Mat210 Advanced Discrete Mathematics Notes

Thobias Høivik

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Contents

1 Pre-Semester Start – Cardinality

The following chapter contains notes based on what I think the course will cover in the first week (week 33). According to the syllabus, cardinality is mentioned early, so this section will review some basics.

Definition 1.1: Cardinality

Let A and B be sets. We say A and B have the same *cardinality*, written $|A| = |B|$, if there exists a bijection $f : A \to B$. If no such bijection exists, the sets have different cardinalities.

Example 1.1

Let $A = \{1,2\}$, $B = \{3,4\}$. While this is a trivial example, we can show that there are as many elements in A as in B by constructing a function $f : A \to B$ and showing that f is a bijection.

Proof that $|A| = |B|$. Let $f : A \to B$ be defined by

$$f(n) = n + 2.$$

Let $x, y \in A$ and suppose $f(x) = f(y)$. Then

$$\begin{aligned} f(x) &= f(y) \\ x + 2 &= y + 2 \\ x &= y. \end{aligned}$$

Thus, f is injective.
Now let $b \in B$. Then $b - 2 \in A$, since $B = \{3,4\}$ and subtracting 2 yields values in $A = \{1,2\}$. So for every $b \in B$, there exists $a = b - 2 \in A$ such that $f(a) = b$. Hence, f is surjective.
Since f is both injective and surjective, it is a bijection, and therefore $|A| = |B|$. $\square$

Definition 1.2: Finite and Infinite Sets

A set A is *finite* if there exists a natural number $n \in \mathbb{N}$ such that $|A| = |\{1,2,\dots,n\}|$. Otherwise, A is *infinite*.

Definition 1.3: Countably Infinite

A set A is *countably infinite* if there exists a bijection $f : \mathbb{N} \to A$. A set is *countable* if it is finite or countably infinite.

Definition 1.4: Uncountable Set

A set A is *uncountable* if it is not countable; that is, there does not exist a bijection from $\mathbb{N}$ to A.

Example 1.2

The set $\mathbb{R}$ is famously uncountable, as is rigorously demonstrated in any introductory analysis course (e.g., via Cantor's diagonal argument).

Definition 1.5: Power Set

Let A be a set. The *power set* of A, denoted $\mathscr{P}(A)$, is the set of all subsets of A.

Theorem 1.1: Cantor's Theorem

For any set A, we have $|\mathscr{P}(A)| > |A|$. In particular, there is no surjection from A onto $\mathscr{P}(A)$.

Proof. It suffices to show that there cannot exist a surjective function $f : A \to \mathscr{P}(A)$. Suppose, for contradiction, that such a surjective function f exists. Define the set

$$B = \{a \in A \mid a \notin f(a)\}.$$

Then $B \subseteq A$, so $B \in \mathscr{P}(A)$. Since f is surjective, there exists $b \in A$ such that $f(b) = B$. We now ask: is $b \in B$?

- If $b \in B$, then by the definition of B, $b \notin f(b) = B$, a contradiction.
- If $b \notin B$, then by the definition of B, $b \in f(b) = B$, again a contradiction.

In either case, we reach a contradiction. Therefore, our assumption that f is surjective must be false. Hence, there is no surjection from A onto $\mathscr{P}(A)$, and so

$$|\mathscr{P}(A)| > |A|.$$

□

After showing that the power set is strictly larger, we usually demonstrate that

$$|\mathscr{P}(A)| = 2^{|A|} > |A|$$

even for infinite sets. However, for infinite cardinals, exponentiation behaves differently than for finite numbers. For example, $2^{\aleph_0} = \mathfrak{c} = |\mathbb{R}|$.

Problem 1.1

Prove that $|\mathbb{N}| = |\mathbb{Z}|$, assuming $0 \in \mathbb{N}$.

Proof of Problem 1.1. We will construct a bijection $f : \mathbb{N} \to \mathbb{Z}$.

Define:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

We first show that f is injective. Suppose $f(x) = f(y)$.

Case 1: Both x and y are even. Then:

$$\frac{x}{2} = \frac{y}{2} \Rightarrow x = y.$$

Case 2: Both x and y are odd. Then:

$$-\frac{x+1}{2} = -\frac{y+1}{2} \Rightarrow x+1 = y+1 \Rightarrow x = y.$$

Case 3: One is even, one is odd. Then $f(x) \in \mathbb{Z}_{\geq 0}$, $f(y) \in \mathbb{Z}_{<0}$, so $f(x) \neq f(y)$. Hence, f is injective.

Now we show that f is surjective. Let $z \in \mathbb{Z}$. We find $n \in \mathbb{N}$ such that $f(n) = z$:

Case 1: $z \geq 0$. Then let $n = 2z$. Since $z \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{N}$, and $f(n) = z$.

Case 2: $z < 0$. Then let $n = -2z - 1$. Since $z \in \mathbb{Z}_{<0}$, $n \in \mathbb{N}$, and:

$$f(n) = -\frac{n+1}{2} = -\frac{(-2z-1+1)}{2} = -\frac{-2z}{2} = z.$$

In both cases, such an $n \in \mathbb{N}$ exists, so f is surjective.

Thus, f is a bijection and $|\mathbb{N}| = |\mathbb{Z}|$. $\square$

2 Tasks from 7.4

Problem 2.1: Task 17

Show that $\mathbb{Q}$ is dense along the number line by showing that given two rational numbers r_1 and r_2 with $r_1 < r_2$, there exists a rational number x such that $r_1 < x < r_2$.

Proof. Let $r_1, r_2 \in \mathbb{Q}$ such that $r_1 < r_2$. Consider that average of these numbers

$$\begin{aligned} x &= \frac{r_1 + r_2}{2} \\ &= \frac{\frac{a}{b} + \frac{c}{d}}{2} \\ &= \frac{a + c}{2bd} \end{aligned}$$

Clearly, x is a rational number since if $a, b, c, d \in \mathbb{Z}$ then $a + c \in \mathbb{Z}$ and $2bd \in \mathbb{Z}$. Furthermore

$$2r_1 < r_1 + r_2 \Rightarrow r_1 < \frac{r_1 + r_2}{2} = x$$
$$r_1 + r_2 < 2r_2 \Rightarrow \frac{r_1 + r_2}{2} = x < r_2$$

Thus we have that x is a rational number satisfying the desired property. Hence $\mathbb{Q}$ is dense along the number line. $\square$

Problem 2.2: Task 26

Prove that any uncountably infinite set A has a countably infinite subset.

Proof. Let A be a set such that $|A| > \aleph_0$. To construct a countably infinite subset we proceed by induction as follows:

Let $a_0 \in A$ be the first element. Then for our next element choose some element $a_1 \in A \setminus \{a_0\}$. We know $A \setminus \{a_0\}$ is non-empty since A is infinite. If we have n elements in our subset take the subsequent element to be

$$a_{n+1} \in A \setminus \{a_0, a_1, \ldots, a_n\}$$

As mentioned earlier, A take away $\{a_1, \ldots, a_n\}$ leaves a non-empty set and a_{n+1} is an available element of this set, meaning we can introduce it to our subset. Then, by mathematical induction, we get a sequence which is itself a type of subset $\{a_i : i \in \mathbb{N}\}$. Clearly we can construct a bijection

$$f : \mathbb{N} \to \{a_i : i \in \mathbb{N}\}$$

such that $f(i) = a_i$. Note that this procedure of making infinitely many choices, means using a weak form of the Axiom of Choice. $\square$

Problem 2.3: Task 27

Let A and B be sets such that $|A| = \aleph_0$. Prove that if there exists some $g : A \to B$ surjection, then B is countable.

Proof. We will proceed by proving that if there exists some surjection from one set Γ to another set Δ, then $|\Gamma| \geq |\Delta|$. With this it follows that B is countable, assuming the conditions set in the problem description. Suppose $\phi : \Gamma \to \Delta$ is surjective, i.e.

$$\forall \delta \in \Delta, \exists \gamma \in \Gamma \text{ s.t } \phi(\gamma) = \delta$$

Since we assume ϕ is well-defined, $\phi(\gamma)$ goes to one and only one $\delta \in \Delta$. Since ϕ is surjective, for any $\delta \in \Delta$ there must be at least one γ mapped to δ. As stated, no γ can map to more than one δ. Therefore, for each δ to have some γ which maps to it there must be at least as many $\gamma \in \Gamma$ as there are $\delta \in \Delta$. In other words,

$$|\Gamma| \geq |\Delta|$$

With this fact, and given that we have sets A, B where $|A| = \aleph_0$ and a surjection $g : A \to B$ it must be the case that

$$|B| \leq |A| = \aleph_0$$

which is what it means to be countable.

$\square$

Problem 2.4: Task 32

Prove that the cartesian product of $\mathbb{Z}$ with itself, $\mathbb{Z} \times \mathbb{Z}$, is countably infinite.

Proof. To show that $\mathbb{Z}^2$ is countably infinite we must show that it is infinite ($|\mathbb{Z}^2| \geq \aleph_0$), and it is countable ($|\mathbb{Z}^2| \leq \aleph_0$), in other words,

$$|\mathbb{Z}^2| = \aleph_0$$

First we show $\mathbb{Z}^2$ is infinite. This should be obvious since $\mathbb{Z}$ is infinite, but to demonstrate this rigorously consider the function $\pi_1 : \mathbb{Z}^2 \to \mathbb{Z}$, defined as follows:

$$\pi_1(a, b) = a$$

Clearly, π_1 is well-defined, since (a, b) is mapped to a unique $a \in \mathbb{Z}$. Also, π_1 is surjective, since for any $a \in \mathbb{Z}$, there exists an infinite amount of elements in $\mathbb{Z}^2$ such that $\pi_1(a, b) = a$. Thus we have shown that we can project $\mathbb{Z}^2$ onto an infinite set $\mathbb{Z}$. Hence $\mathbb{Z}^2$ is infinite. In other words: $|\mathbb{Z}^2| \geq \aleph_0$.

Now we show that there is a surjection from the naturals to $\mathbb{Z}^2$. First define a bijection $h : \mathbb{Z} \to \mathbb{N}$ by

$$h(n) = \begin{cases} 2n, & n \geq 0, \\ -2n - 1, & n < 0. \end{cases}$$

Let $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the Cantor pairing function

$$\pi(a, b) = \frac{(a+b)(a+b+1)}{2} + b,$$

which is a bijection. Its inverse $\pi^{-1} : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ can be written explicitly: for $n \in \mathbb{N}$ set

$$w = \left\lfloor \frac{\sqrt{8n+1} - 1}{2} \right\rfloor, \qquad t = \frac{w(w+1)}{2}, \qquad b = n - t, \qquad a = w - b,$$

so $\pi^{-1}(n) = (a, b)$.

Now define $s : \mathbb{N} \to \mathbb{Z}^2$ by

$$s(n) = \left(h^{-1}(a), h^{-1}(b)\right) \quad \text{where } (a, b) = \pi^{-1}(n).$$

(Here $h^{-1} : \mathbb{N} \to \mathbb{Z}$ exists because h is a bijection.)

To see s is surjective, take any $(x, y) \in \mathbb{Z}^2$. Let $a = h(x)$ and $b = h(y)$. Put $m = \pi(a, b) \in \mathbb{N}$. Then $\pi^{-1}(m) = (a, b)$, hence

$$s(m) = \left(h^{-1}(a), h^{-1}(b)\right) = (x, y).$$

Thus every element of $\mathbb{Z}^2$ has a preimage under s, so s is surjective.

Consequently $|\mathbb{Z}^2| \leq |\mathbb{N}| = \aleph_0$. (Since $\mathbb{Z}^2$ projects onto $\mathbb{Z}$, we also have $|\mathbb{Z}^2| \geq \aleph_0$, so in fact $|\mathbb{Z}^2| = \aleph_0$.)

$\square$

Problem 2.5: Task 38

Suppose $A_1, A_2, \ldots$ is an infinite sequence of countable sets. Prove that

$$\bigcup_{i=1}^{\infty} A_i$$

is countable.

Proof. We intend to show that the countably infinite union of countable sets is countable.

Let $A_1, A_2, \ldots$ be a sequence of countable sets.

Recall that

$$\bigcup_{i=1}^{\infty} A_i = \{x : x \in A_i, i \in \mathbb{Z}_+\}.$$

Since A_i is countable and $\mathbb{Z}_+$ is countable, there exists a surjection $g_i : \mathbb{Z}_+ \to A_i$. Recall also that $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable. Therefore if we can construct a surjective $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \bigcup_{i=1}^{\infty} A_i$, it follows that $\bigcup_{i=1}^{\infty} A_i$ is countable.

Define $f(n, m) = g_n(m)$, where g_n denotes the surjection from $\mathbb{Z}_+$ to A_n. To check surjectivity, let $x \in \bigcup_{i=1}^{\infty} A_i$. Then there exists some $k \in \mathbb{Z}_+$ such that $x \in A_k$. Since g_k is surjective, there exists $m \in \mathbb{Z}_+$ such that $g_k(m) = x$. Hence $f(k, m) = x$. Therefore f is surjective.

Since $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable and f is surjective, it follows that $\bigcup_{i=1}^{\infty} A_i$ is countable. $\square$

3 Multiplication principle

Definition 3.1: Multiplication Principle

If a task can be performed in a sequence of k steps, and the first step can be performed in n_1 ways, the second in n_2 ways, and so on, then the entire task can be performed in

$$n_1 \times n_2 \times \cdots \times n_k$$

ways.

Theorem 3.1: Multiplication Principle

Suppose an experiment consists of two successive stages. If the first stage can be performed in m ways and, for each of these, the second stage can be performed in n ways, then the experiment can be performed in

$$m \times n$$

ways. More generally, if there are k stages with n_i possible outcomes for stage i, then the total number of possible outcomes is

$$\prod_{i=1}^{k} n_i.$$

Example 3.1: Outfits

Suppose you have 3 shirts and 2 pairs of pants. Each shirt can be paired with any pair of pants, so the total number of possible outfits is

$$3 \times 2 = 6.$$

Example 3.2: License Plates

A license plate consists of 3 letters followed by 3 digits. There are 26^3 choices for the letters and 10^3 choices for the digits. Hence, the total number of license plates is

$$26^3 \times 10^3.$$

Example 3.3: Coin and Die

Suppose you flip a coin and then roll a die. The coin has 2 possible outcomes and the die has 6. By the multiplication principle, the total number of outcomes is

$$2 \times 6 = 12.$$

4 Addition principle

Theorem 4.1: Addition Principle, Two Sets

Let A and B be finite and disjoint sets. Then

$$|A \cup B| = |A| + |B|.$$

Proof. By definition of union,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Since A and B are disjoint, every element of A is distinct from every element of B. Thus, counting the elements of A and the elements of B counts all the elements of $A \cup B$ without overlap. Therefore, the total number of elements in $A \cup B$ is $|A| + |B|$. □

Theorem 4.2: Addition Principle, General Form

Let $A_1, A_2, \ldots, A_n$ be pairwise disjoint finite sets. Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|.$$

Proof. We proceed by induction on n. **Base case:** $n = 2$ holds by the previous theorem. **Inductive step:** Assume the statement holds for $n = k$. Consider $n = k + 1$. Then

$$\left|\bigcup_{i=1}^{k+1} A_i\right| = \left|\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}\right|.$$

Since the sets are pairwise disjoint, $\bigcup_{i=1}^{k} A_i$ is disjoint from A_{k+1}. Thus, by the two-set addition principle,

$$\left|\bigcup_{i=1}^{k+1} A_i\right| = \left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}|.$$

By the induction hypothesis,

$$\left|\bigcup_{i=1}^{k} A_i\right| = \sum_{i=1}^{k} |A_i|,$$

so

$$\left|\bigcup_{i=1}^{k+1} A_i\right| = \sum_{i=1}^{k} |A_i| + |A_{k+1}| = \sum_{i=1}^{k+1} |A_i|.$$

Hence, by induction, the theorem holds for all $n \geq 2$. □

Example 4.1

A cafeteria offers:

- 3 types of sandwiches: ham, turkey, or veggie,
- 2 types of salads: Greek or Caesar.

A student may choose either a sandwich or a salad, but not both.
Let S be the set of sandwiches and T the set of salads. Then $|S| = 3$, $|T| = 2$, and $S \cap T = \varnothing$. By the addition principle,

$$|S \cup T| = |S| + |T| = 3 + 2 = 5.$$

Thus, the student has 5 possible choices.

Example 4.2

A college course allows students to choose exactly one project topic from three disjoint categories:

Artificial Intelligence (5 topics), Networking (4 topics), Databases (6 topics).

By the general addition principle, the number of possible project choices is

$$5 + 4 + 6 = 15.$$

4.1 Addition principle for non-disjoint sets

Suppose we have sets A, B such that $A \cap B \neq \emptyset$. Then $|A \cup B| \neq |A| + |B|$ since we would count at least one element twice. We would have to take away one times the number of instances of elements that are in both A and B.

Theorem 4.3

Suppose A and B are sets such that $A \cap B \neq \emptyset$. Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof. Let A, B be sets such that $A \cap B = C \neq \emptyset$. Then $|A| + |B|$ would be the number of elements in $|A \cup B| +$ an extra counting of the elements that are common between them, namely C. Hence we have to take away the number of elements in C.

I.e.

$$|A \cup B| = |A| + |B| - C = |A| + |B| - |A \cap B|$$

$\square$

Theorem 4.4

Let $A = A_1 \cup \cdots \cup A_n$.
Then

$$|A| = \left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_j| - \cdots + (-1)^{n+1} \left|\bigcap_{i=1}^{n} A_i\right|$$

5 Pigeonhole principle

Theorem 5.1: Pigeonhole principle

Let n and m be positive integers. If $n > m$, then any function

$$f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$$

is not injective. Equivalently, if n objects (pigeons) are placed into m boxes (pigeonholes) with $n > m$, then at least one box contains at least two objects.

Example 5.1

Suppose there are 13 people at a party. Each person was born in one of the 12 months of the year. By the pigeonhole principle, at least two people must share a birth month.

Example 5.2

Consider 27 pairs of socks distributed among 26 drawers. By the pigeonhole principle, at least one drawer must contain at least two pairs of socks.

Example 5.3

Let S be a set of 6 integers. If we reduce each integer modulo 5, we obtain elements in $\{0, 1, 2, 3, 4\}$. Since there are 6 integers and only 5 possible remainders, by the pigeonhole principle, at least two integers in S must have the same remainder when divided by 5.