TCPCP problem:

$$\min_{\mathcal{L},\mathcal{S}} \|\mathcal{L}\|_* + \lambda\|\mathcal{S}\|_1, s.t. P_\Omega(\mathcal{S}+\mathcal{L}) = P_\Omega\mathcal{X}_0.$$

Rewrite the constraint into $G(\mathcal{S}(:)+\mathcal{L}(:)) = G\mathcal{X}_0(:) = g$, then we want to solve the following optimization problem,

$$\min_{\mathcal{L},\mathcal{S}} \|\mathcal{L}\|_* + \lambda\|\mathcal{S}\|_1 + \frac{\lambda_1}{2}\|G(\mathcal{S}(:)+\mathcal{L}(:)) - g\|_F^2.$$

In order to solve this problem by ADMM, we introduce $\mathcal{P}, \mathcal{Q}$ and then obtain the following optimization problem,

$$\min_{\mathcal{S},\mathcal{L},\mathcal{P},\mathcal{Q}} \|\mathcal{L}\|_* + \lambda\|\mathcal{S}\|_1 + \frac{\lambda_1}{2}\|G(\mathcal{S}(:)+\mathcal{L}(:)) - g\|_F^2 + \frac{\lambda_2}{2}\|(\mathcal{L}-\mathcal{P})\|_F^2 + \frac{\lambda_3}{2}\|(\mathcal{S}-\mathcal{Q})\|_F^2$$

Solve this by ADMM by introducing two dual problems, the algorithm can be shown as follows,

Algorithm: Solve TCPCP by ADMM

1: Initialization $\mathcal{S} = \mathcal{L} = \mathcal{P} = \mathcal{Q} = \mathcal{Z}_1 = \mathcal{Z}_2 = zeros(dim(X_0))$
2: **for** $k = 1, 2, \cdots$ **do**
3: update $\mathcal{P}$: $\mathcal{P} = D_{\frac{1}{\lambda_2}}(\mathcal{L} + \mathcal{Z}_1)$
4: update $\mathcal{Q}$: $\mathcal{P} = S_{\frac{\lambda}{\lambda_3}}(\mathcal{S} + \mathcal{Z}_2)$
5: update $\mathcal{L}$: $\mathcal{L}(:) = (\lambda_1 G^T G + \lambda_2 I)^{-1}(\lambda_1 G g + \lambda_2 \mathcal{P}(:) - \lambda_1 G^T G \mathcal{S}(:))$
6: update $\mathcal{S}$: $\mathcal{S}(:) = (\lambda_1 G^T G + \lambda_3 I)^{-1}(\lambda_1 G g + \lambda_3 \mathcal{Q}(:) - \lambda_1 G^T G \mathcal{L}(:))$
7: dual update $\mathcal{Z}_1$: $\mathcal{Z}_1 = \mathcal{Z}_1 + \mathcal{L} - \mathcal{P}$
8: dual update $\mathcal{Z}_2$: $\mathcal{Z}_2 = \mathcal{Z}_2 + \mathcal{S} - \mathcal{Q}$
9: **end for**

where the updates for $\mathcal{Z}_1$, $\mathcal{Z}_2$ are dual ascent for sub-optimization problem:

- $\mathcal{Z}_1$: $\min_{\mathcal{P}} \|\mathcal{P}\|_* + \frac{\lambda_2}{2}\|\mathcal{L}-\mathcal{P}\|_F^2$
- $\mathcal{Z}_2$: $\min_{\mathcal{Q}} \|\mathcal{Q}\|_1 + \frac{\lambda_2}{2}\|\mathcal{S}-\mathcal{Q}\|_F^2$