Experiments in Paper Soccer

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1 Rules of Paper Soccer

Paper soccer (or paper hockey) is an abstract strategy game played on a square grid representing a soccer or hockey field. (Wikipedia, needs citation)

1.1 Rules

etc.

2 Formalizing the game

Definition 2.1: Board

Let the *pitch* be an $m \times n$ grid of lattice points

$$V = \{(i, j) \mid 0 \leq i \leq m, 0 \leq j \leq n\} \subset \mathbb{Z}^2,$$

where m and n are odd integers. Let the goal vertices be

$$g_1 = (\lceil m/2 \rceil, n) \quad \text{and} \quad g_2 = (\lceil m/2 \rceil, 0),$$

each corresponding to the opponent's goal line. Two vertices $p, q \in V$ are said to be *adjacent* if $\|p - q\|_2 = 1$ or $\|p - q\|_2 = \sqrt{2}$. Let

$$E(V) = \big\{\{p, q\} \subseteq V \mid p, q \text{ are adjacent}\big\}$$

be the set of all possible edges between adjacent vertices.

Here we make a simplifying assumption by constraining the game to a rectangular grid, whereas the goals typically lie "outside" the grid (usually 1×2 on either end of the pitch).

Definition 2.2: Ball

Let $b_0 = (\lceil m/2 \rceil, \lceil n/2 \rceil)$ denote the initial position of the ball. After t plies (see Definition below), let b_t be the current position of the ball. When the specific index is unimportant, we write simply b.

Definition 2.3: Game state

A state is (G_s, b, p, H) where

- G_s is an undirected graph on V whose edges are the set of drawn segments so far.
- $b \in V$ is the current ball vertex.
- $p \in \{1, 2\}$ is the player whose turn it is to move.
- $H = (b_0, b_1, \ldots, b_n)$ is the sequence of ball positions thus far.

Usually we can determine whose turn it is to move by counting the number of moves played. In the cases where it won't lead to confusion we may denote a game state by G, G', G_i, etc.

Definition 2.4: Ply

A ply consists of choosing an adjacent vertex u (orthogonal or diagonal) such that $\{b, u\} \not\subseteq E(G_s)$ and $\{b, u\}$ is within V. After drawing $\{b, u\}$, we set $E(G_s) := E(G_s) \cup \big\{\{b, u\}\big\}$ and $b := u$.

Definition 2.5: Bounce

If the new ball vertex u satisfies $\deg_{G_s}(u) \geq 2$ after adding the edge $\{b, u\}$, then the same player continues, and another ply is forced. A player's *turn* is the maximal sequence of plies performed by the same player until the ball lands on a vertex v with $\deg_{G_s}(v) = 1$.

Definition 2.6: Winning conditions

If the ball lands on g_1, player 2 wins. If it lands on g_2, player 1 wins.
If no legal ply exists for the player to move, that player loses. (Equivalently, the other player wins.) If all edges have been used, we may optionally declare a draw; however, this cannot occur before all possible plies are exhausted.

Definition 2.7: Weak equivalence

Two game states $G_1 = (G_{s_1}, b_1, p_1, H_1)$ and $G_2 = (G_{s_2}, b_2, p_2, H_2)$ are *weakly equivalent*, written $G_1 \simeq G_2$, if they share the same drawn edges:

$$E(G_{s_1}) = E(G_{s_2}).$$

Definition 2.8: Strong equivalence

Two game states are *strongly equivalent*, written $G_1 \cong G_2$, if they are weakly equivalent and the ball occupies the same vertex:

$$E(G_{s_1}) = E(G_{s_2}) \quad \text{and} \quad b_1 = b_2.$$

Definition 2.9: Equality

Let $G_1 = (G_{s_1}, b_1, p_1, H_1)$ and $G_2 = (G_{s_2}, b_2, p_2, H_2)$ be two game states. We say that G_1 and G_2 are *equal*, written $G_1 = G_2$, if and only if they are strongly equivalent and have identical ball histories.
In particular, if

$$H_1 = (b_{1,0}, b_{1,1}, \ldots, b_{1,n}), \quad H_2 = (b_{2,0}, b_{2,1}, \ldots, b_{2,n}),$$

then $G_1 = G_2$ precisely when

$$b_{1,i} = b_{2,i} \quad \forall\, i \in \{0, 1, \ldots, n\}, \quad \text{and} \quad G_1 \cong G_2.$$

3 Immediate results

Notation.

If G_1 and G_2 are gamestates, we may write

$$E_{G_1} \quad \text{and} \quad E_{G_2}$$

to denote their respective edges.

Theorem 3.1: Equivalence relations

The weak- ($\simeq$) and strong ($\cong$) equivalences are equivalence relations.

Proof. We verify the three defining properties.

(Reflexivity.) For any game state G_a, we have $E_{G_a} = E_{G_a}$ and $b_{G_a} = b_{G_a}$, so $G_a \cong G_a$ and hence $G_a \simeq G_a$.

(Symmetry.) If $G_a \cong G_b$, then by definition $b_{G_a} = b_{G_b}$ and $E_{G_a} = E_{G_b}$. Thus also $b_{G_b} = b_{G_a}$ and $E_{G_b} = E_{G_a}$, so $G_b \cong G_a$ and therefore $G_b \simeq G_a$.

(Transitivity.) If $G_a \cong G_b$ and $G_b \cong G_c$, then $b_{G_a} = b_{G_b} = b_{G_c}$ and $E_{G_a} = E_{G_b} = E_{G_c}$, hence $G_a \cong G_c$ and consequently $G_a \simeq G_c$. $\square$

Definition 3.1: Equivalence classes

The *weak equivalence class* of a game state G is $[G]_{\simeq} = \{H \mid H \simeq G\}$. Likewise, the *strong equivalence class* of G is $[G]_{\cong} = \{H \mid H \cong G\}$.

Theorem 3.2: Equality is determined by history

Let

$$G_1 = (G_{s_1}, b_1, p_1, H_1), \qquad G_2 = (G_{s_2}, b_2, p_2, H_2)$$

be two full game states (including histories). Then G_1 and G_2 are equal if and only if they are history-equivalent, i.e. $H_1 = H_2$. In particular, history-equivalence implies (strong) equivalence.

Proof. Recall the definitions:

- A history $H = (b_0, b_1, \dots, b_n)$ determines the ordered list of plies (edges) $e_i = \{b_{i-1}, b_i\}$ for $i = 1, \dots, n$.
- A full game state is the quadruple (G_s, b, p, H).
- By definition, two game states are *equal* precisely when they are strongly equivalent and have identical histories.

($\Rightarrow$) Suppose G_1 and G_2 are equal. By the definition of equality they have identical histories, so $H_1 = H_2$. This direction is immediate from the definition.

($\Leftarrow$) Conversely, suppose $H_1 = H_2 =: H = (b_0, b_1, \dots, b_n)$. From the history H we can reconstruct the exact ordered list of edges played:

$$E(H) = \{\{b_{i-1}, b_i\} \mid i = 1, \dots, n\}.$$

Hence the edge sets of the two game states satisfy $E(G_{s_1}) = E(G_{s_2}) = E(H)$. Moreover, the current ball position in each state is the last entry of the history, so $b_1 = b_2 = b_n$. Finally, the player to move is determined by the parity of the number of plies n (for instance, if Player 1 starts, then Player 1 moves when n is even, Player 2 when n is odd), so $p_1 = p_2$. Therefore G_1 and G_2 are strongly equivalent and have identical histories; by the definition of equality we conclude $G_1 = G_2$.

The final remark follows immediately: since equality (equivalently, identical history) implies identical edge set and ball position, history-equivalence implies (strong) equivalence. □

Theorem 3.3: Boundedness of Paper Soccer

The game is bounded. I.e. it will end in a finite number of plies.

Proof. Every ply strictly increases $|E(G_s)|$ by 1. If $E(G_s) = E(V)$ (every edge is colored), there are not any plies left, and the player whose turn it is loses. Thus the game ends in at most $|E(V)|$ plies. □

Lemma 3.1

Let G_1, G_2 be game states such that $G_1 \simeq G_2$. Then they must have been reached in the same number of plies.

Proof. Since $G_1 \simeq G_2$, we have $E_{G_1} = E_{G_2}$. Each ply adds exactly one new edge, so the number of plies is determined uniquely by $|E_G|$. Hence G_1 and G_2 were reached after the same number of plies. □

Theorem 3.4

Let G_1, G_2 be game states such that $G_1 \cong G_2$. Then they must have been reached in the same number of plies.

Proof. Since $G_1 \cong G_2$ implies $G_1 \simeq G_2$, the claim follows immediately from the previous lemma. □

3.1 The early-game

Lemma 3.2: Early turns have no bounces

No player's turn can contain more than one ply during the first two turns of the game.

Proof. Initially, the only vertex with nonzero degree is the center vertex b_0, where the ball starts.

Turn 1. There are 8 possible moves, corresponding to the adjacent vertices of b_0. After performing any of these, the edge $\{b_0, b_1\}$ is added, giving $\deg(b_0) = \deg(b_1) = 1$. Thus the new vertex b_1 has degree 1, and by the definition of a bounce, no additional ply is possible.

Turn 2. From b_1, there are at most 7 legal moves (since $\{b_0, b_1\}$ is already occupied). Every reachable vertex u must satisfy $\deg(u) = 0$, because the only vertices with degree ≥ 1 are b_0 and b_1, neither of which are reachable this turn. Hence any resulting position again has the ball on a vertex of degree 1, and no bounce can occur.

Therefore, during the first two turns, every move consists of exactly one ply. □

Lemma 3.3

No non-equal equivalent games exist in the first two turns of the game.

Proof. Let $G_1 \neq G_2$ be games where 2 turns or less have occured.

By the above lemma, 2 plies or less have occured. We ignore the trivial case where G_1 and G_2 have non-equal ply, since they cannot then be equivalent by Lemma 4.1 (REMEMBER CIT/LINKS LATER). Since we're assuming $G_1 \neq G_2$, we cannot be on ply 0 as there only exists one 0 ply game state: $(\emptyset, b_0, 1)$.

—Remember to finish me.—

□

3.2 Importance of bounce-moves

INTUITION: Bounces HAVE to create cyclic subgraphs. No equivalent game states can exist without having a bounce. I.e. equivalent graphs must have a cyclic subgraph. Moreover we may be able to use the fact that if we have a game whose equivalence class is itself (then it must have no cyclic subgraphs) and we can use the fact that for a graph with no cycles there only exists one path from the start to the end?

Lemma 3.4: Characterization of bounce moves

A ply is a bounce move if and only if it creates a cyclic subgraph of the current game-state graph.

Proof. Let the current game-state graph before the ply be $G_s = (V, E_s)$, and suppose the ball is at vertex b. The ply consists of choosing an adjacent vertex u such that $\{b, u\} \notin E_s$ and forming the new graph $G_s' = (V, E_s \cup \{\{b, u\}\})$.

($\Rightarrow$) Suppose the ply is a bounce move. By definition, this means that after the move, the new ball vertex u satisfies $\deg_{G_s'}(u) \geq 2$. Since degrees only increase by 1 when an edge is added, it follows that $\deg_{G_s}(u) \geq 1$. Therefore, there exists a vertex v such that $\{u, v\} \in E_s$.

Because $\{b, u\} \notin E_s$, the new edge $\{b, u\}$ connects two vertices that were already connected in G_s by the path $b \rightsquigarrow v - u$. Hence G_s' contains a cycle.

($\Leftarrow$) Conversely, suppose adding $\{b, u\}$ creates a cycle in G_s'. Then b and u were already connected by some path in G_s, and so $\deg_{G_s}(u) \geq 1$. After the ply, $\deg_{G_s'}(u) = \deg_{G_s}(u) + 1 \geq 2$, so u is a vertex of degree at least 2 in the updated graph. By the definition of the game, this triggers a bounce, and the same player continues.

Hence a ply is a bounce move if and only if it creates a cyclic subgraph. □

Theorem 3.5

Let G_s be a game-state graph reachable from the start-position b_0. If G_s is acyclic (i.e. a forest), then there is exactly one play-history

$$H = (b_0, b_1, \ldots, b_t)$$

that produces E_s (where E_s denotes the edges of G_s). Equivalently, the weak- (and strong) equivalence class determine by E_s contains exactly one state (G_s, b_t, p, H).
In other words: if no bounce occured during the play that produced G_s, then the order in which edges were played is uniquely determined.

Proof. Proceed by induction of $t = |E_s|$, the number of plies in the history that produced G_s.

If $t = 0$, the graph is empty and the unique history is $H = (b_0)$. If $t = 1$, the single edge must be $\{b_0, b_1\}$, so the unique history is $H = (b_0, b_1)$. So the claim holds for $t \in \{0, 1\}$.

Assume the claim holds for all numbers of plies $< t$. Let G_s be an acyclic graph with $|E_s| = t$, and suppose $H = (b_0, \ldots, b_t)$ is a history producing E_s.

Because G_s is a finite forest, it has at least one leaf vertex (a vertex of degree 1). Let ℓ be a leaf of G_s. The unique edge incident to ℓ is some $e = \{\ell, v\} \in E_s$. We claim $\ell = b_t$.

To see this, note that the last ply added in the history is the edge $e = \{b_t, b_{t-1}\}$. Since e is incident to b_t, the endpoint b_t has degree 1 in G_s if and only if it was never and endpoint of any earlier edge, precisely the definition of a leaf created by the last move. Thus the leaf ℓ that corresponds to the unique edge played last must equal b_t.

Remove the last edge e to form the graph $G'_s = (V, E_s \setminus \{e\}$. Then $|E(G'_s)| = t - 1$ and G'_s is still acyclic. The truncated history $H' = (b_0, \ldots, b_{t-1}$ produces G'_s. By the induction hypothesis, H' is the unique history that produces G'. SInce the last edge e must be b_{t-1}, ℓ and ℓ is the uniqe leaf added in the last step, the full history H is uniquely obtained by appending ℓ to H'. Hence H is unique.

This proves uniqueness for t, completing the induction. Therefore any acyclic game-state graph determines exactly one history and, given the current player to move (which follows deterministically from t), exactly one full game state. $\square$

Remarks & corollaries.

If a weak-equivalence class has more than one full game-state, then the corresponding edge-set contains a cycle.