MAT-INF3600 Assignment

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1 Problem 1

Solution for (a). Let $A = \{0, 1\}$, $c^{\mathfrak{A}}$ be any element in A and $R^{\mathfrak{A}}(x, y)$ if $x = y$. Let $f^{\mathfrak{A}}(x) = x$ be the identity function on A.

Then (i) is satisfied and (ii), $(\forall x)[R(x, f(x)]$ is satisfied. □

Solution for (b). Let $A = \{\}$ □

Solution and proof of (d). Let $A = \mathbb{N}$ (0 included) with $c^{\mathfrak{A}} = 0$, $R = \emptyset$ (choice is arbitrary) and $f^{\mathfrak{A}}(n) = n + 1$ be the successor function.

Then $\forall x[f(x) \neq c]$ since $f^{\mathfrak{A}}(c^{\mathfrak{A}}) = 0 + 1$, $f^{\mathfrak{A}}(1) = 1 + 1$, and so on. By the peano axioms, 0 is not the successor of any natural number. Furthermore the second condition of f being injective is satisfied since

$$
\begin{aligned}
f(x) &= f(y) \\
x + 1 &= y + 1 \\
x &= y
\end{aligned}
$$

Hence $\mathfrak{A} \models \Gamma$.

Now to prove that any model of Γ has an infinite universe.

Suppose we have some model of Γ with a finite universe $A = \{c^{\mathfrak{A}}, x_1, x_2, \ldots, x_n\}$. We require $f : A \to A \setminus \{c^{\mathfrak{A}}\}$ and for it to be injective. Since A is finite we have an injective map from a set of size $n + 1$ to a set of size n which is not possible by the pigeonhole principle, thus we arrive at a contradiction.

To vizualize this more clearly we can attempt to construct an injection $f : A \to A$.

$$
\begin{aligned}
f^{\mathfrak{A}}(c^{\mathfrak{A}}) &= x_{i_1} \text{ where } x_{i_1} \neq c^{\mathfrak{A}} \\
f^{\mathfrak{A}}(x_1) &= x_{i_2} \text{ where } x_{i_2} \neq x_{i_1}, \text{ and } x_{i_2} \neq c \\
&\vdots \\
f^{\mathfrak{A}}(x_{n-1}) &= x_{i_n} \text{ where } x_{i_n} \neq x_{i_{n-1}}, \ldots, x_{i_1}, \text{ and } x_{i_n} \neq c^{\mathfrak{A}}
\end{aligned}
$$

But now we arrive at $f(x_n)$ which cannot go to $c^{\mathfrak{A}}$ as that violates $f(x) \neq c$ and $f(x_n)$ cannot go to any x_i as that would violate injectivity. So we cannot construct a well-defined injection that satisfies $f(x) \neq c$ for all x given a finite universe.

Hence any model of Γ necessarily has an infinite universe. □