x is a k-node, let $x_1 \leq \cdots \leq x_{k-1}$ be the values in x, let $x_0 = -\infty$, and let $x_k = \infty$; then $x_{i-1} < v < x_i$ for some i. Let c be the i'th child of x. Then repeat the process from the start, taking $x = c$.

Inserting a value *v*: First attempt to find v as above, **split**ting any 4-nodes encountered (including the root). After reaching a leaf L, and splitting it if it is a 4-node, add v to L.

Deleting from a leaf: Dealing with 2-nodes

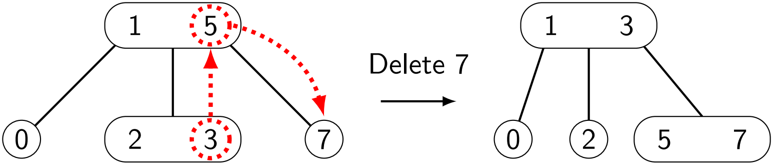
First suppose the value we're trying to delete is a leaf.

If it's in a 3-node or a 4-node... we just remove it:



If it's in a 2-node v, this would break perfect balance. So like with insertion, we need to first turn v into a 3-node or a 4-node.

If v's left (or right) sibling is **not** a 2-node, then we can **transfer** its rightmost (or leftmost) value to v:



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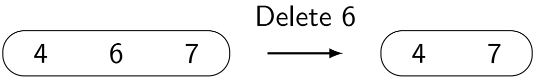

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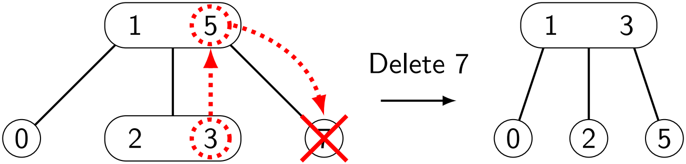

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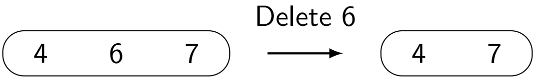

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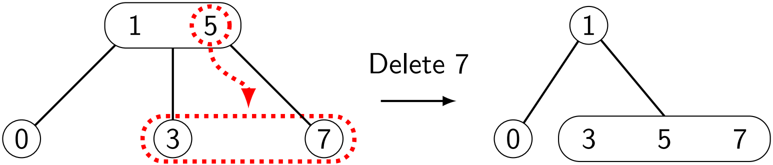

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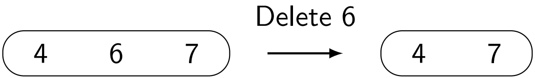

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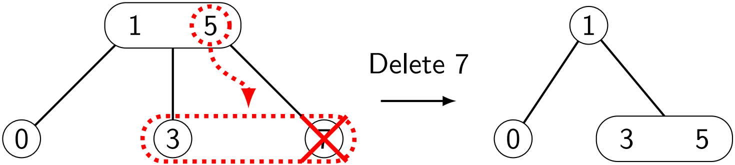

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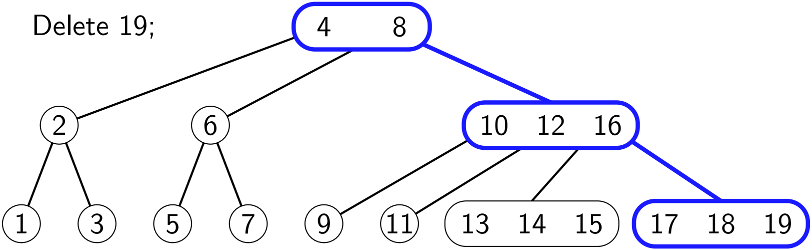

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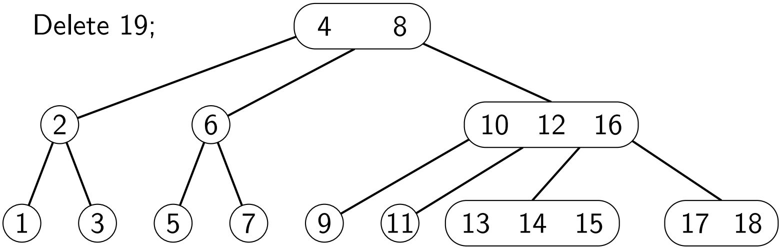

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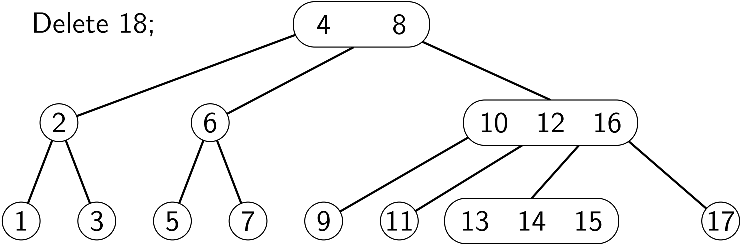

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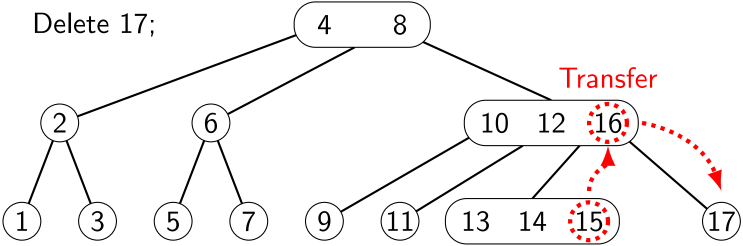

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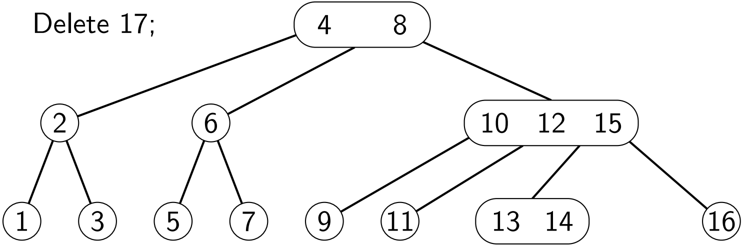

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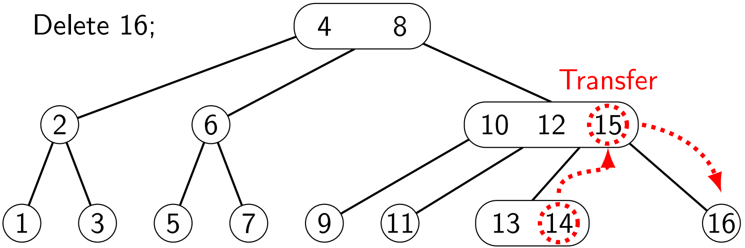

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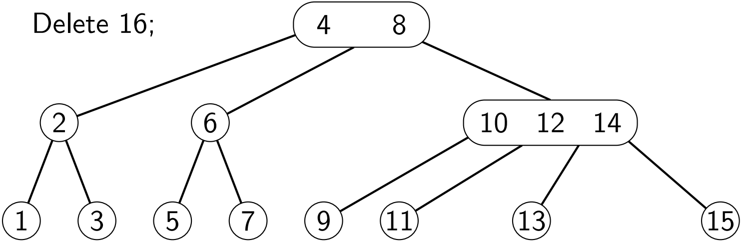

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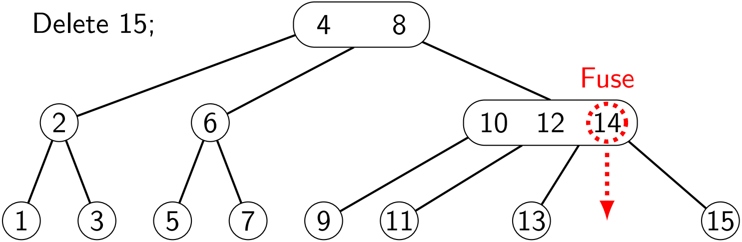

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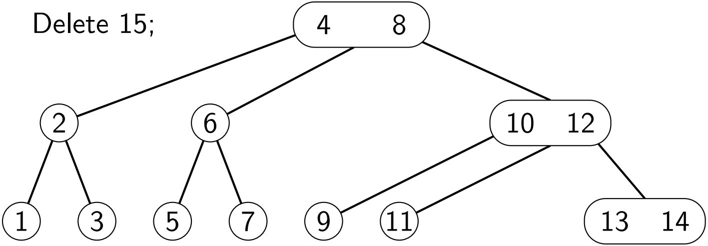

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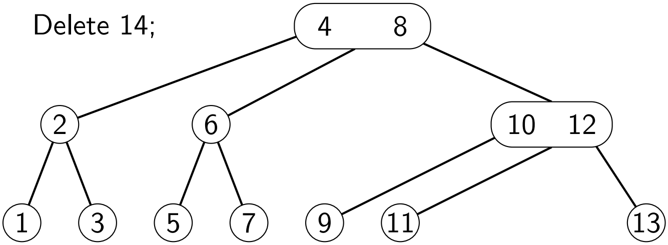

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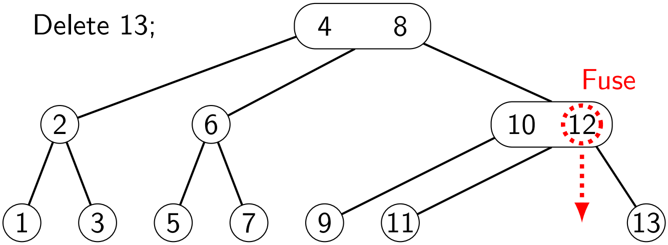

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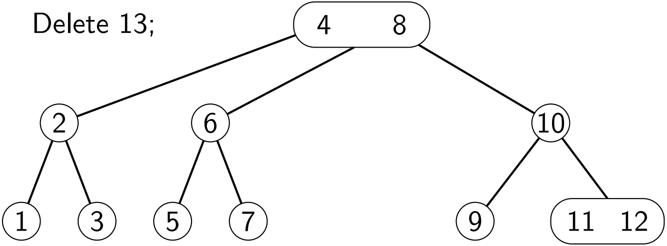

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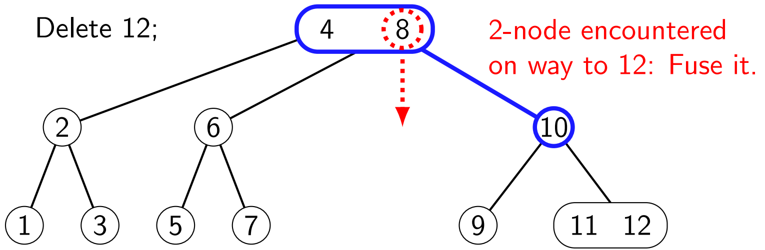

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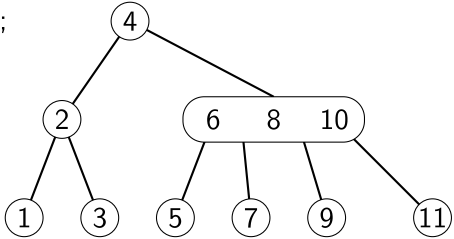

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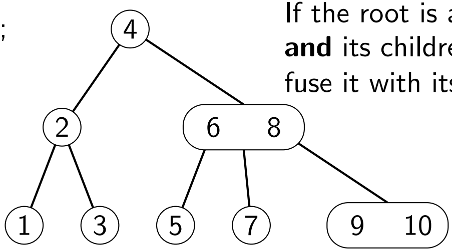

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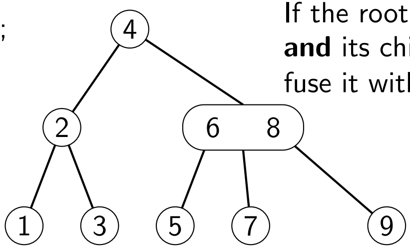

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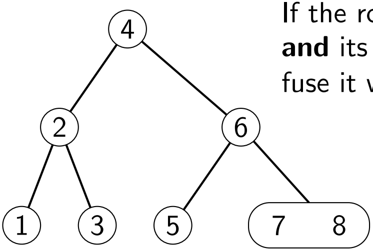

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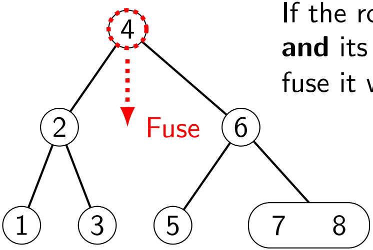

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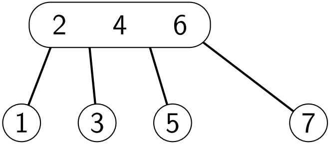

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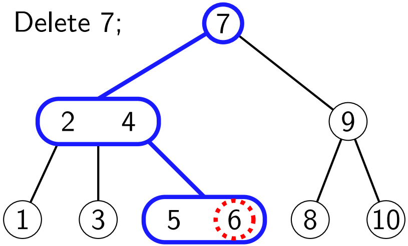

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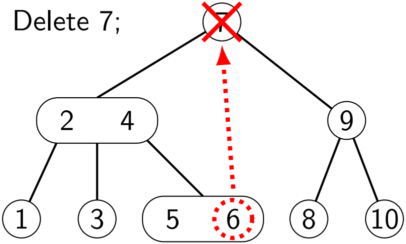

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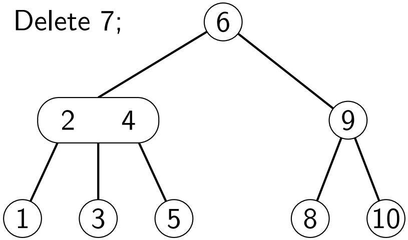

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In general, this will be its own delete operation, and might require fusing/transferring 2-nodes as normal.

Summary of 2-3-4 tree operations

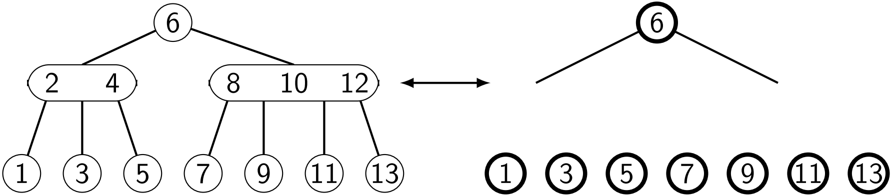
- `Find`(v):
  - Descend the tree recursively, using the rule that all children between x and y have values between x and y. If you reach a leaf not containing v, return `Not found`.
- `Insert`(v):
  - Apply Find(v) until reaching the leaf ℓ where v would be if it was already in the tree.
  - If ℓ is a 2-node or a 3-node, add v to ℓ.
  - Otherwise, **split** ℓ into 2-nodes and add v to the appropriate new leaf.
  - To avoid ℓ being a 4-node with a 4-node parent, split 4-nodes on the way down (including the root).
- `Delete`(v):
  - Apply Find(v) to find the vertex ℓ containing v.
  - If ℓ is not a leaf, find v's predecessor w, overwrite v with w, and `Delete`(w).
  - Otherwise, if ℓ is a 3-node or a 4-node, delete v from ℓ.
  - Otherwise, if ℓ's left or right sibling is a 3-node or 4-node, **transfer** from it to make ℓ a 3-node, then delete v.
  - Otherwise, **fuse** ℓ with its 2-node sibling to make ℓ a 4-node, then delete v.
  - To avoid ℓ being a 2-node with a 2-node parent, fuse or transfer 2-nodes on the way down (including the root).

All operations take $O(d)$ time and maintain perfect balance. Perfect balance implies that in an n-element tree, $d \in O(\log n)$ (exercise!), so we're done.

Red-black trees

Red-black trees are often used over 2-3-4 trees in practice, because they are slightly faster to implement... But they are secretly the same thing!

- Red-black trees are binary search trees where every non-leaf has exactly 2 children, and vertices are coloured red or black.
- Every root-leaf path has the same number of black vertices.
- No red vertex has a red child.



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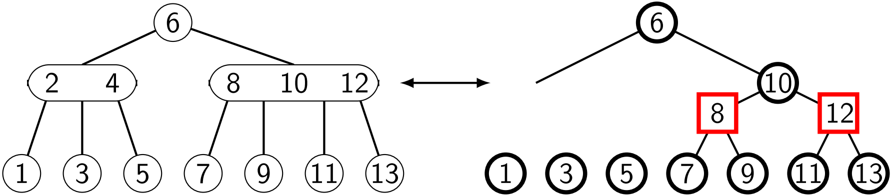

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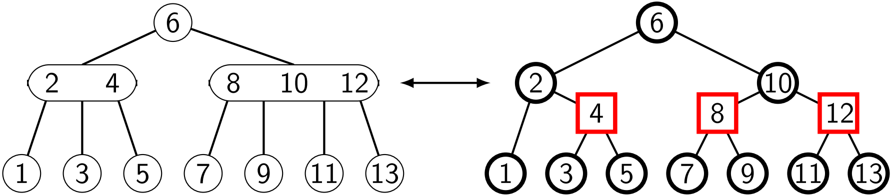

We can turn a 2-3-4 tree into a red-black tree (or vice versa) by: replacing each 2-node by a black node; each 4-node by a black node with two red children; and each 3-node by a black node with one red child.