UNIT 2
SETS AND FUNCTIONS

Contents:

Unit Introduction

In the last unit we discuss the basic concept and principles of the algebra. In this unit you have to study further concept and principles of the algebra. The knowledge of algebra very useful for solving equation. Inequalities, and analyzing basic properties of the function.
In this unit there are 5 number of sessions.

1. Basic Concept of Sets
2. Inequalities
3. Basic Concept of Relation and Function
4. Polynomials and Partial Fractions
5. Sequences, Series and Binomial Theorem

The knowledge of the above topics you can very useful for your education background

Session 6

Basic Concepts of Sets

Contents:

Introduction

The concepts of set are fundamental in all the branches in Mathematics. Sets are the most useful basic tools of Mathematics. Which are extensively used in developing the foundation of relations, functions, logic theory and probability theory?

Most of the concepts and results in mathematics could be expressed in the set theoretic language.

The set theory was developed by the German mathematician, George Canter (1845-1918). In this study session, you are going to learn some basic definitions, and operations involving sets.

6.1 Definition of a set

A set is a collection of distinct objects. We can represent sets in nature by groups of objects such as a stock of seeds, collection of followers, group of farmers, group of cultivated items etc., depending on the characteristic of objects.

Therefore, a set is a well-defined collection of objects. “Well defined”, means that the objects follow a given rule or rules. Using this rule or rules, we will be able to decide whether given object belongs to this set or not.

Example 1

Some examples for sets are

(i) The set of odd numbers {1,3,5,7,9}

(ii) The set of cultivated items {paddy, cinnamon, coconut, rubber, tea}

(iii) The Irrigations projects in Sri Lanka {Mahaweli, Valawe, Galoya, Moragahakanda…}

Notations

Sets are usually denoted by the capital letters A, B, P, Q etc., and their elements (the objects that are in a set) by small letters a, b, x, y etc.

Let X be any set of objects and ‘x’ be a member (element) of X;

It is denoted by$x \in X$ and read it as “x belongs to X” or “x is an element of X”.

If y is not an element (object) of X, then we denote it as $y \notin X$ and read it as “y does not belong to X” or “y is not an element of X” or “y is not a member of X”.

Representation of sets

A set can be represented in two ways. They are

(a) Tabular Form or Roster Form

(b) Set Builder Form or Rule Method

Tabular Form

In this method, we list all the members of the set separating them by means of commas and enclosing them in curly brackets { }.This method is also known as Roster Form

Example 2

Let X be the set consisting of the number multiple of 3, between 2 and 16.

$$X = \{3, 6, 9, 12, 15\}$$

Set Builder Form

In this method, instead of listing all elements, we write the set by some special property or properties satisfied by the elements and write it as

$$X = \{x: P(x)\} \text{ or } A = \{x |\ x \text{ has the property } P(x)\}$$

and read it as "X is the set of all elements x such that x has the property P".

This form is also known as the rule method.

Example 3

Let X be the set that consists of the numbers, which lie between 2 and 16, and can be divided by 3.

$$X = \{x: 2 < x < 16, x = 3n, n\ is\ an\ integer\}$$

Activity 1

(a) Write down the following sets in the roster form

(i) $A = \{x: x$ is two digits number such that the sum of its digits is 10}

(ii) B ={The set of all letters that occurs in ENGINEERING TECNOLOGY}

(iii) $C = \{x: x$ is an integer of multiple of 4 and $4 \leq x \leq 48$ }

(b) Write down the following sets in the builder form

(i) $A = \{13,26,39 \ldots 91\}$

(ii) $B = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots, 97\}$

(iii) $C = \{1, {}^1/_2, {}^1/_3, {}^1/_4, \ldots\}$

6.2 Some special sets

In this section, you can learn some special sets.

(I) Finite and Infinite Sets

A set which have no elements or a finite number of elements is called a finite set.

Example 4

Examples for finite sets

i. A = The set of prime number less than 20

$$A = \{2,3,5,7,11,13,19\}$$

ii. B =The set of vowels that occur in the words ENGINEERING TECHNOLOGY

$$B = \{E, I, O\}$$

iii. C $=\{x|\ x$ is a divisor of 72$\}$

C $=\{1,2,3,4,6,8,9,12,18,24,36,72\}$

All the sets in above example contain finite number of elements, and hence they are finite sets.

If X $=\{3, 6, 9, 12, 15, 18\}$, then $n(X) = 6$.

A set having infinite number of elements is called an infinite set. Thus, in an infinite set, if the elements are counted one by one, the counting process never comes to an end.

Example 5

Examples for finite sets

(i) The set of all the natural numbers $= \{1,2,3, \ldots\}$

(ii) The set of all the points on a given plane

(iii) $\{x : x \in \mathbb{R}, 10 < x < 11\}$

All the sets in above example contain infinite number of elements, and hence they are infinite sets.

Note that all infinite sets cannot be described in the roster form. For an example, the set of all the points on a given plane.

Cardinal Number

If X is a finite set, then the number of distinct elements in the set X is called the Cardinal number of the set X. If X is an infinite set the Cardinal number of X is ininfinty. The Cardinal number of the set X is denoted by $n(X)$.

(II) Null set

A set that contains no element is called the Null set or Empty set.

The null set is denoted by ϕ or { }.

The empty set is also a finite set.

Example 6

Examples for finite sets

(i) The set of odd numbers divisible by 2 is a null set

(ii) $\{x : x \in \mathbb{Z} \text{ and } x^2 = 3\} = \phi$

Because there is no integer whose square is 3

(iii) $\{x : x \in \mathbb{R} \text{ and} x^2 + 3 = 0\} = \phi$

Because the square of a real number is never negative

All the sets in above example contain no elements, and hence they are empty sets.

(III) **Singleton**

A set containing only one element is called a Singleton.

Example 7

Examples for Singleton

(i) $A = $ {The set of even prime numbers}

$$\therefore A = \{2\}$$

(ii) $B = \{x: x \text{ in an integer and } 3 < x < 5\}$

$B = \{4\}$ All the sets in above example contain only one element, and hence they are Singleton.

(IV) EQUAL SETS

Two sets A and B are said to be equal if they consist of the same elements. Then we write $A = B$.

$A = B$ if and only if every element of A is an element of B and every element of B is an element of A. This statement can be written in symbolic form as follows.

$A = B \Leftrightarrow x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$.

We denote $X \neq Y$ to indicate that two sets X and Y are not equal.

Example 8

Example for equal sets.

If $A = \{x: x \in \mathbb{N}\ and\ |x| < 5\}$ and $B = \{1,2,3,4\}$

$A = \{1,2,3,4\}$ and $B = \{1,2,3,4\}$

$\therefore\ A = B$

Activity 2

(a) State whether the following sets are finite or infinite.

i. $P = \{x: x \in \mathbb{Z} \text{ and } x^2 - 2x - 3 = 0\}$

ii. $Q =$ The set of natural numbers which are divisible by 5

iii. .$R =$ The set of points that lie on a circle.

iv. $S = \{x: x \in \mathbb{Z} \text{ and } x^2 = 64\}$.

(b) Which of the following sets are null sets?

i. $A = \{x: x$ is a natural number and $x < 7$ and $x > 9\}$

ii. $B = \{$set of even prime number$\}$

iii. $C = \{$set of odd numbers divisible by $4\}$

iv. $D = \{x: x$ is a point common to any two parallel lines$\}$

(c) Are the following sets equal?

$A = \{x: x$ is the letter in the word WOLF$\}$

$B = \{ x: x$ is the letter in the word FOLLOW$\}$

$C = \{ x: x$ is the letter in the word LOOWF $\}$

(d) Show that the following sets are equal.

$A = \{2,1\}$ $B = \{2,1,1,2,1\}$

$C = \{ x: x^2 - 3x + 2 = 0\}$

(e) Are the following sets equal ?

$A =$ The set of letters in the word MATHEMATICS

$B =$ The set of letters in the word MATCHES

(V) Sub set of a set

If A and B are two sets, then B is called a subset of A if every element of B is also an element of A. It is denoted by $B \subseteq A$, or $A \supseteq B$.

(i) $B \subseteq A$is read as A contains B or A is super-set B.

Proper Subset of a set

A set B is said to be a proper subset of the set A if every element of set B is an element of A, and there exists an element of Awhich is not an element of B. It is denoted by $B \subset A$.

We read it as "B is a proper subset of A".

Therefore, B is a proper subset of A if every element of B is an element of A, and there exists at least one element in A which is not in B.

Example 9

(i) If $A = \{1,2,5\}$ $B = \{1,2,3,4,5\}$

$\therefore A \subset B$ [A is a proper subset of B]

(ii) $\mathbb{N} = \{$All the natural number$\}$

$\mathbb{Z} = \{$All integers$\}$

$$\mathbb{N} = \{1,2,3 \ldots\}, \mathbb{Z} = \{0,1,2,3 \ldots\}$$

$$\therefore \mathbb{N} \subset \mathbb{Z} \text{ and } \mathbb{N} \neq \mathbb{Z}$$

(VI) Universal set

A universal set is the collection of all the objects in a particular context or theory. Universal set is denoted by $\mathcal{U}$.

Example 10

Let $A = \{2,4,6,8,12,14\}$

$B = \{3,6,9,12,15\}$

$C = \{4,8,12\}$ and

$\mathcal{U} = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14\}$ be given sets.

Now we can see that, A, B and C are subsets of $\mathcal{U}$.

$$A \subset \mathcal{U}, \quad B \subset \mathcal{U}, \quad C \subset \mathcal{U}$$

Hence the set $\mathcal{U}$ can be taken as the universal set.

Activity 3

1) Replace ($\star$) by $\subseteq$ or $\nsubseteq$ to make the statement correct in the following.

a) (3,9,12) $\star$ (2,4,9,12)

b) (6,7,8,9,10) $\star$ (5,6,7,8,9,10,11)

c) $\{x: x \in \mathbb{N}$ and $x \geq 7\} \star \{6,7,8,9,10,11\}$

d) $\{x: x$ is a quadrilateral with equal sides$\} \star \{x: x$ is a square$\}$

e) $\{x: x$ is a square$\} \star \{x: x$ is a quadrilateral with equal sides$\}$

2) *Let* $A = \{0,1,2,3,4,5\}$

$B = \{0,1,2,3,4\}$

$C = \{2,4\}$

Find all sets X such that

(a) $X \subset B$ *and* $X \subset C$

(b) $X \subset A$ *and* $X \nsubseteq B$

3)

a) List all the subsets of $X = \{a, b, c, d\}$

b) List all the proper subsets of $Y = \{1,2,3,4\}$

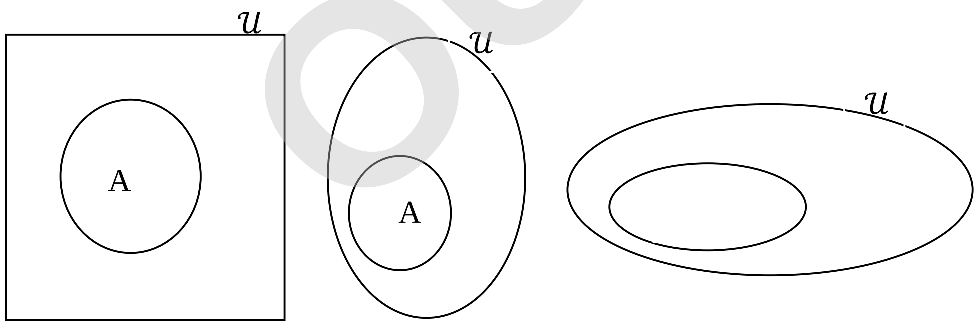
6.3 Venn-diagrams

In order to visualize and illustrate any property or theorem relating to universal sets, their subsets and certain operations on sets, John Venn, a British Mathematician developed pictorial representations called "Venn diagrams".

Illustrations of Certain Relationships between Sets by Venn Diagrams

(i) If $\mathcal{U}$ be set of letters of English alphabets and A the set of vowels. Then, $A \subset \mathcal{U}$.

If $A \subset \mathcal{U}$ and $A \neq \mathcal{U}$ then A and $\mathcal{U}$ can be represented by either of the below diagrams Figure 6.3.1(a), 6.3.1(b), 6.3.1(c)



(a) (b) (c

Figure 6.3.1 the venn-diagram when X and Y are not comparable

If the sets X and Y are not comparable, then neither of X or B is a subset of the other. This fact can be represented by either of the below diagrams the other. This fact can be represented by either of the below diagrams

Suppose X and Y are not comparable. Then X and Y can be represented by the diagram on the right if they are disjoint, or the diagram on the left if they are not disjoint.

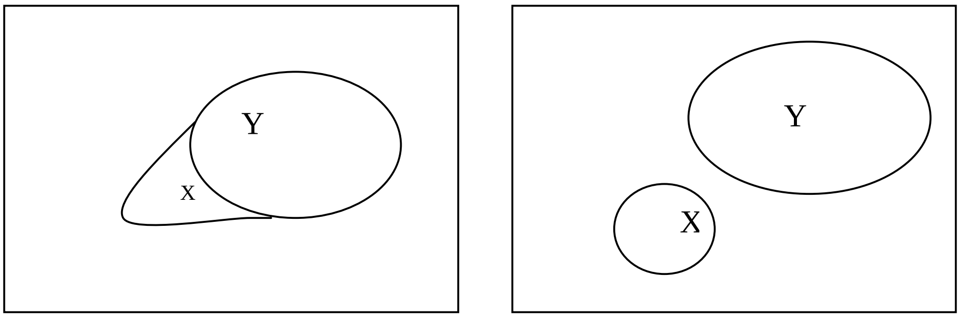


Figure 6.3.2 Venn diagrams when A & B are not comparable

If $X = \{1,2,3,4,5\}$ and $Y = \{7,8,9\}$ then X and Y are disjoint. These can be illustrated by Venn diagram figure 1.3.3

$\therefore\ X \nsubseteq Y\ ,\ Y \nsubseteq X.$

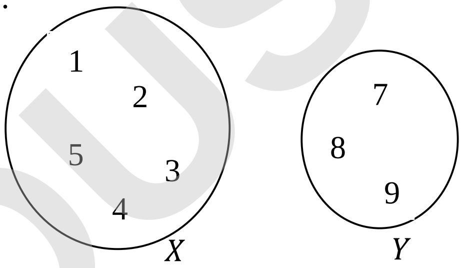
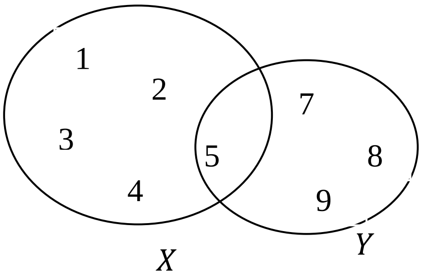


Figure 6.3.3

If X ={1,2,3,4,5} and $Y = \{5,7,8,9\}$, then itcan be illustrated by Venn diagram figure 1.3.4



Here, $\nsubseteq Y\ ,\ Y \nsubseteq X$

Figure 6.3.4

Compliment set

Let $X \subseteq \mathcal{U}$.The set consists of the elements in the universal set but not in the set X is called the compliment set of the set X.

The complement of the subset X in $\mathcal{U}$ is denoted by "X'" or "X^c".

$$\therefore X' = X^c = \{ x : x \in \mathcal{U}, x \notin X\}$$

Notice that $\mathcal{U}^c = \emptyset$ and $\emptyset^c = \mathcal{U}$.

Let $\mathcal{U}$ ={1,2,3,4,5,6,7,8,9}

X ={2,3,4}

$X' = X^c$ ={1,5,6,7,8,9}

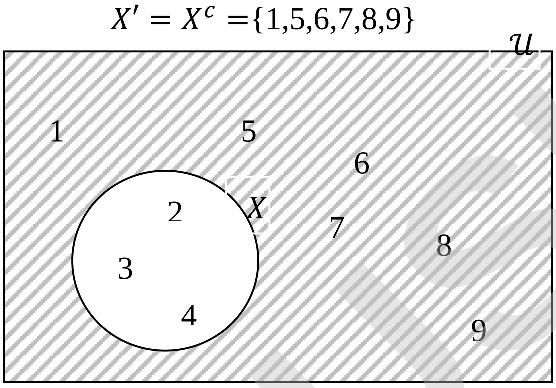


Figure 6.3.5 (Representation of X' by Venn diagram)

6.4 Operations on sets

Union of sets

Let A and B be two given sets, Then the union of A and B is the set of all those elements which belong to either A or B or both.

The union of the sets A and B is denoted by "$\boldsymbol{A} \cup \boldsymbol{B}$" and is read as "***A* union *B***". $\therefore A \cup B = \{ x$:either $x \in A$ or $x \in B\}$

Note that the union set contains all the elements of A and B, except that the common elements of both A and B are exhibited only once.

Example 11

If $A = \{1,9,16,18,21,22,23\}$

B ={9,18,21,24}

Find $A \cup B$ and illustrate $A \cup B$ in a Venn diagram.

Answer

A = {1,9,16,18,21,22,23} and B ={9,18,21,24}

$A \cup B$ = {1,9,16,18,21,22,23,24}

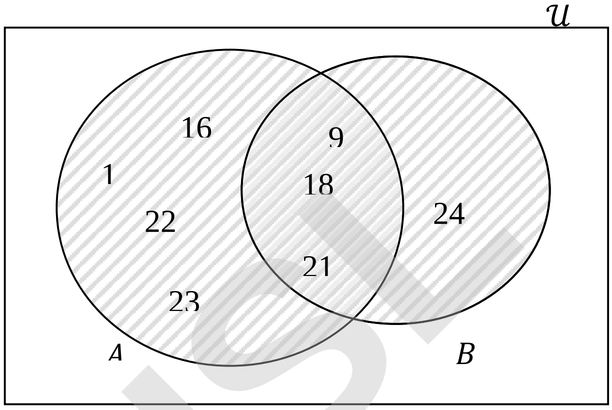


Figure 6.4.1 (Representation of $A \cup B$ by Venn diagram)

Intersection of sets

Let A and B be two given sets. Then the intersection of A and B is the set of elements which belong to both A and B. In other words, the intersection of A and B is the set of common members of A and B.

The intersection of A and B is denoted by "$A \cap B$" and is read as "A intersection B".

$$\therefore A \cap B = \{ x : x \in A \text{ and } x \in B\}$$

Example 12

Let A = *{6,8,9,10,12,16}*

B ={2,5,6,10,16}

$\therefore A \cap B$ ={6,10,16}

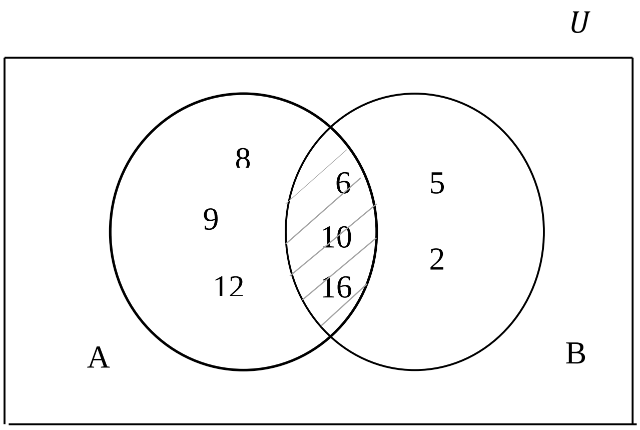


Figure 6.4.2 (Representation of $A \cap B$ by Venn diagram)

Disjoint Sets

If $A \subseteq \mathcal{U}$ and $B \subseteq \mathcal{U}$ and $A \cap B = \phi$, then A and B are said to be disjoint sets. For an example

$A = \{2,3,4\}, B = \{5,8,9\}$

Then A and B are disjoint sets since there are no common elements in A and B sets.

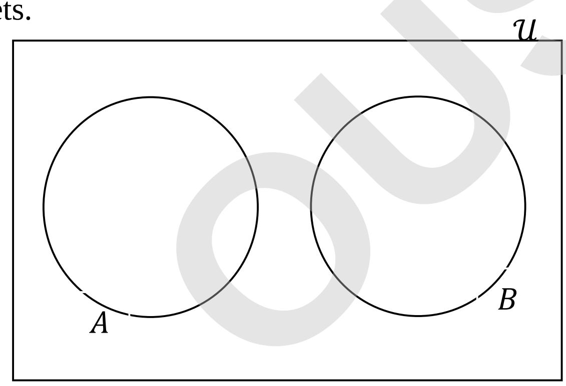


Figure 6.4.3 (Representation of disjoint sets A and B)

Difference of Sets

Let A and B be two given sets. The difference of sets A and B is the set of elements which belongs to A but not in B.

It is denoted by $A - B$ and read it as "A difference B",

$\therefore A - B = \{ x: x \in A \text{ and } x \notin B\}$

$B - A = \{ x: x \in B \text{ and } x \notin A\}$

In general, $A - B \neq B - A$.

Example 12

Let $A = \{9,12,15,17,20,21\}$

$B = \{9,12,14,16,18,21\}$

and $C = \{15,17,18,22\}$

Find $A - B$, $B - C$, $C - A$ and $B - A$.

$A - B = \{15, 17, 20\}$

$B - C = \{9,12,14,16,21\}$

$C - A = \{18,22\}$

$B - A = \{14,16,18\}$

<u>Representation of $(A - B)$ by Venn Diagram</u>

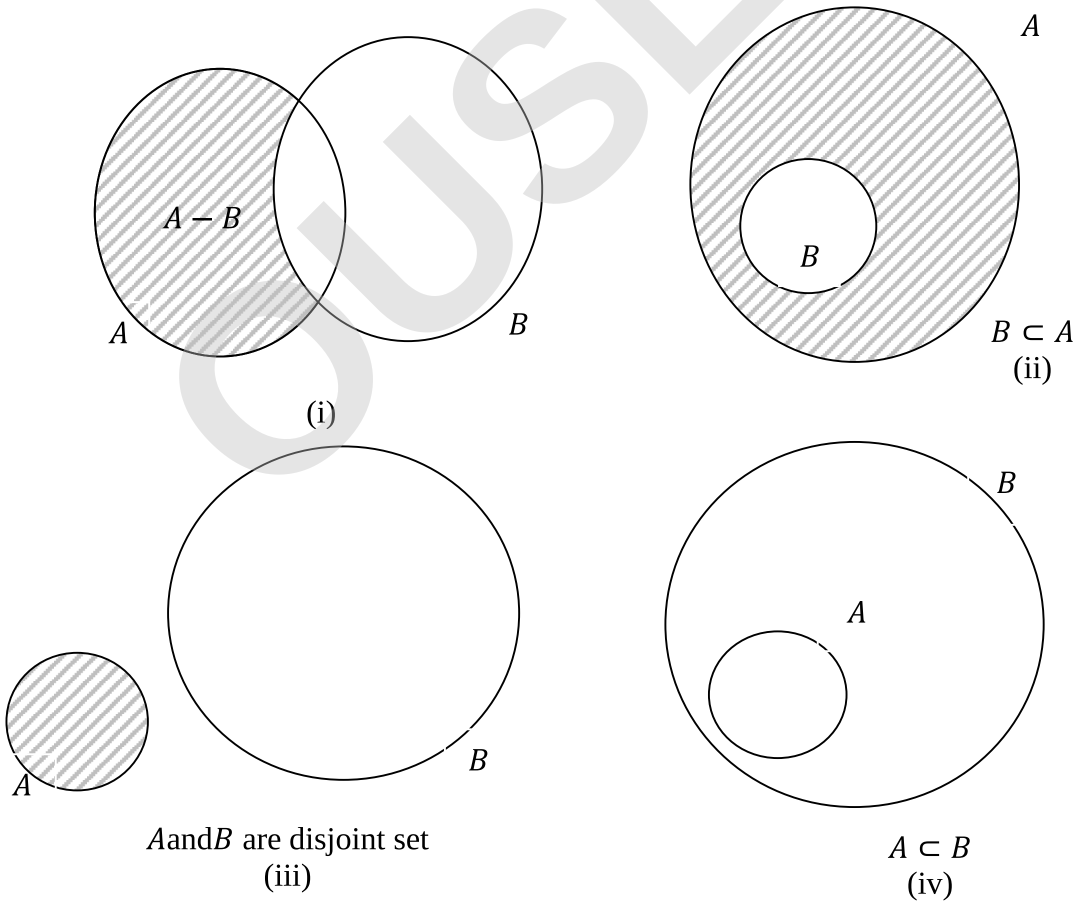


Figure 6.4.4

(i) $A \cap B \neq \phi$

(ii) $B \subset A$

(iii) A and B are disjoint set. $A - B = A$

(iv) $A \subset B$. Does not exist $A - B = \phi$.

6.5 Laws of Algebra of sets

Operations on sets satisfy the following laws of algebra.

(a) **Identity Laws**

For any $A \subseteq \mathcal{U}$

(i) $A \cup \phi = A$ (ii) $A \cap \mathcal{U} = A$

(b) **Idempotent Laws**

For any $A \subseteq \mathcal{U}$

(i) $A \cup A = A$ (ii) $A \cap A = A$

(c) **Commutative Laws**

For any two sets A and B

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

(d) **Associative Laws**

For any three sets A, B and C

(i) $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$

(e) **Distributive Laws**

For any subsets A, B and C of $\mathcal{U}$

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(f) De Morgan's rule

For any two sets $A \subseteq \mathcal{U}$ and $B \subseteq \mathcal{U}$

(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

Example 13

1. If $A = \{1,2,3,4\}$ $B = \{3,4,5\}$ $C = \{4,5,7,8\}$

   Find

   (a) $A \cap B$

   (b) $B \cap C$

   (c) $B - C$

2. Let $\mathcal{U} = \{1,2,3,4,5,6,7,8\}$

   $A = \{1,2,3,4\}$ $B = \{3,4,6\}$ $C = \{5,6,7,8\}$

   Verify the following.

(a) $A \cup (B \cup C) = (A \cup B) \cup C$

(b) $A \cap (B \cap C) = (A \cap B) \cap C$

(c) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(d) $(A \cap B)' = (A' \cup B')$

(e) $(A \cup B)' = A' \cap B'$

(f) $A' \cap B = B - A$

3. Shade the following sets in the Venn diagram.

(i) $A' \cap (B \cup C)$

(ii) (ii) $A' \cap (C - B)$

4. Let $A = \{x : x \text{ is a natural number}\}$

$B = \{x : x \text{is an even natural number}\}$

$C = \{x : x \text{is an odd natural number}\}$

$D = \{ x : x \text{is a prime number}\}$

Find

(a) $A \cap B$ (b) $A \cap C$ (c) $A \cap D$

(d) $B \cap C$ (e) $B \cap D$ (f) $C \cap D$

Answers

1. $A = \{1,2,3,4\}$ $B = \{3,4,5\}$ $C = \{4,5,7,8\}$
   (a) $A \cap B = \{3,4\}$
   (b) $B \cap C = \{4,5\}$
   (c) $B - C = \{3\}$
2. $\mathcal{U} = \{1,2,3,4,5,6,7,8\}$

   $A = \{1,2,3,4\}$ $B = \{3,4,6\}$ $C = \{5,6,7,8\}$

   (a) $B \cup C = \{3,4,5,6,7,8\}$

   $A \cup (B \cup C) = \{1,2,3,4,5,6,7,8\} = \mathcal{U}$

   $A \cup B = \{1,2,3,4,6\}$

   $\therefore (A \cup B) \cup C = \{1,2,3,4,5,6,7,8\} = \mathcal{U}$

   $\therefore A \cup (B \cup C) = (A \cup B) \cup C = \mathcal{U}$

   (b) $B \cap C = \{6\}$ $\therefore A \cap (B \cap C) = \{\quad\} = \phi$

   $A \cap B = \{3,4\}$ $\therefore (A \cap B) \cap C = \{\quad\}$

   $\therefore A \cap (B \cap C) = (A \cap B) \cap C = \phi$

   (c) $A \cup (B \cap C) = \{1,2,3,4,6\}$

   $A \cup B = \{1,2,3,4,6\}$

   $A \cup C = \{1,2,3,4,5,6,7,8\}$

   $\therefore (A \cup B) \cap (A \cup C) = \{1,2,3,4,6\}$

   $\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(d) $(A \cap B)' = \{1,2,5,6,7,8\}$

$A' = \{5,6,7,8\}$ $\quad$ $B' = \{1,2,5,7,8\}$

$A' \cup B' = \{1, 2, 5, 6, 7, 8\}$

$\therefore (A \cap B)' = (A' \cup B')$

(e) $(A \cup B)' = \{7,8\}$

$A' \cap B' = \{7,8\}$

$\therefore (A \cup B)' = A' \cap B'$

(f) $A' = \{5,6,7,8\}$ $\quad$ $B - A = \{6\}$

$A' \cap B = \{6\}$

$\therefore A' \cap B = B - A$

3.

(i) $A' \cap (B \cup C)$

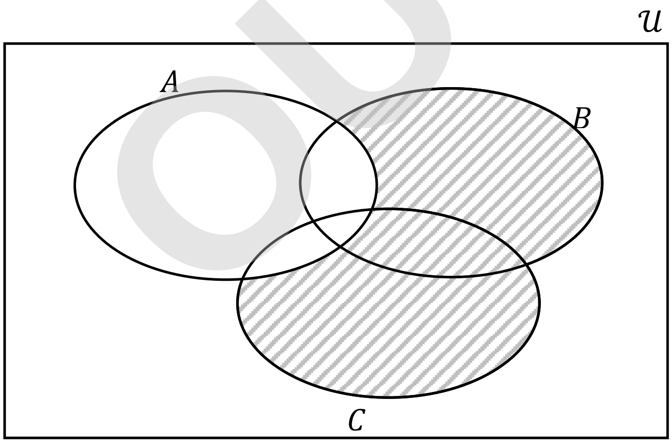


Figure 6.5.1 Venn diagram of $A' \cap (B \cup C)$

ii. $A' \cap (C - B)$

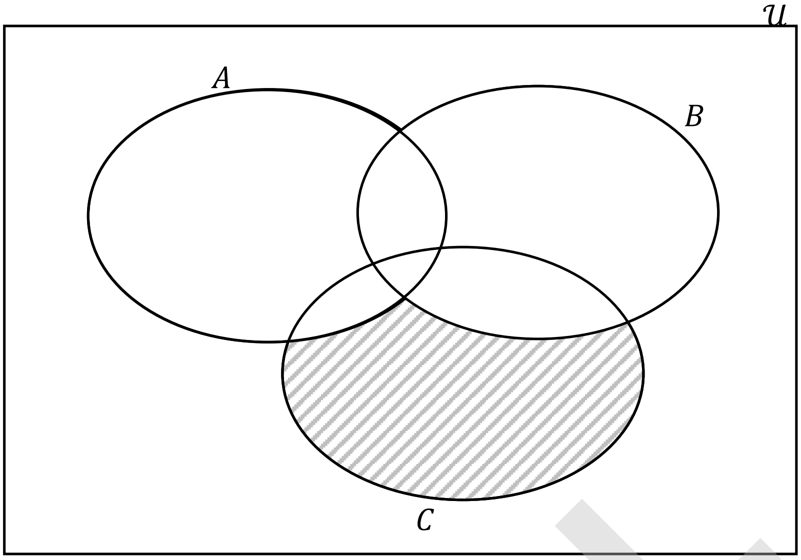


Figure 6.5.2 Venn diagram of $A' \cap (C - B)$

4. $A =$ {1, 2, 3, 4, 5, 6, 7, …}

   $B =$ {2, 4, 6, …}

   $C =$ {1, 3, 5, 7,}

   $D =$ {2, 3, 5, 7,}

(a) $A \cap B = B$

(b) $A \cap C = C$

(c) $A \cap D = D$

(d) $B \cap C = \phi$

(e) $B \cap D = \{2\}$

(f) $C \cap D = \{x : x$ is an odd prime number$\}$

Activity 4

1) If $\mathcal{U}$ = {1,2,3,4,5,6,7,8,9}, find the complements of the following sets

   (i) $A = \{2,4,6,8\}$ (ii) $B = \{1,3,5,7,9\}$

   (iii) $C = \{2,3,5,7\}$ (iv) ϕ

   (v) $\mathcal{U}$

2) If $\mathcal{U} = \{2,3,4,5,6,7,8,9\}$ $A = \{2,4,7\}$ $B = \{3,5,7,9,11\}$

   $C = \{7,8,9,10\}$

   Compute

   (i) $(A \cap \mathcal{U}) \cap (B \cup C)$ (ii) $C - B$

   (iii) $B - C$ (iv) $(B - C)'$

3) Verify $(A \cap B)' = A' \cup B'$ and $(A \cup B)' = A' \cap B'$ where

   $A = \{2,3,4,5,6\}$ $B = \{3,6,7,8\}$ $\mathcal{U} = \{1,2,3,4,5,6,7,8\}$

4) If $A = \{4,5,8,12\}$ $B = \{1,4,6,9\}$ $C = \{1,2,3,4\}$ then find

(i) $A - (B - A)$ (ii) $A - (C - B)$

5) Verify that by using the Venn diagram

a) $(A \cup B) \cup C = A \cup B \cup C$

b) $(A \cap B) \cap C = A \cap B \cap C$

c) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

d) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

e) $(A \cup B)' = A' \cap B'$

f) $(A \cap B)' = A' \cup B'$

6.6 Applications of sets

Up to now we were discussing in this session, few operations on an abstract set. We shall now discuss the practical applications of these operations of set theory.

Formula for $n(A \cup B)$

Let A and B two sets

(a) A and B are disjoint $A \cap B = \phi$

(b) A and B are not disjoint $A \cap B \neq \phi$

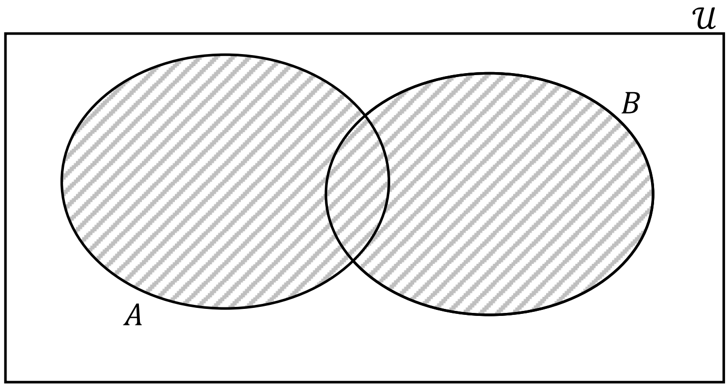


Figure 6.6.1

The number of elements in a set X is denoted by $n(X)$. From the Venn diagram [figure 6.6.1],

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

If A and B are disjoint sets, then $n(A \cup B) = n(A) + n(B)$

Example 14

In a society of farmers, 153 farmers cultivated paddy, 78 farmers cultivated vegetables, and 35 farmers cultivated both paddy and vegetables. Find the total number of farmers in the society.

Answer

Let V be the set of famers who cultivate vegetables.

Let P be the set of famers who cultivate paddy and
$\mathcal{U}$ be the set of farmers.

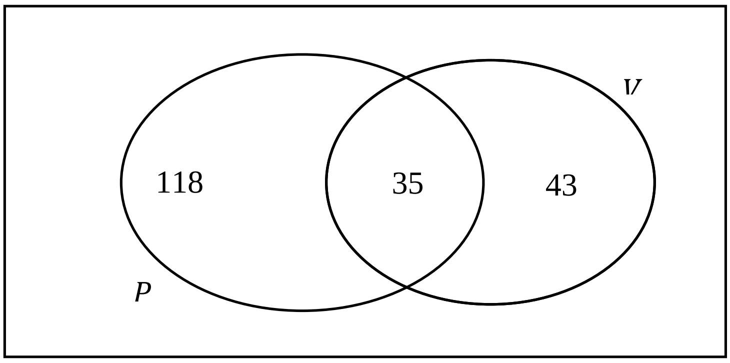


Figure 6.6.2 : Venn diagram for Example 14

Total number of farmers = $n(P \cup V)$

$$n(P \cup V) = n(P) + n(V) - n(P \cap V)$$

$$= 153 + 78 - 35$$

$$= 196$$

The number of farmers cultivated only paddy $= n(P - V) = 118$

The number of farmers cultivated only vegetable $= n(V - P) = 43$

Formula for $n(A \cup B \cup C)$

Let A,B and C are any three sets. Then,

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

Example 15

Every student in a class of 42 students, studies at least one of the subjects Plant Diversity, Animal Farming and Hybrid Technology for fruits. 14 students study Plant Diversity, 20 students study Animal Farming and 24 Hybrid Technology for fruits. 3 students study Plant Diversity and Animal Farming. 2 students study Animal Farming and Hybrid Technology for fruits. There is no student who studies all the three subjects. Find the number of students who study Plant Diversity but not Hybrid Technology.

Answer

Let P = the event which is related to Plant Diversity, A= the event which is related to Animal farming and H = the event which is related to Hybrid Technology.

$n(P) = 14$ $n(A) = 20$ $n(H) = 24$

$n(P \cap A) = 3$ $n(A \cap H) = 2$ $n(P \cap H) = ?$

$n(P \cap A \cap H) = 0$ $n(P \cup A \cup H) = 42$

$$n(P \cup A \cup H) = n(P) + n(A) + n(H) - n(P \cap A) - n(A \cap H) - n(H \cap P) + n(P \cap A \cap H)$$

42 = 14 + 20 + 24 – 3 – 2 – $n(P \cap H)$ + 0

$x = n(P \cap H) =$ 53 – 42 = 11

Given that $n(P \cap H \cap A) = 0$

∴ $n(P \cap H \cap A') = 11$ and $n(P \cap A \cap H') = 3$

∴ The number of students who study only Plant diversity

= 14 – (3+x) = 0

∴ The number of students who study Plant Diversity but not Hibrid Technology = 14 – 11 = 3

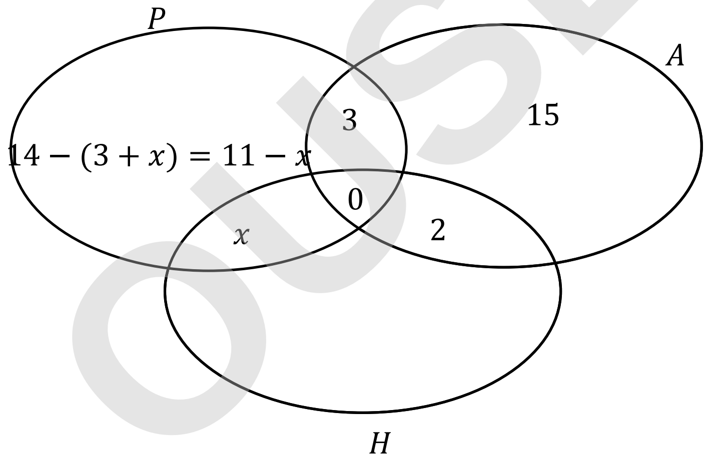


Figure 6.6.3 : the venn diagram for the example 15

Where $n(P \cap H) = x$

Total number = 42 = 11 – x + 3 + x + 15 + 24 – (2 + x) = 51 – x

x = 51 – 42

x = 11

∴ $n(P \cap H) = n(P \cap H \cap A') = 11$

But, $n(P) = 14$

$n(P \cap A \cap H') = 3$

$n(P \cap H \cap A') = x = 11$

$\therefore n(P \cap H' \cap A') = 14 - (3 + 11) = 0$

$\therefore n(P \cap H') = 3$

Activity 5

1) In a survey of 600 students in a college 150, were listed as drinking tea, 225 as drinking coffee and 100 were listed as both drinking tea as well as coffee. Find how many students were drinking neither tea nor coffee.
2) In a survey of 25 students, it was found that 15 had taken Biology, 12 had taken Agriculture and 11 had taken Chemistry. 5 had taken Biology and Chemistry. 9had taken Biology and Agriculture. 4 had taken Agriculture and Physics and 3 had taken all the three subjects. Find the number of students that had

i. Chemistry
ii. Biology
iii. Agriculture
iv. Agriculture and Chemistry but not Biology
v. Biology and agriculture but not Chemistry
vi. Only one of the subjects
vii. At least one of the three subjects
viii. None of the subjects

Solutions to Activities

Activity 1

(a) (i) $A = \{19,28,37,46,55,64,73,82,91\}$

(ii) $B = \{ E, N, G, I, T, C, O, L, Y \}$

(iii) $C = \{4,8,12,16,20,24,28,32,36,40,44,48\}$

(b) (i) $A = \{x: x$ is a multiple of $13, 1 < x < 100\}$

(ii) $B = \{x: x$ is a prime number less than $100\}$

(i) $C = \{x : x = \frac{1}{n}$, where n is a natural number$\}$

Activity 2

(a)

i. P ={-3,1} is finite

ii. Q ={5,10,15 …}is infinite

iii. R ={The set of all the points lie on a circle} infinite

iv. S ={-8,8} is a finite set.

(b)

i. A = Null set = { }

ii. B ={2} is not a null set

iii. C ={ } is a null set

iv. D ={ } is a null set

(c) A ={W,O,L,F}

B ={W, O, L, F}

C ={W, O, L, F}

$\therefore\ A = B = C;$

(d) A ={2,1}B ={2,1}C ={2,1}

$A = B = C$

(e) A ={M,A,T,H,I,C,S}

B ={M, A,T,H,E,S}

$\therefore\ A \neq B$

Activity 3

1)

(a) We cannot say anything regarding (3, 9, 12) and (2,4,9,12) since they not sets.

(b) We cannot say anything regarding (6,7,8,9,10) and (5,6,7,8,9,10,11) as they are not sets.

(c) $A = \{\, x : x \in \mathbb{N}$ and $x \geq 7\}$ = {7,8,9,10,11 …}

B = {6,7,8,9,10,11}

$\therefore \{\, x : x \in \mathbb{N}andx \geq 7\} \not\subseteq$ {6,7,8,9,10,11}

(d) $A = \{\, x : x$ is a quadrilateral with equal sides} = {Rhombus, Square}

$B = \{\, x : x$is a square} = {Square}

$\therefore$ $\{x : x$ is a quadrilateral with equal sides$\} \nsubseteq \{x : x$ is a square$\}$

But,

(e) $\{x : x$ is a square$\} \subseteq \{x : x$ is a quadrilateral with equal sides$\}$

2) $A = \{0,1,2,3,4,5\}$

$B = \{0,1,2,3,4\}$

$C = \{2,4\}$

(a) $X \subset B$ and $X \subset C$

$\therefore X = \{2\}$ or $X = \{4\}$

(b) $X \subset A$ and $X \nsubseteq B$

$X$is a subset of A and is not a subset of B

$A = \{0,1,2,3,4,5\}$ $B = \{0,1,2,3,4\}$

{0,5},{1,5},{2,5},{3,5},{4,5},{0,1,5},{0,2,5},{0,3,5},{0,4,5},{1,2,5},{1,3,5},{1,4,5},{2,3,5},{2,4,5},{3,4,5},{0,1,2,5},{0,1,3,5},{0,1,4,5},{0,2,4,5},{0,3,4,5},{1,3,4,5},{1,2,4,5},{2,3,4,5},{0,2,3,5},{0,1,2,3,5},{1,2,3,4,5},{0,2,3,4,5},{0,1,2,4,5},{0,1,2,3,5},{0,1,2,4,5}

3)

a) The subsets of X

$\{\ \}, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}$

b) The proper subsets of Y

{ },{1},{2},{3},{4},{1,2},{1,3},{1,4},{2,3},{2,4},{3,4},{1,2,3},{1,2,4},{2,3,4},{1,3,4}

Activity 4

1)

(i) $A' = \{1,3,5,7,9\}$

(ii) $B' = \{2,4,6,8\}$

(iii) $C' = \{1,4,6,8,9\}$

(iv) $\phi' = \mathcal{U}$ (v) $\mathcal{U}' = \phi$

2)

(i) $(A \cap \mathcal{U}) \cap (B \cup C) = \{7\}$

(ii) $C - B = \{8,10\}$

(iii) $B - C = \{3,5\}$

(iv) $(B - C)' = \{2,4,6,7,8,9,10,11\}$

4)

(ii) $A - (B - A) = \{4,5,8,12\}$

(iii) $A - (C - B) = \{4,5,8,12\}$

Activity 5

1) 325

2)

i. 5	ii. 4	iii. 2
iv. 1	v. 6	vi. 11
vii. 23	viii. 2	

Summary

- Set is a collection of distinct objects.
- Representation of sets are two ways.
  - a) Tabular form (Roster Form)
  - b) Set builder or Rule Method

- Sets can be grouped following ways.
  - a) Finite and Infinite sets
  - b) Null set
  - c) Singleton
  - d) Equal sets
  - e) Subset of a set
  - f) Universal set

- By using a Venn diagram, we can illustrate the certain relationship between sets i.e.

Complement of a set

Union of sets

Intersection of sets

Disjoint sets

Difference of sets

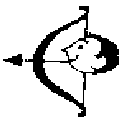
- There are six basic laws and rule in the set theory

1) Identity law
2) Idempotent law
3) Commutative law
4) Associative law
5) Distributive law
6) De Morgan's Rule

- A, B and C any three sets and $n(X)$ denoted by the number of elements of the set X.

$$\therefore n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$



Learning outcomes

On completion of this study session you should be able to

- Represent sets
- Identify the various sets and their properties
- Illustrate the relationships between sets by using the Venn diagram
- Solve problem by using the laws and rules of set algebra

Session 7

Inequalities

Contents

Introduction

In the mathematics studies will frequently meet with relations between unequal quantities, and we now indicate the fundamental laws which govern them. In the engineering practice, we have to use the principle of inequalities. In several situations, we have to apply one or more than number of variables and therefore finally we have to solve inequalities by using the basic principle of inequalities.

7.1 Basic concepts of inequality

Let a and b are two real numbers and a is greater than $b$if and only if $a-b$ is positive. It is denoted by $a>b$.The symbol $>$ for greater than and the symbol $<$ are used.

For an example $a=-6$ and $b=-10$

$\therefore -6>-10$Since$-6+10>0-6+10=4$

$-6-(-10)$positive.

If the numbers are represented by on a line $X'OX$, the origin representing zero. Points on the right side of the point O representing positive.

The left side of the point O representing negative numbers. Their distances from0,measured in terms of a suitable unit, giving the magnitudes of the numbers.

If $a>b$, then the point A is to the right of B where the points A and B are represented the numbers a and b.

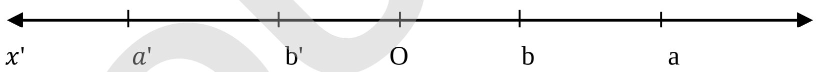


Figure 7.1.1

The points A' and B' are the images of the points $A$and B. The points A' and B' represented the values $-a$and $-b$. You can see that the point B' is right of the point A'. Thus$-b>-a$.

Now we have

If $a>b$ then $-b>-a$

$$\therefore a>b \Leftrightarrow -b>-a$$

Let p and q are real numbers. Then one of the following relations is true.

(i) $p>q$

(ii) $p=q$

(iii) $p<q$

If $c > 0$ and $d > 0$ then obviously $c + d > 0$and $cd > 0$.

If we write $x \geq y$

Then

(i) x is either greater than or equal to y

(ii) x is not less than y

y is not greater than x

7.2 Important results

Let$a$, b, c are real numbers;

(i) If $a > b$ and $b > c$ then $a > c$

(ii) If $a > b$ then $a + c > b + c$

(iii) If $a > b$ and $c > 0$ then $ac > bc$

(iv) If $a > b$ and $c < 0$ then $ac < bc$

(v) If $a > b$ then $\frac{1}{a} < \frac{1}{b}$

(vi) If $a > b$ and $c > 0$ then $\frac{a}{c} > \frac{b}{c}$

(vii) If $a > b$ and $c < 0$ then $\frac{a}{c} < \frac{b}{c}$

7.3 Inequalities in one variable

In this section, we shall study some methods for finding the solution sets of inequalities in one variable.

Example 1

Find the solution sets the following inequalities;

a) $(x - 3)(x + 2) \leq 0$

b) $x^2 - 5x + 4 > 0$

c) $(x - 2)(x + 1)(x + 3) \leq 0$

d) $(x - 4)^2 < 9$

e) $9x - 3x^2 \geq 4(1 + x + x^2)$

f) $\frac{(x+1)(x-2)}{(x-4)(x-3)} \leq 0, x \neq 4, x \neq 3$

g) $x^2 - 4x - 8 \leq 0$

Solution

a) $(x-3)(x+2) \leq 0$

Let$f(x) = (x-3)(x+2)$. You can see that when x changes through at $x = -2$ and $x = 3$ the sign of $f(x)$ will change.

So, we have to consider the following cases. When

(i) $x \leq -2 - 2 < x \leq 3$

(ii) $x > 3$

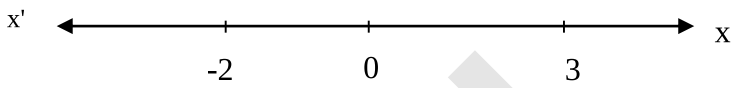


Figure 7.3.1

(i) We can select a number for x such that $x \leq -2$ (it is obvious $x < 3$). Therefore when $x < -2$ sign of $(x+2)$ and $(x-3)$ both are negative. $\therefore$ $(x+2)(x-3) > 0$. But at $x = -2 \Leftrightarrow (x+2) = 0. \therefore$ $(x-3)(x+2) = 0$. $\therefore$ $f(x) \geq 0$ for$-2 \leq x$.

(ii) Now we can select a number for x such that $-2 < x < 3 \Leftrightarrow (x+2) > 0$ and $x - 3 < 0$. Therefore, the sign of $(x+2)$ is positive and $(x-3)$ is negative. But at $x = 3, (x-3) = 0. \therefore$at $(x \leq 3)$, $(x-3) \leq 0$. $\therefore$when $-2 < x \leq 3, (x-3)(x+2) \leq 0$.

(iii) When $x > 3$ then $(x+2) > 2$ and $(x-3) > 0$. $\therefore$ thesign of $(x-3)(x+2) > 0$.

$$\therefore x > 3, (x-3)(x+2) > 0$$

$$\therefore -2 \leq x \leq 3, (x-3)(x+2) \leq 0.$$

Now we can represent the above result on a number line;

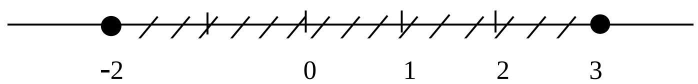


Figure 7.3.2

By using the notations of Set Theory we can express the set of the solution of the above inequality by

$$\{x: x \in \mathbb{R}, x \in [-2,3]\}.$$

b) $x^2 - 5x + 4 > 0$ $G(x) = x^2 - 5x + 4$

$$(x-4)(x-1) > 0 \therefore G(x) = (x-1)(x-4)$$

By using the above (a) part method,

$$x < 1 \quad (x-1)(x-4) > 0$$

$$x > 4 \quad (x-1)(x-4) > 0$$

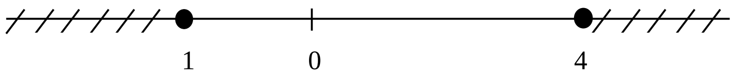


Figure 7.3.3

$\therefore x^2 - 5x + 4 > 0$ When $x < 1$ or $x > 4$.

In the set notation the solutions set= $\{x: x \in \mathbb{R}, x \in (-\infty, 1) \cup (4, \infty)\}$.

c) $(x-2)(x+1)(x+3) \leq 0$

In this case we have to examine the ranges

(i) $-3 \geq x$

(ii) $-1 \geq x > -3$

(iii) $2 \geq x > -1$

(iv) $x > 2$

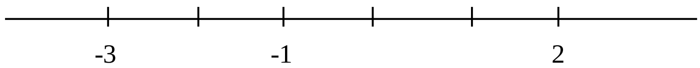


Figure 7.3.4

When $-3 \geq x, \Rightarrow \ (x-2)(x+1)(x+3) \leq 0$

$When\ -1 \geq x > -3, \quad \Rightarrow \ (x-2)(x+1)(x+3) \geq 0$

When $\ 2 \geq x > -1, \Rightarrow \ (x-2)(x+1)(x+3) \leq 0$

$When\ \ x > 2, \ \Rightarrow \ (x-2)(x+1)(x+3) > 0.$

Therefore,

the range which satisfies the inequality$(x-2)(x+1)(x+3) \leq 0$ is

$-3 \geq x,\ or\ 2 \geq x > -1.$

The set of solutions $= \{x: x \in \mathbb{R}, x \in (-\infty, -3] \cup (-1, 2]\}$.

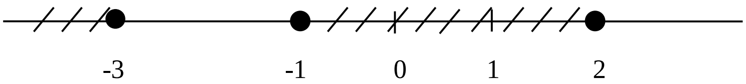


Figure 7.3.5

d) $(x-4)^2 < 9$

$$(x-4)^2 - 3^2 < 0$$

$$(x-4+3)(x-4-3) < 0$$

$$(x-1)(x-7) < 0$$

$$x < 1 \qquad (x-1)(x-7) > 0$$

$$1 < x < 7 \qquad (x-1)(x-7) < 0$$

$$7 < x(x-1)(x-7) > 0$$

$$\therefore (x-4)^2 < 9 \implies 1 < x < 7$$

Figure 7.3.6

In the set notation, the solutions set= $\{x: x \in \mathbb{R}, x \in (-\infty, 1] \cup (7, \infty]\}$.

e) $9x - 3x^2 \geq 4(1 + x + x^2)$

$$9x - 3x^2 \geq 4 + 4x + 4x^2$$

$$0 \geq 7x^2 - 5x + 4$$

Let $f(x) = 7x^2 - 5x + 4$

$$\Delta_x = (-5)^2 - 4(7)(4) = -87 < 0$$

So, there are no real values of x which satisfy the inequality

$7x^2 - 5x + 4 \leq 0$

i.e. there are no real values for x which satisfy the inequality

$$9x - 3x^2 \geq 4(1 + x + x^2)$$

.

f) $\frac{(x+1)(x-2)}{(x-4)(x-3)} \leq 0$

$$\frac{(x+1)(x-2)}{(x-4)(x-3)} = \frac{(x+1)(x-2)(x-4)(x-3)}{(x-4)^2(x-3)^2}$$

$\therefore \frac{(x+1)(x-2)}{(x-4)(x-3)} \leq 0 \iff \frac{(x+1)(x-2)(x-4)(x-3)}{(x-4)^2(x-3)^2} \leq 0.$

Since $(x-4)^2(x-3)^2$ is positive, then we can write the inequality as follows.

$\therefore \frac{(x+1)(x-2)}{(x-4)(x-3)} \leq 0 \iff (x+1)(x-2)(x-4)(x-3) \leq 0.$

Note that $x \neq 4$ and $x \neq 3$.

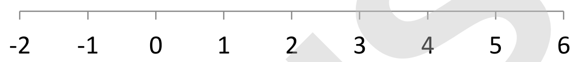


Figure 7.3.7

When $x < -1 \;\Rightarrow\; (x+1)(x-2)(x-4)(x-3) > 0$

When $-1 \leq x \leq 2 \;\Rightarrow\; (x+1)(x-2)(x-4)(x-3) \leq 0$

When $2 < x < 3 \;\Rightarrow\; (x+1)(x-2)(x-4)(x-3) > 0$

When $3 < x < 4 \;\Rightarrow\; (x+1)(x-2)(x-4)(x-3) < 0$

When $4 < x \;\Rightarrow (x+1)(x-2)(x-4)(x-3) > 0$.

$\therefore$ *the set of solutions which satisfy the inequality* $\frac{(x+1)(x-2)}{(x-4)(x-3)} \leq 0$ is

$$-1 \leq x \leq 2 \text{ or } 3 < x < 4$$

In the set notation, the solutions set satisfying the inequality $\frac{(x+1)(x-2)}{(x-4)(x-3)} \leq 0$ is $\{x : x \in \mathbb{R}, x \in [-1,2] \cup (3,4)\}$.

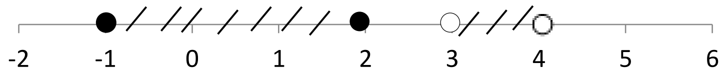


Figure 7.3.8

g) $x^2 - 4x - 8 \leq 0$

$$x^2 - 4x \leq 8$$

$x^2 - 4x + 4 \leq 8 + 4$ (Completing square of the left hand side)

$$(x-2)^2 \leq 12$$

The solution of $(x-a)^2 \leq b^2$ are

$$(x-a) \leq \pm b$$

$$x - a \leq b \quad \text{or} \quad x - a \geq -b$$

$$x \leq a + b \text{or} x \geq a - b$$

Assume that both a and b are positive $a + b \geq x \geq a - b$

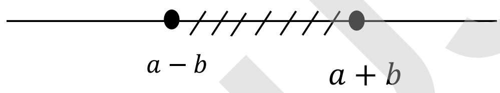


Figure 7.3.9

Also, the solution of $(x-c)^2 \geq d^2$ are

$$x - c \geq d \quad \text{or} \quad x - c \leq -d$$

Let assume the both c and d are positive

$$x \geq c + d \text{ } or \text{ } x \leq c - d$$

Figure 7.3.10

We have $(x-2)^2 \leq 12 \iff (x-2)^2 \leq (2\sqrt{3})^2$

$$x - 2 \leq 2\sqrt{3} \quad \text{or} \quad x - 2 \geq -2\sqrt{3}$$

$$x \leq 2\left(1 + \sqrt{3}\right) \text{ } or \text{ } x \geq -2(\sqrt{3} - 1)$$

$$x^2 - 4x - 8 \leq 0 \implies -2(\sqrt{3} - 1) \leq x \leq 2\left(1 + \sqrt{3}\right)$$

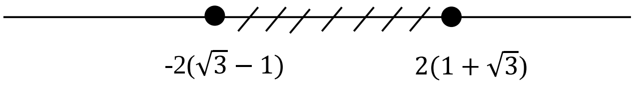


Figure 7.3.11

7.4 Inequalities with one variable and modulus

We can illustrate the solutions of this type of cases by solving the following,

Example 2

Find the solution of the following inequalities.

(a) $|x-4| > 4$

(b) $|x-3| < 2$

(c) $|x^2 - x| > 20$

(d) $|x^2 - 5| \geq 4$

(e) $|x-3| \geq |x+2|$

(f) $|x^2 - 3x| \geq |x+2|$

(g) $\left|\frac{x-2}{x+3}\right| \leq 4$

(h) $1 \leq |x-3| \leq 2$

Solution

(a) $|x-4| > 4$

$$(x-4) > 4 \quad \text{or} \quad (x-4) < -4$$

$x > 8$ or x < 0 Or

$$|x-4| > 4 \Longrightarrow (x-4)^2 > 4^2$$

$$(x-4)^2 - 4^2 > 0$$

$$(x-4+4)(x-4-4) > 0$$

$$x(x-8) > 0, \ \therefore x < 0 \ \text{ or } x > 8$$

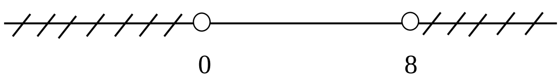


Figure 7.4.1

The set of solutions is

$\{x: x \in \mathbb{R}, x \in (-\infty, 0) \cup (8, \infty)\}$.

(b) $|x-3| < 2$

$$\therefore x-3 < 2 \quad \text{or} \quad x-3 > -2$$

$$x < 5 \qquad\qquad x > 1$$

$$\therefore 5 > x > 1.$$

Or

$$|x-3| < 2 \Rightarrow (x-3)^2 < 2^2$$

$$(x-3)^2 - 2^2 < 0$$

$$(x-3-2)(x-3+2) < 0$$

$$(x-5)(x-1) < 0$$

$$1 < x < 5.$$

$$\therefore |x-3| < 2 \qquad \therefore 1 < x < 5.$$

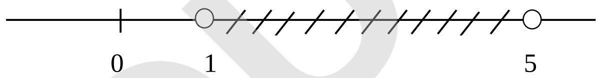


Figure 7.4.2

(c) $|x^2 - x| > 20$

$$x^2 - x > 20 \qquad \text{or} \qquad x^2 - x < -20$$

$$x^2 - x - 20 > 0 \quad \text{or} \; x^2 - x + 20 < 0$$

$$(x-5)(x+4) > 0 \qquad \text{or} \; \left(x - \frac{1}{2}\right)^2 + 20 - \frac{1}{4} < 0$$

$\left(x-\frac{1}{2}\right)^2 + \frac{79}{4} < 0$ there are no real values for x such that $\left(x-\frac{1}{2}\right)^2 + \frac{79}{4} < 0$.

Now, consider $(x-5)(x+4) > 0$

$$\therefore x < -4 \quad \text{or} \quad x > 5.$$

Or

$$|x^2 - x| > 20 \Longrightarrow (x^2 - x)^2 > 20^2$$

$$(x^2 - x)^2 - 20^2 > 0$$

$$(x^2 - x - 20)(x^2 - x + 20) > 0$$

$$(x-5)(x+4)\left\{\left(x-\frac{1}{2}\right)^2 + \frac{79}{4}\right\} > 0$$

$$\text{For all } \ x \in \mathbb{R}, \left(x-\frac{1}{2}\right)^2 + \frac{79}{4} > 0$$

$$\therefore -4 < x \text{ or } \ \mathrm{x} > 5$$

The set of solutions is

$$\{x: x \in \mathbb{R}, x \in (-\infty, -4) \cup (5, \infty)\}.$$

Figure 7.4.3

(d) $|x^2 - 5| \geq 4$

$$x^2 - 5 \geq 4 \qquad \text{and} \qquad x^2 - 5 \leq -4$$

$$x^2 \geq 9 \qquad\qquad x^2 \leq 1$$

$$x^2 \geq 3^2 x^2 \leq 1$$

$$x \geq 3, x \leq -3 \qquad -1 \leq x \leq 1$$

Or

$$|x^2 - 5| \geq 4$$

$$(x^2 - 5)^2 \geq 4^2$$

$$(x^2 - 5)^2 - 4^2 \geq 0$$

$$(x^2 - 5 - 4)(x^2 - 5 + 4) \geq 0$$

$$(x^2 - 9)(x^2 - 1) \geq 0$$

$$(x+3)(x+1)(x-1)(x-3) \geq 0$$

$$x \le -3, \quad -1 \le x \le 1, \quad x \ge 3.$$

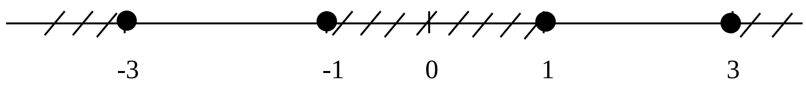


Figure 7.4.4

$$x = \{x: x \in \mathbb{R}, x \in (-\infty, -3] \cup [-1,1] \cup [3, \infty)\}.$$

(e) $|x^2 - 3| \ge |x + 2|$

$$x^2 - 3 \ge x + 2 \qquad \text{or} \qquad x^2 - 3 \le -(x + 2)$$

$$x^2 - x - 5 \ge 0 \qquad\qquad x^2 + x - 1 \le 0$$

$$\left(x - \frac{1}{2}\right)^2 - 5 - \frac{1}{4} \ge 0 \qquad \left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{4} + 1\right) \le 0$$

$$\left(x - \frac{1}{2}\right)^2 - \frac{21}{4} \ge 0 \qquad \left(x + \frac{1}{2}\right)^2 - \frac{5}{4} \le 0$$

$$\left(x - \frac{1}{2}\right)^2 - \left(\frac{\sqrt{21}}{2}\right)^2 \ge 0$$

$$\left(x - \frac{1}{2} - \frac{\sqrt{21}}{2}\right)\left(x - \frac{1}{2} + \frac{\sqrt{21}}{2}\right) \ge 0$$

$$\left\{x - \frac{1 + \sqrt{21}}{2}\right\}\left\{x - \frac{1 - \sqrt{21}}{2}\right\} \ge 0$$

$$x \le -\frac{\sqrt{21} - 1}{2} \text{and} \qquad x \ge \frac{1 + \sqrt{21}}{2}$$

$$\left(x + \frac{1}{2}\right)^2 - \frac{5}{4} \le 0$$

$$\left(x + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2 \le 0$$

$$\left(x + \frac{1}{2} - \frac{\sqrt{5}}{2}\right)\left(x + \frac{1}{2} + \frac{\sqrt{5}}{2}\right) \le 0$$

$$\left\{x - \frac{\sqrt{5} - 1}{2}\right\}\left\{x - \frac{-\sqrt{5} - 1}{2}\right\} \le 0$$

$$\frac{-\sqrt{5}-1}{2} \le x \le \frac{\sqrt{5}-1}{2}.$$

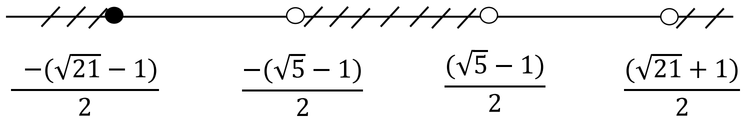


Figure 7.4.5

$$x \le -\frac{\sqrt{21}-1}{2}; \qquad \frac{-\sqrt{5}-1}{2} \le x \le \frac{\sqrt{5}-1}{2}; \qquad x \ge \frac{1+\sqrt{21}}{2}$$

The set of solution is

$$\left\{x: x \in \mathbb{R}, x \in \left(-\infty, -\frac{\sqrt{21}-1}{2}\right] \cup \left[\frac{-\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}\right] \cup \left[\frac{1+\sqrt{21}}{2}, \infty\right)\right\}.$$

(f) $|x^2 - 3x| \ge |x+2|$

$$\therefore (x^2-3x)^2 \ge (x+2)^2$$

$$(x^2-3x)^2 - (x+2)^2 \ge 0$$

$$\{(x^2-3x)+(x+2)\}\{(x^2-3x)-(x+2)\} \ge 0$$

$$(x^2-2x+2)(x^2-4x-2) \ge 0$$

$$\{(x-1)^2+1\}\{(x^2-4x+4-2-4\} \ge 0$$

$$\{(x-1)^2+1\}\left\{(x-2)^2 - \left(\sqrt{6}\right)^2\right\} \ge 0$$

$$\{(x-1)^2+1\}\{(x-2)+\sqrt{6}\}\{(x-2)-\sqrt{6}\} \ge 0$$

$$x \in \mathbb{R}, \qquad (x-1)^2+1 > 0$$

$$\{x-(2-\sqrt{6})\}\{x-(2+\sqrt{6})\} \ge 0$$

$$\{x-(-\sqrt{6}+2)\}\{x-(\sqrt{6}+2)\} \ge 0$$

When $x \le -\left(\sqrt{6}-2\right) \Rightarrow \{x-(-\sqrt{6}+2)\}\{x-(\sqrt{6}+2)\} \ge 0$

$When\ -\left(\sqrt{6}-2\right) \le x \le \left(\sqrt{6}+2\right) \Rightarrow \{x-(-\sqrt{6}+2)\}\{x-(\sqrt{6}+2)\}$
≤ 0

When $\sqrt{6}+2 \le x \Rightarrow \{x-(-\sqrt{6}+2)\}\{x-(\sqrt{6}+2)\} \ge 0$

$\therefore$ When $x \le -(\sqrt{6}-2)$ or $\sqrt{6}+2 \le x$

$$\Rightarrow \{x-(-\sqrt{6}+2)\}\{x-(\sqrt{6}+2)\} \ge 0.$$

$\therefore$ $|x^2-3x| \ge |x+2|$ when $x \le -(\sqrt{6}-2)$ or $\sqrt{6}+2 \le x$.

Figure 7.4.6

$$x = \{x : x \in \mathbb{R}, x \in (-\infty, -(\sqrt{6}-2)] \cup [\sqrt{6}+2, \infty)\}.$$

(g) $\left|\frac{x-2}{x+3}\right| \le 4$

$$\Rightarrow |x-2| \le 4|x+3|$$

$$\Rightarrow (x-2)^2 \le 4^2(x+3)^2$$

$$\Rightarrow (x-2)^2 - 4^2(x+3)^2 \le 0$$

$\Rightarrow \{x-2-4(x+3)\}\{x-2+4(x+3)\} \le 0$

$\Rightarrow (-3x-14)(5x+10) \le 0$

$\Rightarrow \left(x+\frac{14}{3}\right)(x+2) \ge 0$

$\Rightarrow \left[x-\left(-\frac{14}{3}\right)\right][x-(-2)] \ge 0$

When, $x \le -\frac{14}{3} \Rightarrow \left[x-\left(-\frac{14}{3}\right)\right][x-(-2)] \ge 0$

When $-\frac{14}{3} \le x \le -2 \quad \Rightarrow \quad \left[x-\left(-\frac{14}{3}\right)\right][x-(-2)] \le 0$

When $-2 \le x \Rightarrow \left[x-\left(-\frac{14}{3}\right)\right][x-(-2)] \ge 0$

$$\therefore \left|\frac{x-2}{x+3}\right| \le 4 \qquad x \le -\frac{14}{3} \text{ or } -2 \le x$$

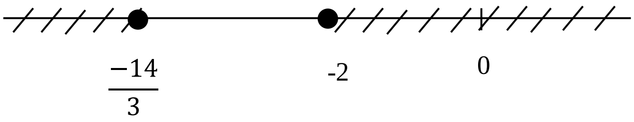


Figure 7.4.7

$$x = \left\{x : x \in \mathbb{R}, x \in \left(-\infty, -\frac{14}{3}\right] \cup [-2, \infty)\right\}.$$

(h) $1 \le |x-3| \le 2$

$$1 \le |x-3|$$

$$x - 3 \ge 1 \quad \text{or} \quad x - 3 \le -1$$

$$x \ge 4 \qquad x \le 2$$

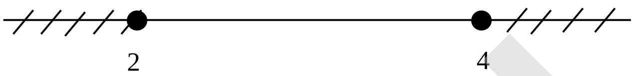


Figure 7.4.8

$$|x-3| \le 2$$

$$x - 3 \le 2 \qquad x - 3 \ge -2$$

$$x \le 5 \qquad x \ge 1 \Rightarrow 5 \ge x \ge 1$$

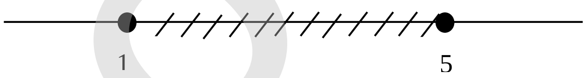
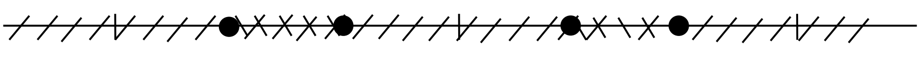


Figure 7.4.8

Now we have to consider common range for satisfying the both inequalities.



F re 7.4

$$\therefore 1 \le x \le 2 \qquad 4 \le x \le 5$$

$$\therefore\ 1 \le |x-3| \le 2 \quad 1 \le x \le 2, 4 \le x \le 5$$

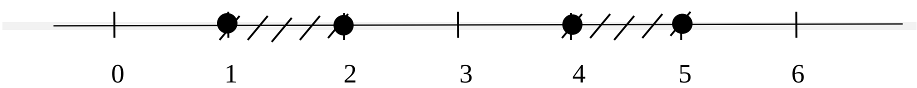


Figure 7.4.10

$\therefore$The set of solutions satisfying the inequality $1 \leq |x-3| \leq 2$ is

$\{x: x \in \mathbb{R}, x \in [1,2] \cup [4,5]\}$.

7.5 Inequalities involving in algebraic fraction

For the solutions of inequalities involving in algebraic functions, we have to use some special method.

Example 3

Find the solutions of the following inequalities.

(a) $\frac{2}{2x-1} \leq 1$

(b) $\frac{2x-3}{4x-5} + 2 \leq 0$

(c) $1 \leq \frac{4x-4}{2x+3} < 3$

(d) $\frac{2}{2x-1} < 1$

(e) $\frac{1}{2-x} < \frac{1}{x-3}$

Solution

(a) $\frac{2}{2x-1} \leq 1 \therefore \frac{2(2x-1)}{(2x-1)^2} \leq 1$

Now the denomination always positive. So we can write this as follows.$\therefore$

$2(2x-1) \leq (2x-1)^2$

$$(2x-1)^2 - 2(2x-1) \geq 0$$

$$(2x-1)\{2x-1-2\} \geq 0$$

$$(2x-1)(2x-3) \geq 0$$

$$\text{If} \frac{1}{2} > x, \text{ then} \qquad (2x-1)(2x-3) > 0$$

$$\text{If} \frac{3}{2} \geq x > \frac{1}{2}, \text{ then} \qquad (2x-1)(2x-3) \leq 0$$

$$\text{If } x \geq \frac{3}{2}, \text{then} \qquad (2x-1)(2x-3) \geq 0$$

$$\therefore \frac{2}{2x-1} \leq 1 \Longrightarrow \frac{1}{2} > x \text{ or } x \geq \frac{3}{2}$$

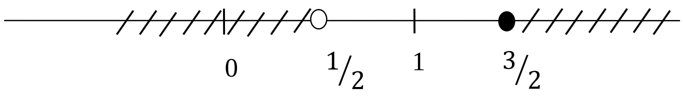


Figure 7.5.1

$$\text{The set of solutions} = \left\{x: x \in \mathbb{R}, x \in \left(-\infty, \frac{1}{2}\right) \cup \left[\frac{3}{2}, \infty\right)\right\}.$$

(b) $\frac{2x-3}{4x-5} + 2 \leq 0$

$$\frac{2x-3+2(4x-5)}{4x-5} \leq 0$$

$$\frac{2x-3+8x-10)}{4x-5} \leq 0$$

$$\frac{(10x-13)(4x-5)}{(4x-5)^2} \leq 0$$

$$(10x-13)(4x-5) \leq 0;\ (4x-5) \neq 0$$

$$\left(x-\frac{13}{10}\right)\left(x-\frac{5}{4}\right) \leq 0$$

$$\text{If } x < \frac{5}{4}, \text{then} \left(x-\frac{13}{10}\right)\left(x-\frac{5}{4}\right) > 0$$

$$\text{If } \frac{5}{4} < x \leq \frac{13}{10}, \text{then} \left(x-\frac{13}{10}\right)\left(x-\frac{5}{4}\right) \leq 0$$

$$\text{If} \frac{13}{10} < x, \text{then} \left(x-\frac{13}{10}\right)\left(x-\frac{5}{4}\right) > 0$$

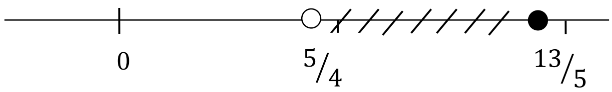


Figure 7.5.2

$$\text{The set of solutions} = \left\{x: x \in \mathbb{R}, x \in \left(\frac{5}{4}, \frac{13}{10}\right]\right\}.$$

(c) $1 \leq \frac{4x-4}{2x+3} < 3$

$$1 \leq \frac{4x-4}{2x+3} \quad ; \quad \frac{4x-4}{2x+3} \leq 3$$

$$1 - \frac{4x-4}{2x+3} \leq 0$$

$$\frac{2x+3-(4x-4)}{(2x+3)} \leq 0$$

$$\left(\frac{7-2x}{2x+3}\right) \leq 0 \quad 2x+3 \neq 0$$

$$(7-2x)(2x+3) \leq 0$$

$$\left[x - \left(-\frac{3}{2}\right)\right]\left[x - \frac{7}{2}\right] \geq 0$$

$$\therefore x < -\frac{3}{2}; \quad x \geq \frac{7}{2}$$

$$\frac{7-2x}{2x+3} \leq 0 \Rightarrow x < -\frac{3}{2}, x \geq \frac{7}{2} \longrightarrow (1)$$

$$\frac{4x-4}{2x+3} < 3$$

$$\frac{4x-4}{2x+3} - 3 < 0$$

$$\frac{4x-4-3(2x+3)}{2x+3} < 0$$

$$\frac{13+2x}{2x+3} > 0 \qquad (13+2x)(2x+3) < 0$$

$$\left(x - \left(-\frac{13}{2}\right)\right)\left(x - \left(-\frac{3}{2}\right)\right) < 0$$

$$\therefore \frac{13+2x}{2x+3} > 0 \Rightarrow \frac{-13}{2} < x < -\frac{3}{2} \longrightarrow (2)$$

Now we have to consider the common interval of (1) and (2).

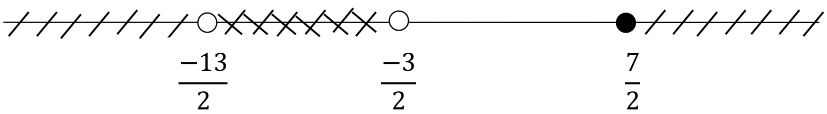


Figure 7.5.3

$$\therefore \; 1 \le \frac{4x-4}{2x+3} < 3 \qquad \frac{-13}{2} < x < -\frac{3}{2}$$

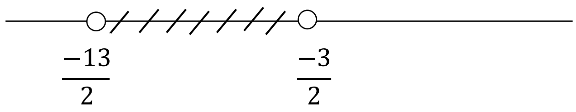


Figure 7.5.4

The set of solutions $= \left\{x: x \in \mathbb{R}, x \in \left(\frac{-13}{2}, \frac{-3}{2}\right)\right\}$.

(d) $\frac{2}{2x-1} < 1 \Longrightarrow \frac{2}{2x-1} - 1 < 0$

$$\frac{2-(2x-1)}{2x-1} < 0$$

$$\frac{(3-2x)}{(2x-1)} < 0 \Longrightarrow (3-2x)(2x-1) < 0$$

$$\left(x-\frac{3}{2}\right)\left(x-\frac{1}{2}\right) > 0$$

$$\therefore x < \frac{1}{2} \text{ or } x > \frac{3}{2}$$

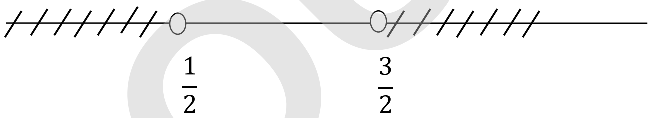


Figure 7.5.5

The set of solutions $= \left\{x: x \in \mathbb{R}, x \in \left(-\infty, \frac{1}{2}\right) \cup \left(\frac{3}{2}, \infty\right)\right\}$

(e) $\frac{1}{2-x} < \frac{1}{x-3} \Longrightarrow \frac{1}{2-x} - \frac{1}{x-3} < 0$

$$\frac{(x-3)-(2-x)}{(2-x)(x-3)} < 0$$

$$(2x-5)(2-x)(x-3) < 0$$

$$(x-2)\left(x-\frac{5}{2}\right)(x-3) > 0$$

$$\therefore 2 < x < \frac{5}{2}, x > 3$$

$$\frac{1}{2-x} < \frac{1}{x-3} \Longrightarrow 2 < x < \frac{5}{2} \text{ or } x > 3$$

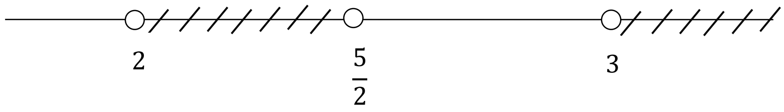


Figure 7.5.6

$$\text{The set of solutions} = \left\{x : x \in \mathbb{R}, x \in \left(2, \frac{5}{2}\right) \cup (3, \infty)\right\}$$

Activity 1

Find the values of x and give the value of set satisfying the following inequalities.

(a) $|x+1| > 2|x-1|$

(b) $\frac{1}{x} > \frac{x}{x+2}$

(c) $|x^2-7| > |x^2-1|$

(d) $\frac{1}{2}|x-1| > |x-4|$

(e) $|3x-4| > 2-5x$

(f) $x - 4x(x-4) \leq 5$

(g) $\frac{1}{x+1} + \frac{2}{x+2} > \frac{3}{x+3}$

(h) $\frac{x}{x-1} < \frac{x}{x-2}$

(i) $\frac{7}{x-5} < x+1$

(j) $\frac{2}{x-3} > \frac{3}{x-2}$

Answers to Activities

Activity 1

(a) $x < -3$ or $x > \frac{1}{3}$ $i.e.\ x = \left\{x : x \in \mathbb{R}, x \in (-\infty, -3) \cup \left(\frac{1}{3}, \infty\right)\right\}$

(b) $-2 < x < -1$ or $0 < x < 2$ $i.e.\quad x = \{x : x \in \mathbb{R}, x \in (-2, -1) \cup (0, 2)\}$

(c) $2 > x > -2$ $i.e.\quad x = \{x : x \in \mathbb{R}, x \in (-2, 2)\}$

(d) $3 < x < 7$ $i.e.$ $x = \{x: x \in \mathbb{R}, x \in (3,7)\}$

(e) $-1 > x$ or $x > \frac{7}{4}$ $i.e. \left\{x: x \in \mathbb{R}, x \in (-\infty, -1) \cup \left(\frac{7}{4}, \infty\right)\right\}$

(f) $-1 \le x \le 1$ or $4 \le x \le 5$ $i.e.$ $x = \{x: x \in \mathbb{R}, x \in [-1,1] \cup [4,5]\}$

(g) $x < -3$ or $-2 < x < -1$ or $x > \frac{3}{2}$

$$i.e. \left\{x: x \in \mathbb{R}, x \in (-\infty, -3) \cup (-2, -1) \cup \left(\frac{3}{2}, \infty\right)\right\}$$

(h) $0 > x$ or $1 < x < 2$ $i.e.$ $x = \{x: x \in \mathbb{R}, x \in (-\infty, 0) \cup (1,2)\}$

(i) $x < 5$ or $x > 6$ $i.e.$ $x = \{x: x \in \mathbb{R}, x \in (-\infty, 5) \cup (6, \infty)\}$

(j) $x < 2$ or $3 < x < 5$ $i.e.$ $x = \{x: x \in \mathbb{R}, x \in (-\infty, 2) \cup (3,5)\}$

Summary

- If $(a - b)$ is positive a is said to be algebraically greater than b. If $a - b$ is negative, a is said to be algebraically less than b.
- If $a > b$ and $b > c$ then $a > c$
- If $a > b$ and $c > 0$ then $ac > bc$
- If $a > b$ and $c < 0$ then $ac < bc$
- If $a > b$ then $\frac{1}{a} < \frac{1}{b}$
- If $a > b$ and $c > 0$ then $\frac{a}{c} > \frac{b}{c}$
- If $a > b$ and $c < 0$ then $\frac{a}{c} < \frac{b}{c}$
- If $(x - a)(x - b) > 0$ and $a > b$ then $x < b$ or $x > a$
- If $(x - a)(x - b) < 0$ and $a > b$ then $b < x < a$

Learning outcomes

On completion of this session you should be able to

- Describe the basic rules of inequalities.
- Apply the concepts of algebra to solve inequalities.
- Solve the problems of inequalities involving with one variable and modulus.

Session 8

Basic Concepts of Relation and Function

Contents

Introduction

We can list different types of algebraic expressions into the two groups. One of them is called Relation and the other one is Functions.

The difference between in these two groups are going to discuss in this study session.

The concept of Function based on modern Algebra

The idea on function is very useful to handle the limit of function and the derivative of a function.

8.1 Relation

First,we have to recall that in a given ordered pair (x, y). x is called the first component, and y is called the second component of the ordered pair. Let X and Y be two sets.

Cartesian product

The set of ordered pairs $\{(x, y): x \in X,\ y \in Y\}$ is called the Cartesian product of X and Y and is denoted by $X \times Y$.

In general, $f: X \to Y$ be a relation from X to Y.

Example 1

Given that X = [-2, 3] and Y =[0, 9]. Sketch on the Cartesian plane the graphs of the relation

(i) $f: x \to y$ where $(x, y) \in X \times Y$ and $y = x^2$

(ii) $g: x \to y$ where $(x, y) \in Y \times X$ and $y^2 = x$

Solution

(i) The graph of f is $\{(x, y) \in X \times Y,\ y = x^2\}$ which is a part of a parabola as shown in the figure 8.1.1.

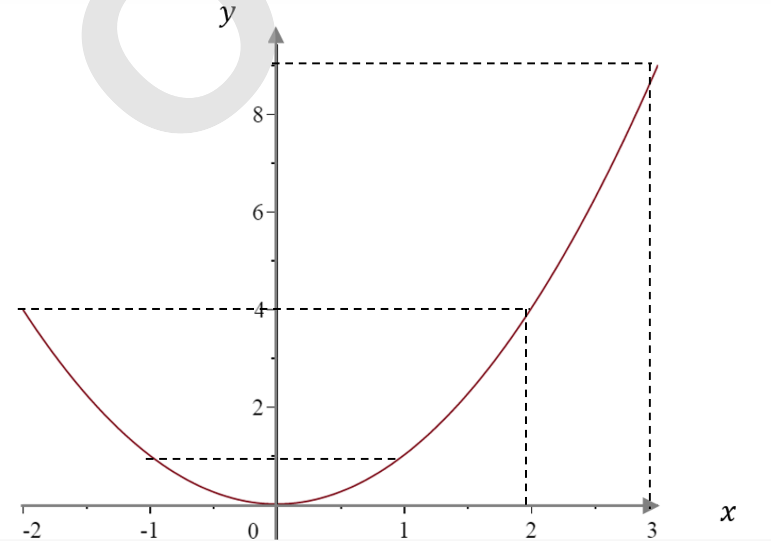


Figure 8.1.1

Example 2

Let $X = \{1,3,5\}$ and $Y = \{0,1, \text{-}1\}$

Then $X \times Y = \{(1,0), (1,1), (1, -1), (3,0), (3,1), (3, -1), (5,0), (5,1), (5, -1)\}$

For any ordered pair $(x, y) \in X \times Y$, let us say that y is related to x if and only if $y = 4 - x$,

Then the set of (x, y) for which $y = 4 - x$ is $\{x, y \in X, Y;\ y = 4 - x\} = \{(3,1), (5, -1)\}$ which forms a relation from X to Y.

We may write that relation as $\boldsymbol{y}\ \ \boldsymbol{f}\ \ \boldsymbol{x}$ for $y = 4 - x$ and say that f is a relation from X to Y, in symbol.

$f: X \to Y$ or $f: x \to y$ where $x \in X, y \in Y$

Note that we are only interested in those ordered pair $(x, y) \in X \times Y$ for which $y\ \ f\ \ x$ i.e. $y = 4 - x$ and we just ignore those for which $y\ \ \not{f}\ \ x$ i.e.$y \neq 4 - x$

(ii) The graph of g is $\{(x, y) \in Y \times X,\ y^2 = x\}$ which is a part of a parabola as shown in the figure 3.1.2.

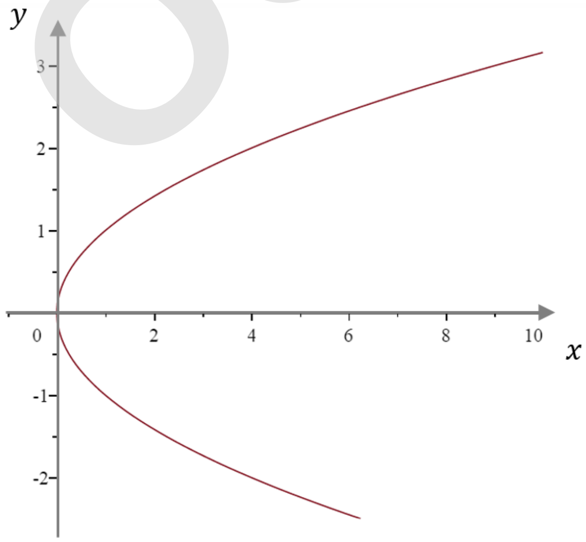


Figure 8.1.2

8.2 Functions

In the example 3.1.1 (i) we notice that for each x of which $-2 \leq x \leq 3$ there is a unique y (i.e.$y = x^2$) such that $y \;\; f \;\; x$. However, the above property does not hold for the relation g in the example 3.1.1 (ii). There is more than one value of y for an example 2 and -2 such that $(-2)^2 = 2^2 = 4$.

To distinguish f and g we call f is a function and g is a relation and is not a function.

Definition

A function $f: X \to Y$ is a rule (or relation) which associate with each element $x \in X$ a unique element $y \in Y$ such that $y \;\; f \;\; x$.

Usually we denote the element $y \in Y$ for which $y \;\; f \;\; x$ by "$f(x)$" i.e. $y = f(x)$.

The element y or $f(x)$ in Y is called the f image of x or the image of x under f.

The set X is called the <u>Domain</u> of f and the set Y is called the <u>Codomain</u> of f.

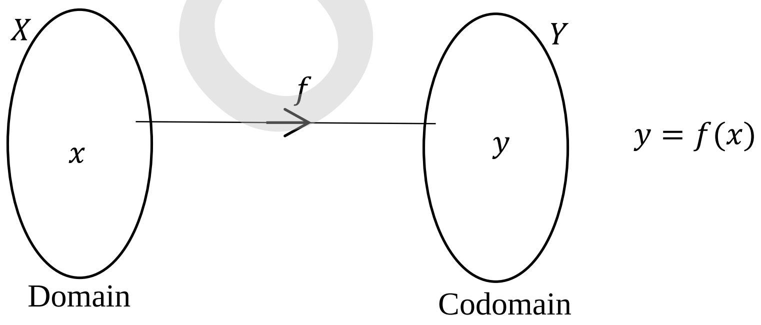


Figure 8.2.1

8.3 Range of the function

Let $X = \{0, \pm 1, \pm 2\}$ and $Y = \{0,1,4,5,6\}$

Then the function $f: X \to Y$ given by $f(x) = x^2$.

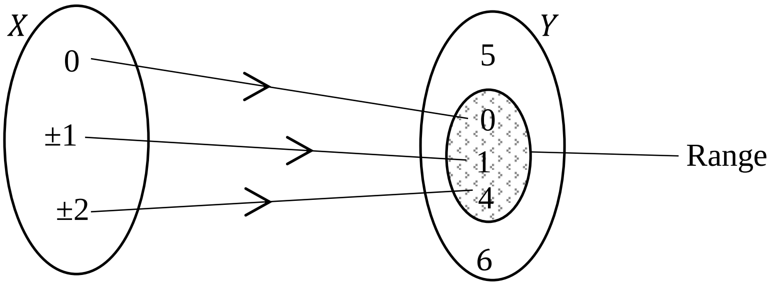


Figure 8.3.1

$f: X \to Y$ be a function. The set Y is the codomain of the function f. Very often, we are interested only in those elements of Y which are f- images of the elements in the domain of X.The domain of a function f is denoted by D_f,

The set of such f- images in Y is called the range of f.The range of a function f is denoted by R_f,

∴ Domain of the above function $D_f = X = \{0, \pm 1, \pm 2\}$

Codomain of the above function$= Y = \{0,1,4,5,6\}$

Range of the above function = R_f = {1,0,4}

Clearly, the range of f must be a subset of the codomain of the function f.

∴ $R_f \subseteq Y$.

8.4 Intervals in Real Numbers

The set of all real numbers lying between the two given real numbers is called an interval in ℝ.

Let 'a' and 'b' be any two real numbers such that $a < b$; Then we can define the following types of intervals.

(i) Closed Interval $[a, b]$

$[a, b]$ = Closed interval from a to b

= $\{x: x \in \mathbb{R}\ a \leq x \leq b\}$

= $\{x: x \in \mathbb{R},\ x \in [a, b]\}$

= set of all real numbers lying between a and b including the end points (values) a and b

Figure 8.4.1

(ii) Open Interval from a to b , $]a, b[$

$]a, b[$ = Open interval from a to b

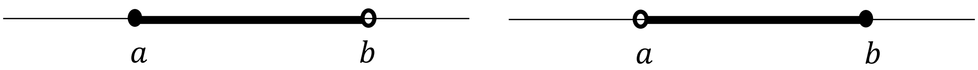
= $\{x: x \in \mathbb{R}, a < x < b\}$

= $\{x: x \in \mathbb{R},\ x \in (a, b)\}$

= set of all real numbers lying between a and b excluding the end points (values) a and b

Figure 8.4.2

(iii) Closed-Open Interval $[a, b[$ and Open-Closed Interval $]a, b]$



$[a, b[$ = $\{x: x \in \mathbb{R}, a \leq x < b\}$
= $\{x: x \in \mathbb{R},\ x \in [a, b)\}$

$]a, b]$ = $\{x: x \in \mathbb{R}, a < x \leq b\}$
= $\{x: x \in \mathbb{R},\ x \in (a, b]\}$

Figure 8.4.3

(iv) Real number set $\mathbb{R}$ as an open interval

We introduce two special numbers -∞ and ∞

where $-\infty$ = a number less than any real number

$+\infty$ = a number greater than any real number

$-\infty < x$ for all $x \in \mathbb{R}$ and $x < \infty$ for all $x \in \mathbb{R}$.

$$\therefore \mathbb{R} =]-\infty, \infty[\; = \{x: x \in \mathbb{R}, -\infty < x < \infty\}$$
$$= \{x: x \in \mathbb{R}, x \in (-\infty, \infty)\}$$

$$]-\infty, a[\; = \{x: x \in \mathbb{R}, -\infty < x < a\}$$
$$= \{x: x \in \mathbb{R}, x \in (-\infty, a)\}$$
$$]-\infty, a] \; = \{x: x \in \mathbb{R}, -\infty < x \leq a\}$$

$$[a, \infty[= \{x: x \in \mathbb{R}, a \leq x < \infty\}$$

8.5 Some Basic Important Algebraic Functions

(i) Constant Function

A function $f_c: \mathbb{R} \to \mathbb{R}$ defined by $f_c: x \to c$ for all $x \in \mathbb{R}$, where c is a constant is called a constant function.

Its domain is $\mathbb{R}$ and the range of f_c simply$\{c\}$.

The graph of f_c is a straight line parallel to x axis.

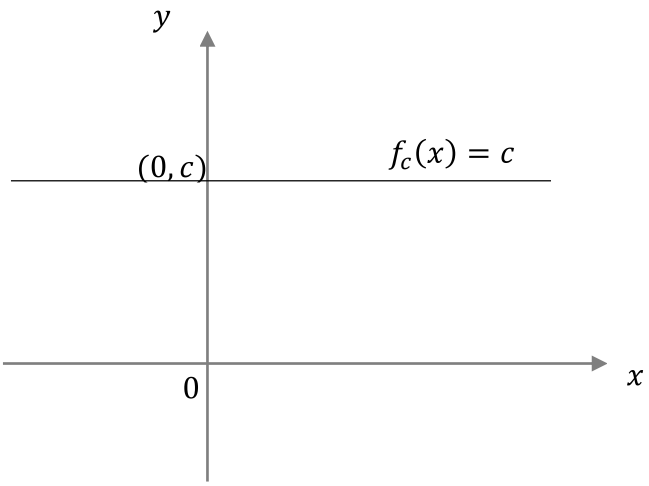


Figure 8.5.1

(ii) **Identity Function**

Let the $i : \mathbb{R} \to \mathbb{R}$ is defined by

$i : x \to x$.

$i(x) = x$ is called the identity function. The domain of i is $\mathbb{R}$ and the range is also $\mathbb{R}$.

The graph of the identity function i is a straight line passing through the origin and inclined at the angle of 45^0 with the x axis (unit gradient).

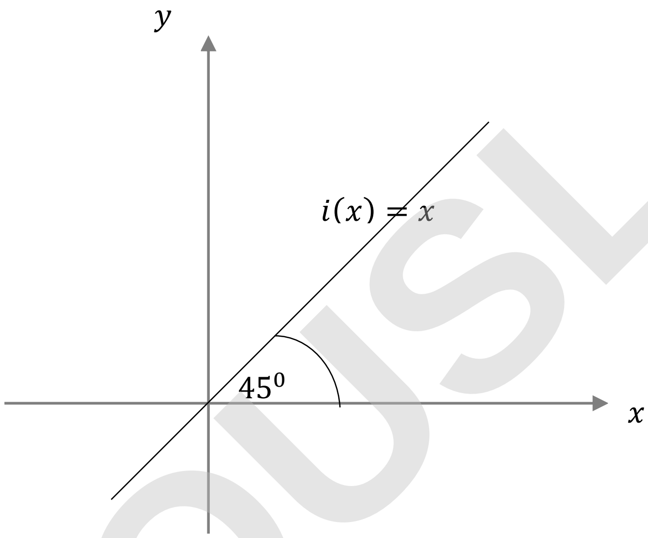


Figure 8.5.2

(iii) **Modulus Functions** (Absolute Value Function)

The function $f_m : \mathbb{R} \to \mathbb{R}_0^+$ defined by

$$f_m : x \to |x|$$

$$fm(x) = |x| = \begin{cases} x & when\ x > 0 \\ 0 & when\ x = 0 \\ -x & when\ x < 0 \end{cases}$$

f_m is also called the absolute value function or modulus functions. The domain of f_m is $\mathbb{R}$ and the range is $\mathbb{R}_0^+$.

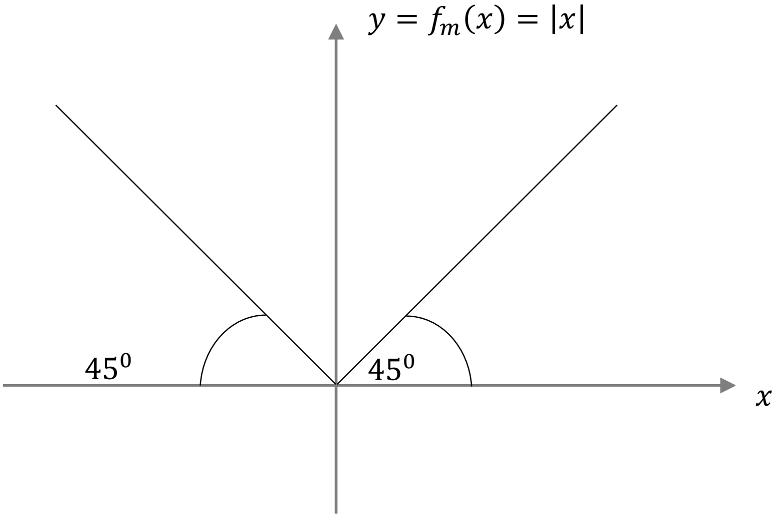


Figure 8.5.3 The graph of $f_m(x) = |x|$

The relationship between f_m and i

$f_m(x) = i(x)$ when $x \geq 0$

$f_m(x) = -i(x)$ when $x < 0$

(iv) **Linear Function**

The function $l: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$l: x \rightarrow mx + c$$

(where $m \neq 0$ and c is any given real constant) is called a linear function.

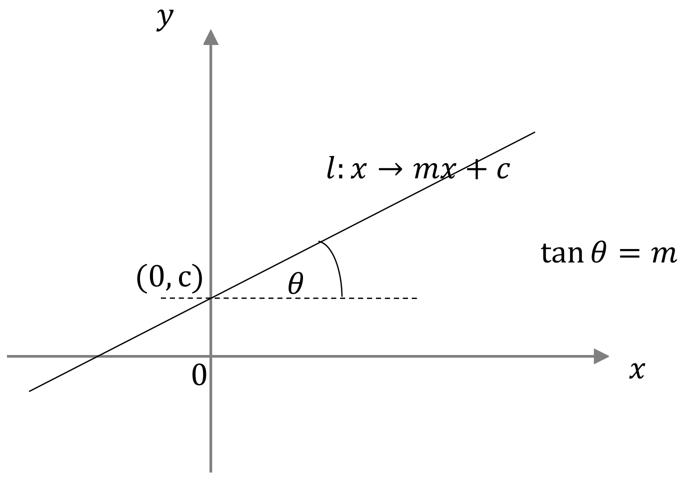


Figure 8.5.4: The graph of linear function

Its graph is a straight line of gradient m and y intercept c.

The domain of $l$is $\mathbb{R}$.

The range of $l$is $\mathbb{R}$.

(v) **Quadratic Function**

Let the function q on $\mathbb{R}$ is defined by

$$q: x \to ax^2 + bx + c$$

Where $a \neq 0$ and a, b, c are real constants.

Then q is called a quadratic function {for more details of the quadratic function please refer Block 01, unit 01 session 04}. The graphs and ranges of the function q you can see that from the above session.

(vi) **Reciprocal Function**

The function $f_R: \mathbb{R} - \{0\} \to \mathbb{R}$

$$f: x \to \frac{1}{x}$$

Where $x \neq 0$ is called the reciprocal function.

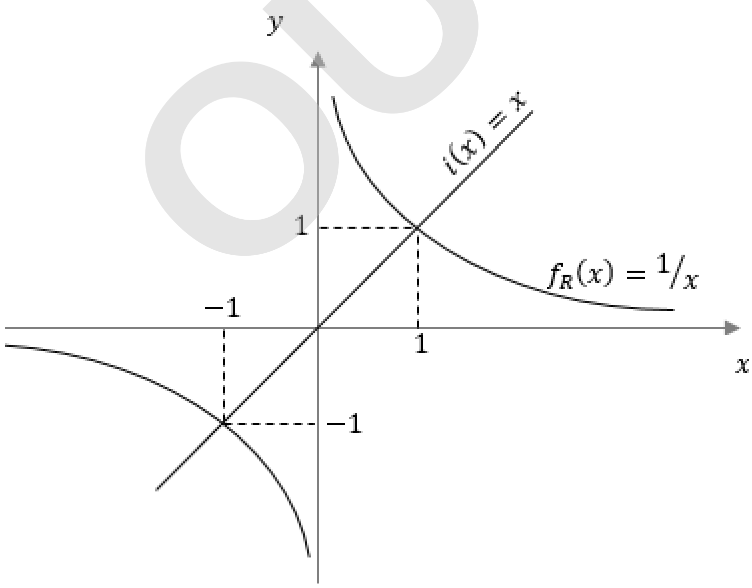


Figure 8.5.5: The graphs of the Reciprocal function and Identity function

Its domain as well as range is $\mathbb{R} - \{0\}$.

The relationship between f_R and i (identity function) is

$f_R(x).\,i(x) = 1;\, x \neq 0$.

f_R is actually the reciprocal function of i and vice versa.

The graph of f is a rectangular hyperbola in the first and third quadrants of the Cartesian plane.

Clearly if $x > 0$ we can see that

I. $f_R(1) = i(1) = 1$

II. $f_R(x) > 1$ and $0 < i(x) < 1$ for $1 > x > 0$

III. $0 < f_R(x) < 1$ and $i(x) > 1$ for $x > 1$

IV. $f_R(x) \to \infty$ as $i(x) \to 0$

V. $f_R(x) \to 0$ as $i(x) \to \infty$

(vii) **Exponential Function**

Let a be a positive real number. Then the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a^x$ (where $a \neq 1$) is called the exponential function.

Its domain is $\mathbb{R}$ and the range $(0, \infty)$.

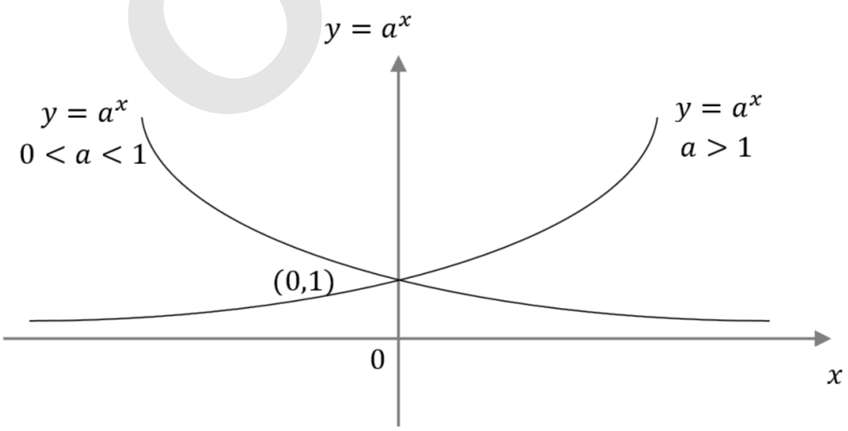


Figure 8.5.6: The graph of exponential function

(viii) **Logarithmic Function**

Let $a(\neq 1)$ be a positive real number. Then the function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \log_a x$ is called the logarithmic function.

Its domain is $\mathbb{R}^+$ and the range is $\mathbb{R}$.

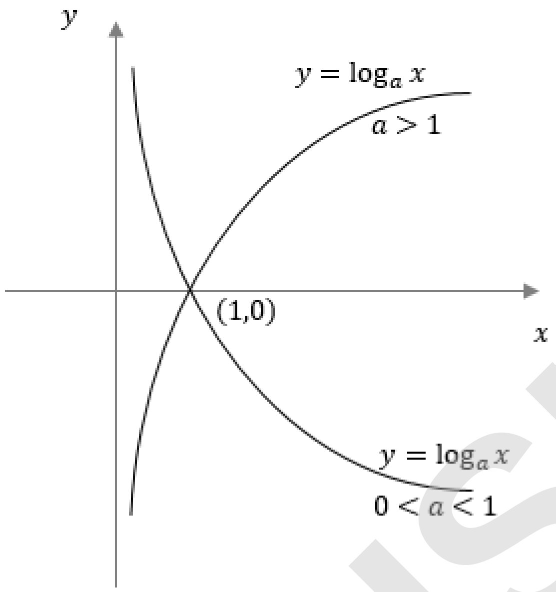


Figure 8.5.7: The graph of logarithmic function

8.6 Largest Possible Domain and the Range of a Function

We can say that a function is not completely specified if the domain is not indicated, since different choices of domain will give different functions.

However, when a domain is not specified it is usually understood that the domain of the largest possible domain for which the function is defined.

We have to select the largest set in the real number set for x variable such that the image of the function is in the real number set.

Working rule for finding the Domain of a functions

In order to find the domain of a function $y = f(x)$ find all those real values of x for which y is defined ($\therefore y$ is real)

I. For Algebraic Functions

a) Denominator should be non – zero

b) Expression under the even root should be non - negative

II. For Trigonometric Functions

a) $\sin(x)$ and $\cos(x)$ are defined for all real x

b) $\tan(x)$ and $\sec(x)$ are defined for all real x except

$x = (2n+1)\frac{\pi}{2}, \qquad (n \in \mathbb{Z})$

c) $\cot(x)$ and $\operatorname{cosec}(x)$ are defined for all real x except $x = n\pi$ $(n \in \mathbb{Z})$

III. For Logarithmic Functions

a) $\log_b a$ defined $a > 0$, $b > 0$ and $b \neq 1$

IV. For Exponential Functions

a) a^x is defined for all real x where $a > 0$

Working Hints for Solving Problems on domain of a function

(i) $(x-a)(x-b) > 0 \Rightarrow x < a$ or $x > b$ {Hence $a < b$}

(ii) $(x-a)(x-b) < 0 \Rightarrow a < x < b$ {Hence $a < b$}

(iii) $|x| < a \Rightarrow -a < x < a$

(iv) $|x| > a \Rightarrow x < -a$ or $x > a$

(v) $\log_b a > k \Rightarrow \begin{cases} a = b^k \ if \ b > 1 \\ a < b^k \ if \ b < 1 \end{cases}$

(vi) $\sqrt{x^2} = |x|$

How to find the range of the given function

We can give the method of the determination of the range of the functions as follows.

Find the domain of the given function $y = f(x)$

i. If the domain is an infinite interval, solve the equation $y = f(x)$ and find x in terms of y.

We assume that $x = g(y)$.

Then we have to find y values for which x is real.

The set of values of y so obtained constitutes the range of y.

ii. If the domain of the function is a finite interval, find the least and greatest values of y, for values of x in the domain. Let us assume that the p and q are the least and greatest values of y respectively.

Example 3

Find the domain of the following functions.

i. $y = f(x) = \frac{x}{\sqrt{(x+1)(x-2)}}$

ii. $y = f(x) = \sqrt{\frac{x+3}{x-1}}$

iii. $y = f(x) = \log_3[x^2 + 2x - 8]$

iv. $y = f(x) = \log_{10}(3x - 1)$

v. $y = f(x) = \log_{10}(x^2 - 9)$

vi. $y = \log_{10}|x|$

vii. $y = f(x) = \frac{1}{\sqrt{2+|x|}}$

viii. $y = f(x) = \frac{1}{\log_{10}(3-x)} + \sqrt{x - 2}$

ix. $y = f(x) = \sqrt{\frac{(x-1)(x+3)}{x-2}}$

x. $y = f(x) = \sqrt{x^2 - 9} + \sqrt{x^2 - 4}$

Solution

i. $f(x) = \frac{x}{\sqrt{(x+1)(x-2)}}$

$\therefore \sqrt{(x + 1)(x - 2)}$ must be real and it is not equal to zero.

$\therefore (x + 1)(x - 2) > 0 \qquad x < -1$ or $x > 2$

$$x \in (-\infty, -1) \cup (2, \infty)$$

The domain of $f = \{x: x \in \mathbb{R}, x \in (-\infty, -1) \cup (2, \infty)\}$

ii. $f(x) = \sqrt{\frac{x+3}{x-1}} = \frac{\sqrt{(x+3)(x-1)}}{x-1}$

$\therefore$ we require $x - 1 \neq 0$ and $(x+3)(x-1) > 0$

$x \neq 1$ and $x < -3$ or $x > 1$

$\therefore$the domain of $f = \{x: x \in \mathbb{R}, x \in (-\infty, -3) \cup (1, \infty)\}$

iii. $y = f(x) = \log_3[x^2 + 2x - 8] = \log_3[(x+4)(x-2)]$

We know that $\log(g(x))$ is defined only $g(x) > 0$.

$\therefore (x+4)(x-2) > 0 \Rightarrow x < -4$ or $x > 2$

The domain of f= $\{x: x \in \mathbb{R}, x \in (-\infty, -4) \cup (2, \infty)\}$

iv. $f(x) = \log_{10}(3x - 1)$

$\therefore 3x - 1 > 0$ $\quad x > {}^1/_3$

The domain of f= $\{x: x \in \mathbb{R}, x \in ({}^1/_3, \infty)\}$

v. $f(x) = \log_{10}(x^2 - 9) \quad \Rightarrow \quad x^2 - 9 > 0$

$(x-3)(x+3) > 0$

$x < -3$ or $x > 3$

The domain of f= $\{x: x \in \mathbb{R}, x \in (-\infty, -3) \cup (3, \infty)\}$

vi. $y = \log_{10}|x|$

$\log|x|$ is defined for all x, except zero.

The domain of $f = \{x: x \in \mathbb{R}, x \notin 0\}$

vii. $y = f(x) = \frac{1}{\sqrt{2+|x|}}$

We know that for all real $x, 2 + |x| > 0$

$\therefore$The domain of $f = \{x: x \in \mathbb{R}\}$

viii. $y = f(x) = \frac{1}{\log_{10}(3-x)} + \sqrt{x - 2}$

$\log_{10}(3 - x)$ is defined only $3 - x > 0 \Rightarrow x < 3$

$\sqrt{x-2}$ is defined only $x - 2 > 0 \qquad x > 2$

$\therefore f(x)$ is defined only $2 < x < 3$

The domain of *f*$= \{x: x \in \mathbb{R}, x \in (2,3)\}$

ix. $\quad y = f(x) = \sqrt{\frac{(x-1)(x+3)}{x-2}}$

$$y = f(x) = \frac{\sqrt{(x-1)(x-2)(x+3)}}{(x-2)}$$

$f(x)$is defined only $(x-1)(x-2)(x+3) \geq 0$ and $x \neq 2$

Figure 8.6.1

The domain of *f*$= \{x: x \in \mathbb{R}, x \in [-3,1] \cup (2,\infty)\}$

x. $\quad y = f(x) = \sqrt{x^2-9} + \sqrt{x^2-4}$

$\therefore\ x^2 - 9 \geq 0 \qquad\qquad x^2 - 4 \geq 0$

$x^2 \geq 9 \qquad\qquad x^2 \geq 4$

$x \geq 3, x \leq -3 \qquad\qquad x \geq 2, x \leq -2$

The common interval $3 \leq x$or $x \leq -3$.

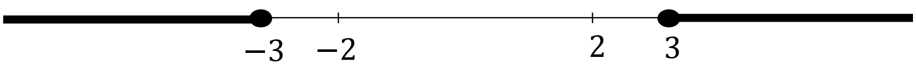


Figure 8.6.2

The domain of *f*$= \{x: x \in \mathbb{R}, x \in (-\infty,-3] \cup [3,\infty)\}$

Example 4

Find the ranges of the following functions.

(i) $f(x) = 2x^2 - 3x - 5$ (ii) $f(x) = \frac{x^2}{1+2x^2}$

(iii) $f(x) = \frac{x}{1+2x}$ (iv) $f(x) = \frac{2x}{1+x^2}$

(v) $f(x) = \frac{1}{2+\cos(3x)}$ (vi) $f(x) = \ln(4x^2 + 4x - 3)$

(vii) $f(x) = \sqrt{4x^2 + 4x - 3}$ (viii) $f(x) = \frac{x^2+x-6}{x^2+3x+2}$

(ix) $f(x) = \sqrt{\frac{1-x}{x-2}}$ (x) $f(x) = \frac{x+2}{x^2-8x+16}$

Solutions

(i) $f(x) = 2x^2 - 3x - 5$

∴the domain of $f\ is\ all\ x \in \mathbb{R}$

$$y = f(x) \quad = 2x^2 - 3x - 5 = 2\left[x^2 - {}^3/_2\, x - {}^5/_2\right]$$

$$= 2[x^2 - {}^3/_2\, x + ({}^3/_4)^2 - {}^5/_2 - {}^9/_{16}]$$

$$= 2\left[(x - {}^3/_2)^2 - {}^{49}/_{16}\right]$$

$$= 2(x - {}^3/_2)^2 - {}^{49}/_8$$

$$\therefore x \in \mathbb{R} \quad y \geq -{}^{49}/_8$$

∴the range of f $= \{y \in \mathbb{R}, y \in [-{}^{49}/_8, \infty)\}$

(ii) $f(x) = \frac{x^2}{1+2x^2}$

The domain of *f*= $\{x: x \in \mathbb{R}\}$

$$y = \frac{x^2}{1+2x^2}$$

$$y(1 + 2x^2) = x^2$$

$$x^2(1 - 2y) = y$$

$$x^2 = \frac{y}{1-2y}$$

$$x^2 \geq 0 \qquad \frac{y}{1-2y} \geq 0$$

$y(1 - 2y) \geq 0$ and $1 - 2y \neq 0$

$$y\left(y - {}^{1}\!/_{2}\right) \leq 0$$

$0 \leq y < {}^{1}\!/_{2}$ since$y \neq {}^{1}\!/_{2}$

$\therefore$the range of $f = \{y: y \in \mathbb{R}, y \in [0, {}^{1}\!/_{2})\}$

(iv) $f(x) = \frac{2x}{1+x^2}, D_f = \{x: x \in \mathbb{R}\}$

$$y = \frac{2x}{1+x^2} \Rightarrow (1 + x^2)y = 2x$$

$$yx^2 - 2x + y = 0 \text{ and } y \neq 0$$

since $x \in \mathbb{R}$, $\quad (-2)^2 - 4(y)(y) \geq 0$

$$y^2 \leq 1$$

$$-1 \leq y \leq 1$$

$\therefore$the range of f; $\quad R_f = \{y: y \in \mathbb{R}, y \in [-1,1], y \neq 0\}$

(v) $f(x) = \frac{1}{2+\cos(3x)}$

The domain of f, $D_f = \{x: x \in \mathbb{R}\}$

$$y = \frac{1}{2 + \cos(3x)} \Rightarrow (2 + \cos(3x))y = 1$$

$$\cos(3x) = \frac{1 - 2y}{y}$$

Since $-1 \leq \cos(3x) \leq 1$

$-1 \leq \frac{1-2y}{y} \leq 1$ $\qquad$ we know that $y > 0$

$\because -1 \leq \cos(3x) \leq 1$

$-y \leq 1 - 2y \leq y$

$y \leq 1 \qquad 3y \geq 1$

$y \leq 1 \qquad y \geq {}^{1}\!/_{3}$

$\therefore$The range of f; $\quad R_f = \{y: y \in \mathbb{R}, y \in [{}^{1}\!/_{3}, 1]\}$

(vi) $f(x) = \ln(4x^2 + 4x - 3)$

$$4x^2 + 4x - 3 > 0$$

$$(2x + 1)^2 - 2^2 > 0$$

$$(2x+1+2)(2x+1-2) > 0$$

$$(2x+3)(2x-1) > 0$$

$$\left(x+{}^{3}/_{2}\right)\left(x-{}^{1}/_{2}\right) > 0$$

$$x < -{}^{3}/_{2}\,, x > {}^{1}/_{2}$$

∴Domain of $D_f = \left\{x: x \in \mathbb{R}, x \in \left\{\left(-\infty, -{}^{3}/_{2}\right) \cup \left({}^{1}/_{2}\,, \infty\right)\right\}\right\}$

$$y = \ln(4x^2+4x-3) \Leftrightarrow e^y = 4x^2+4x-3$$

$$4x^2+4x-(3+e^y) = 0$$

$$x = \frac{-4 \pm \sqrt{4^2+4\times 4(3+e^y)}}{8} = \frac{-1 \pm \sqrt{4+e^y}}{2}$$

$x = -\frac{1}{2}\left(1+\sqrt{4+e^y}\right)$

$x < -{}^{3}/_{2}$

$-\frac{1}{2}\left(1+\sqrt{4+e^y}\right) < -{}^{3}/_{2}$

$3 < 1+\sqrt{4+e^y}$

$\sqrt{4+e^y} > 2$

$4+e^y > 4$

$e^y > 0$

$\therefore y \in \mathbb{R}$

$x = -\frac{1}{2}\left(1-\sqrt{4+e^y}\right)$

$x > {}^{1}/_{2}$

$\frac{1}{2}\left(\sqrt{4+e^y}-1\right) > {}^{1}/_{2}$

$\sqrt{4+e^y} > 2$

$e^y > 0$

$y \in \mathbb{R}$

∴the range of f; $R_f = \{y: y \in \mathbb{R}\}$

(vii) $f(x) = \sqrt{4x^2+4x-3}$

$= \sqrt{(2x+3)(2x-1)}$

$\therefore (2x+3)(2x-1) \geq 0$

$x \geq {}^{1}/_{2} \qquad x \leq -{}^{3}/_{2}$

$y = \sqrt{4x^2+4x-3}$

$4x^2+4x = 3+y^2$

$(2x+1)^2 = 4+y^2$

$4+y^2 \geq 0$

It is true for all real y.

∴the range of f; $R_f = \{y: y \in \mathbb{R}\}$

(viii) $f(x) = \frac{x^2+x-6}{x^2+3x+2} = \frac{x^2+x-6}{(x+2)(x+1)}$

$\therefore D_f = \{x: x \in \mathbb{R}, x \neq -2, x \neq -1\}$

$$y = \frac{x^2+x-6}{x^2+3x+2}$$

$$y(x^2+3x+2) = x^2+x-6$$

$$(y-1)x^2 + (3y-1)x + 6 + 2y = 0$$

$x \in \mathbb{R} \quad \therefore (3y-1)^2 - 4(y-1)(6+2y) \geq 0$

$$9y^2 - 6y + 1 - 8y^2 - 16y + 24 \geq 0$$

$$y^2 - 22y + 25 \geq 0$$

$$(y-11)^2 \geq 121 - 25$$

$$(y-11)^2 \geq 96$$

$y - 11 \geq 4\sqrt{6}, \qquad y - 11 \leq -4\sqrt{6}$

$y \geq 11 + 4\sqrt{6} \qquad y \leq -\left(4\sqrt{6} - 11\right)$

$R_f = \{y: y \in \mathbb{R}, y \in \left[(-\infty, -(4\sqrt{6}-11) \cup (11+4\sqrt{6}, \infty)\right]\}$

(ix) $f(x) = \sqrt{\frac{1-x}{x-2}} = \frac{\sqrt{(1-x)(x-2)}}{(x-2)}$

$(1-x)(x-2) \geq 0$ and $x - 2 \neq 0$

$$(x-1)(x-2) \leq 0$$

$$1 \leq x < 2$$

$D_f = \{x: x \in \mathbb{R}, x \in [1,2)\}$

$$y = \frac{\sqrt{(1-x)(x-2)}}{(x-2)}$$

$$y^2(x-2)^2 = (1-x)(x-2)$$

$$(x-2)\{y^2(x-2) + (x-1)\} = 0$$

$x \neq 2, \quad y^2(x-2) + (x-1) = 0$

$$x(y^2+1) = 2y + 1$$

$$x = \frac{2y+1}{y^2+1}$$

Since $x \in \mathbb{R}, \ y \in \mathbb{R}$.

$\therefore$the range of f; $R_f = \{y: y \in \mathbb{R}\}$

(x) $f(x) = \frac{x+2}{x^2-8x+16}$

$$y = \frac{x+2}{(x-4)^2} \qquad x \neq 4$$

$\therefore D_f = \{x: x \in \mathbb{R}, x \neq 4\}$

$$y = \frac{x+2}{(x-4)^2}$$

$$y(x^2 - 8x + 16) = x + 2$$

$$yx^2 - (8y+1)x + 16y - 2 = 0$$

$$\therefore (8y+1)^2 - 4y(16y-2) \geq 0$$

$$24y \geq -1$$

$$y \geq -{}^1/_{24}$$

$\therefore$the range of f; $R_f = \left\{y: y \in \mathbb{R}, y \in \left[-{}^1/_{24}, \infty\right)\right\}$

Example 5

1) (i) If $f(x) = \frac{x - \tan x}{x + \cos x}$ find $f({}^\pi/_4)$.

(ii) $f(x) = \frac{x^2+1}{x}$ find $f(\tan\theta)$.

Also show that $f\left({}^1/_x\right) = f(x)$.

2) If $f(x) = \frac{a^x - a^{-x}}{a^x + a^{-x}}$ show that

$$f(x+y) = \frac{f(x) + f(y)}{1 + f(x).f(y)}$$

3) If $f(x) = {}^{|x|}/_x \; x \neq 0$ show that $|f(\alpha) - f(-\alpha)| = 2$ where $\alpha \neq 0$.

4) If $f(x) = \frac{x+1}{x-1}$ find $f(x)$.

Solution

1) (i) $f(x) = \frac{x - \tan x}{x + \cos x}$

$$f\left({}^\pi/_4\right) = \frac{{}^\pi/_4 - \tan{}^\pi/_4}{{}^\pi/_4 + \cos{}^\pi/_4} = \frac{\frac{\pi}{4} - 1}{\frac{\pi}{4} + \frac{1}{\sqrt{2}}} = \frac{\pi - 4}{\pi + 4\sqrt{2}}$$

(ii) $f(x) = \frac{x^2+1}{x}$

$$f(\tan\theta) = \frac{\tan^2\theta + 1}{\tan\theta} = \frac{\sec^2\theta}{\tan\theta} = \frac{\sec\theta}{\sin\theta} = \sec\theta\csc\theta$$

$$f(1/x) = \frac{(1/x)^2 + 1}{1/x} = \frac{1+x^2}{x}$$

$$\therefore f(x) = f(1/x)$$

2) $f(x) = \frac{a^x - a^{-x}}{a^x + a^{-x}}$

$$f(x) = \frac{a^x - \frac{1}{a^x}}{a^x + \frac{1}{a^x}} = \frac{a^{2x} - 1}{a^{2x} + 1}$$

$$\therefore f(x) + f(y) = \left(\frac{a^{2x} - 1}{a^{2x} + 1}\right) + \left(\frac{a^{2y} - 1}{a^{2y} + 1}\right)$$

$$= \frac{(a^{2y}+1)(a^{2x}-1) + (a^{2y}-1)(a^{2x}+1)}{(a^{2x}+1)(a^{2y}+1)}$$

$$= \frac{2\left(a^{2(x+y)} - 1\right)}{(a^{2x}+1)(a^{2y}+1)}$$

$$f(x).f(y) = \frac{(a^{2x}-1)(a^{2y}-1)}{(a^{2x}+1)(a^{2y}+1)}$$

$$1 + f(x).f(y) = 1 + \frac{(a^{2x}-1)(a^{2y}-1)}{(a^{2x}+1)(a^{2y}+1)}$$

$$= \frac{(a^{2x}+1)(a^{2y}+1) + (a^{2x}-1)(a^{2y}-1)}{(a^{2x}+1)(a^{2y}+1)}$$

$$= \frac{2\left\{a^{2(x+y)} + 1\right\}}{(a^{2x}+1)(a^{2y}+1)}$$

$$\therefore \frac{f(x) + f(y)}{1 + f(x).f(y)} = \frac{2\left\{a^{2(x+y)} - 1\right\} \Big/ (a^{2x}+1)(a^{2y}+1)}{2\{a^{2(x+y)} + 1\} \Big/ (a^{2x}+1)(a^{2y}+1)}$$

$$= \frac{a^{2(x+y)} - 1}{(a^{2(x+y)} + 1)}$$

$$= f(x+y)$$

$$\therefore f(x+y) = \frac{f(x) + f(y)}{1 + f(x).f(y)}$$

3) $f(x) = {}^{|x|}/_{x}\,,\ x \neq 0$

When $\alpha > 0$,

$$f(\alpha) = \frac{\alpha}{\alpha} = 1$$

$$f(-\alpha) = \frac{\alpha}{-\alpha} = -1$$

$$|f(\alpha) - f(-\alpha)| = |1 - (-1)| = 2$$

when$\alpha < 0$

$$\alpha = -m \quad m > 0 \quad f(\alpha) = \frac{m}{-m} = -1$$

$$f(-\alpha) = \frac{m}{m} = 1$$

$$|f(\alpha) - f(-\alpha)| = |-1 - 1| = 2$$

$$\therefore |f(\alpha) - f(-\alpha)| = 2; \qquad \alpha \neq 0$$

4) $f(x) = \frac{x+1}{x-1}$

$$f\big(f(x)\big) = \frac{f(x)+1}{f(x)-1} = \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} = \frac{(x+1)+(x-1)}{(x+1)-(x-1)} = \frac{2x}{2}$$

$$f(f(x)) = x$$

Activity 1

(1) Find the domain, D_f of the following functions where D_f is a subset of $\mathbb{R}$.

(a) $f{:}\,x \to \sqrt{1-x^2}$ (b) $f{:}\,x \to \sqrt{x^2-9}$

(c) $f{:}\,x \to \left|\frac{1}{x-2}\right|$ (d) $f{:}\,x \to \frac{4x+1}{4x-1}$

(e) $f{:}\,x \to x|x|$ (f) $f{:}\,x \to \frac{1}{\sqrt{x+|x|}}$

(g) $f{:}\,x \to \frac{x}{x^2-3x+2}$ (h) $f{:}\,x \to \frac{x}{\ln(1+x)}$

(i) $f{:}\,x \to \sqrt{x-3} + \sqrt{1-x}$ (j) $f{:}\,x \to \frac{x^2-5x+3}{x^2-1}$

(2) Find the range R_f of the following functions; where R_f is a subset of $\mathbb{R}$.

(a) $f: x \to \frac{1}{\sqrt{4+3\sin x}}$ (b) $f: x \to sin^2 x^3 + cos^2 x^3$

(c) $f: x \to \frac{x^2-9}{x-3}, x \neq 3$ (d) $f: x \to \frac{4-x}{x-4}, x \neq 4$

(e) $f: x \to \frac{x+2}{x^2-8x+2}$ (f) $f: x \to x^2 + 2x + 7$

(g) $f: x \to 10 - 4x - x^2, x \geq 1$ (h) $f: x \to |x^2 - 4|, x \in \mathbb{R}$

(i) $f: x \to 4 - |2x|, x \in \mathbb{R}$ (j) $f: x \to \frac{1}{x-2}, x \in \mathbb{R}, x \neq 2$

(3) (i) If $f(t) = \frac{1+t^2}{1-t^2} t \neq 1, t \neq -1$

find $f(1/x)$.

(ii) If $f(x) = \frac{5x^2+1}{2-x}$

find $f(3x), f(x^3), 3f(x), [f(x)]^3$.

(iii) If $(x) = \frac{1+x}{1-x}$, show that $\frac{f(x)f(x^2)}{1+(f(x))^2} = \frac{1}{2}$ and

$f(f(x)) = -\frac{1}{x}$.

(iv) If $f(x) = x^2 - 3x + 4$ then find the value of x satisfying the equation $f(x) = f(2x+1)$.

(v) If $y = f(x) = \frac{2x-3}{5x-2}$

show that $x = f(y)$.

Solution to Activity

Activity 1

(1) (a) $D_f = \{x: x \in \mathbb{R}, x \in [-1,1]\}$

(b) $D_f = \{x: x \in \mathbb{R}, x \in (-\infty, -3] \cup [3, \infty)\}$

(c) $D_f = \{x: x \in \mathbb{R}, x \notin 2\}$

(d) $D_f = \{x: x \in \mathbb{R}, x \neq 1/4\}$

(e) $D_f = \{x: x \in \mathbb{R}\}$

(f) $D_f = \{x: x \in \mathbb{R}^+\}$

(g) $D_f = \{x: x \in \mathbb{R}, x \neq 2, x \neq 1\}$

(h) $D_f = \{x : x \in \mathbb{R}, x \neq 0, x \geq -1\}$

(i) This function is not defined.

(j) $D_f = \{x : x \in \mathbb{R}, x \neq 1, x \neq -1\}$

(2) (a) $R_f = \left\{y : y \in \mathbb{R}, y \in \left({}^{1}/_{\sqrt{7}}, 1\right)\right\}$

(b) $R_f = \{y : y = 1\}$

(c) $R_f = \{y : y \in \mathbb{R}, y \neq 6\}$

(d) $R_f = \{y : y \in \mathbb{R}, y = -1\}$

(e) $R_f = \{y : y \in \mathbb{R}, y \in (-\infty, {}^{1}/_{4}) \cup (-{}^{1}/_{20}, \infty)\}$

(f) $R_f = \{y : y \in \mathbb{R}, y \geq 6\}$

(g) $R_f = \{y : y \in \mathbb{R}, y \leq 5\}$

(h) $R_f = \{y : y \in \mathbb{R}_0^+\}$

(i) $R_f = \{y : y \in \mathbb{R}, y \leq 4\}$

(j) $R_f = \{y : y \in \mathbb{R}, y \neq 0\}$

(3) (i) $\frac{x^2+1}{x^2-1}$ (ii) $\frac{45x^2+1}{2-3x}, \frac{5x^6+1}{2-x^3}, \frac{15x^2+3}{2-x}, \left(\frac{5x^2+1}{2-x}\right)^3$

(iv) $x = -1$ or $x = {}^{2}/_{3}$

Summary

A function $f : X \to Y$ is a rule (or relation) which associates with elements $x \in X$ and unique $y \in Y$ such that $y\ f\ x$.

The set X is called the domain of f.

The set Y is called the codomain of f.

The elements of Y which are f images of the elements in the domain X, the set of such f images in Y is called the range of f.

Learning outcomes

On completion of this study session you should be able to

- Identify the difference between the function and the relation
- Sketch the graph of the given specific function
- Determine of the domain and the range of the given function

Session 9

Polynomials and Partial Fractions

Contents:

Introduction

In this session, we discuss the polynomials that has important role in the study of algebra. Most concepts of algebra are based on polynomials. We present the remainder theorem and factor theorem that relate to polynomials. Some problems involving these theorems are solved.

In the Unit 01 of the session 2, you have studied the simplifications of the algebraic fractions. In the process by adding and subtracting of two or more algebraic fractions, we have formed a single fraction.

For example,

$$\frac{1}{x+1}+\frac{2}{x-2}+\frac{3}{x-3}=\frac{6x^2-12x-6}{(x+1)(x-2)(x-3)} \quad ;$$

In this session, you can learn how to express a single fractioning to addition finite number of algebraic fractions. These fractions are called Partial Fractions.

9.1 Definition of a Polynomial

Let $f: x \to a_0 + a_1x + a_2x^2 + \cdots a_rx^r + \cdots a_nx^n$ be function.

If $x \in \mathbb{R}$ and $a_0, a_1, a_2 \ldots\ldots a_r \ldots a_n$ are all real constants and $a_n \neq 0$ and n **is a positive integer**, then in general, $f(x)$ is a polynomial in x.

$$\therefore\ f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_rx^r + a_nx^n$$

The positive integer 'n' is called the degree of the polynomial.

For example, $f_1(x) = 6x^5 + 4x^3 + 8$

$$f_2(x) = 1 + 2x$$

$$f_3(x) = 3x^2 + 9x^8$$

Here, $f_1(x)$ is a polynomial of degree five, $f_2(x)$ is a polynomial of degree one, and $f_3(x)$ is a polynomial of degree eight.

Also, $g(x) = \sqrt{1-x^2}$,$h(x) = 3x^{-2} + 5$, $k(x) = \frac{x^2+1}{x-1}$, $l(x) = 3x^{-\frac{1}{2}} + 9x + 12$ are not polynomials in x.

Algebraic Operation on Polynomials

Sum, Difference and Product of Two Polynomials

Let $F(x) = 9x^6 + 8x^4 + 3x + 1$ and $G(x) = 8x^3 + 6x^2 + 7x + 1$

We know that $F(x)$ and $G(x)$ are polynomials of degree6 and degree 3 respectively.

The sum of the polynomials $F(x)$and $G(x)$ is given by the polynomial

$$F(x) + G(x) = 9x^6 + 8x^4 + 3x + 1 + 8x^3 + 6x^2 + 7x + 1$$
$$= 9x^6 + 8x^4 + 8x^3 + 6x^2 + 10x + 2$$

$\therefore$ $F(x) + G(x)$ is a polynomial of degree 6.

The difference of the polynomials $F(x)$ and $G(x)$ is given by the polynomial

$$F(x) - G(x) = 9x^6 + 8x^4 + 3x + 1 - (8x^3 + 6x^2 + 7x + 1)$$
$$= 9x^6 + 8x^4 - 8x^3 - 6x^2 - 4x$$

$\therefore$ $F(x) - G(x)$ is a polynomial of degree 6.

The product of the polynomials $f(x)$ and $g(x)$is given by the polynomial

$$F(x)G(x) = (9x^6 + 8x^4 + 3x + 1)(8x^3 + 6x^2 + 7x + 1)$$
$$= 72x^9 + 48x^8 + 64x^7 + 57x^6 + 56x^5 + 32x^4 + 32x^3 + 27x^2 + 10x + 1$$

$\therefore$ $F(x)G(x)$is a polynomial of degree 9.

In general, $F(x) = a_0 + a_1x + a_2x^2 + \cdots + a_rx^r + a_nx^n$

$$G(x) = b_0 + b_1x + b_2x^2 + \cdots + b_rx^r + b_mx^m$$

Where $n > m$,

$$\therefore F(x) + G(x)$$
$$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_r + b_r)x^r + \cdots (a_m + b_m)x^m + a_{m+1}x^{m+1} + \cdots a_nx^n$$

$\therefore F(x) + G(x)$ is a polynomial of degree n.

$$F(x) - G(x) = (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)x^2 + \cdots + (a_r - b_r)x^r + \cdots (a_m - b_m)x^m + a_{m+1}x^{m+1} + \cdots a_nx^n$$

$\therefore F(x) - G(x)$ is also a polynomial of degree n.

$$F(x).G(x) = \{a_0 + a_1x + a_2x^2 + \cdots + a_rx^r + a_nx^n\} \times \{b_0 + b_1x + b_2x^2 + \cdots + b_rx^r + b_mx^m\}$$
$$= a_0b_0 + (a_1b_0 + a_0b_1)x + (a_1b_1 + a_2b_0 + a_0b_2)x^2 + \cdots (a_0b_3 + a_1b_2 + a_2b_3 + a_3b_0)x^3 + \cdots (a_nb_m)x^{m+n}$$

$\therefore F(x).G(x)$ is a polynomial of degree $m + n$.

Given two polynomials $F(x)$ and $G(x)$, we can consider their sum $F(x) + G(x)$, their difference $F(x) - G(x)$ and their product$F(x).G(x)$, Which are also polynomials.

9.2 Division Algorithm

Dividing one polynomial by a non-zero polynomial

Now we are going to discuss dividing one polynomial by a non-zero polynomial. To perform the division of two polynomials, we use the "long division method".

Let $F(x) = 12x^5 + 8x^4 - 2x^2 + 6,\ G(x) = x^2 + x + 2$

$F(x)$ is the divident, $G(x)$ is the divisor.

(We studied this long division method in session no.2 unit 01.)

$$\frac{F(x)}{G(x)} = Q(x) + \frac{R(x)}{G(x)}$$

Here, $Q(x)$ is called the **quotient** and $R(x)$ is called the **remainder.**

$$\therefore F(x) = Q(x).G(x) + R(x)$$

This expression is called the **Division algorithm**.

Now we can consider the given example.

$$\frac{12x^5 + 8x^4 - 2x^2 + 6}{x^2 + x + 2}$$

$$12x^3 - 4x^2 - 20x + 26$$

$$\begin{array}{l|l}
x^2 + x + 2 & 12x^5 + 8x^4 \ - 2x^2 + 6 \\
 & \underline{12x^5 + 12x^4 - 24x^3} \\
 & \quad -4x^4 + 24x^3 - 2x^2 + 6 \\
 & \quad \underline{-4x^4 + 4x^3 - 8x^2} \\
 & \qquad -20x^3 + 6x^2 + 6 \\
 & \qquad \underline{-20x^3 - 20x^2 - 40x} \\
 & \qquad\quad 26x^2 + 40x + 6 \\
 & \qquad\quad \underline{26x^2 + 26x + 52} \\
 & \qquad\qquad 14x - 46
\end{array}$$

$\boldsymbol{Q(x) = 12x^3 - 4x^2 - 20x + 26}$

$R(x) = 14x - 46$

$By\ the\ division\ algorithm, we\ have$ $12x^5 + 8x^4 \ - 2x^2 + 6 = (12x^3 - 4x^2 - 20x + 26)(x^2 + x + 2)$

$+ 14x - 46$

If the degree of $Q(x)$ is ‘n’ and the degree of $G(x)$ is ‘m’, then in general, the degree of $Q(x)$ is $(n - m)$ and the degree of $R(x)$ is $(m - 1)$.

9.3 Remainder Theorem and Factor Theorem

Consider the polynomial $F(x) = 12x^5 - 4x^4 - 3x^2 + 2x + 1$ and divide by $G(x) = (x + 2)$

Let us divide$F(x)$ by $G(x)$ using the long division method.

We can say that the quotient $Q(x)$ is $(5 - 1) = 4$ and the degree of the remainder is $(1 - 1) = 0$ i.e. a constant term.

$$
\begin{array}{r|l}
 & 12x^4 + 20x^3 - 40x^2 - 77x - 152 \\
\hline
x + 2 & 12x^5 - 4x^4 - 3x^2 + 2x + 1 \\
 & \underline{12x^5 - 24x^4} \\
 & \quad -20x^4 - 3x^2 + 2x + 1 \\
 & \quad \underline{-20x^4 + 40x^3} \\
 & \quad\quad -40x^3 - 3x^2 + 2x + 1 \\
 & \quad\quad \underline{-40x^3 - 80x^2} \\
 & \quad\quad\quad 77x^2 + 2x + 1 \\
 & \quad\quad\quad \underline{77x^2 + 154x} \\
 & \quad\quad\quad\quad -152x + 1 \\
 & \quad\quad\quad\quad \underline{-152x - 304} \\
 & \quad\quad\quad\quad\quad 305
\end{array}
$$

$\therefore$ the quotient is $12x^4 + 20x^3 - 40x^2 - 77x - 152$ and the remainder is 305.

By the division algorithm, we have, $12x^5 - 4x^4 - 3x^2 + 2x + 1 = (12x^4 + 20x^3 - 40x^2 - 77x - 152)(x + 2) + 305$

We have

$$F(x) = 12x^5 - 4x^4 - 3x^2 + 2x + 1$$

$$
\begin{aligned}
F(-2) &= 12(-2)^5 + 4(-2)^4 + 3(-2)^2 + 2(-2) + 1 \\
&= 12\times 32 - 4 \times 16 - 3 \times 4 - 4 + 1 \\
&= 384 - 64 - 12 - 3
\end{aligned}
$$

$F(-2) = 305$ which is the above remainder.

$\therefore$ When $F(x)$ is divided by $(x+2)$, then the remainder is $F(-2)$.

This example illustrates the following result known as the remainder theorem,

> If a polynomial $F(x)$ is divided by $(x-\alpha)$, then the remainder is equal to $F(\alpha)$. This is known as the remainder theorem.

Proof of the Remainder Theorem

Let $Q(x)$ be the Quotient, when $F(x)$ is divided by $(x-\alpha)$ and the remainder is R.

$\therefore$by using the Division Algorithm,

$F(x) = Q(x)(x-\alpha) + R$

Now we can see if we substitute value for α, the $Q(x)$ term vanishes from the above identity.

$F(\alpha) = Q(\alpha)(\alpha-\alpha) + R$

$$\therefore R = F(\alpha)$$

Example 1

1. Find the remainder when,
   (i) $4x^3+3x^2-5x-2$ is divided by $(x-2)$
   (ii) $6x^6+5x^2+3x+8$ is divided by $(x+2)$
   (iii) $3x^4+4x^3+6x-8$ is divided by $(x+1)$

2. (a) If the polynomial x^3-ax^2-bx+1 is divided by $(x+3)$, then the remainder is 19. If it is divided by $(x+1)$, then the remainder is 9. Find the values of a and b.

   (b) The polynomials x^3+4x^2-2x-1 and x^3+3x^2-3x+5 leave the same remainder when they are divided by $(x-p)$. Find the possible values of p.

Solution:

(1) (i).$f(x) = 4x^3 + 3x^2 - 5x - 2$, divided by $(x - 2)$

∴The remainder $\boldsymbol{f(2) = 4(2)^3 + 3(2)^2 - 5(2) - 2}$

$= 32 + 12 - 12 = 32$

∴The remainder $f(2) = 32$

(ii) $f(x) = 6x^6 + 5x^2 - 3x - 8$, divided by $(x + 2)$

∴The remainder $f(-2)$

$= 6(-2)^6 + 5(-2)^2 - 3(-2) - 8$

$= \boldsymbol{6 \times 64 + 5 \times 4 + 6 - 8}$

$= \boldsymbol{404 + 18}$

$= 422$ ∴The remainder $f(-2) = 422$

(iii) $f(x) = 3x^4 + 4x^3 + 6x - 8$, divided by $(x + 1)$

∴The remainder $f(-1) = 3(-1)^4 + 4(-1)^3 + 6(-1) - 8$

$= 3 + 4 - 6 - 8$

$= -2$

∴ The remainder $= -2$

(2) (a) Let $f(x) = x^3 - ax^2 - bx + 1$

∴The remainder $f(-3) = (-3)^3 - a(-3)^2 - b(-3) + 1 = 19$

$9a - 3b = -27 + 1 - 19$

$3a - b = 15 \quad - (1)$

When $f(x)$ is divided by $(x + 1)$, then the remainder

$f(-1) = (-1)^3 - a(-1)^2 - b(-1) + 1 = 9$

$b - a = 9 \quad - (2)$

$(1) + (2)$

$2a = 24$

$a = 12\,, \qquad b = 21$

(b) Let $f(x) = x^3 + 4x^2 - 2x - 1$

$g(x) = x^3 + 3x^2 - 3x + 5$

Since the $f(x)$ and $g(x)$ is divided by $(x-p)$, the remainder are the same $f(p)=g(p)$

$$p^3+4p^2-2p-1=p^3+3p^2-3p+5$$

$p^2+p-6=0$

$(p-2)(p+3)=0$

$\therefore$ the possible p values,$\Rightarrow p\ =2$ and $p=-3$

Factor Theorem

If $f(\alpha)=0\ where\ \alpha\ is\ a\ constant$, $(x-\alpha)$ is a factor of $f(x)$.

.

Proof.

Suppose that $Q(x)$ and $R(x)$ are the quotient and the remainder respectively when $F(x)$ is divided by $(x-\alpha)$.

By the division algorithm, we can write as $F(x)=(x-\alpha)Q(x)+R(x)$

If $f(\alpha)=0$, $\therefore F(x)=(x-\alpha)Q(x)$

It says if $F(\alpha)=0$, then $(x-\alpha)$ is a factor of $F(x)$.

Note

Let $F(x)$ be a polynomial and α is a real constant. then $(x-\alpha)$ is a factor of $F(x)\ if\ and\ only\ if\ F(\alpha)=0$, which is the factor theorem.

Example 2

(1)

Prove that $(x-2)$ and $(x+3)$ are factors of the polynomial $3x^4+7x^3-15x^2-25x+6$ and find the other quadratic factor.

(a) If $F(x)=6x^5-7x^4-20x^3+15x^2+px+q$ has factors of $(x+1)$ and $(x-2)$, find the values of p and q. Hence obtain the remaining factors.

(b) Factorize the following polynomials.

(i) x^4-x^2-72

(ii) $3x^3-2x^2-7x-2$

(iii) $4y^3-13y+6$

(iv) $4y^4-4yx^3-9y^2+y+2$

(2)

(a) Find what values of p must have in order $(x+p)$ may be a factor of,

$4x^3+(3p-2)x^2-(p^2-1)x+3.$

Write down the remaining expression corresponding to each value of $p.$

(b) Prove that two polynomials $f(x)$ and $g(x)$ have a common linear factor $(x-k)$. $(x-k)$is a factor of the polynomial $f(x)-g(x)$. Hence show that if the equations $bx^3+4x^2-5x-10=0$ and $bx^3-9x-2=0$ have a common root then find the values of b.

Solution:

1. (a) let $f(x)=3x^4+7x^3-15x^2-25x+6$

$$f(2)=3(2)^4+7(2)^3-15(2)^2-25(2)+6$$
$$=48+56-60-50+6=0$$

From the factor theorem $(x-2)$ is a factor of $f(x)$.

$$f(-3)=3(-3)^4+7(-3)^3-15(-3)^2-25(-3)+6$$
$$=243-189-135+75+6=0$$

∴ from the factor theorem $(x+3)$ is a factor of $f(x)$,

$$\therefore\ f(x)\equiv 3x^4+7x^3-15x^2-25x+6$$
$$\equiv (x-2)(x+3)g(x)$$

$g(x)$ is the remaining factor of the polynomial $f(x)$. we know that $f(x)$ is 4^{th}degree and $(x-2)(x+3)$ is a 2^{nd}degree.

∴ $g(x)$ is a 2^{nd}degree polynomial.

The general form of $g(x)$ is $(\lambda x^2+\mu x+r)$ where $\lambda\,,\mu,r$ constants to be determined.

$$f(x)\equiv 3x^4+7x^3-15x^2-25x+6 \quad -(1)$$
$$\equiv (x-2)(x+3)(\lambda x^2+\mu x+r)$$

$$\equiv (x^2 + x - 6)(\lambda x^2 + \mu x + r)$$

$$f(x) \equiv \lambda x^4 + (\lambda + \mu)x^3 + (\mu - 6\lambda + r)x^2 + (r - 6\mu)x - 6r$$
$$- (2)$$

Equating coefficients between (1) and (2),

$$\lambda = 3$$

$$\lambda + \mu = 7$$
$$3 + \mu = 7$$
$$\mu = 4$$

$$-6r = 6$$
$$r = -1$$

You can see that these values of λ, μ and r satisfies the coefficients of x^2 and x.

$f(x) = (x^2 + x - 6)(3x^2 + 4x - 1)$

$(3x^2 + 4x - 1)$ is a quadratic factor of $f(x)$.

(b) $F(x) = 6x^5 - 7x^4 - 20x^3 + 15x^2 + px + q$

Since $(x + 1)$ and $(x - 2)$ are factors of $F(x)$,

$F(-1) = 6(-1)^5 - 7(-1)^4 - 20(-1)^3 + 15(-1)^2 + p(-1) + q = 0$

$= -6 - 7 + 20 + 15 - p + q$

$= 0$

$p - q = 22 \quad - (1)$

$F(2) = 6(2)^5 - 7(2)^4 - 20(2)^3 + 15(2)^2 + p(2) + q = 0$

$192 - 112 - 160 + 60 + 2p + q = 0$

$2p + q = 20 \quad - (2)$

$(1) + (2)$

$$3p = 42$$

$\therefore p = 14$ and $q = -8$

$\therefore F(x) = 6x^5 - 7x^4 - 20x^3 + 15x^2 + px + q$

$=(x-2)(x+1)G(x)$

$= (x^2 - x - 2)G(x)$

$F(x)$ is a factor polynomial of degree 5.

∴The other factor $g(x)$ is a polynomial of degree 3.

$g(x) = \lambda x^3 + \mu x^2 + rx + \delta$

$F(x) = 6x^5 - 7x^4 - 20x^3 + 15x^2 + 14x - 8 \quad -(3)$

$= (x^2 - x - 2)(\lambda x^3 + \mu x^2 + rx + \delta)$

$F(x) = \lambda x^5 + (\mu - \lambda)x^4 + (r - \mu - 2\lambda)x^3 + (\delta - r - 2\mu)x^2 - (\delta + 2r)x - 2\delta \quad -(4)$

Equating coefficients between (3) and (4)

$\lambda = 6 \quad \mu - \lambda = -7 \qquad r - \mu - 2\lambda = -20$

$\mu - 6 = -7 \qquad r = -20 - 1 + 2 \times 6$

$\mu = 6 - 7 = -1 \qquad r = -9$

$\delta - r - 2\mu = 15$

$\delta = 15 + r + 2\mu = 15 - 2 - 9 = 4$

$\therefore g(x) = 6x^3 - x^2 - 9x + 4$

$g(1) = 6(1)^3 - (1)^2 - 9(1) + 4 = 0$

$\therefore$ $(x-1)$ is a factor of $g(x)$.

$\therefore g(x) \equiv 6x^3 - x^2 - 9x + 4 = (x-1)\{6x^2 + wx - 4\}$

Considering coefficient of x^2 $we\ have\ -1 = -6 + w$

$\Rightarrow w = 5$

$$g(x) = (x-1)\{6x^2 + 5x - 4\}$$
$$= (x-1)(3x+4)(2x-1)$$

$\therefore f(x) \equiv 6x^5 - 7x^4 - 20x^3 + 15x^2 + 14x - 8$

$\equiv (x+1)(x-1)(x-2)(3x+4)(2x-1)$

(c) (i) $f(x) \equiv x^4 - x^2 - 72$

$\equiv (x^2 - 9)(x^2 + 8)$

$\equiv (x-3)(x+3)(x^2 + 8)$

(ii) $g(x)$=$3x^3 - 2x^2 - 7x - 2$

$g(-1)$ =$-3 - 2 + 7 - 2 = 0$

$(x + 1)$ is a factor of $g(x)$

$g(x) = \ 3x^3 - 2x^2 - 7x - 2$

$= (x + 1)\{3x^2 + \lambda x - 2\}$

$x^2; \quad \lambda + 3 = -2$

$\lambda = -5$

$g(x) \equiv (x + 1)(3x^2 + \lambda x - 2)$

$\equiv (x + 1)(3x + 1)(x - 2)$

(iii) $g(y) = 4y^3 - 13y + 6$

$g(-2) = \ 4(-2)^3 - 13(-2) + 6$

$= -32 + 26 + 6 = 0$

$\therefore$ $(y + 2)$is a factor of $g(y)$

$g(y) = 4y^3 - 13y + 6 \quad \equiv (y + 2)\{4y^2 + \lambda y + 3\}$

$y^2; \;\; 0 = 8 + \lambda$

$\therefore \lambda = -8$

$g(y) = 4y^3 - 13y + 6 \quad \equiv (y + 2)\{4y^2 - 8y + 3\}$

$g(y) = 4y^3 - 13y + 6$

$= (y + 2)(\ 4y^2 - 8y + 3)$

$= (y + 2)(2y - 1)(2y - 3)$

(iv) $f(y) = 4y^4 - 4y^3 - 9y^2 + y + 2$

$f(2) = 4(2)^4 - 4(2)^3 - 9(2)^2 + 2 + 2$

$= 64 - 32 - 36 + 4$

$= 0$

$\therefore$ $(y - 2)$ is a factor of $f(x)$

$f\left(\frac{1}{2}\right) = 4\left(^1/_2\right)^4 - 4\left(^1/_2\right)^3 - 9\left(^1/_2\right)^2 + {}^1/_2 + 2$

$$= \frac{1}{4} - \frac{1}{2} - \frac{9}{4} + \frac{5}{2}$$

$$= 0$$

$\therefore (2y - 1)$ is a factor of $f(y)$

$f(y) = 4y^4 - 4y^3 - 9y^2 + y + 2 \equiv (y-2)(2y-1)[\lambda y^2 + \mu y + \gamma]$

$f(y) = 4y^4 - 4y^3 - 9y^2 + y + 2$

$\quad = (2y^2 - 5y + 2)(\lambda y^2 + \mu y + \gamma)$

$2\lambda = 4$

$\lambda = 2$

$-5\lambda + 2\mu = -4$

$-10 + 2\mu = -4$

$\mu = 3$

$2\gamma = 2$

$\gamma = 2$

$f(y) = (y-2)(2y-1)(2y^2 + 3y + 1)$

$\quad = (y-2)(2y-1)(2y+1)(y+1)$

(3) (a) $f(x) = 4x^3 + (3p-2)x^2 - (p^2-1)x + 3$

Since $(x + p)$ is a factor of $f(x)$,

$f(-p) = 0$

$$-4p^3 + (3p-2)p^2 - (p^2-1)(-p) + 3 = 0$$

$$-2p^2 - p + 3 = 0$$

$$(2p+3)(p-1) = 0$$

$p = 1$ or $p = {}^{-3}/_{2}$

When $p = 1$

$f(x) = 4x^3 + (3 \times 1 - 2)x^2 - (1-1)x + 3$

$\quad = 4x^3 + x^2 + 3$

$= (x+1)\{4x^2 + \lambda x + 3\}$

Since $(x+p)$ is a factor of $f(x)$

$= 4x^3 + (4+\lambda)x^2 + (\lambda+3)x + 3$

$\therefore\ 4 + \lambda = 1, \qquad \lambda + 3 = 0$

$\lambda = -3$

$f(x) = (x+1)\{4x^2 - 3x + 3\}$

when $p = {}^{-3}/_{2}$

$f(x) = 4x^3 + \left({}^{-9}/_{2} - 2\right)x^2 - \left({}^{9}/_{4} - 1\right)x + 3$

$= 4x^3 - {}^{13}/_{2}\,x^2 - {}^{5}/_{4}\,x + 3$

We have $\left(x - {}^{3}/_{2}\right)$ is a factor of $f(x)$

$f(x) = 4x^3 - {}^{13}/_{2}\,x^2 - {}^{5}/_{4}\,x + 3$

$= \left(x - {}^{3}/_{2}\right)\left\{4x^2 + \frac{x}{2} - 2\right\}$

(b) Given that the polynomial $F(x)$ and $G(x)$ have a common linear factor.

Let's assume that the common factor $(x-\alpha)$

$\therefore$ From the factor theorem we have

$F(\alpha) = 0 \qquad -(1)$ and $G(\alpha) = 0 \qquad -(2)$

Let's assume that $H(x) = F(x) - G(x)$

$\therefore\ H(\alpha) = F(\alpha) - G(\alpha)$; which means from (1) and (2) $H(\alpha) = 0$

$\therefore$ $(x-\alpha)$is also a linear factor of $H(x) = F(x) - G(x)$.

$F(x) = bx^3 + 4x^2 - 5x - 10$

$G(x) = bx^3 + ax - 2$

If $F(x), G(x)$ have a common linear factor, it is also a linear factor of $F(x) - G(x)$

$H(x) = F(x) - G(x) = (\,bx^3 + 4x^2 - 5x - 10) - (bx^3 + ax - 2)$

$= 4x^2 + 4x - 8$

$= 4(x^2 + x - 2)$

$= 4(x+2)(x-1)$

$\therefore$ The linear factor $(x+2)$ or $(x-1)$

If it is $(x+2)$ Then $F(-2)$ and $G(-2)=0$

$$F(-2) = b(-2)^3 + 4(-2)^2 - 5(-2) - 10 = 0$$

$$-8b + 16 + 10 - 10 = 0$$

$$\therefore b = 2$$

$$G(-2) = b(-2)^3 + a(-2) - 2 = 0$$

$$-8b - 2a = 3$$

$$a = \frac{19}{2}$$

If the common factor $(x-1)$, Then $F(1)$, and $G(1) = 0$

$$F(1) = b(1)^3 + 4(1)^2 - 5(1) - 10 = 0$$

$$b = 11$$

$$G(1) = b(1)^3 + a(1) - 2 = 0$$

$$a = -9 \quad \therefore \quad b = 2 \quad or \quad b = 11$$

Activity 1

(1) Given that $2x^3 + bx^2 + cx + 18$ has the factor $(x-2), (x+3)$ find b, and c and compile the factorization.

(2) Find the remainder

(a) When $x^4 - 3x^3 + 4x^2 - 6x + 7$ is divisible by $(x-1)$

(b) When $x^3 + x^2 + x + 1$ is divisible by $(x+3)$

(c) When $x^4 - 7x^3 - 17x^2 - 6$ is divisible by $(x+1)$

(d) When $5x^4 + 4x^3 + 5$ is divisible by $(x-2)$

(3) When the polynomial $G(x) = x^8 + px^2 + qx + r$ is divisible by $(x-1), (x+1)$and x the remainder are $2, 5$ and 2 respectively. Find the remainder when $G(x)$ is divisible by $(x-2)$

(4) Let $f(x) \equiv x^4 - bx^3 - 11x^2 + 4(b+1)x + a$ where a, b are constant given that$f(x)$is complete square of an quadratic expression and $(x+2)$ is a factor of. $f(x)$ Find the values of a and b and factorized $f(x)$.

(5) Let $f(x) \equiv 2x^3 + 3x^2 - 3x + q$ where q is a non-zero integer, find the value of q such that $(x - q)$ is a factor of the above $f(x)$. Also find the a, b, c values such that $f(x) = (x - a)(2x - 1)(x + 2) + bx + c$

(6) (a) Given that $(x^2 - 4)$ is a factor of the function $f(x) = x^3 + cx^2 + dx - 12$. Find the values of c and d and hence factorize $f(x)$.
(b) By using the factor theorem find one of the factors of the function $f(x)$. Where $f(x) = 2x^3 - 9x^2 + 7x + 6$. Hence factorize $f(x)$ into its linear factors.
(c)Let $f(x) = 2x^3 + x^2 - 8x - 4$
 (i) Show that $(2x + 1)$ is a factor of $f(x)$
 (ii) Factories $f(x)$ completely
 (iv) Hence find the values $f(x) = 0$

9.4 Definition of Algebraic Functions

Rational Function (fraction)

An expression of the form $\frac{f(x)}{g(x)}$ Where $f(x)$ and $g(x)$ are polynomials in x is called a rationed function or fraction.

Proper fraction

A fraction $\frac{f(X)}{g(x)}$ is a proper fraction; if the degree of $f(x)$ less than the degree $g(x)$.

Improper fraction

A fraction $\frac{F(x)}{g(x)}$ is an improper fraction; if the degree of $f(x)$ is greater than or equal to the degree of $g(x)$.

Example 3

(a) The fractions $\frac{x^3+3x^2+6x+8}{x^5+x^4+2}$, $\frac{x^2-2x+9}{x^3+2x}$ are proper fractions.
$\frac{x^3-3x^2+9x+6}{x^2+2}$, $\frac{2x^3+2x^2+1}{x^3+1}$ are examples for the improper fractions.

(b) Express the following as improper fractions

(i) $\frac{3x^4+2x^3-x^2+x-1}{x^3+3x^2-x-2}$

(ii) $\frac{x^3+5x^2+4x-9}{x^3+3x^2+9x+2}$

Solution:

$$\frac{3x^4+2x^3-x^2+x-1}{x^3+3x^2-x-2} \equiv (3x-7)+\frac{(23x^2-15)}{x^3+3x^2-x-2}$$

$$\frac{x^3+5x^2+4x-9}{x^3+3x^2+9x+2} = \frac{x^3+3x^2+9x+2+2x^2-5x+7}{x^3+3x^2-x+2}$$

$$= 1+\frac{2x^2-5x+7}{x^3+3x^2-x+2}$$

9.5 Partial fractions

Partial fractions of distinct linear factions in the denominator of a proper fraction

We can say that, every linear factor $(Px+Q)$ in the denominator of a proper fraction, the corresponds a partial fraction of the form $\frac{A}{Px+Q}$

where A is a constant.

Example 4

Express (a) $F(x)=\frac{2x^2+x+1}{(x-2)(x-1)(x+1)}$

(b)$G(x)=\frac{3x^2-1}{(x^2-1)(x+2)}$

Solution:

Since $F(x)$ and $G(x)$ are proper fractions, we can proceed directly to express $f(x)$ in partial fractions.

(a) $\frac{2x^2+x+1}{(x-2)(x-1)(x+1)} \equiv \frac{A}{(x-2)}+\frac{B}{(x-1)}+\frac{C}{(x+1)}$

$2x^2+x+1 \equiv A(x-1)(x+1)+B(x-2)(x+1)+C(x-2)(x-1)$

Putting $x = 2$; $2(2)^2 + 2 + 1 = A(2-1)(2+1)$

$$11 = 3A$$

$$\therefore A = {}^{11}/_{3}$$

Putting $x = 1$; $2(1)^2 + 1 + 1 = B(1-2)(1+1)$

$$\therefore B = -2$$

Putting $x = -1$; $2(-1)^2 - 1 + 1 = C(-1-2)(-1-1)$

$$6C = 2$$

$$\therefore C = {}^{1}/_{3}$$

$$\frac{2x^2 + x + 1}{(x-2)(x-1)(x+1)} \equiv \frac{{}^{11}/_{3}}{(x-2)} - \frac{2}{(x-1)} + \frac{{}^{1}/_{3}}{(x+1)}$$

Alternative method,

We have

$2x^2 + x + 1 \equiv A(x-1)(x+1) + B(x-2)(x+1) + C(x-2)(x-1)$

$2x^2 + x + 1 \equiv (A + B + C)x^2 - (B + 3C)x + (2C - 2B - A)$

$\therefore$ Equating coefficient of

x^2; $A + B + C = 2 \quad -(1)$

x ; $-(B + 3C) = 1 \quad -(2)$

x^0; $2C - 2B - A = 1 \quad -(3)$

$(1) + (3) \quad 3C - B = 3 \quad -(4)$

$(2) + (4) \quad -2B = 4$

$$B = -2$$

$$3C = 1$$

$$C = {}^{1}/_{3}$$

From (1) $A = 2 - (B + C)$

$$= 2 - \left(-2 + {}^{1}/_{3}\right)$$

$$= 4 - {}^{1}\!/_{3}$$
$$= {}^{11}\!/_{3}$$

(b) $G(x) = \frac{3x^2-1}{(x^2-1)(x+2)} = \frac{3x^2-1}{(x-1)(x+1)(x+2)}$

$$\frac{3x^2 - 1}{(x-1)(x+1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+2)}$$

$$3x^2 - 1 \equiv A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)$$

When $x = 1$; $3 - 1 = A(1+1)(1+2)$

$$\therefore A = {}^{1}\!/_{3}$$

When $x = -1$; $3 - 1 = B(-1-1)(-1+2)$

$$\therefore B = -1$$

When $x = -2$; $3(-2)^2 - 1 = C(-2-1)(-2+1)$

$$11 = 3C$$

$$\therefore C = {}^{11}\!/_{3}$$

$$\frac{3x^2 - 1}{(x-1)(x+1)(x+2)} = \frac{{}^{1}\!/_{3}}{(x-1)} - \frac{1}{(x+1)} + \frac{{}^{11}\!/_{3}}{(x+2)}$$

Alternating method

We have

$3x^2 - 1 \equiv A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)$

$3x^2 - 1 = (A + B + C)x^2 + (3A + B)x + (2A - 2B - C)$

$\therefore$ Equating coefficient of

x^2; $A + B + C = 3 \quad -(1)^{/}$

x ; $\quad 3A + B = 0 \quad -(2)^{/}$

x^0; $2A - 2B - C = -1 \quad -(3)^{/}$

$(1)^{/} + (3)^{/} \quad 3A - B = 2 \quad -(4)^{/}$

$(2)^{/} + (4)^{/} \qquad 6A = 2$

$A = {}^{1}/_{3}$

$B = -1$

$C = 3 - (A + B)$

$C = 3 - \left({}^{1}/_{3} - 1\right)$

$C = 4 - {}^{1}/_{3}$

$C = {}^{11}/_{3}$

Partial fractions of repeated liner fraction in the denominator of a proper fraction

We can say that, to every linear factor $(Px + Q)$ repeated 'n' times in the denominator, there corresponds the sum of 'n' partial fractions.

$$\therefore \frac{a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots \ldots \ldots + a_1 x \quad + a_0}{(Px + Q)^2}$$

$$\equiv \frac{A_1}{Px + Q} + \frac{A_2}{(Px + Q)^2} + \cdots \ldots \ldots + \frac{A_r}{(Px + Q)^r} + \cdots . \frac{A_n}{(Px + Q)^n}$$

Example 5

(a) $f(x) = \frac{x^2 - 3x + 6}{(x-2)^2(x-1)}$

(b) $g(x) = \frac{x^3 + 8x^2 + 9x + 2}{(x+1)^3(x-2)}$

Solution:

(a) $f(x) = \frac{x^2 - 3x + 6}{(x-2)^2(x-1)} \equiv \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-1)}$

$\therefore x^2 - 3x + 6 \equiv A(x - 2)(x - 1) + B(x - 1) + C(x - 2)^2 \ - (1)$

When $x = 2$;

$(2)^2 - 3 \times 2 + 6 = B(2 - 1)$

$\therefore B = 4$

$x = 1$

$1^2 - 3(1) + 6 = C(1-2)^2$

$C = 4$

Coefficient of x^2; $\quad A + C = 1 \; -(2)$

$$A = 1 - C = -3$$

Or coefficient of x; $\quad -3 = A(-3) + B - 4C \; -(3)$

x^0; $\quad 6 = 2A - B + 4C \; -(4)$

$(3) + (4) \quad 3 = -A \qquad \therefore C = 1 + 3$

$$A = -3 \qquad C = 4$$

$$B = 2A + 4C - 6$$
$$= 2 \times (-3) + 4 \times 4 - 6$$
$$= 4$$

$$f(x) = \frac{-3}{(x-2)} + \frac{4}{(x-2)^2} + \frac{4}{(x-1)}$$

(b) $g(x) = \frac{x^3+8x^2+9x+2}{(x+1)^3(x-2)} \equiv \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{(x-2)}$

$$x^3 + 8x^2 + 9x + 2 = A(x+1)^2(x-2) + B(x+1)\;(x-2)$$
$$+C(x-2) + D(x+1)^3 \; -(1)$$

$$x^3 + 8x^2 + 9x + 2 = A[x^2 + 2x + 1][x-2] + B[x^2 - x - 2] + C(x-2)$$
$$+D[x^3 + 3x^2 + 3x + 1]$$

$$x^3 + 8x^2 + 9x + 2 = A[x^3 - 3x - 2] + B[x^2 - x - 2] + C(x-2)$$
$$+D[x^3 + 3x^2 + 3x + 1] \quad -(2)$$

Equating the coefficient of

$x^3; A + D = 1 \quad -(3)$

$x^2; B + 3D = 8 \quad -(4)$

$x^1;\ -3A - B + C + 3D = 9 \quad -(5)$

$x^0;\ -2A - 2B - 2C + D = -12 \quad -(6)$

$(5) \times 2 + (6) \quad -8A - 4B + 7D = 6 \quad -(7)$

$(4) \times 4 + (7) \quad -8A + 19D = 38 \quad -(8)$

$(3) \times 8 + (8) \quad 27D = 46$

$$D = {}^{46}\!/_{27}$$

$$\therefore B = 8 - \frac{3 \times 46}{27}$$

$$B = \frac{216 - 138}{27}$$

$$B = \frac{78}{27}$$

$$B = \frac{26}{9}$$

$$A = 1 - \frac{46}{27}$$

$$A = \frac{-19}{27}$$

$$C = 9 + 3A + B - 3D$$

$$= 9 - \frac{19}{9} + \frac{26}{9} - \frac{46}{9}$$

$$= \frac{81 + 26 - 65}{9}$$

$$= \frac{42}{9}$$

$$= \frac{14}{3}$$

$$\therefore A = -\frac{19}{9}, B = \frac{26}{9}, C = \frac{14}{3}, D = \frac{46}{27}$$

$$g(x) = \frac{x^3 + 8x^2 + 9x + 2}{(x+1)^3(x-2)} \equiv \frac{{}^{-19}\!/_{27}}{x+1} + \frac{{}^{26}\!/_{9}}{(x+1)^2} + \frac{{}^{14}\!/_{3}}{(x+1)^3} + \frac{{}^{46}\!/_{27}}{(x-2)}$$

Partial fractions of distinct quadratic fractions in the denominator of a proper fraction

By using the definition of the proper fraction, we can say that, every quadratic factor $ax^2 + bx + c$ in the denominator there corresponds a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$

Express $f(x) = \frac{x^2+x+1}{(x^3+1)(x-1)}$ $g(x) = \frac{x^3+x^2-1}{(x-2)^2(x^2+2)}$

$$\frac{x^2 + x + 1}{(x^3 + 1)(x - 1)} \equiv \frac{x^2 + x + 1}{(x + 1)(x^2 - x + 1)(x - 1)}$$

$$\frac{x^2 + x + 1}{(x^3 + 1)(x - 1)} \equiv \frac{A}{x + 1} + \frac{B}{x - 1} + \frac{Cx + D}{x^2 - x + 1}$$

$$x^2 + x + 1 \equiv A(x - 1)(x^2 - x + 1) + B(x + 1)(x^2 - x + 1)$$
$$+(Cx + D)(x + 1)(x - 1)$$

When

$x = 1;\quad 3 = B(1 + 1)(1 - 1 + 1)$

$\therefore B = {}^{3}/_{2}$

$x = -1;\quad 1 - 1 + 1 = A(-2)(1 + 1 + 1)$

$\therefore A = {}^{-1}/_{6}$

$x = 0;\quad 1 = A + B - D$

$D = A + B - 1$

$D = {}^{-1}/_{6} + {}^{3}/_{2} - 1$

$D = \frac{9 - 1 - 6}{6}$

$D = {}^{1}/_{3}$

Coefficient of x^3; $\quad A + B + C = 1$

$$C = 1 - (A + B)$$
$$= 1 - \left({}^{3}/_{2} - {}^{1}/_{6}\right)$$
$$= \frac{6 - 9 + 1}{6}$$
$$= \frac{-2}{6}$$
$$= {}^{-1}/_{3}$$

$$\frac{x^2 + x + 1}{(x^3 + 1)(x - 1)} \equiv \frac{{}^{-1}/_{6}}{x + 1} + \frac{{}^{3}/_{2}}{x - 1} + \frac{{}^{1}/_{3}\,(1 - x)}{x^2 - x + 1}$$

$$g(x) = \frac{x^3 + x^2 - 1}{(x - 2)^2(x^2 + 2)} = \frac{A}{x - 2} + \frac{B}{(x - 2)^2} + \frac{Cx + D}{x^2 + 2}$$

$$x^3 + x^2 - 1 = A(x^2 + 2)(x - 2) + B(x^2 + 2) + (Cx + D)(x - 2)^2$$

When $x = 2$; $\quad 2^3 + 2^2 - 1 = B(2^2 + 2)$

$$6B = 11$$
$$B = \frac{11}{6}$$

Equations Coefficient

x^3; $\quad 1 = A + C \quad -(1)$

x^0; $\quad -1 = -4A + 2B + 4D \quad -(2)$

x^2; $\quad 1 = -2A + B + D - 4C \quad -(3)$

$(2) - 4 \times (3) \qquad -5 = 4A - 2B + 16C$

$$-5 = 4A - {}^{11}/_{3} + 16C$$
$$4A + 16C = {}^{11}/_{3} - 5 = {}^{-4}/_{3}$$
$$A + 4C = {}^{-1}/_{3} \quad -(4)$$

$$(1) - (4) \qquad -3C = 1 + {}^{1}/_{3} = {}^{4}/_{3}$$

$$C = {}^{-4}/_{9}$$

$$A = {}^{13}/_{9}$$

$$D = 1 + 2A + 4C - B$$

$$D = 1 + \frac{26}{9} - \frac{16}{9} - \frac{11}{6}$$

$$D = {}^{5}/_{18}$$

$$\therefore g_{(x)} = \frac{x^3 + x^2 - 1}{(x-2)^2(x^2+2)} = \frac{{}^{13}/_{9}}{x-2} + \frac{{}^{11}/_{6}}{(x-2)^2} + \frac{{}^{1}/_{18}(5-8x)}{x^2+2}$$

Activity 2

Express each of the following

(i) $\frac{x}{(2x-1)(3x-1)}$

(ii) $\frac{(x-3)^2}{(x-4)^2}$

(iii) $\frac{12x-34}{(x-2)(x+3)(x-4)}$

(iv) $\frac{x^2+x+1}{(x^3+1)}$

(v) $\frac{9x}{(x+2)(2x+1)^2}$

Solutions to Activities

Activity 1

(1) $b = -1, c = -15\,, f(x) = (x-2)(x+3)(2x-3)$

(2) (a) 3, (b) -20, (c) 0, (d) 0

(3) $b = {}^{1}/_{2}\,, c = {}^{-3}/_{2}\,, r = 0, 257$

(4) $a = 36\,, b = 2, \{(x-3)(x+2)\}^2$

(5) $q = -2, (x+2)(2x+1)(x-1\}; b = -1, c = -2, a = 0$

(6) (a) $c = 3,\; d = -4, (x-2)(x+2)(x+3)$

(b) $(x-3)(x-2)(2x+1)$

(c)$(2x+1)(x-2)(x+2); x = \pm 2$, or $x = {}^{-1}/_{2}$

Activity 2

(i) $\frac{1}{2x-1} - \frac{1}{3x-1}$

(ii) $\frac{1}{(x-1)^2} - \frac{4}{(x-1)^3} + \frac{4}{(x-1)^4}$

(iii) $\frac{1}{x-2} - \frac{2}{x+3} + \frac{1}{x-4}$

(iv) $\frac{{}^{1}/_{3}}{x+1} + \frac{{}^{2}/_{3}(x+1)}{x^2-x+1}$

(v) $\frac{-2}{2-x} + \frac{4}{2x+1} - \frac{3}{(2x+1)^2}$

Summary

The form of the polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where n is a positive integer and $a_0, a_1, \ldots, a_n$are real constants, are called coefficients of the polynomial and $a_n \neq 0$.

The integer ‘n’ is called the degree of the polynomial.

Given two polynomials $F(x)$and $Q(x)$, then $F(x)$+$Q(x)$, $F(x) - Q(x)$ and $F(x).Q(x)$,are also polynomials.

If a polynomial $F(x)$is divided by $(x-\alpha)$ then the remainder is $f(\alpha)$.

This is known as the remainder theorem. Also, when $f(\alpha)=0$ then $(x-\alpha)$is a factor of$F(x)$, which is known as the factor theorem.

$$\frac{F(x)}{G(x)} = Q(x) + \frac{R(x)}{G(x)}$$

In this $Q(x)$ is a polynomial and $\frac{R(x)}{G(x)}$ is the fraction.

We can say that to every linear factor $(px+q)$ in the denominator of a proper fraction of the form $\frac{A}{px+q}$ where A is a constant.

We can say that to every linear factor $(px+q)$ repeated 'n' times in the denominator, there corresponds the sum of 'n' partial fractions.

That is, we have,

$$\frac{a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots a_0}{(px+q)^n}$$

$$= \frac{A_1}{(px+q)} + \frac{A_2}{(px+q)^2} + \cdots \frac{A_r}{(px+q)^r} + \cdots \frac{A_n}{(px+q)^n}$$

Also, for every quadratic factor (ax^2+bx+c)in the denominator there is a corresponds partial fraction form $\frac{Ax+B}{a^2+bx+c}$.

Learning outcomes

On completion of this session you should be able to,

- Identify the polynomial expression.
- Find factors of the polynomial by using remainder theorem.
- Find factors of the polynomial by using factor theorem.
- Understand different kinds of fractions.
- Find partial fractions of several types of proper fractions.

Session 10

Sequences, Series and Binomial Theorem

Contents:

Introduction

In this session, we discuss the concepts of sequence, series and binomial expansion. We present the difference between the sequences and the series in detail. We may think that you have learned Arithmetic and Geometric Series in your education. The theory of binomial expansion is very useful to simplify the powers of algebraic expressions. We hope that you have known about expansions of $(a \pm \mathrm{b})^2$and $(a \pm \mathrm{b})^3$. Using the basic rules of algebra, it is difficult to find out the expansions of higher power of binomial like $(a \pm \mathrm{b})^{23}$. Therefore, we look for a general formula that help us finding the higher powers of a binomial.

10.1 Sequence

A sequence is a function whose domain is the set natural numbers. A sequence whose range is a subset of real numbers is called a real sequence. In this course, we discuss only the real sequence.

Notation

The terms of a sequence are usually denoted by $u_1, u_2, u_3,\cdots\cdots\cdots$ or $t_1, t_2, t_3,\cdots\cdots\cdots$ or $x_1, x_2, x_3,\cdots\cdots$ or $s_1, s_2, s_3,\cdots\cdots$ the number occurring at the n^{th} place of a sequence i.e. $\mathcal{U}_n$ is called the general term of the sequence. where $n \in \mathbb{Z}^+$.

A sequence is said to be finite or infinite according as it has finite or infinite numbers of terms.

Examples for sequences

(i) 3, 5, 7, 9, ………21,

(ii) 8, 5, 2, -1, -4, ……… -21

(iii) 3, 9, 27, 81, ………

(iv) $1, {}^{1}/_{4}, {}^{1}/_{16}, {}^{1}/_{64}$ ……

We can observe that each term (except first) in (i) is formed by adding 2 to the preceding term; each term in (ii) is formed by subtracting 3 from the preceding term, each term in (iii) is formed by multiplying the preceding term by 3; each term in (iv) is formed by dividing the preceding term by 4; moreover (i), and (ii) are infinite sequences.

However, to define a sequence we need not always have an explicit formula for the n^{th} term.

For example, for the infinite sequence 2, 3, 5, 7, 11, 13, 17, ……… of all positive prime numbers, we may not be able to give an explicit formula for the n^{th} them.

Example 1

Write down the first five terms of the sequences as defined below.

(i) $U_n = (2n - 1)$ (ii) $V_n = n^2 + 1$

(iii) $t_n = \frac{2n-1}{2n+1}$

Solution:

(i) $u_n = 2n - 1$

$u_1 = 2 \times 1 - 1 = 1$

$u_2 = 2 \times 2 - 1 = 3$

$u_3 = 2 \times 3 - 1 = 5$

$u_4 = 2 \times 4 - 1 = 7$

$u_5 = 2 \times 5 - 1 = 9$

∴The first five terms of the sequence

$u_n = 2n - 1$ are $1, 3, 5, 7$ and 9

(ii) $v_n = n^2 + 1$

$v_1 = 1^2 + 1 = 2$

$v_2 = 2^2 + 1 = 5$

$v_3 = 3^2 + 1 = 10$

$v_4 = 4^2 + 1 = 17$

$v_5 = 5^2 + 1 = 26$

∴ The first five terms of the sequences

$v_n = n^2 + 1$ are 2,5,10,17 and 26

(i) $t_n = \frac{2n-1}{2n+1}$

$t_1 = \frac{2 \times 1-1}{2 \times 1+1} = {}^1/_3$

$t_2 = \frac{2 \times 2-1}{2 \times 2+1} = {}^3/_5$

$t_3 = \frac{2 \times 3-1}{2 \times 3+1} = {}^5/_7$

$t_4 = \frac{2 \times 4-1}{2 \times 4+1} = {}^7/_9$

$t_5 = \frac{2 \times 5-1}{2 \times 5+1} = {}^9/_{11}$

∴ The first five terms of the sequences

$t_n = \frac{2n-1}{2n+1}$ are ${}^1/_3$, ${}^3/_5$, ${}^5/_7$, ${}^7/_9$, and ${}^9/_{11}$.

Example 2

Find the 20th and 23rd terms of the sequences defined by

$$Zn = \begin{cases} n(2n+1), & \text{if } \text{n is even natural number} \\ \dfrac{2n+1}{n}, if & n \text{ is odd number} \end{cases}$$

Solution:

When $n \ = \ 20$ is even

$Z_{20} = 20(2 \ \times \ 20 + 1) = 20 \ \times \ 41 = 820$

When n = 23 is an odd number

$Z_{21} = \frac{2(21)+1}{21} = \frac{43}{21}$

Activity 1

(1) Write the first three terms of the sequences whose n^{th} term Z_n is given by

(a) $4n + 1$, (b) $2n^2 - 1$, (c) $\frac{n+3}{n+2}$

(2) Find the19th and 20th terms of the sequence defined by

$$Un = \begin{cases} 3(n+1), & \text{Where n is even} \\ 3(n-1), & \text{Where n is odd} \end{cases}$$

10.2 Series

By adding the terms of a sequence, we obtain a series. A series is finite or infinite according as the number of terms in the corresponding sequence is finite or infinite.

	Sequence	Series
(i)	3, 5, 7, 9, ………. 21	3+5+7+9+………. +21
(ii)	8, 5, 2, -1, ………. -21	8+5+2+(-1) + ………. + (-21)
(iii)	3, 9, 27, 81, ……….	3+9+27+81+……….
(iv)	1, $^1/_4$, $^1/_{16}$, $^1/_{64}$ ……….	1+$^1/_4$+$^1/_{16}$+$^1/_{64}$ + ……….

10.3 Arithmetic Series

A sequence whose terms increase or decrease by a fixed number is called an arithmetic series. Arithmetic Series is also known as Arithmetic Progression. In general, an arithmetic series is denoted by AP. The fixed number is called the common difference of the AP. In an AP we usually denote the first term by *a*, the common difference by *d* and the n^{th} term T_n.

$$\therefore d = T_n - T_{n-1}$$

∴ Thus, the terms of an AP can be written as

$$a\ , a + d\ , a + 2d, \ldots \ldots . . a + (n-1)d$$

$$T_n = a + (n-1)d$$

∴ The general term of an AP

$$\boxed{T_n = a + (n-1)d}$$

Sum of n terms of an AP

Let a be the first term, d the common difference and the S_n be the sum of the first $n$number of terms;

$$S_n = a + (a+d) + (a+2d) + \cdots \ldots \ldots . +[a + (n-2)d]$$
$$+ [a + (n-1)d] \quad -(1)$$

Re-writing S_n in the reverse order we have

$$S_n = [a + (n-1)d] + [a + (n-2)d] + [a + (n-3)d\}$$
$$+ \cdots . . \ldots . +[a + d] + a \quad -(2)$$

(1) + (2)

$$2S_n = [2a + (n-1)d] + [2a + (n-1)d] + \cdots \ldots [2a + (n-1)d]$$
$$[2a + (n-1)d]\text{There are n terms}$$

$$2S_n = n[2a + (n-1)d]$$

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

If $T_n = l = a + (n-1)d$

The last terms of the given AP

$$S_n = \frac{n}{2}\,[a + l]$$

Activity 2

(a) A sequence $\{U_n\}$ is given by the formula $U_n = 10\text{-}3n$; Prove that it is an AP. Find 20th terms of the sequence and find the sum of the series with 20 terms.

(b) The 2nd, 31st and last terms of an AP are $7\,{}^{3}/_{4}$, ${}^{1}/_{2}$ and $-6\,{}^{1}/_{2}$ respectively. find the first term, common difference and the number of terms of the series. Find the sum of the series.

Arithmetic Mean (AM)

(a) Single Arithmetic Mean

A number $\overline{x}$ said to be single AM between two given numbers a and b

If $a, \overline{x}, b$ are in AP

$\overline{x} - a = b - \overline{x}$ [common difference in AP]

$2\overline{x} = a + b$

$$\overline{x} = \frac{a+b}{2}$$

(b) n- Arithmetic mean

The numbers $X_1, X_2, \ldots \ldots \ldots X_n$ are said to be the n-Arithmetic mean between two given number a and b.

$$\therefore a, X_1, X_2, \ldots \ldots \ldots \ldots X_n, b$$

$\therefore b$ is the $(n+2)$ terms of the AP.

$T_n = a + (n-1)d$

$T_{n+2} = a + (n+2-1)d$

$T_{n+2} = a + (n+1)d$

$b = a + (n+1)d$

$d = \dfrac{(b-a)}{n+1}$

$\therefore\ X_1 = a + d = a + \dfrac{1}{n+1}(b-a) = \dfrac{1}{n+1}(na+b)$

$X_2 = a + 2d = a + \dfrac{2}{n+1}(b-a) = \dfrac{2b+(n-1)a}{n+1}$

Therefore, we can find remaining terms $X_3, X_4, \dots \dots \dots \dots X_n$.

$T_{n+1} = X_n = a + nd = a + \dfrac{n}{n+1}(b-a)$

$X_n = \dfrac{1}{n+1}(a+nb)$

Example 3

If n Arithmetic means are inserted between 1 and 31 such that the ratio of the mean and nth mean is 3:29, find the value of n?

Solution

(a) $1, X_1, X_2, \dots \dots \dots . X_n, 31$

$T_{n+2} = 31 = 1 + (n+1)d$

$d = \dfrac{30}{n+1}$

$$\therefore\ X_1 = 1 + d = 1 + \frac{30}{n+1} = \frac{n+31}{n+1}$$

$X_n = a + nd$

$X_n = 1 + \dfrac{n(30)}{n+1} = \dfrac{31n+1}{n+1}$

Given that $\frac{X_1}{X_n} = \frac{3}{29}$ $\quad 29X_1 = 3X_n$

$29\left[\dfrac{n+31}{n+1}\right] = 3\left[\dfrac{31n+1}{n+1}\right]$

$$31 \text{ X } 29 - 3 = (93 - 29)n$$

$$n(64) = (30+1)(30-1) - 3$$
$$n(64) = 900 - 4$$

$$n = \tfrac{896}{64} = 14$$

$n = 14$

Example 4

Find six arithmetic means between 2 and 16. Also prove that their sum is 6 times of the AM between 2 and 16.

Solution

$2, X_1, X_2, X_3, X_4, X_5, X_6, 16$

$T_8 = a + 7d$

$16 = 2 + 7d$

$\therefore d = 2$

∴ Required AM are 4, 6, 8, 10, 12, 14

The sum of the six AM = 4+6+……+14

$= \frac{6}{2}(4+14) = 54$

The AM of 2 and 16 $= \frac{1}{2}(2+16) = 9$

$$54 = 6 \times 9$$

∴ Sum of six AM = 9 times of the between 2 and 16

Example 5

(a) Find the maximum sum of the AP;

40, 38, 36, 34………………

(b) The first, second and the last terms of AP are a, b and 2a respectively show that it's sum is $\frac{3ab}{2(b-a)}$

(c) If the ratio of the sum of *m* terms and *n* terms of an AP is m^2: n^2. Prove that the ratio of its $(m+n)^{th}$ and n^{th} terms is $(2m-1):(2n-1)$

(d) Find the sum to *n* terms of the sequence *log a*, *log ar*, *log ar*^{2,}…

Solution

(a) 40, 38, 36, 34,…………..

We can see that the common difference = -2

∴The maximum sum of the AP is

$S_n = 40 + 38 + \ldots\ldots\ldots. +2$

$l = n + (n-1)d$

$2 = 40 + (n-1)(-2)$

$2n = 40$

$n = 20$

∴The maximum sum of the AP

$$S_{20} = \frac{20}{2}(40 + 2) = 420$$

(b) *a*, *b* …………………… 2*a*

∴The common differenced = (*b*-*a*)

$l = a + (n-1)d$

$2a = a + (n-1)(b-a)$

$2a = a + (n-1)b - na + a$

$n(b-a) = b$

$\therefore n = \dfrac{b}{b-a}$

$\therefore S_n = \dfrac{n}{2}[a+l]$

$$S_n = \frac{b}{2(b-a)}[a+2a] = \frac{3ab}{2(b-a)}$$

(c) $S_m = \frac{m}{2}[2a+(m+1)d]$

$S_n = \dfrac{n}{2}[2a+(n+1)d]$

$\dfrac{\frac{m}{2}\,[2a+(m+1)d]}{\frac{n}{2}[2a+(n+1)d]} = \dfrac{m^2}{n^2}$

$\dfrac{2a+(m+1)d}{2a+(n+1)d} = \dfrac{m}{n}$

$2an + n(m-1)d = 2am + m(n-1)d$

$d(nm - n - mn + m) = 2a(m-n)$

$d(m-n) = 2a(m-n)$

$d = 2a$

$T_m = a + (m-1)d = a + (m-1)2a = a(2m-1)$

$T_n = a + (n-1)d = a + (n-1)2a = a(2n-1)$

$\dfrac{T_m}{T_n} = \dfrac{a(2m-1)}{a(2n-1)}$

$\therefore\ \dfrac{T_m}{T_n} = \dfrac{2m-1}{2n-1}$

(d) log a, log ar, log ar^2 …………… log ar^{n-1}

log a + (log a + log r) + (log a + log r^2) + ………, (log a + log r^{n-1})

log a + (log a + log r) + (log a + 2 log r) +……………… {log a + (n-1) log r}

∴ The given sequence represents AP

∴ The common difference of the given AP = log r

$$S_n = \frac{n}{2}\{\,2a + (n-1)d\}$$

$$S_n = \frac{n}{2}\{\,2\log a + (n-1)\log r\}$$

Activity 3

1.

(a) How many terms of the sequence 18, 16, 14, ………. should be taken so that their sum is zero.

(b) Find the sum of all the natural numbers between 100 and 1000 which are multiple of 5.

(c) The sum of an AP is 525, if it's first terms is 3 and the last term is 39, find the common difference.

(d) If the first and the last terms of an AP are *-4* and *146* and the sum of the AP is *7171* find the number of terms and the common difference of the AP.

(e) Find the least value of *n* such that 1+3+5+7+………. to *n* terms 7,500.

2.

(a) Find the n^{th} term of the series $\frac{1}{n} + \frac{n+1}{n} + \frac{2n+1}{n} + \ldots\ldots\ldots$

(b) Show that $(a-b)^2$, (a^2+b^2) , and $(a+b)^2$ are in AP

(c) Show that the sequence $\log a\,, \log({a^2}/{b}\,), \log({a^3}/{b^2}), \log({a^4}/{b^3}) \ldots\ldots\ldots$ forms an AP

(d) If a^2, b^2, c^2 are in an AP. Prove that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are also in an AP

(e) If $\log 2\,, \log(2^x-1)$, and $\log(2^x+3)$are in AP, Find *x*.

10.4 Geometric Progression

A sequence (finite or infinite) of non-zero numbers in which very term, except the first one nears a constant ratio with its preceding term, is called a geometric series. Geometric series is also known as Geometric progression. In general, a geometric series in denoted by GP

Let a be the first term and $r(\neq 0)$ be the common ratio of a GP.

Let $T_1, T_2, T_3, \ldots\ldots\ldots\ldots, T_n$ denote 1^{st}, 2^{nd}, 3^{rd}, …………….. n^{th} terms respectively.

Then we have $T_1 = a$

$T_2 = T_1.r = ar = r$

$T_3 = T_2.r = ar^2 = r^2$

$T_4 = T_3.r = ar^3 = r^3$

$T_n = T_{n-1} = ar^{n-1}$

$T_n = ar^{n-1}$

The general term of a GP.

$$\boxed{T_n = ar^{n-1}}$$

Sum of the first n terms of a GP

Let a be the first term and $r(\neq 0)$ be the common ratio of the given GP. If S_n denotes the sum of n terms, then

$$S_n = a + ar + ar^2 + \cdots \ldots \ldots . + ar^{n-1} \quad -(1)$$

Multiplying both sides of (1) by r

We set

$$rS_n = ar + ar^2 + \cdots \ldots \ldots . + ar^{n-1} + ar^n \quad -(2)$$

(1) – (2) $(1-r)S_n = a - ar^n$

$$S_n = \frac{a(1-r^n)}{(1-r)}$$

$$\left.\begin{array}{l} r < 1 \;\; s_n = \frac{a(1-r^n)}{(1-r)} \\ r > 1 \;\; s_n = \frac{a(r^n-1)}{(r-1)} \end{array}\right\} r \neq 1$$

If $r=1$ then $S_n = na$

Example 6

(a) Find the sum of the n terms of the series $(a+b) + (a^2+2b) + (a^3+3b) + \ldots \ldots \ldots$

(b) Find the sum of the series $^2/_9 + {}^1/_3 + {}^1/_2 + \ldots \ldots \ldots \ldots {}^{81}/_{32}$

(c) How many terms of the GP 3, $^3/_2$, $^3/_4$, … … … … …. are needed to give the sum $\frac{3069}{512}$?

(d) Find the sum to n terms the series whose n^{th} term is $2^n + 3n$

(e) The sum of first three terms of a GP is to the sum of first six terms as 125:152 Find the common ratio of the GP.

Solution

$$S_n = (a + b) + (a^2 + 2b) + (a^3 + 3b) + \cdots \ldots \ldots \ldots .. + (a^n + nb)$$

$$S_n = (a + a^2 + a^3 + \cdots \ldots \ldots \ldots + a^n) + b(1 + 2 + 3 + \cdots . n)$$

$$S_n = a(1 + a + a^2 + \cdots \ldots \ldots \ldots + a^{n-1}) + b(1 + 2 + 3 + \cdots . n)$$

First point of this series $(1 + a + a^2 + \cdots \ldots \ldots \ldots + a^{n-1})$ is GP with first term 1 and common ratio *a*

$$(1 + a + a^2 + \cdots \ldots \ldots \ldots + a^{n-1}) = \frac{(1 - a^n)}{(1 - a)}$$

The second point of this series (1+2+3+…………+ n) is AP with the first term 1 and the common difference 1

$$1 + 2 + 3 + \cdots \ldots \ldots \ldots \ldots . n = \frac{n}{2}\ (1 + n)$$

$$S_n = a\frac{(1 - a^n)}{(1 - a)} + \frac{bn}{2}\ (1 - n)$$

$\frac{2}{9} + \frac{1}{3} + \frac{1}{2} + \cdots \ldots \ldots \ldots \ldots .. + \frac{81}{32}$ we can see that $\frac{1/3}{2/9} = \frac{1/2}{1/3} = \frac{3}{2}$ is the common ratio of the above series

∴ The given series is GP with 1st term $2/9$ and the common ratio $3/2$

∴ The last term

$$T_n = ar^{n-1} = 81/32$$

$$\frac{2}{9}(3/2)^{n-1} = 81/32$$

$$\frac{1}{3}\left(\frac{3^{n-1}}{2^{n-1}}\right) = 81/32$$

$$\left(\frac{3}{2}\right)^{n-1} = \left(\frac{243}{64}\right)$$

$$\left(\frac{3}{2}\right)^{n-1} = \left(\frac{3}{2}\right)^6$$

$$n - 1 = 6 \qquad \therefore n = 7$$

$$\therefore \ S_n = \frac{2}{9}\frac{\left[\left({}^3\!/_2\right)^7 - 1\right]}{\left[{}^3\!/_2 - 1\right]} = \frac{2}{9} \times \frac{2}{1}\left[\left({}^3\!/_2\right)^7 - 1\right]$$

$$= \frac{4}{9}\frac{[37-27]}{[27]}$$

$$= \frac{2059}{288}$$

3, ${}^3\!/_2$, ${}^3\!/_4$ …… the first term= 3 and the common ratio =${}^1\!/_2$

$$S_n = \frac{3069}{512} = 3\frac{\left[1 - \left({}^1\!/_2\right)^n\right]}{\left[1 - {}^1\!/_{2^n}\right]}$$

$$\frac{3069}{512} = 6(\ 1 - {}^1\!/_{2^n})$$

$$\frac{1023}{1024} = \ 1 - {}^1\!/_{2^n}$$

$${}^1\!/_{2^n} = 1 - \frac{1023}{1024} = \frac{1}{1024}$$

$$\Rightarrow 2^n = 1024 = 2^{10}$$

$$n = 10$$

$$T_n = 2^n + 3n$$

$$T_1 = 2^1 + 3 \times 1$$

$$T_2 = 2^2 + 3 \times 2$$

$$T_{n-1} = 2^{n-1} + 3(n-1)$$

$$T_n = 2^n + 3n$$

$$S_n = \sum_{r=1}^{n} T_r = 2(1 + 2 + 2^2 + \cdots \dots\ldots + 2^{n-1}) + 3(1 + 2 + 3 + \cdots$$
$$+ n)$$
$$= 2\frac{(2^n - 1)}{(2 - 1)} + 3\left(\frac{n}{2}\right)(n + 1)$$
$$S_n = 2(2^n - 1) + \left(\frac{3n}{2}\right)(n + 1)$$

$$\frac{S_3}{S_6} = \frac{125}{152}$$

$$S_3 = \frac{a(1-r^3)}{(1-r)} S_6 = \frac{a(1-r^6)}{(1-r)}$$

$$\frac{S_3}{S_6} = \frac{a(1-r^3)/_{1-r}}{a(1-r^6)/_{1-r}} = \frac{1-r^3}{1-r^6}$$

$$\frac{125}{152} = \frac{1}{1+r^3}$$

$$125 + 125r^3 = 152$$

$$125r^3 = 27$$

$$r^3 = \frac{27}{125} = \left(^3/_5\right)^3$$

$\therefore$ *the common ratio* = $^3/_5$

Example 7

1.

(a) Determine the number of terms of geometric progression $\{T_n\}$ if T_1=3 and T_n=96 and S_n=189

(b) How many terms of the sequence $^2/_9, ^{-1}/_3, ^1/_2 \ldots \ldots\ldots$ are needed to give the sum $^{55}/_{72}$

(c) The sum of three terms of a GP is ${}^{13}/_{12}$ and their product is -1. Find the terms.

Solution

(a) 6

(b) (b) 5

(c) ${}^{3}/_{4}, -1, {}^{4}/_{3}, \dots or\ {}^{4}/_{3}, -1, {}^{3}/_{4}, \dots$

Geometric mean GM

A number G is said to be the single geometric mean between two given numbers a and b if a, G, b are in GP.

$\therefore\ \frac{G}{a} = \frac{b}{G}$

$\therefore\ G^2 = ab$

$$G = \pm\sqrt{ab}$$

n-Geometric Means

The numbers G_1, G_2, ……………G_n are said to be the n Geometric means between a and b

$$a\ , G_1 G_2 \dots \dots \dots \dots \dots \dots G_n, b$$

$b = ar^{n+1}$

$\therefore\ r^{n+1} = \left({}^{b}/_{a}\right)$

$r = \left({}^{b}/_{a}\right)^{\frac{1}{n+1}}$

$\therefore$ we can find

$$G_1 = ar, G_2 = ar^2 \dots \dots \dots \dots \dots .. G_n = ar^n$$

$$G_1 = a\left(\frac{b}{a}\right)^{n+1} \quad G_2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$$

$G_n = a\left(\frac{b}{a}\right)^{n/_{n+1}}$

Example 8

1. Find 4 geometric mean between ${}^{4}/_{9}$ and ${}^{27}/_{8}$
2. Find 5 geometric mean between 9 and ${}^{64}/_{81}$

Solution

$$^{4}/_{9}\,, G_1, G_2, G_3, G_4, {}^{27}/_{8}$$

The common ratio r, $\quad ar^5 = {}^{27}/_{8}$

${}^{4}/_{9}\, r^5 = {}^{27}/_{8}$

$r^5 = {}^{3^5}/_{2^5}$

$$\therefore r = \frac{3}{2}$$

$G_1 = \frac{4}{9} \times \frac{3}{2} = \frac{2}{3}$

$G_2 = 1$

$G_3 = \frac{3}{2}$

$G_4 = \frac{9}{4}$

${}^{2}/_{3}\,, 1\,, {}^{3}/_{2}\,, {}^{9}/_{4}$

Then, 9 $\;G_1 \;\; G_2 \;\; G_3 \;\; G_4 \;\; G_5 \;\; \frac{64}{81}$

The common ratio = r

$ar^6 = \frac{64}{81}$

$9r^6 = \frac{64}{81}$

$r^6 = \frac{2^6}{3^6}$

$$r = \frac{2}{3}$$

$6, 4, {}^{8}/_{3}\ , {}^{16}/_{9}\ , {}^{32}/_{27}$

10.5 Summation of Other Basic Series

Sum of the first n terms integers

$$S_n = 1 + 2 + 3 + \cdots \ldots \ldots \ldots (n-2) + (n-1) + n \quad -(1)$$

$$S_n = n + (n-1) + (n-2) + \cdots \ldots \ldots \ldots 2 + 1 \quad -(2)$$

$$2\,S_n = (n+1)n$$

$$S_n = \frac{n}{2}(n+1)$$

$\sum$ notation

The Expression $U_1 + U_2 + U_3 + \cdots + U_n$

$$S_n = \sum_{r=1}^{n} U_r = U_1 + U_2 + U_3 + \ldots + U_n$$

$$S_n = \sum_{r=1}^{n} ar^{n-1} = a + ar + ar^2 + \cdots \ldots \ldots \ldots + ar^{n-2} + ar^{n-1}$$

$\therefore$ S_n of the above series

$$S_n = \sum_{r=1}^{n} r_n = 1 + 2 + \cdots ..(n-1) + n = \frac{n}{2}(n+1)$$

Sum of the square of the first n terms of positive integers

$$S_n = 1^2 + 2^2 + \cdots \ldots \ldots \ldots \ldots ..(n-1)^2 + n^2 = \sum_{r=1}^{n} r^2$$

Without proof, we can apply

$$\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$$

$$\sum_{r=1}^{5} r^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

$$= \frac{5}{6}(5+1)(10+1)$$

$$= 55$$

Sum of the cubes of the first n terms of positive integers

$$S_n = 1^3 + 2^3 + \cdots \ldots \ldots \ldots \ldots . . (n-1)^3 + n^3 = \sum_{r=1}^{n} r^3$$

Without proof , we can apply

$$\sum_{r=1}^{n} r^3 = \frac{n^2}{4}\,(n+1)^2$$

$$\sum_{r=1}^{6} r^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3$$

$$\sum_{r=1}^{6} r^3 = \frac{6^2}{4}\,(6+1)^2 \; = 441$$

10.6 Binomial Expansion

In this section, we discuss the expansion $(a+b)^n$, where n is a non-negative integer.

We can show by the ordinary multiplication that

$(a+b)^0 = 1$

$(a+b)^1 = 1a + 1b$

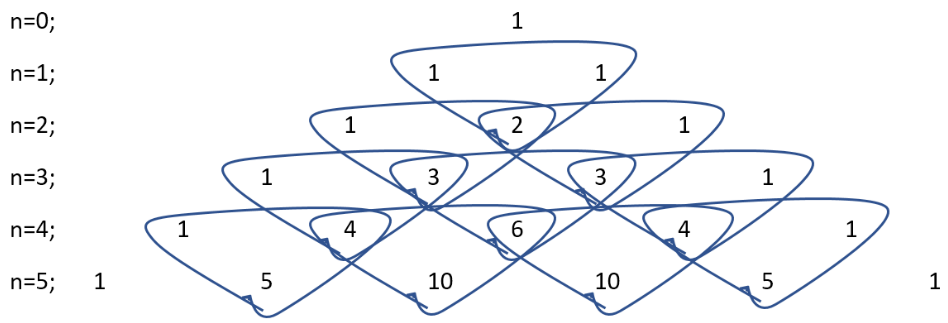
$(a+b)^2 = 1a^2 + 2ab + 1b^2$

$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$

$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

$(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$

Notice that the coefficients of terms in the above expressions from what is known as Pascal’s triangle.



Each row of the Pascal's Triangle starts and ends with 1. Others can be obtained by adding the terms on either side of it in the preceding row.

If n is a positive integer. , then

$$(a+x)^n \equiv \ {}^nC_0a^n + {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 + \cdots \ldots . + {}^nC_ra^{n-r}x^r$$
$$+ {}^nC_{n-1}a \quad x^{n-1} + {}^nC_nx^n$$

Where

$${}^nC_r = \frac{n!}{(n-r)! \ \ r!} \qquad 0 \le r \le n$$

$n! = 1 \times 2 \times 3 \times \ldots \ldots \ldots \ldots \ldots (n-1) \times n \quad and \ \ 0! = 1$

$n!$ Called Factorial 'n'

$$2! = 1 \times 2 = 2$$

$3! = 1 \times 2 \times 3 = 6$

$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$

$\therefore \ {}^nC_1 = {}^nC_n = 1$

You can observe that Pascal's Triangle can be written as,

1

${}^1C_0$ ${}^1C_1$

1 1

${}^2C_0$ ${}^2C_1$ ${}^2C_2$

1 2 1

${}^3C_0$ ${}^3C_1$ ${}^3C_2$ ${}^3C_3$

1 3 3 1

${}^4C_0$ ${}^4C_1$ ${}^4C_2$ ${}^4C_3$ ${}^4C_4$

1 4 6 4 1

Also, you can observe that the binomial coefficients are related to

$$\boxed{{}^{n}C_{r-1} + {}^{n}C_{r} = {}^{n+1}C_{r}}$$

$\therefore$

$$
\begin{aligned}
{}^{1}C_0 + {}^{1}C_1 &= {}^{2}C_1(1+1) = 2\\
{}^{2}C_0 + {}^{2}C_1 &= {}^{3}C_1(1+2) = 3\\
{}^{2}C_1 + {}^{2}C_2 &= {}^{3}C_2(2+1) = 3\\
{}^{3}C_0 + {}^{3}C_1 &= {}^{4}C_1(1+3) = 4\\
{}^{3}C_1 + {}^{3}C_2 &= {}^{4}C_2(3+3) = 6\\
{}^{3}C_2 + {}^{3}C_3 &= {}^{4}C_3(3+1) = 4
\end{aligned}
$$

$$
\begin{aligned}
\{x+a\}^n \equiv\ & {}^{n}C_0 x^n + {}^{n}C_1 a x^{n-1} + {}^{n}C_2 a^2 x^{n-2} + \cdots \ldots. + {}^{n}C_r a^r x^{n-r}\\
& + \cdots. + {}^{n}C_{n-1} a^{n-1} x + {}^{n}C_n a^n
\end{aligned}
$$

Observations of the above expansion;

(i) The number of terms in the expansion is one more than the index n; i.e. $(n+1)$

(ii) Power of x goes on decreasing by one and that of a goes on increasing by one.

(iii) In each term is the sum of the indices of x and a is n

(iv) The coefficients of the terms are ${}^{n}C_0, {}^{n}C_1, {}^{n}C_2, \ldots\ldots\ldots.., {}^{n}C_r \ldots\ldots {}^{n}C_{n-1}, {}^{n}C_n$

(v) Since ${}^{n}C_r = {}^{n}C_{n-r}$

$\therefore\ {}^{n}C_n = {}^{n}C_0 = 1$

Hence the coefficients of terms which are equidistant from the beginning and end are equal.

(vi) The notation

$$\sum_{r=0}^{n} {}^{n}C_r x^{n-r} a^r$$

Stands for

${}^{n}C_0 x^n + {}^{n}C_1 x^{n-1} a + {}^{n}C_2 x^{n-2} a^2 + \cdots \ldots. + {}^{n}C_n a^n$

Particular Cases of the Binomial Expansion

(1) Expansion of $(x-a)^n$ Changing a in to $-a$ in the expansion, we have

$$(x-a)^n = {}^nC_0x^n - {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 - {}^nC_3x^{n-3}a^3 + \cdots (-1)^r\,{}^nC_rx^{n-r}a^r + \cdots (-1)^n\,{}^nC_na^n$$

(2) Expansion of $(1-x)^n$, Changing x in to 1 and a in to x

$$(1+x)^n \equiv {}^nC_01^n + {}^nC_11^{n-1}x + {}^nC_21^{n-2}x^2 + \cdots \dots . + {}^nC_r1^{n-r}x^r + \cdots . + {}^nC_nx^n$$

$$(1+x)^n \equiv {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \cdots \dots . + {}^nC_rx^r + \cdots . + {}^nC_nx^n$$

$${}^nC_1 = \frac{n!}{(n-1)!\quad 1!} = \frac{1\times 2\times 3\times \dots ..(n-1)\times n}{1\times 2\times 3 \dots \dots \dots .(n-1)} = n$$

$${}^nC_2 = \frac{n!}{(n-2)!\quad 2!} = \frac{\cancel{1\times 2\times 3\times \dots ..(n-2)}(n-1)\,n}{\cancel{1\times 2\times 3 \dots \dots \dots .(n-2)}\times 2!} = \frac{n(n-1)}{2}$$

$${}^nC_3 = \frac{n(n-1)(n-2)}{3!}$$

$$\therefore\ (1+x)^n = 1 + nx + \frac{n}{2!}(n-1)x^2 + \frac{n}{3!}(n-1)(n-2)x^3 + \cdots \dots \dots .. + x^n$$

(3) Expansion of $(1-x)^n$

$$(1-x)^n = 1 - nx + \frac{n}{2!}(n-1)x^2 - \frac{n}{3!}(n-1)(n-2)x^3 + \cdots \dots \dots ..(-1)^nx^n$$

Example 9

Expand the following

(a) $(3x+2y)^5$

(b) $({}^{2x}/_{3} - {}^{3}/_{2x})^4$

(c) $(1+x+x^2)^3$

(d) $(x + {}^{1}/_{x})^5$

(e) $(3x - {}^{y}/_{3})^3$

Solution

$$(3x+2y)^5 = {}^5C_0(3x)^5 + {}^5C_1(3x)^4(2y)^1 + {}^5C_2(3x)^3(2y)^2$$
$$+ {}^5C_3(3x)^2(2y)^3 + \qquad {}^5C_4(3x)^1(2y)^4$$
$$+ {}^5C_5(2y)^5$$

$${}^5C_0 = {}^5C_5 = 1$$

$${}^5C_1 = {}^5C_4 = \frac{5!}{(5-1)! \quad 1!} = \frac{5!}{4!} = 5$$

$${}^5C_2 = {}^5C_3 = \frac{5!}{(5-2)! \quad 2!} = \frac{5!}{3!\,2!} = \frac{4 \times 5}{2} = 10$$

$$(3x+2y)^5 = 3^5x^5 + 5.3^4.2.x^4y + 10.3^3.2^2.x^3.y^2$$
$$+ 10.3^2.2^3.x^2.y^3 + 5.3 \quad .2^4.x \quad .y^4 + 2^5.y^5$$

$$(3x+2y)^5 = 243x^5 + 810x^4y + 1080x^3y^2 + 720x^2y^3 + 240x \quad y^4$$
$$+ 32y^5$$

$$\left\{\frac{2x}{3} - \frac{3}{2x}\right\}^4 = {}^4C_0\left(\frac{2x}{3}\right)^4 + {}^4C_1\left(\frac{2x}{3}\right)^3\left(\frac{3}{2x}\right)^1 + {}^4C_2\left(\frac{2x}{3}\right)^2\left(\frac{3}{2x}\right)^2$$
$$+ {}^4C_3\left(\frac{2x}{3}\right)^1\left(\frac{3}{2x}\right)^3 + {}^5C_4\left(\frac{3}{2x}\right)^4$$

$$\left\{\frac{2x}{3} - \frac{3}{2x}\right\}^4 = \frac{24}{34}x^4 - 4.\frac{23}{33}.\frac{3}{2}.\frac{x^3}{x} + 6.\frac{2^2}{3^2}.\frac{3^2}{2^2}.\frac{x^2}{x^2} - 4\frac{2}{3}.\frac{33}{23}.\frac{x}{x^3}$$
$$+ \frac{34}{24\,x^4}$$

$$\left\{\frac{2x}{3} - \frac{3}{2x}\right\}^4 = \frac{16}{81}x^4 - \frac{16}{9}x^2 + 6 - \frac{9}{16\,x^2} + \frac{81}{16\,x^4}$$

$$(1+x+x^2)^3 = {}^3C_0 1^3 + {}^3C_1 1^2(x+x^2)^1 + {}^3C_2 1^3(x+x^2)^2 +$$
$${}^3C_3 1^4(x+x^2)^3$$
$$= 1 + 3(x+x^2) + 3(x^2+2x^3+x^4) + \big({}^3C_0x^3 +$$
$${}^3C_1x^2x^2 + {}^3C_2x \quad x^4 + {}^3C_3x^6\big)$$
$$= 1 + 3(x+x^2) + 3(x^2+2x^3+x^4) + (x^3 + 3x^4 + 3x^5 + x^6)$$

$$= 1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6$$

$$(x + {}^1/_x)^5 = \ {}^5C_0x^5 + \ {}^5C_1x^4\frac{1}{x} + \ {}^5C_2x^3\frac{1}{x^2} + \ {}^5C_3x^2\frac{1}{x^3} + \ {}^5C_4x \quad \frac{1}{x^4}$$
$$+ \ {}^5C_5\frac{1}{x^5} \quad =$$

$$(x + {}^1/_x)^5 = x^5 + 5x^3 + 10x + 10\ \frac{1}{x} + 5\frac{1}{x^3} + \frac{1}{x^5}$$

$$(x + {}^1/_x)^5 = \left(x^5 + {}^1/_{x^5}\right) + 5\left(x^3 + {}^1/_{x^3}\right) + 10\left(x \quad + {}^1/_x \quad\right)$$

$$(3x - {}^y/_3)^3 = \ {}^3C_0(3x)^3 - \ {}^3C_1(3x)^2({}^y/_3) \quad + \ {}^3C_2(3x) \quad ({}^y/_3)^2$$
$$- \qquad {}^3C_3({}^y/_3)^3$$

$$(3x - {}^y/_3)^3 = 27x^3 - 3 \times \left({}^{x^2y}/_3\right) \quad + 3 \times \left({}^{3xy^2}/_9\right) - \left({}^{y^3}/_{27}\right)$$

$$(3x - {}^y/_3)^3 = 27x^3 - 9x^2y + xy^2 - \left({}^{y^3}/_{27}\right)$$

Example 10

Expand $(x + y)^6 - (x - y)^6$

Hence find the value $(3 + \sqrt{5})^6 - \left(3 - \sqrt{5}\right)^6$

Solution

$$(x + y)^6 = \ {}^6C_0x^6 + \ {}^6C_1x^5y + \ {}^6C_2x^4y^2 + \ {}^6C_3x^3y^3 + \ {}^6C_4x^2y^4$$
$$+ \ {}^6C_5x \quad y^5 + \ {}^6C_6y^6$$

$${}^6C_0 = \ {}^6C_6 = 1$$

$${}^6C_1 = \ {}^6C_5 = \frac{61}{51} = 6$$

$${}^6C_2 = \ {}^6C_4 = \frac{6!}{4!\,2!} = 15$$

$${}^6C_3 = \frac{6!}{3!\,3!} = 20$$

$$\therefore\ (x + y)^6 \equiv x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6x \quad y^5$$
$$+ y^6 \qquad \rightarrow (1)$$

$$(x-y)^6 \equiv x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6x\ \ y^5 + y^6 \quad \rightarrow (2)$$

(1) + (2)

$$(x+y)^6 - (x-y)^6 = 2[x^6 + 15x^4y^2 + 15x^2y^4 + y^6]$$

$$(3+\sqrt{5})^6 - \left(3-\sqrt{5}\right)^6$$

$$x = 3\ and\ y = \sqrt{5}$$

$$(3+\sqrt{5})^6 - \left(3-\sqrt{5}\right)^6 = 2[3^6 + 15.3^4.\sqrt{5}^2 + 15.3^2.\sqrt{5}^4 + \sqrt{5}^6]$$

$$(3+\sqrt{5})^6 - \left(3-\sqrt{5}\right)^6 = 2[729 + 6075 + 3375 + 125]$$

$$(3+\sqrt{5})^6 - \left(3-\sqrt{5}\right)^6 = 20608$$

General term of a Binomial Expansion in the expansion $(x+a)^n$, $(r-1)^{th}$ term $T_{r+1} = {}^nC_r x^{n-r} a^r$ is the general term of the expansion Example 5.6.3 write the general terms of expansion of

(a) $(3x^2 - 2y^3)^6$

(b) (b) $(3 - x^3)^{10}$

(c) $(2x^2 - {}^3/_x)^8 \quad x \neq 0$

Solution

(a)

$$(3x^2 - 2y^3)^6 = \sum_{r=0}^{6} {}^6C_r\,(3x^2)^{(6-r)}(-2y^3)^r$$

∴ The general term $T_{r+1} = {}^6C_r 3^{(6-r)}(-1)^r 2^r x^{12-2r} y^{3r}$

$T_{r+1} = (-1)^r\ {}^6C_r 3^{(6-r)} 2^r x^{12-2r} y^{3r}$

(b)

$$(3 - x^3)^{10} = \sum_{r=0}^{10} {}^{10}C_r\,3^{(10-r)}(-x^3)^r$$

The general terms $(-1)^{r\,10}C_r 3^{(10-r)} 2^r x^{3r}$

(c)

$$(2x^2 - {}^{3}/_{x})^8 = \sum_{r=0}^{8} {}^{8}C_r\,(2x^2)^{(8-r)}(-{}^{3}/_{x})^r$$

$$(2x^2 - {}^{3}/_{x})^8 = \sum_{r=0}^{8} \quad (-1)^r\,{}^{8}C_r 2^{(8-r)} 3^r \frac{x^{(8-r)2}}{x^r}$$

$$(2x^2 - {}^{3}/_{x})^8 = \sum_{r=0}^{8} \quad (-1)^r\,{}^{8}C_r 2^{(8-r)} 3^r x^{16-3r}$$

$\therefore$ The general term $(-1)^r\,{}^{8}C_r 2^{8-r} 3^r x^{16-3r}$

Example 11

a. Find the 7th term from end in $\left\{\frac{x^3}{2} - \frac{2}{x^2}\right\}^{10}$

b. Find the 7th term of $\left\{\frac{x}{3} + 6y\right\}^{10}$ and $\left\{\frac{4x}{3} - \frac{3}{2x}\right\}^{10}$

c. Find the middle term on $\left\{\frac{2x^3}{2} + \frac{3}{2x^2}\right\}^{10}$

d. Find the two middle terms of $\left\{3x - \frac{x^3}{6}\right\}^7$

Solution

$\left\{\frac{x^3}{2} - \frac{2}{x^2}\right\}^{11}$ This expansion has 11+1 (12 Terms)

$\therefore$ 7th term from the end = the 5th term from the beginning

the general term of the expansion,

$$T_{r+1} = {}^{11}C_r \left(\frac{x^3}{2}\right)^{11-r} \left(\frac{-2}{x}\right)^r$$

$\therefore$ we have to find T_5 $\quad\therefore r = 4$

$$T_5 = {}^{11}C_5 \left(\frac{x^3}{2}\right)^{11-5} \left(\frac{-2}{x}\right)^5$$

$$T_5 = {}^{11}C_5 (-1)^5 \left(\frac{1}{2}\right)^{11-5} (2)^5 (x^3)^{11-5} \left(\frac{1}{x^5}\right)$$

$$T_5 = -{}^{11}C_5 \left(\frac{1}{2}\right) \quad (x^3)^{13}$$

$$T_5 = -\frac{1}{2}\frac{11!}{5!\,6!}x^{13}$$

$$T_5 = -\frac{1}{\cancel{2}}\frac{\cancel{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times} 7 \times \cancel{8 \times} 9 \times \cancel{10 \times} 11}{1 \times \cancel{2 \times} 3 \times \cancel{4 \times\ 5 \times\ 6!}}x^{13}$$

$$T_5 = -(7 \times 3 \times 11)x^{13}$$

$T_5 = -231\, x^{13}$

(i)

$$\left\{\frac{x}{3} + 6y\right\}^{10} = \sum_{r=0}^{10} {}^{10}C_r\, ({}^{x}/_{3})^{(10-r)}(6y)^r$$

$$\therefore T_7 = {}^{10}C_6({}^{x}/_{3})^{(10-6)}(6y)^6 = {}^{10}C_6 x^4\frac{1}{3^4}x^6y^6$$

$$\therefore T_7 = \frac{10!}{6!\quad 4!}\,.2^4.6^2.x^4y^6 = \frac{7.8.9.10}{1.2.3.4}\,.16.36.x^4y^6$$

$$\therefore T_7 \ = 120960\, x^4y^6$$

$$\left\{\frac{4x}{3} - \frac{3}{2x}\right\}^{10} = \sum_{r=0}^{10} {}^{10}C_r\, ({}^{4x}/_{3})^{(10-r)}({}^{-3}/_{2x})^r$$

$$\left\{\frac{4x}{3} - \frac{3}{2x}\right\}^{10} = \sum_{r=0}^{10} {}^{10}C_r\, (-1)^r\frac{4^{10-r}}{3^{10-r}}\frac{3^r}{2^r}\frac{x^{10-r}}{x^r}$$

$$\left\{\frac{4x}{3} - \frac{3}{2x}\right\}^{10} = \sum_{r=0}^{10} {}^{10}C_r\, (-1)^r\frac{2^{9-3r}}{3^{10-2r}}x^{10-2r}$$

We have to find T_7 $\therefore r = 6$

$$T_7 = (-1)^6\, {}^{10}C_6\frac{2^{20-3\times 6}}{3^{10-2\times 6}}x^{10-2\times 6}$$

$$= -\ {}^{10}C_6\frac{2^2}{3^{-2}}x^{-2}$$

$$= {}^{10}C_6 \times 36 \times \frac{1}{x^2}$$

$$= \frac{10!}{6!\quad 4!}\frac{36}{x^2}$$

$$= \frac{210 \times 36}{x^2}$$
$$= \frac{7560}{x^2}$$

The expansion $\left\{\frac{2x^3}{2} + \frac{3}{2x^2}\right\}^{10}$ has (10+1) = (11) terms. Therefore, the middle term is the 6th term from the beginning.

$$\left\{\frac{2x^3}{2} + \frac{3}{2x^2}\right\}^{10} = \sum_{r=0}^{10} {}^{10}C_r \, ({2x^3}/{3})^{(10-r)}({3}/{2x^3})^r$$

$$\therefore T_{r+1} = {}^{10}C_r({2}/{3})^{(10-r)}({3}/{2})^r \frac{(x^3)^{10-r}}{x^{2r}}$$
$$= {}^{10}C_r({2}/{3})^{(10-2r)}x^{(30-5r)}$$
$$\therefore T_6 = {}^{10}C_5({2}/{3})^{(10-2\times5)}x^{(30-5\times5)}$$
$$= {}^{10}C_5 x^5$$
$$= \frac{10!}{5! \quad 5!}x^5$$
$$= 252\, x^5$$

The expansion $\left\{3x - \frac{x^3}{6}\right\}^7$ has 8 terms. Therefore the 4th and 5th terms are two middle terms of the expansion.

$$\left\{3x - \frac{x^3}{6}\right\}^7 = \sum_{r=0}^{7} {}^7C_r \, (3x)^{(7-r)}({-x^3}/{6})^r$$
$$= \sum_{r=0}^{7} {}^7C_r \, (-1)^r \frac{3^{7-r}}{6^r} \quad x^{7-r+3r}$$
$$= \sum_{r=0}^{7} (-1)^r \, {}^7C_r \frac{3^{7-r}}{6^r} \quad x^{7+2r}$$

$$T_{r+1} = (-1)^r \, {}^7C_r \frac{3^{7-r}}{6^r} \quad x^{7+2r}$$

$$T_5 = (-1)^4\ {}^7C_4 \frac{3^{7-4}}{6^4} \quad x^{7+2X4}$$

$$= {}^7C_4 \frac{3^3}{6^4} \quad x^{15}$$

$$= \frac{35}{8 \times 6} \quad x^{15} = \frac{35}{48} \quad x^{15}$$

$$T_4 = (-1)^3\ {}^7C_3 \frac{3^{7-3}}{6^3} \quad x^{7+2X3}$$

$$= -\ {}^7C_4 \frac{3^4}{6^3} \quad x^{13}$$

$$= -35 \times \frac{3}{8} \quad x^{13}$$

$$= -\frac{105}{8} \quad x^{13}$$

$\therefore$ The middle terms of the expansion $\frac{35}{48} \quad x^{15}$ and $-\frac{105}{8} \quad x^{13}$

Example 12

(a) Find the coefficient of x^{10} in the binomial expansion $\left\{2x^2 - \frac{3}{x}\right\}^{11}$

(b) Find the coefficient of x^{16}in the expansion $\{x^2 - 2x\}^{10}$

(c) Find the coefficient the term involving x^{32} and x^{17}in the expansion of $\left\{\frac{x^4}{6} - \frac{5}{x^3}\right\}^{15}$

(d) Find the term independent of x in the expansion of $\left\{3x^2 - \frac{5}{x^4}\right\}^{12}$

(e) Find the term independent of x in the expansion of $\left\{\sqrt{\frac{x}{3}} - \frac{3\sqrt{3}}{x^3}\right\}^{14}$

Solution:

(a)

$$\left\{2x^2 - \frac{3}{x}\right\}^{11} = \sum_{r=0}^{11} {}^{11}C_r\,(2x^2)^{(11-r)}\left(-3/_x\right)^r$$

$$= \sum_{r=0}^{11} (-1)^r \, {}^{11}C_r \, 3^r 2^{11-r} x^{(22-3r)}$$

$\therefore$ For x^{10} 22-3r=10

r =10

$\therefore$ The coefficient of $x^{10} (-1)^4 . {}^{11}C_4 3^4 2^{11-4}$

$$= {}^{11}C_4 \,. 3^4 . 2^7$$

$$= 3421440$$

(b)

$$\{x^2 - 2x\}^{10} = \sum_{r=0}^{10} {}^{10}C_r (x^2)^{(10-r)} (-2x \quad)^r$$

$$= \sum_{r=0}^{10} {}^{10}C_r (-1)^r 2^r \, x^{(20-r)}$$

For $x^{16} \Rightarrow 20 - r = 16$

$$r = 4$$

The coefficient of x^{16} in the expansion

The coefficient of x^{16} in the expansion $\{x^2 - 2x\}^{10} ==$

$$= {}^{10}C_4 (-1)^4 2^4 = 3360$$

(c)

$$\left\{\frac{x^4}{6} - \frac{5}{x^3}\right\}^{15} = \sum_{r=0}^{15} \left(\frac{x^4}{6}\right)^{(15-r)} (-1)^r \left(\frac{5}{x^3}\right)^r$$

$$= \sum_{r=0}^{15} (-1)^r \frac{5^r}{6^{15-r}} x^{(60-4r-3r)}$$

$$= \sum_{r=0}^{15} (-1)^r \frac{5^r}{6^{15-r}} x^{(60-7r)}$$

For the term x^{32}

$\therefore 60 - 7r = 32$

$r = \frac{60-32}{7} = 4$

$\therefore$ the Coefficient of x^{32} term,

$$(-1)^4 \frac{5^4}{6^{15-4}} = \frac{5^4}{6^{11}}$$

(d) (i)

$$\left\{3x^2 - \frac{2}{x^4}\right\}^{12} = \sum_{r=0}^{12} {}^{12}C_r (3x^2)^{(12-r)} \left(\frac{-2}{x^4}\right)^r$$

$$= \sum_{r=0}^{12} (-1)^r \, {}^{12}C_r . 3^{(12-r)} . 2^r \, x^{24-2r-4r}$$

$$= \sum_{r=0}^{12} (-1)^r \, {}^{12}C_r . 3^{(12-r)} . 2^r \, x^{24-6r}$$

It will be independent of x of

$$24 - 6r = 0$$

$$r = 4$$

$\therefore$ T_5 is the term which is independent of x

$$= (-1)^4 \, {}^{12}C_4 . 3^{(12-4)} x^4$$

$$= {}^{12}C_4 . 3^8 x^4$$

$$= 51963120$$

(e) (ii)

$$\left\{\sqrt{\frac{x}{3}} - \frac{3\sqrt{3}}{x^3}\right\}^{14} = \sum_{r=0}^{14} {}^{14}C_r \left(\sqrt{\frac{x}{3}}\right)^{14-r} \left(\sqrt{\frac{-3\sqrt{3}}{x^3}}\right)^r$$

$$= \sum_{r=0}^{14} (-1)^r \, {}^{14}C_r \left(x^{1/2}\right)^{14-r} x^{-3r} \left(3\sqrt{3}\right)^r$$

$$= \sum_{r=0}^{14} (-1)^r \, {}^{14}C_r \, x^{\frac{(14-7r)}{2}} \frac{\left(3\sqrt{3}\right)^r}{\sqrt{3}^{14-r}}$$

$$= \sum_{r=0}^{14} (-1)^{r\,14}C_r\, x^{\frac{(14-7r)}{2}} 3^r \sqrt{3}^{(2r-14)}$$

It will be independent of x if $\frac{14-7r}{2} = 0$

$$T_3 = (-1)^{2\,14}C_2 3^2 \left(\sqrt{3}\right)^{-10} = {}^{14}C_2 \frac{1}{3^6} = \frac{91}{729}$$

Example 13

a. The 2nd, 3rd and 4th terms in the expansion $(x + y)^n$ are 240, 720 and 1080 respectively. Find *x*, *y* and *n*.

b. If the coefficients of $(2r + 4)^{th}$ and $(r - 2)^{th}$ terms in the expansion of $(1 + x)^{18}$ are equal find the value of r.

Solution

$$(x + y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + {}^nC_3 x^{n-3} y^3 + \cdots \ldots\ldots\ldots\ldots.. + {}^nC_n y^n$$

$${}^nC_0 = 1, \quad {}^nC_1 = n\,, \quad {}^nC_2 = \frac{n}{2}(n-1), \quad {}^nC_3 = \frac{n}{6}(n-1)(n-2)$$

$$\therefore n x^{n-1} y = 240 \quad -(1)$$

$$\frac{n}{2}(n-1) x^{n-2} y^2 = 720 \quad -(2)$$

$$\frac{n}{6}(n-1)(n-2) x^{n-3} y^3 = 1080 \quad -(3)$$

$(1) \times (3)$

$$\frac{n^2}{6}(n-1)(n-2) x^{2n-4} y^4 = 240\ X\ 1080 \quad -(4)$$

$(2)^2$

$$\frac{n^2}{4}(n-1)^2 x^{2n-4} y^4 = 720^2 \quad -(5)$$

$(4)/(5)$

$$\frac{4}{6}\frac{(n-2)}{(n-1)} = \frac{240 \times 1080}{720 \times 720} = \frac{1}{2}$$

$$\frac{2}{3}\frac{(n-2)}{(n-1)} = \frac{1}{2}$$

$$4(n-2) = 3(n-1)$$

$n = 5$

$\therefore$ from (1) $5x^4y = 240$

$$x^4y = 48$$

From (2)

$\frac{5}{2}(4)x^3y^2 = 720$

$x^3y^2 = 72$

$$y = {}^{48}/_{x^4}$$

$x^3\left(\frac{48}{x^4}\right)^2 = 72$

$x^5 = \frac{48^2}{72}$

$x^5 = 32$

$\therefore x = 2, \quad y = {}^{48}/_{16} = 3$

$(1+x)^{18} = \sum_{k=0}^{18} {}^{18}C_k\, x^k$

$$\therefore T_{k+1} = {}^{18}C_k x^k$$

$\therefore$ the coefficient of $(k+1)^{th}$ term $= {}^{18}C_k$

$\therefore$ (2r+4) term's coefficient = ${}^{18}C_{2r+3}$

(r-2) term's coefficient $= {}^{18}C_{r-3}$

We know that ${}^{n}C_k = {}^{n}C_{n-k}$

$\therefore 2r + 3 = 18 - (r - 3)$

$3r = 18$

$r = 6$

Activity 4

(a) Expand each of the following

(i) $(2x + 3x^2)^5$ (ii) $\left\{3xy - \frac{2x}{y}\right\}^5$ (iii) $\left(x + \frac{1}{y}\right)^{11}$ (iv) $(y^2 + 3x)^8$

(b) Find

(i) 10^{th} terms of $\left(2x^2 + {}^1/_x\right)^{12}$

(ii) 7^{th} terms of $\left({}^{x^2}/_2 + {}^2/_{x^2}\right)^9$

(iii) General terms of $(x^2 - yx)^{12}$

(c) Find

(i) 4^{th} terms from the end in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$

(ii) the middle terms of the expansion $\left(3x + {}^{x^3}/_6\right)^9$

(iii) the middle terms of the expansion $\left({}^{2x^2}/_3 + {}^3/_{2x^2}\right)^{10}$

(d) Find the coefficient of

(i) x^5 and x^{-2} in the expansion of $\left(x + {}^1/_{x^3}\right)^{17}$

(ii) x^{10} and x^{-2} in the expansion of $\left(2x^2 - {}^3/_x\right)^{11}$

(e) Find the value of a if the coefficients of x^2 and x^3 in the expansion of $(3 - ax)^9$ are Equal

Solutions to Activities

Activity 1

(1) (a) $5, 9, 13$ (b) $1, 7, 17$ (c) $\frac{4}{3}, \frac{5}{4}, \frac{6}{5}$

(2) $54, 63$

Activity 2

(a) $U_n = 10 - 3n$

$U_{n-1} = 10 - 3(n-1)$

$$U_n - U_{n-1} = 10 - 3n - (10 - 3n + 3) = -3$$

$\therefore U_n - U_{n-1}$ is a constant value

$\therefore U_n$ is an AP.

When $n = 1$, $\quad U_1 = (10 - 3) = 7$

It's the first term is 7 and the common differences is -3

$U_{20} = 10 - 3(20) = -50$

$S_{20} = \frac{20}{2}\{7 + (-50)\} = -430$

(b) $T_2 = 7\ {}^3/_4 = a + d \qquad a + d = {}^{31}/_4 \quad -(1)$

$T_{31} = {}^1/_2 = a + 30d \qquad a + 30d = {}^1/_2 \quad -(2)$

$\therefore (2) - (1)\ \ 29d = {}^1/_2 - {}^{31}/_4 = -{}^{29}/_4$

$d = -\ {}^1/_4$

$\therefore$ The common difference of $AP = -{}^1/_4$

$a = {}^{31}/_4 + {}^1/_4 = {}^{32}/_4 = 8$

The first term of the series = 8

$l = a + (n-1)d = -6\ {}^1/_2$

$\Rightarrow 8 + (n-1)(-{}^1/_4) = -{}^{13}/_2$

$\Rightarrow ({}^1/_4)(n-1) = 8 + {}^{13}/_2$

$\Rightarrow ({}^n/_4) = 8 + {}^{26}/_4 + {}^1/_4 = {}^{59}/_4$

$\Rightarrow n = 59$

$\therefore$ The number of terms of the above series = 59

$S_n = \frac{n}{2}\ [a + l]$

$S_{59} = \frac{59}{2}\ [8 + \frac{13}{2}]$

$S_{59} = [\frac{59}{2} \times \frac{3}{2}] = \frac{177}{4}$

Sum of the terms of the $AP = {}^{177}/_{4}$

Activity 3

1. (a) 19 (b) 98450 (c) ${}^{3}/_{2}$ (d) ${}^{3}/_{2}$ (e) 23
2. (a) $\frac{1}{n} + (n-1)$ (e) $log_2 5$

Activity 4

(a) (i) $32x^5 + 240x^6 + 720x^7 + 1080x^8 + 810x^9 + 243x^{10}$

(ii) $x^5 \left\{ 243y^5 - 810y^3 + 1080y - \frac{720}{y} + \frac{240}{y^3} - \frac{32}{y^4} \right\}$

(iii) $x^{11} + \frac{11x^{10}}{y} + \frac{55\,x^9}{y^2} + \frac{165\,x^8}{y^3} + \frac{330\,x^7}{y^4} + \frac{462\,x^6}{y^5} + \frac{462\,x^5}{y^6} + \frac{330\,x^4}{y^7} + \frac{165\,x^3}{y^8} + \frac{55\,x^2}{y^9} +$
$\frac{11\,x}{y^{10}} + \frac{1}{y^{11}}$

(iv)$y^6 + 24y^{14}x + 252y^{12}x^2 + 1512y^{10}x^3 + 5670y^8x^4 + 1360y^6x^5 +$
$20412y^4x^6 + 17496y^2x^7 + 6561x^8$

(b) (i) $\frac{1760}{x^3}$ (ii) $\frac{672}{x^6}$ (iii) $(-1)^r\,{}^{12}C_r x^{24-2r} y^r$

(c) (i)$\frac{7}{18}$ (ii)$-\frac{21}{16}x^{19}$, $\frac{189}{8}x^{17}$ (iii) -101376

(d) (i) 680, no term (ii) $2^8.3^5.5.11$, no term

(e) 9/7

Summary

A sequence is a function whose domain is the set of natural numbers. By adding of the terms of a sequence we obtain a series. A sequence whose terms increase or decrease by a fixed number is called an arithmetic series. A sequence (finite or infinite) of non-zero numbers in which very term except a first one, bears a constant ratio with its preceding term is called geometric series.

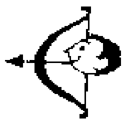
The binomial expansion for positive integral index given by

$$(x+a)^n = {}^nC_0x^n + {}^nC_1ax^{n-1} + {}^nC_2a^2x^{n-2} + \cdots .\ {}^nC_ra^rx^{n-r} + \cdots + {}^nC_na^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}a^r$$

Where ${}^nC_r = \frac{n!}{(n-r)!\ \ r!}$ $\qquad 0 \le r \le n$

$$n! = 1.2.3 \ldots (n-1)n \qquad 0! = 1$$

Learning outcomes



- On completion of this study session you should be able to Identify sequences and series and the properties of arithmetic series and geometric series.

- Find binomial expansion by using the Pascal Triangle, and the coefficient of the given terms in the expansion $(ax+by)^n$.