Professor Dave's Differential Equations

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Separable First-Order Differential Equations

This is the simplest type of differential equation. Separable differential equations can be written in the form

$$g(y)dy = f(x)dx$$

where y is a function of x, and f and g are general functions.

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \Rightarrow g(y)dy = f(x)dx$$

$$\int g(y)dy = \int f(x)dx$$

Example

Consider the differential equation

$$\frac{dy}{dx} = \frac{1+\sin x}{15y^4}$$

cross multiply to get

$$15y^4 dy = (1+\sin x)dx$$

$$\int 15y^4 dy = \int (1+\sin x)dx$$

$$3y^5 + C_1 = x - \cos x + C_2$$

$$3y^5 = x - \cos x + C, \quad C = C_2 - C_1$$

an implicit general solution of the differential equation. The explicit general solution would be

$$\boxed{y = \sqrt[5]{\frac{x - \cos x + C}{3}}}$$

Assume now that we are given the boundary conditions $x = 0 \land y(0) = 0$.

$$3(0)^5 = 0 - \cos 0 + C \Rightarrow 0 = -1 + C \Rightarrow C = 1$$

$$\boxed{y = \sqrt[5]{\frac{x - \cos x + 1}{3}}}$$

If we were given the boundary condition earlier we could instead

$$\int_0^y 15y^4 dy = \int_0^x (1 + \sin x) dx$$

where the lower bound is the boundary of the variable in the upper bound. This works well when boundary conditions are simple.

Example

We are given the DE

$$\frac{dy}{dx} = \frac{2x}{y(1+x^2)}, \; y = 0 \text{ when } x = 1$$

$$y dy = \frac{2x}{1+x^2} dx$$

$$\int_0^y y dy = \int_1^x \frac{2x}{1+x^2} dx$$

$$\text{Let } u = 1 + x^2 \Rightarrow du = 2x dx$$

$$\Rightarrow \int_2^{1+x^2} \frac{du}{u} = \ln(1+x^2) - \ln 2$$

$$\frac{y^2}{2} = \ln \frac{1+x^2}{2}$$

$$\boxed{y^2 = 2 \ln \frac{1+x^2}{2}}$$