Lecture Notes: Abstract Algebra (Course by: Alvaro Lozano-Robledo)

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Lagrange's Theorem

Let G be a finite group and let H be a subgroup of G.
Then the order of H is a divisor of G $\rightarrow |G| = k|H|, k \in \mathbb{Z} \Leftrightarrow |G| \equiv 0 \bmod |H|$

Examples

$\{0, 3\} \leq \mathbb{Z}/6\mathbb{Z} \wedge \{0, 2, 4\} \leq \mathbb{Z}/6\mathbb{Z}$
$|\{0, 3\}| = 2 \wedge |\{0, 2, 4\}| = 3$
$\{1, 6\} \leq (\mathbb{Z}/7\mathbb{Z})^x$
and many more.

Coset Definition

Let $< G, \star >$ be a group and let H be a subgroup of G.
A left coset of H with representative $g \in G$ is $gH = g \star H = \{g \star h | h \in H\}$
and a right coset is $\{h \star g | h \in H\}$.
Trivially, G abelian $\Rightarrow gH = Hg \ \forall H \leq G$, $gH = Hg$ means H is a **normal subgroup**.

Partition Theorem

Let G be a group and H a subgroup of G. The left cosets of H in G,

$$gH = \{gh \mid h \in H\}, \quad g \in G,$$

form a partition of G. That is:

1. Every element of G belongs to some left coset of H.
2. Any two left cosets of H are either disjoint or identical.

Proof:
(1) Every element of G belongs to some coset: Let $g \in G$. Since $e \in H$, we have

$$g = ge \in gH.$$

Thus, every element of G is contained in some left coset of H.
(2) Cosets are either disjoint or identical: Suppose $g_1H \cap g_2H \neq \emptyset$, meaning there exists some $x \in g_1H \cap g_2H$. Then we can write:

$$x = g_1h_1 = g_2h_2, \quad \text{for some } h_1, h_2 \in H.$$

Rearranging gives

$$g_2^{-1}g_1 = h_2h_1^{-1} \in H,$$

so g_1 and g_2 belong to the same left coset of H, implying

$$g_1H = g_2H.$$

Therefore, the left cosets of H form a partition of G. □

The Index of H in G Definition

Let G be a group and H a subgroup. The index of H in G, denoted by $[G : H]$ is the number of disjoint left-cosets of H in G.

Example

$G = \mathbb{Z}/6\mathbb{Z} \leq \{0, 2, 4\} = H$, $[G : H] = 2 \because (0 + H) \sqcup (1 + H) = G$

Proposition

Let H be a subgroup of G. Then every coset of H has the same number of elements.
Proof:
Let H be a subgroup of G, and let $g \in G$, consider gH.
Let $\lambda : H- > gH, \lambda(h) = g \star h$

1. Well defined: $h \in H \Rightarrow g \star h \in gH$
2. Injective: $\lambda(g) = \lambda(g^*) \Rightarrow gh = gh^* \Rightarrow h = h^*$
3. Surjective: $f \in gH \Rightarrow \exists h \in H$ s.t $f = gh \Rightarrow \lambda(h) = gh = f$

Hence λ is a bijection $\rightarrow |H| = |gH| = |g^*H|$

Proof of Lagrange's Theorem

Let G be a finite group and H a subgroup of G.
Then, $[G : H] = \frac{|G|}{|H|}$ is the number of distinct cosets of H in G.
In particular $|H| \times [G : H] = |G|$.
Thus $|H|$ divides $|G|$.

$H \leq G$.
Then $G = g_1H \sqcup g_2H \sqcup \cdots \sqcup g_nH$, where $n = [H : G]$
Prop $\rightarrow |g_1H| = |g_2H| = \cdots |g_nH|$ and $\therefore |G| = \sum_{i=1}^{n} |g_iH| = |H| \times n = |H| \times [G : H]$
In particular $|G| = |H| \times [G : H] \rightarrow [G : H] = \frac{|G|}{|H|}$ $\square$