

# MAT-INF3600 Exam Practice

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## 1 Problem set

**Problem 1.1.** Let  $\mathcal{L}$  be the language with a binary relation symbol  $R$ . Consider the sentence  $\phi$ :

$$\forall x \neg R(x, x) \wedge \forall x, y, z [R(x, y) \wedge R(y, z) \rightarrow R(x, z)]$$

*Prove or disprove:* If a structure  $\mathcal{M}$  satisfies  $\phi$  with a finite domain  $M$ , then there must exist  $a \in M$  such that  $(a, b) \notin R, \forall b \in M$ .

*Proof.* The claim is true. Assume  $\mathcal{M} \models \phi$  with  $|M| = |\{m_1, \dots, m_n\}| < \infty$ . Suppose, for a contradiction, that there is some  $m_j$  for every  $m_i$  such that  $(m_i, m_j) \in R$ .

Then we have  $(m_{i_1}, m_{i_2}) \in R$ .  $R$  is irreflexive so  $m_{i_2}$  must relate to some element other than  $m_{i_1}$  (as otherwise we would get  $(m_{i_1}, m_{i_1}) \in R$  through transitivity), thus  $(m_{i_2}, m_{i_3}) \in R$ . Extrapolating we get  $(m_{i_{n-1}}, m_{i_n}) \in R$ , but now  $m_{i_n}$  must relate to some element, but this would lead to  $(m_{i_n}, m_{i_n}) \in R$  through transitivity so  $\mathcal{M} \not\models \phi$ , a contradiction.  $\square$

**Problem 1.2.** Provide a formal derivation of:

$$(\neg P \rightarrow \neg Q) \vdash (Q \rightarrow P)$$

*Proof.* The deduction, using L & K's system, is trivial.

Let  $\Sigma = \{\neg P \rightarrow \neg Q\}$ .

$$\begin{array}{ll} 1. \neg P \rightarrow \neg Q & (\Sigma) \\ 2. Q \rightarrow P & 1, (PC) \end{array}$$

We could also use the deduction theorem where we have  $\phi \vdash \psi$  if and only if  $\vdash \phi \rightarrow \psi$  for formula  $\psi$  and sentence  $\phi$ .  $\square$

**Problem 1.3.** Let  $\Sigma$  be a set of first-order sentences. Suppose that for every natural number  $n$ , there exists a model  $\mathcal{M}_n$  of  $\Sigma$  such that  $\mathcal{M}_n$  has  $n$  elements. Prove  $\Sigma$  has an infinite model.

*Proof.* Assume  $\mathcal{M}_n \models \Sigma, |M| = n$  for every  $n \in \mathbb{N}$ .

Consider the expanded set  $\Sigma' \supseteq \Sigma$  defined as

$$\Sigma' := \Sigma \cup \left\{ \exists x_1, \dots, \exists x_n \bigwedge_{i < j} x_i \neq x_j : n \in \mathbb{N} \right\}$$

This set is essentially  $\Sigma$  as well as "there are at least  $n$  elements" (for every  $n$ ). Consider the set

$$\Delta \subset \Sigma'$$

such that  $\phi_k \in \Delta$  where  $\phi_k$  is the statement "there are at least  $k$  elements". It is then straightforward that every finite subset of  $\Sigma'$  is satisfiable so  $\Sigma'$  is as well, and in particular, it's model is infinite.  $\Sigma' \supseteq \Sigma$  so this model also satisfies  $\Sigma$ .  $\square$

## 2 Problem set

**Problem 2.1.** Show that the class of all finite structures (in a language with at least one relation) is not first-order axiomatizable.

*Sketch proof.* Classifying all finite structures includes structures that are arbitrarily large. It is straightforward to show that if some set of sentences has arbitrarily large finite models then it also has an infinite model.

As seen in problem 1.3, defining "at least  $n$  elements" for every natural  $n$  and having that sentence satisfied for every  $n$  means that it is satisfied by an infinite structure.  $\square$