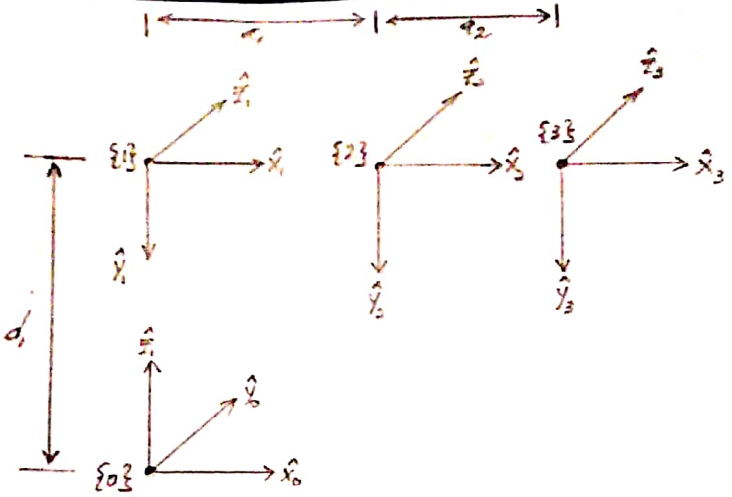
FORWARD KINEMATICS



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	$-90°$	0	d_1	θ_1
2	0	a_1	0	θ_2
3	0	a_2	0	θ_3

$\therefore$ By using the formula to determine the transpose :

$$
{}^{i-1}_{i}T = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\therefore {}^{0}_{1}T = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \quad {}^{1}_{2}T = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & a_1\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & a_1\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}_{3}T = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & a_2\cos\theta_3 \\ \sin\theta_3 & \cos\theta_3 & 0 & a_2\sin\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\therefore {}^{0}_{3}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T
$$

$$
{}^{0}_{1}T\,{}^{1}_{2}T = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & a_1\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & a_1\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}_{1}T\,{}^{1}_{2}T = \begin{bmatrix} \cos\theta_1\cos\theta_2 & -\cos\theta_1\sin\theta_2 & -\sin\theta_1 & a_1\cos\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 & -\sin\theta_1\sin\theta_2 & \cos\theta_1 & a_1\sin\theta_1\cos\theta_2 \\ -\sin\theta_2 & -\cos\theta_2 & 0 & -a_1\sin\theta_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$\therefore\ {}^0_1T\,{}^1_2T\,{}^2_3T = \begin{bmatrix} c\theta_1 c\theta_2 & -c\theta_1 s\theta_2 & -s\theta_1 & a_1 c\theta_1 c\theta_2 \\ s\theta_1 c\theta_2 & -s\theta_1 s\theta_2 & c\theta_1 & a_1 s\theta_1 c\theta_2 \\ -s\theta_2 & -c\theta_2 & 0 & -a_1 s\theta_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_2 c\theta_3 \\ s\theta_3 & c\theta_3 & 0 & a_2 s\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0_1T\,{}^1_2T\,{}^2_3T = \begin{bmatrix} c\theta_1 c\theta_2 c\theta_3 & -c\theta_1 s\theta_2 s\theta_3 & -s\theta_1 & a_2 c\theta_1 c\theta_2 c\theta_3 - a_2 c\theta_1 s\theta_2 s\theta_3 + a_1 c\theta_1 c\theta_2 \\ s\theta_1 c\theta_2 c\theta_3 - s\theta_1 s\theta_2 s\theta_3 & -s\theta_1 c\theta_2 s\theta_3 - s\theta_1 s\theta_2 c\theta_3 & c\theta_1 & a_2 s\theta_1 c\theta_2 c\theta_3 - a_2 s\theta_1 s\theta_2 s\theta_3 + a_1 s\theta_1 c\theta_2 \\ -s\theta_2 c\theta_3 - c\theta_2 s\theta_3 & s\theta_2 s\theta_3 - c\theta_2 c\theta_3 & 0 & -a_2 s\theta_2 c\theta_3 - a_2 c\theta_2 s\theta_3 - a_1 s\theta_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By using trigonometric identities:

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$$\therefore\ {}^0_1T\,{}^1_2T\,{}^2_3T = \begin{bmatrix} c\theta_1 c\theta_2 c\theta_3 & -c\theta_1 s\theta_2 s\theta_3 & -s\theta_1 & a_2 c\theta_1 c(\theta_2+\theta_3) + a_1 c\theta_1 c\theta_2 \\ s\theta_1 c(\theta_2+\theta_3) & -s\theta_1 s(\theta_2+\theta_3) & c\theta_1 & a_2 s\theta_1 c(\theta_2+\theta_3) + a_1 s\theta_1 c\theta_2 \\ -s(\theta_2+\theta_3) & -c(\theta_2+\theta_3) & 0 & -a_2 s(\theta_2+\theta_3) - a_1 s\theta_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let's:

$c(\theta_2+\theta_3) = c\theta_{23}$

$s(\theta_2+\theta_3) = s\theta_{23}$

$$\therefore\ {}^0_1T\,{}^1_2T\,{}^2_3T = \begin{bmatrix} c\theta_1 c\theta_2 c\theta_3 & -c\theta_1 s\theta_2 s\theta_3 & -s\theta_1 & a_2 c\theta_1 c\theta_{23} + a_1 c\theta_1 c\theta_2 \\ s\theta_1 c\theta_{23} & -s\theta_1 s\theta_{23} & c\theta_1 & a_2 s\theta_1 c\theta_{23} + a_1 s\theta_1 c\theta_2 \\ -s\theta_{23} & -c\theta_{23} & 0 & -a_2 s\theta_{23} - a_1 s\theta_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

INVERSE KINEMATICS

From the Forward Kinematic ${}^0_1T\,{}^1_2T\,{}^2_3T$; the positions are given as:

$$P_x = a_3 c\theta_1 c\theta_{23} + a_1 c\theta_1 c\theta_2$$

$$P_y = a_3 s\theta_1 c\theta_{23} + a_1 s\theta_1 c\theta_2$$

$$P_z = -a_3 s\theta_{23} - a_1 s\theta_2 + d_1$$

$$r = \sqrt{P_x^2 + P_y^2}$$

$$r = \sqrt{(a_3 c\theta_1 c\theta_{23} + a_1 c\theta_1 c\theta_2)^2 + (a_3 s\theta_1 c\theta_{23} + a_1 s\theta_1 c\theta_2)^2}$$

$$r = a_1^2 c^2\theta_1 c^2\theta_2 + a_3^2 c^2\theta_1 c^2\theta_{23} + 2a_1a_3 c^2\theta_1 c\theta_2 c\theta_{23} + a_3^2 s^2\theta_1 c^2\theta_{23} + a_1^2 s^2\theta_1 c^2\theta_2 + 2a_1a_2 s^2\theta_1 c\theta_2 c\theta_{23}$$

$$r = \sqrt{(a_1 c\theta_2 + a_3 c\theta_{23})^2} = a_1 c\theta_2 + a_3 c\theta_{23} \quad \cdots \text{①}$$

$$h = P_z - d_1$$

$$h = -(a_1 s\theta_2 + a_3 s\theta_{23}) \quad \cdots \text{②}$$

$$\therefore P_x^2 + P_y^2 + (P_z - d_1)^2 = r^2 + h^2 \quad \cdots \text{③}$$

$$r^2 + h^2 = (a_1 c\theta_2 + a_3 c\theta_{23})^2 + (-a_1 s\theta_2 - a_3 s\theta_{23})^2$$

$$= a_1^2 c^2\theta_2 + a_3^2 c^2\theta_{23} + 2a_1a_3 c\theta_2 c\theta_{23} + a_1^2 s^2\theta_2 + a_3^2 s^2\theta_{23} + 2a_1a_3 s\theta_2 s\theta_{23}$$

$$= a_1^2 (c^2\theta_2 + s^2\theta_2) + a_3^2 (c^2\theta_{23} + s^2\theta_{23}) + 2a_1a_3 (c\theta_2 c\theta_{23} + s\theta_2 s\theta_{23})$$

$$= a_1^2 + a_3^2 + 2a_1a_3 c(\theta_2 + \theta_3 - \theta_2)$$

$$= a_1^2 + a_3^2 + 2a_1a_3 c\theta_3$$

$\therefore$ From ③:

$$P_x^2 + P_y^2 + (P_z - d_1)^2 = a_1^2 + a_3^2 + 2a_1a_3 c\theta_3$$

$$c\theta_3 = \frac{P_x^2 + P_y^2 + (P_z - d_1)^2 - a_1^2 - a_3^2}{2a_1a_2}$$

$$\therefore \theta_3 = \mathrm{ATAN2}(s\theta_3, c\theta_3)$$

From the trigonometric identities:

$$c^2\theta_3 + s^2\theta_3 = 1$$

$$\therefore s\theta_3 = \pm\sqrt{1 - c^2\theta_3}$$

∴ From ①:

$$r = a_1 c\theta_2 + a_2 c\theta_{23}$$

$$r = a_1 c\theta_2 + a_2 (c\theta_2 c\theta_3 - s\theta_2 s\theta_3)$$

$$r = a_1 c\theta_2 + a_2 c\theta_2 c\theta_3 - a_2 s\theta_2 s\theta_3$$

∴ From ②:

$$h = -(a_1 s\theta_2 + a_2 s\theta_{23})$$

$$h = -a_1 s\theta_2 - a_2 (s\theta_2 c\theta_3 + c\theta_2 s\theta_3)$$

$$h = -a_1 s\theta_2 - a_2 s\theta_2 c\theta_3 - a_2 c\theta_2 s\theta_3$$

let's $k_1 = a_1 + a_2 c\theta_3$ and $k_2 = a_2 s\theta_3$

$$\therefore r = c\theta_2 (a_1 + a_2 c\theta_3) - a_2 s\theta_2 s\theta_3$$

$$r = k_1 c\theta_2 - k_2 s\theta_2 \quad \cdots ④$$

$$\therefore h = -(a_1 + a_2 c\theta_3) s\theta_2 - a_2 s\theta_3 c\theta_2$$

$$h = -k_1 s\theta_2 - k_2 c\theta_2 \quad \cdots ⑤$$

$$D = +\sqrt{k_1^2 + k_2^2} \quad ; \quad \gamma = \tan^{-1}\left(\frac{k_2}{k_1}\right)$$

$$k_1 = D\cos\gamma \quad ; \quad k_2 = D\sin\gamma$$

∴ From ④:

$$r = D c\gamma c\theta_2 - D s\gamma s\theta_2$$

$$\frac{r}{D} = c\gamma c\theta_2 - s\gamma s\theta_2$$

$$\frac{r}{D} = c(\gamma + \theta_2) \quad \cdots ⑥$$

∴ From ⑤:

$$h = -D c\gamma s\theta_2 - D s\gamma c\theta_2$$

$$\frac{h}{D} = -(c\gamma s\theta_2 + s\gamma c\theta_2)$$

$$\frac{h}{D} = -\sin(\gamma + \theta_2) \quad \cdots ⑦$$

$$\frac{⑦}{⑥} : \frac{h}{r} = -\frac{\sin(\gamma+\theta_2)}{\cos(\gamma+\theta_2)} = -\tan(\gamma+\theta_2)$$

$$\tan(\gamma+\theta_2) = -\frac{h}{r}$$

$$\gamma + \theta_2 = \mathrm{ATAN2}\left(-\frac{h}{r}\right)$$

$$\theta_2 = \mathrm{ATAN2}\left(-\frac{h}{r}\right) - \gamma$$

$$\theta_2 = \mathrm{ATAN2}\left(-\frac{h}{r}\right) - \mathrm{ATAN2}\left(\frac{k_2}{k_1}\right)$$

$$\therefore \theta_1 = \mathrm{ATAN2}\left(\frac{P_y}{P_x}\right)$$

$$\theta_1 = \mathrm{ATAN2}(P_y, P_x)$$

To determine the angle per pulse, first we have to refer to the datasheet to obtain the cycles per main shaft revolution of the desired motor. The motor that we used in the project is SPG30E-60K. The specifications of the motor are:

- Rated voltage : 12VDC
- No load current : <100mA
- No load speed : 75 ± 7.5 RPM
- Rated load torque : 294mN.m (3kgf.cm)
- Rated current : < 600mA
- Rated load speed : 50 ± 5RPM
- Gear Ratio : 60:1
- Shaft size : D-shaped with 6mm diameter, 15.5mm in length
- Resolution of the encoder output:
  - ↳ 7 pulses per rear shaft revolution, single channel output, either channel A or B.
  - ↳ 420 counts per main shaft revolution, single channel output, either channel A or B.
- Quadrature hall effect encoder.

∴ From the specifications:

N = cycles per main shaft revolution × no. of hall effector

$N = 420 \times 4$

$N = 1680$ pulses per main shaft revolution.

Since a complete cycle consisted of 360°; ∴ the angle per pulse per main shaft revolution (AP) is

$$AP = \frac{360^\circ}{N} = \frac{360^\circ}{1680}$$

$AP = 0.2143^\circ$ per pulse per main shaft revolution

From the measurement of the hardware, $a_1 = 20\,cm$; $a_3 = 20\,cm$ and $d_1 = 3\,cm$

Now by considering the $\theta_1 = 45°$; $\theta_2 = -90°$; and $\theta_3 = -45°$

∴ By using forward kinematic equation, we can determine the location of the end effector :

$$\therefore {}^0_1T\,{}^1_2T\,{}^2_3T = \begin{bmatrix} c(45°)c(-90°)c(45°) & -c(45°)s(-90°)s(45°) & -s(45°) & 20c(45°)c(-90°+45°)+20c(45°)c(-90°) \\ s(45°)c(-90°+45°) & -s(45°)s(-90°+45°) & c(45°) & 20s(45°)c(-90°+45°)+20s(45°)c(-90°) \\ -s(-90°+45°) & -c(-90°+45°) & 0 & -20s(-90°+45°)-20s(-90°)+3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.5 & -0.7071 & 10 \\ 0.5 & 0.5 & 0.7071 & 10 \\ 0.7071 & -0.7071 & 0 & 37.1421 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By using the GUI build with Visual C Sharp, we can check if we can obtain the same angle when :

$P_x = 10$

$P_y = 10$

$P_z = 37.1421$

with angles :

$\theta_1 = 45°$

$\theta_2 = -90°$

$\theta_3 = 45°$

Scanned by CamScanner