TCPCP problem:

$$\min_{\mathcal{L},\mathcal{S}} \|\mathcal{L}\|_* + \lambda\|\mathcal{S}\|_1, s.t. P_\Omega(\mathcal{S}+\mathcal{L}) = P_\Omega\mathcal{X}_0. \tag{1}$$

Rewrite the constraint into $G(\mathcal{S}(:)+\mathcal{L}(:)) = G\mathcal{X}_0(:) = g$, then we want to solve the following optimization problem,

$$\min_{\mathcal{L},\mathcal{S}} \|\mathcal{L}\|_* + \lambda\|\mathcal{S}\|_1 + \frac{\mu}{2}\|G(\mathcal{S}(:)+\mathcal{L}(:)) - g\|_F^2. \tag{2}$$

In order to solve this problem by ADMM, we introduce $\mathcal{P},\mathcal{Q}$ and then obtain the following optimization problem,

$$\min_{\mathcal{S},\mathcal{L},\mathcal{P},\mathcal{Q}} \|\mathcal{P}\|_* + \lambda\|\mathcal{Q}\|_1, \quad s.t.\, \mathcal{P} = \mathcal{L},\ \mathcal{Q} = \mathcal{S},\ G(\mathcal{S}(:)+\mathcal{L}(:)) = g. \tag{3}$$

The Lagrangian is augmented as

$$\begin{aligned}
\mathcal{L}_{\rho_1,\rho_2}(\mathcal{S},\mathcal{L},\mathcal{P},\mathcal{Q}) = {} & \|P\|_* + \lambda\|\mathcal{Q}\|_1 && (4)\\
& + \frac{\rho_1}{2}\|\mathcal{L} - \mathcal{P} + \rho_1^{-1}\mathcal{Z}_1\|_F^2 && (5)\\
& + \frac{\rho_2}{2}\|\mathcal{S} - \mathcal{Q} + \rho_2^{-1}\mathcal{Z}_2\|_F^2 && (6)\\
& + \frac{\rho_3}{2}\|G(\mathcal{S}(:)+\mathcal{L}(:)) - g + \rho_3^{-1}\mathcal{Z}_3\|_F^2 && (7)
\end{aligned}$$

Each of the above sub-problems can be solved as the followings,

$$\begin{aligned}
\mathcal{P} &= arg\min_{\mathcal{P}} \frac{1}{2}\|\mathcal{L} - \mathcal{P} + \rho_1^{-1}\mathcal{Z}_1\|_F^2 + \frac{1}{\rho_1}\|P\|_*\\
&= D_{1/\rho_1}(\mathcal{L} + \rho_1^{-1}\mathcal{Z}_1) && (8)\\
\mathcal{Q} &= arg\min_{\mathcal{Q}} \frac{1}{2}\|\mathcal{S} - \mathcal{Q} + \rho_2^{-1}\mathcal{Z}_1\|_F^2 + \frac{\lambda}{\rho_2}\|Q\|_1\\
&= S_{\lambda/\rho_2}(\mathcal{S} + \rho_2^{-1}\mathcal{Z}_2) && (9)\\
\mathcal{L} &= arg\min_{\mathcal{L}} \frac{\rho_1}{2}\|\mathcal{L} - \mathcal{P} + \rho_1^{-1}\mathcal{Z}_1\|_F^2 + \frac{\rho_3}{2}\|G(\mathcal{S}(:)+\mathcal{L}(:)) - g + \rho_3^{-1}\mathcal{Z}_3\|_F^2\\
&= (\rho_3 G^T G + \rho_1 I)^{-1}(\rho_3 G^T g + \rho_1\mathcal{P}(:) - \mathcal{Z}_1 - \rho_3 G^T G\mathcal{S}(:) - G^T Z_3) && (10)\\
\mathcal{S} &= arg\min_{\mathcal{S}} \frac{\rho_2}{2}\|\mathcal{S} - \mathcal{Q} + \rho_2^{-1}\mathcal{Z}_2\|_F^2 + \frac{\rho_3}{2}\|G(\mathcal{S}(:)+\mathcal{L}(:)) - g + \rho_3^{-1}\mathcal{Z}_3\|_F^2\\
&= (\rho_3 G^T G + \rho_1 I)^{-1}(\rho_3 G^T g + \rho_2\mathcal{Q}(:) - \mathcal{Z}_2 - \rho_3 G^T G\mathcal{L}(:) - G^T Z_3) && (11)
\end{aligned}$$

Solve this by ADMM by introducing two dual problems, the algorithm can be shown as follows,

Algorithm: Solve TCPCP by ADMM

Initialization $\mathcal{S} = \mathcal{L} = \mathcal{P} = \mathcal{Q} = \mathcal{Z}_1 = \mathcal{Z}_2 = zeros(dim(X_0))$
for $k = 1, 2, \cdots$ **do**
  update $\mathcal{P}$: $\mathcal{P} = D_{\frac{1}{\rho_1}}(\mathcal{L} + \rho_1^{-1}\mathcal{Z}_1)$
  update $\mathcal{Q}$: $\mathcal{P} = S_{\frac{\lambda}{\rho_2}}(\mathcal{S} + \rho_2^{-1}\mathcal{Z}_2)$
  update $\mathcal{L}$: $\mathcal{L}(:) = (\rho_3 G^T G + \rho_1 I)^{-1}(\rho_3 G^T g + \rho_1 \mathcal{P}(:) - \mathcal{Z}_1 - \rho_3 G^T G \mathcal{S}(:) - G^T Z_3)$
  update $\mathcal{S}$: $\mathcal{S}(:) = (\rho_3 G^T G + \rho_1 I)^{-1}(\rho_3 G^T g + \rho_2 \mathcal{Q}(:) - \mathcal{Z}_2 - \rho_3 G^T G \mathcal{L}(:) - G^T Z_3)$
  dual update $\mathcal{Z}_1$: $\mathcal{Z}_1 = \mathcal{Z}_1 + \rho_1 * (\mathcal{L} - \mathcal{P})$
  dual update $\mathcal{Z}_2$: $\mathcal{Z}_2 = \mathcal{Z}_2 + \rho_2 * (\mathcal{S} - \mathcal{Q})$
  dual update $\mathcal{Z}_3$: $\mathcal{Z}_3 = \mathcal{Z}_3 + \rho_3 * (G(\mathcal{S}(:) + \mathcal{L}(:)) - g)$
end for