A non-examinable sketch proof of the Cook-Levin theorem
COMS20010 2020, Video 10-2

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Lies, damned lies and sketch proofs

Cook-Levin Theorem: Any problem in NP is Cook-reducible to SAT.

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Let X be **any** problem in NP, and let $\vec{x}$ be an instance of X.

Then we will construct a CNF formula $F_{\vec{x}}$, of size polynomial in $|\vec{x}|$, which is satisfiable if and only if $\vec{x}$ is a `Yes` instance. Our Cook reduction then just applies the SAT oracle to $F_{\vec{x}}$ and outputs the result.

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By definition of NP, there is a polynomial-time algorithm `Verify` such that $\vec{x}$ is a `Yes` instance if and only if $\texttt{Verify}(\vec{x}, \vec{w}) = \texttt{Yes}$ for some $\vec{w}$. So we would like to express the statement "$\texttt{Verify}(\vec{x}, \vec{w}) = \texttt{Yes}$" as a CNF formula in $\vec{w}$.

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Problem: We know **nothing** about how `Verify` works. So to do this, we need to be able to express "A computer running program P on input I for t steps outputs `Yes`" as a CNF formula of polynomial size...!

Turing machines

Our goal: Given a program P, we would like a CNF formula $f(P, \vec{I}, t)$ which is satisfiable iff running P on input $\vec{I}$ for t steps outputs `Yes`.

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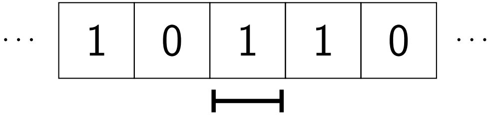

Machine in state 3, head reads 1 $\longrightarrow$ Write 0, move right, enter state 3.

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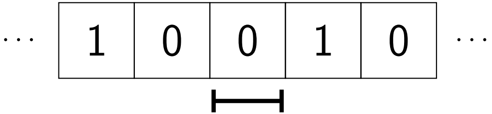

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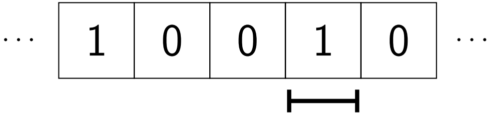

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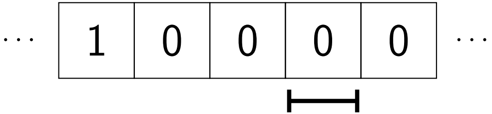

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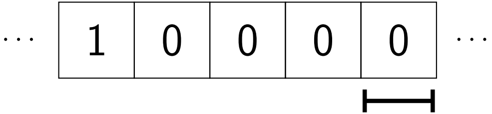

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$\cdots$	1	0	0	0	0	$\cdots$

Machine in state 3, head reads 0 $\longrightarrow$ Write 1, move left, enter state 7.

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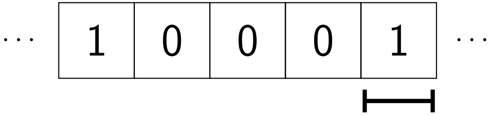

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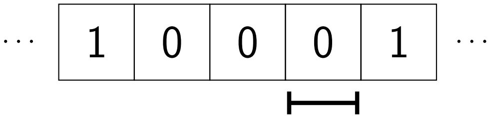

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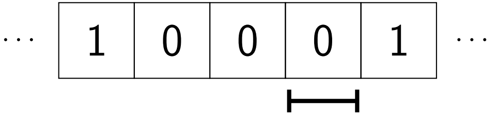

Machine in state 7, head reads 0 $\longrightarrow$ Write 1, move left, enter state 1.

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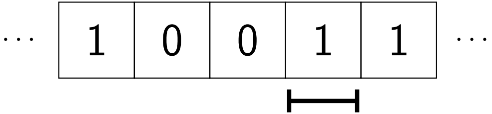

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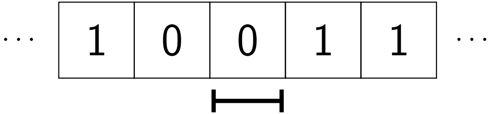

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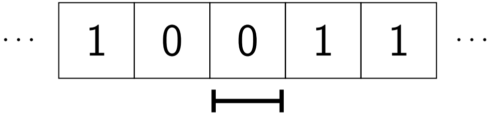

Machine in state 1, head reads 0 $\longrightarrow$ Write 0, move right, enter state 10.

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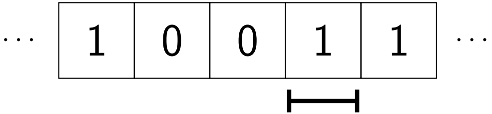

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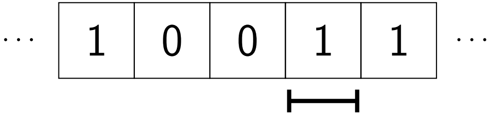

Machine in state 10, head reads 1 $\longrightarrow$ Halt.

Expressing computation with logic

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So we can take our program in the form of a Turing machine, which is much easier to simulate!

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Our new goal: Given a **Turing machine** M, we would like a CNF formula $f(M, \vec{I}, t)$ which is satisfiable iff after running M on input $\vec{I}$ for t steps, it halts with output `Yes`.

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Idea: Model the entire computation from start to finish, step by step.

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Problem: The tape is infinite, so we need infinitely many variables!

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Solution: In time t the tape head can only move t spaces, so the state of the Turing machine is determined entirely by cells $-t, -t+1, \ldots, t$.

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So actually we only need $O(t^2)$ variables.

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Our variables: We want: $C_{p,\tau} =$ contents of cell p at time τ;
$S_{s,\tau} = 1$ iff machine is in state s at time τ;
$P_{p,\tau} = 1$ iff head is at cell p at time τ.

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If we can express these as CNF statements **of length polynomial in t**, we're done since an AND of CNFs is in CNF.

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This is painful but ultimately doable!

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So we check consistency with an AND of **many** clauses that look like:

$$[P_{i,\tau} = 1 \wedge S_{3,\tau} = 1 \wedge C_{i,\tau} = 0 \Rightarrow C_{i,\tau+1} = 1]$$

Each such clause has only 4 variables, so it can be expressed as an $O(1)$-length CNF formula.

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Each such clause has only 4 variables, so it can be expressed as an $O(1)$-length CNF formula. We need $\Theta(t)$ such formulae for every variable in $\mathcal{M}_{\tau+1}$, one for each possible (position, state, cell contents) tuple, so in total our consistency check will have length $\Theta(t^2)$ and $|f(M, \vec{I}, t)| \in \Theta(t^3)$. □