We are interested in determining the density reading of the densitometer at the temperature used for the pycnometer reading such that a 1:1 comparison of the two instruments can be made. To do so, we assume the following first order Taylor series:

$$\rho_d(T_p) = \rho_d(T_d) + \frac{\partial \rho_T}{\partial T}\Big|_x (T_p - T_d),$$

where ρ_d is the density reading of the densitometer, T_p is the pycnometer temperature reading, T_d is the densitometer temperature reading, and *x* is the mole fraction of ethanol-water. In order to find the partial derivative of density w.r.t temperature, we will first derive a different equation, the partial derivative of true proof TP, and recover the desired derivative from there. First, the density of the fluid with a glass instrument effect ρ_T^{AP} is (assume density reading by densitometer doesn't have glass instrument effect and is equal to ρ_T):

$$\rho_T^{AP} = \rho_T[1 + \alpha(T - T_{60})],$$

where ρ_T is the density of the fluid without the glass expansion artifact and α is the thermal expansion coefficient of glass. A general value of α for glass is 25×10^{-5} /℃ . Furthermore, we define function *f* to convert density readings to apparent proof using Table 6:

$$f: \rho \rightarrow AP.$$

We also define function *g* to convert apparent proof and temperature to the true proof using Table 1:

$$g: AP, T \rightarrow TP.$$

All together, we can construct the following relation for determining true proof based on the fluid density at temperature *T*:

$$TP = g(f(\rho_T[1 + \alpha(T - T_{60})]), T).$$

Taking the partial derivative of the above TP function yields:

$$\partial TP = \frac{\partial g}{\partial AP}\Big|_T \frac{\partial f}{\partial \rho}[\partial \rho_T[1 + \alpha(T - T_{60})] + \alpha \rho_T dT] + \frac{\partial g}{\partial T}\Big|_{AP} \partial T = 0.$$

Note that we assume x and TP to be constant. Dividing through by ∂T leads to:

$$\frac{\partial g}{\partial AP}\Big|_T \frac{\partial f}{\partial \rho}\left[\frac{\partial \rho_T}{\partial T}[1 + \alpha(T - T_{60})] + \alpha \rho_T\right] + \frac{\partial g}{\partial T}\Big|_{AP} = 0.$$

We can now begin rearranging the above equation to solve for the partial derivative of density w.r.t temperature as follows:

$$\frac{\partial g}{\partial AP}\Big|_T \frac{\partial f}{\partial \rho}\left[\frac{\partial \rho_T}{\partial T}[1+\alpha(T-T_{60})]+\alpha\rho_T\right] = -\frac{\partial g}{\partial T}\Big|_{AP}.$$

$$\left[\frac{\partial \rho_T}{\partial T}[1+\alpha(T-T_{60})]+\alpha\rho_T\right] = \frac{-\frac{\partial g}{\partial T}\Big|_{AP}}{\frac{\partial g}{\partial AP}\Big|_T \frac{\partial f}{\partial \rho}} = -\frac{\partial g}{\partial T}\Big|_{AP}\frac{\partial AP}{\partial g}\Big|_T \frac{\partial \rho}{\partial f}.$$

Note that we can simplify the partial derivative terms. The cyclic relation is:

$$\frac{\partial g}{\partial T}\Big|_{AP}\frac{\partial AP}{\partial g}\Big|_T\frac{\partial T}{\partial AP}\Big|_g = -1.$$

We can rearrange this to the more useful form (negative sign):

$$\frac{\partial g}{\partial T}\Big|_{AP}\frac{\partial AP}{\partial g}\Big|_T = -\frac{\partial AP}{\partial T}\Big|_g.$$

Applying this relation thus simplifies our expression for the density derivative:

$$\frac{\partial \rho_T}{\partial T} = \frac{-\frac{\partial g}{\partial T}\Big|_{AP}\frac{\partial AP}{\partial g}\Big|_T\frac{\partial \rho}{\partial f} - \alpha\rho_T}{1+\alpha(T-T_{60})}$$

$$= \frac{\frac{\partial AP}{\partial T}\Big|_{TP}\frac{\partial \rho}{\partial AP}}{1+\alpha(T-T_{60})} - \frac{\alpha\rho_T}{1+\alpha(T-T_{60})}.$$

Since we are keeping x constant, it follows that:

$$\frac{\partial \rho_T}{\partial T}\Big|_x = \frac{\frac{\partial AP}{\partial T}\Big|_{TP}\frac{\partial \rho}{\partial AP}\Big|_x}{1+\alpha(T-T_{60})} - \frac{\alpha\rho_T}{1+\alpha(T-T_{60})}.$$

The above equation is appropriate for using densitometer reading ρ_T at temperature $T = T_d$. If the hydrometer (pycnometer) data was used, we would need to substitute ρ_T with the hydrometer density reading (with instrument effect) ρ_T^{AP} at $T = T_p$:

$$\frac{\partial \rho_T}{\partial T}\Big|_x = \frac{\frac{\partial AP}{\partial T}\Big|_{TP}\frac{\partial \rho}{\partial AP}\Big|_x}{1+\alpha(T-T_{60})} - \frac{\alpha\rho_T^{AP}}{[1+\alpha(T-T_{60})]^2}.$$

Example calculation: A densitometer reading of 0.8500 g/mL was found for 0.5191 mole fraction ethanol at temperature 31˚C. The temperature of the pycnometer reading, for comparison purposes, was 28˚C. The relevant terms are, in their algebraic form:

$$x = 0.5191,$$

$$\rho_d(T_d = 31℃) = 0.8500\frac{g}{mL},$$

$$T_p = 28℃.$$

Using the true-proof function on fluid true density and temperature of 62˚F, the apparent proof of the solution is 168.85˚P with an inferred true proof of 159.80˚P. We also need to approximate the partial derivatives $\frac{\partial AP}{\partial T}\big|_{TP}$ and $\frac{\partial \rho}{\partial AP}\big|_{x}$; however the $\frac{\partial AP}{\partial T}\big|_{TP}$ is an intractable quantity. Therefore, we revert back to the cyclic relation to determine this quantity.

$$\frac{\partial AP}{\partial T}\Big|_{TP}\frac{\partial \rho}{\partial AP}\Big|_{x} = -\frac{\partial TP}{\partial T}\Big|_{AP}\frac{\partial AP}{\partial TP}\Big|_{T}\frac{\partial \rho}{\partial AP}\Big|_{x}.$$

The partials are approximated as:

$$\frac{\partial TP}{\partial T}\Big|_{AP} \sim \frac{158.8 - 159.1°P}{31.1111 - 30.5556℃} = -0.5400\frac{°P}{℃}.$$

$$\frac{\partial AP}{\partial TP}\Big|_{T} \sim \frac{169 - 168°P}{158.8 - 159.1°P} = -3.333.$$

$$\frac{\partial \rho}{\partial AP}\Big|_{x} \sim \frac{0.99904 \times (0.86380 - 0.86518)\frac{g}{mL}}{160 - 159°P} = -0.001379\frac{g}{mL\,°P}.$$

The correction for the pycnometer temperature is thus:

$$\frac{\partial \rho_T}{\partial T}\Big|_{x} = \frac{\frac{\partial AP}{\partial T}\Big|_{TP}\frac{\partial \rho}{\partial AP}\Big|_{x}}{1 + \alpha(T - T_{60})} - \frac{\alpha\rho_T}{1 + \alpha(T - T_{60})}$$

$$= \frac{(0.5400)(3.333)(0.001379)\frac{g}{mL\,℃}}{1 + \frac{25 \times 10^{-6}}{℃}(31 - 15.5556℃)} - \frac{\frac{25 \times 10^{-6}}{℃}0.8500\frac{g}{mL}}{1 + \frac{25 \times 10^{-6}}{℃}(31 - 15.5556℃)}$$

$$= 0.002451\frac{g}{mL\,℃}$$

$$\rho_d(28℃) = 0.8500\frac{g}{mL} + 0.002451\frac{g}{mL\,℃}(28 - 31℃) = \mathbf{0.8426\frac{g}{mL}}$$