Lecture Notes: Real Analysis — Convergent Sequences (Course: MIT 18.100A)

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Definition 1. *$\{x_n\}$ converges to x if* $\forall \epsilon > 0, \exists M \in \mathbb{N}$ *such that* $\forall n \geq M : |x_n - x| < \epsilon$.

Notation: $\lim\limits_{x \to \infty} x_n$ or $x_n \to x$.
To find the definition of a non-convergent sequence we can negate the definition:

Definition 2. *A sequence $\{x_n\}$ does not converge to x if* $\exists \epsilon > 0$ *such that* $\forall M \in \mathbb{N}, \exists n \geq M$ *such that* $|x_n - x| \geq \epsilon$.

Examples:
$\lim\limits_{x \to \infty} \dfrac{1}{n^2 + 30n + 1} = 0.$

Proof. We aim to prove that $\lim\limits_{x \to \infty} \dfrac{1}{n^2 + 30n + 1} = 0$. We do this by finding an expression for M in terms of ϵ to satisfy the definition of convergence. First note that

$$\frac{1}{n^2} > \frac{1}{n^2 + 30n + 1}, \forall n \in \mathbb{N} \setminus \{0\}$$

We can prove this claim via induction:
Base case: n = 1

$$\frac{1}{1} > \frac{1}{1 + 30 + 1}$$

Suppose, for our inductive hypothesis, that

$$\frac{1}{n^2} > \frac{1}{n^2 + 30n + 1}$$

Then, for $k = n + 1$

$$\text{LHS: } \frac{1}{k^2} = \frac{1}{n^2 + 2n + 1}$$

$$\text{RHS: } \frac{1}{k^2 + 30k + 1} = \frac{1}{n^2 + 2n + 1 + 30n + 31} = \frac{1}{n^2 + 32n + 32}$$

Suppose, for a contradiction,

$$\frac{1}{n^2+2n+1} < \frac{1}{n^2+32n+32}$$
$$\text{multiply by } -1$$
$$n^2+2n+1 > n^2+32n+32$$
$$0 > 32n+31$$

which is obviously not true since $n \geq 1$. Thus our assumption was false and therefore, by the principles of mathematical induction,

$$\frac{1}{n^2} > \frac{1}{n^2+30n+1}, \forall n \in \mathbb{N} \setminus \{0\}$$

Now to prove that the sequence converges let $\epsilon > 0$ and choose $M > \frac{1}{\sqrt{\epsilon}}$ via Archimedes' principle. Then, using Archimedes' principle again, we can choose any natural number $n > M$ which will have the following property:

$$n > M \Rightarrow n > \frac{1}{\sqrt{\epsilon}}$$

$$\Rightarrow n^2 > \frac{1}{\epsilon}$$
$$\Rightarrow \epsilon n^2 > 1$$
$$\Rightarrow \epsilon > \frac{1}{n^2} > \frac{1}{n^2+30n+1} = |\frac{1}{n^2+30n+1}|$$
$$\Rightarrow \frac{1}{n^2+30n+1} < \epsilon$$

In other words we have shown that

$$\forall \epsilon > 0, \exists M \in \mathbb{N} \text{ s.t } \forall n \geq M : |\frac{1}{n^2+30n+1}| < \epsilon$$

Thus $(n^2+30n+1)^{-1}$ converges to 0. □

Theorem 1. *If $\{x_n\}$ is convergent then $\{x_n\}$ is bounded.*

Proof. Suppose $x_n \to x$. Then $\exists M \in \mathbb{N}$ such that $\forall n \geq M, |x_n - x| < 1$. Which implies $\forall n \geq M$:

$$|x_n| = |x_n - x + x| \leq |x_n - x| + |x| = 1 + |x|$$

Define $B = |x_1| + |x_2| + \cdots + |x_{M-1}| + 1 + |x|$ to be the bound of $\{x_n\}$. Then $\forall n \in \mathbb{N}$,

$$|x_n| \leq B$$

Thus $\{x_n\}$ is bounded by $B = 1 + |x| + \sum_{k=1}^{M-1} |x_k|$. □

Definition 3. *A sequence* $\{x_n\}$ *is* ***monotone increasing*** *if* $\forall n \in \mathbb{N} : x_n \leq x_{n+1}$. *I.e* $x_1 \leq x_2 \leq x_3 \leq \ldots$
A sequence $\{x_n\}$ *is* ***monotone decreasing*** *if* $\forall n \in \mathbb{M} : x_n \geq x_{n+1}$.
If a sequence satisfies one of these conditions we say it is ***monotone*** *or* ***monotonic***.

Theorem 2. *A monotonic sequence* $\{x_n\}$ *is convergent* ***if and only if*** *it is bounded.*

Proof. Let $\{x_n\}$ be a convergent sequence, converging to x. We already know that any convergent series is bounded by

$$B = x_1 + x_2 + \ldots x_{M-1} + 1 + |x| = 1 + |x| + \sum_{k=1}^{M-1} |x_k|$$

so it must hold for convergent monotonic sequences also.
Now to prove that any bounded monotonic sequence, there are two cases to consider.
Case 1: Monotone Increasing
$\{x_n\} \leq \{x_{n+1}\}$ and $\exists B \geq 0 : |x_n| \leq B$. Let $x = \sup\{x_n : n \in \mathbb{N}\}$ which must exist since the set of terms in the sequence is non-empty and bounded by B. Let $\epsilon > 0$. Since $x = \sup\{x_n\}$ there exists a natural number M such that $x - \epsilon < x_M \leq x$. Since the sequence is increasing we know that $\forall n \geq M : x_n \geq x_M > x - \epsilon$ and $x_n \leq x$. Which implies

$$|x_n - x| < \epsilon$$

Case 2: Monotone Decreasing
By using a symmetric argument and picking $x = \inf\{x_n\}$ we reach the same conclusion. □

Theorem 3.

1. *If* $C \in (0,1)$ *then* $\lim_{x\to\infty} C^n = 0$.
2. *If* $C > 1$ *then* $\{C^n\}$ *is unbounded.*

Proof of #2. We need to show that $\forall B \geq 0, \exists n \in \mathbb{N} : c^n > B$. Let $n \in \mathbb{N}$ such that

$$n > \frac{B}{C-1}$$

$$C^n = (1 + (C-1))^n \geq 1 + n(C-1) \geq n(C-1) > \frac{B}{C-1}C - 1 = B$$
$$C^n > B$$

□

Proof of #1. Clearly if $C \in (0,1)$ then $\{C^n\}$ is bounded by 1. If we can then show that $\{C^n\}$ is monotone decreasing, then we know it must be convergent to L. We prove the claim via induction.

We start with our base case

$$0 < c^2 < c$$

Suppose $0 < C^{m+1} < C^m$. Then

$$0 < C^{m+2} < C^{m+1} < C^m$$

So $\{C^n\}$ is monotone decreasing and bounded which implies it converges to L. Now we'll show $L = 0$. Let $\epsilon > 0$. Then $\exists M \in \mathbb{M}$ such that $\forall n \geq M$ $|C^n - L| < (1-c)\epsilon/2$. Then

$$(1-C)|L| = |L - CL| \leq |L - C^{M+1} + C^{M+1} - cL| \leq |L - C^{M+1}| + C|C^M + 1|$$

$$< \frac{\epsilon}{2}(1-C) + C\frac{\epsilon}{2}(1-C) < \epsilon(1-C)$$

Thus

$$|L| < \epsilon \Rightarrow |L| = 0 \Rightarrow L = 0$$

□

Subsequences

Definition 4. *Let $\{x_n\}$ be a sequence, and let $\{n_k\}$ be a sequence of natural numbers such that $n_1 < n_2 < n_3 < \dots$ (strictly decreasing). The sequence $\{x_{n_k}\}$ is called a subsequence of $\{x_n\}$.*

Example:
$1, 2, 3, 4, 5, 6, \dots$ is a sequence with $1, 3, 5, 7, 9, \dots$ being a subsequence of the odd entries. $1, 1, 1, 1, \dots$ would be a **non-example**, since n_k is not strictly increasing.

Theorem 4. *If $\{x_n\}$ converges to x and $\{x_{n_k}\}$ is a subsequence then*

$$\lim_{k\to\infty} x_{n_k} = x$$

Proof. Since $1 \leq n_1 < n_2 < n_3 < \cdots \Rightarrow \forall k \in \mathbb{N} : n_k \geq k$. Let $\epsilon > 0$. Since $x_n \to x$, $\exists M_0 \in \mathbb{N}$ such that $\forall n > M_0, |x_n - x| < \epsilon$. Choose $M = M_0$. If $k \geq M \Rightarrow n_k \geq M = M_0 \Rightarrow$ $|x_{n_k} - x| < \epsilon$

□