Oblig 1

Thobias Høivik

Problem (1). *Let (X, d) be a metric space and $K \subseteq X$ a subset. A function $f : K \to \mathbb{R}$ is* lower semicontinuous *if for all $x \in K$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$y \in K,\ d(x, y) < \delta \Rightarrow f(x) < f(y) + \varepsilon$$

The goal of this problem is to show that if $f : K \to \mathbb{R}$ is lower semicontinuous and K is compact, then f attains a minimum, i.e. there is some $\overline{x} \in K$ such that

$$f(\overline{x}) \leq f(x) \qquad \forall x \in K.$$

You may proceed as follows:
(a) Let $m := \inf\{f(x) : x \in K\}$. Argue that there exists a minimizing sequence *for f, i.e. a sequence $(x_n)_{n\in\mathbb{N}}$ in K such that $f(x_n) \to m$ as $n \to \infty$.*
Assume from now that K is compact.
(b) Show that there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and some $\overline{x} \in K$ such that $x_{n_k} \to \overline{x}$ as $k \to \infty$.
(c) Show that $m \leq f(\overline{x}) \leq m + \varepsilon$ for any $\varepsilon > 0$, and conclude that $\overline{x}$ is a minimum of f.

Proof of (a). Let $f : K \to \mathbb{R}$ be some function and let $m := \inf\{f(x) : x \in K\}$.

By the definition of the infimum we have that, for every $\varepsilon > 0$ there exists $x \in K$ such that

$$f(x) \leq m + \varepsilon$$

In particular for $\varepsilon := \frac{1}{n}$ this must be the case. Then, for $n \in \mathbb{N}$ there exists $x_n \in K$ such that

$$f(x_n) \leq m + \frac{1}{n}$$

but m is a lower bound for all $f(x_n)$ hence

$$m \leq f(x_n) \leq m + \frac{1}{n}$$

so

$$f(x_n) \underset{n\to\infty}{\to} m$$

$\square$

Proof of (b). $(x_n)_{n\in\mathbb{N}} \in K$ as described in proof of (a). Since K is compact, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence which converges to some point $\overline{x} \in K$. $\square$

Proof of (c). By definition of m as the infimum of all values of f we can see that

$$m \leq f(\overline{x})$$

By the virtue of f being lower semicontinuous we may also conclude that for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$d(\overline{x}, y) < \delta \Rightarrow f(\overline{x}) < f(y) + \varepsilon$$

In particular, since $x_{n_k} \to \overline{x}$, we can choose k large enough so that $d(\overline{x}, x_{n_k}) < \delta$. I.e.

$$d(\overline{x}, x_{n_k}) < \delta \Rightarrow f(\overline{x}) < f(x_{n_k}) + \varepsilon$$

As $k \to \infty$ we know that $f(x_{n_k}) \to m$ so taking the limit we get

$$m \leq f(\overline{x}) \leq m + \varepsilon$$

Taking $\varepsilon \to 0$ we conclude that

$$m \leq f(\overline{x}) \leq m$$

so $\overline{x}$ is a minimum of f. □

Problem (2). *Let (X, d) be a metric space and $E \subseteq X$ any subset. Let $f : E \to \mathbb{R}$ be any bounded function (i.e. there is some $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in E$). Define $g : \overline{E} \to \mathbb{R}$ by*

$$g(x) := \liminf_{\substack{y \to x \\ y \in E}} f(y) \qquad \forall x \in \overline{E},$$

that is,

$$g(x) := \lim_{n \to \infty} f_n(x), \qquad \text{where } f_n(x) := \inf\{f(y) : y \in E \cap B(x; 1/n)\}.$$

(Here, $\overline{E} := E \cup \partial E$ is the closure of E.) The function g is called the lower semicontinuous envelope *of f.*
(a) Show that g is well-defined, i.e. that the above limit exists, and that

$$g(x) \leq f(x)$$

for all $x \in E$.
(b) Show that g is lower semicontinuous.
(c) Show that if f is lower semicontinuous on E, then $g(x) = f(x)$ for all $x \in E$.

Proof of (a). Denote $\xi_{x,n} := E \cap B(x; 1/n)$. It is easy to see that $\xi_{x,n+1} \subseteq \xi_{x,n}$ (and that they are nonempty for all $n \in \mathbb{N}$). Notice then that

$$f_n(x) \leq f_{n+1}(x) \leq f_{n+2}(x) \leq \cdots \leq M$$

I.e. we get a monotone increasing sequence which is bounded above, which converges.

It is quite clear that for $x \in E$

$$\lim_{n \to \infty} f_n(x) \leq f(x)$$

as for every n $x \in B(x; \frac{1}{n})$ because $d(x,x) = 0 < 1/n$ and by the definition of the infimum $f_n(x) = \inf\{f(y) : y \in \xi_{x,n}\}$ is a lower bound for the value of every specific $x \in \xi_{x,n}$. □

Proof of (b). We wish to show that for every $x \in \overline{E}$ and $\varepsilon > 0$ we can find $\delta > 0$ such that

$$z \in \overline{E},\ d(x,z) < \delta \Rightarrow g(x) < g(z) + \varepsilon$$

Let $x \in \overline{E}$ and $\varepsilon > 0$. Since $g(x) = \lim_{n\to\infty} f_n(x)$ and the sequence of $f_n(x)$ is non-decreasing, we have $f_n(x) \leq g(x)$ for all $n \in \mathbb{N}$. Thus there exists $N \in \mathbb{N}$ such that

$$g(x) < f_N(x) + \varepsilon$$

Recall that $f_N(x) = \inf\{f(y) : y \in E \cap B(x; 1/N)\}$. Choose $\delta > 0$ such that $B(z;\delta) \subseteq B(x; 1/N)$. For any $M > 1/\delta$ we have $B(z; 1/M) \subseteq B(z;\delta) \subseteq B(x; 1/N)$. As we said in (a), the infimum over a larger set is less than or equal to the infimum over a subset. Therefore

$$f_N(x) \leq f_M(z)$$

As $m \to \infty$ we get

$$f_N(x) \leq g(z)$$

Recalling that

$$g(x) < f_N(x) + \varepsilon$$

we finally conclude that

$$g(x) < g(z) + \varepsilon$$

□

Proof of (c). Let $x \in E \subseteq \overline{E}$. If f is lower semicontinuous then $\forall\varepsilon > 0,\ \exists\delta > 0$ s.t.

$$z \in E,\ d(x,z) < \delta \Rightarrow f(x) < f(y) + \varepsilon \Rightarrow f(x) \leq f(y) + \varepsilon$$

In particular this holds for every $z \in E \cap B(x; 1/N)$ (assume $N \geq 1/\delta$), so it holds for the infimum as well, i.e.

$$d(x,z) < \delta \Rightarrow f(x) \leq f_n(x) + \varepsilon,\ n \geq N$$

Once again, nondecreasing sequence so take the limit to get

$$f(x) \leq g(x) + \varepsilon$$

Together with the results from (a) and (b) we conclude the desired result. □

Problem (3). *Fix some $m \in \mathbb{N}$. We equip $\mathbb{R}^m$ with the metric given by the euclidian norm, $d(x,y) = ||x,y||$, where $||z|| = \sqrt{\sum_{n=1}^{m} |z_n|^2}$ (for $x, y, z \in \mathbb{R}^m$). Define*

$$X := \text{Lip}([0,1], \mathbb{R}^m) = \{x : [0,1] \to \mathbb{R}^m : x \text{ is Lipschitz continuous}\},$$
$$Y := C^1([0,1], \mathbb{R}^m)$$
$$= \left\{ x : [0,1] \to \mathbb{R}^m : \text{both } x \text{ and } \frac{dx}{dt} \text{ exist and are continuous} \right\}.$$

We equip X with the supremum metric,

$$\rho(x,y) := \sup_{t \in [0,1]} |x(t) - y(t)| \quad \text{for } x, y \in X.$$

Fix $x_0, x_1 \in \mathbb{R}$, let $c > 0$, and define

$$E_c := \left\{ x \in Y : x(0) = x_0, x(1) = x_1, \left\| \frac{dx}{dt}(t) \right\| \leq c \ \forall t \in [0,1] \right\}.$$

(a) Show that

1. *every $x \in E_c$ has Lipschitz constant at most c,*
2. *E_c is a subset of X,*
3. *E_c is non-empty when c is large enough.*

(b) Show that E_c is a precompact *subset of X, i.e., every sequence $(x_n)_{n \in \mathbb{N}}$ in E_c has a subsequence that converges to a point in X.*
(c) For some continuous, bounded function $v : \mathbb{R}^m \to [0, \infty)$, define $f : E_c \to \mathbb{R}$ by

$$f(x) := \int_0^1 v(x(t)) \left\| \frac{dx}{dt}(t) \right\| \quad \text{for } x \in E_c$$

Show that f is bounded on E_c.
(d) It is a fact that f is not continuous on E_c, and E_c is not compact, so we cannot apply the Extreme Value Theorem to deduce the existence of a minimum of f.
It can *be shown that f is lower semicontinuous (but you are not asked to show this here). Assuming that this is true, show that there exists some $\overline{x} \in \overline{E_c}$ which "minimizes" f, in the sense that $\overline{x}$ is a minimum of the lowest semicontinuous envelope g of f, and that if $\overline{x} \in E_c$, then $\overline{x}$ is a minimum of f.*

Proof of (a). We begin with the second claim.

Claim 2.

We are essentially asked to prove that $x \in E_c$ is Lipschitz. There is, I belive, a theorem in the book which explicitly states that functions from subsets of the reals to the reals which have bounded derivatives are Lipschitz, but that is not entirely applicable. Assume, for a contradiction, that there exists $x \in E_c$ which is not in X, i.e. x is not Lipschitz.

By definition x and x' are continuous with the latter being bounded by $c \in \mathbb{R}^+$. We're assuming non-Lipschitz so for every $C > 0$, there are points $\alpha_C, \beta_C \in [0,1]$ such that

$$||x(\alpha_C) - x(\beta_C)|| > C||\alpha_C - \beta_C||$$

In particular there exist exist points α_c, β_c such that

$$||x(\alpha_c) - x(\beta_c)|| > c||\alpha_c - \beta_c||$$

so

$$\frac{||x(\alpha_c) - x(\beta_c)||}{||\alpha_c - \beta_c||} > c$$

but as $dx/dt \leq c$ for every $t \in [0,1]$ it must be the case that

$$c \geq \frac{||x(\alpha_c) - x(\beta_c)||}{||\alpha_c - \beta_c||}$$

In particular, $c > c$, a contradiction. Therefore our assumption that there exists $x \in E_c$ not Lipschitz is wrong and thus every $x \in E_c$ is Lipschitz. Summing up we get that

$$E_c \subseteq X$$

Now **Claim 1** follows nicely as we just saw that the assumption that $||x(\alpha_c) - x(\beta_c)|| > c||\alpha_c - \beta_c||$ was wrong.

Claim 3.

Notice that as $x \in E_c$ is continuous, and it's derivative continuous, the average growth between 0 and 1 is $||x_1 - x_0||$. By the Mean-Value-Theorem there must be some point $t \in [0,1]$ such that the momentary growth in that point is $||x_1 - x_0||$. So $\frac{dx}{dt}$ must be at least $||x_1 - x_0||$. Therefore we require $c \geq ||x_1 - x_0||$. □

Proof of (b). Suppose $(x_n)_{n \in \mathbb{N}}$ is a sequence in E_c. □