

# MAT-INF3600 Assignment

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## Problem 1

*Solution for (a).* Let  $A = \{0, 1\}$ ,  $c^{\mathfrak{A}}$  be any element in  $A$  and  $R^{\mathfrak{A}}(x, y)$  if  $x = y$ . Let  $f^{\mathfrak{A}}(x) = x$  be the identity function on  $A$ .

Then (i) is satisfied and (ii),  $(\forall x)[R(x, f(x))]$  is satisfied.  $\square$

*Solution for (b).* Let  $A = \{0, 1\}$ ,  $c^{\mathfrak{A}}$  be any element in  $A$  and  $R^{\mathfrak{A}}(x, y)$  if  $x < y$  with  $f^{\mathfrak{A}}(x) = x$  as before. Then, we get a 2 element universe, but  $R(x, f(x))$  is not satisfied for any  $x$  since 0 is not less than itself and 1 is not less than itself.  $\square$

*Solution for (c).* Now we must construct a 5 element universe with an injective function.

Let  $A = \{1, \dots, 5\}$  with  $c^{\mathfrak{A}} = 1$  (or any other choice),  $R = \emptyset$  (again, arbitrary) and  $f(5) = 1$ ,  $f(x) = x + 1$  otherwise.  $\square$

*Solution and proof of (d).* Let  $A = \mathbb{N}$  (0 included) with  $c^{\mathfrak{A}} = 0$ ,  $R = \emptyset$  (choice is arbitrary) and  $f^{\mathfrak{A}}(n) = n + 1$  be the successor function.

Then  $\forall x[f(x) \neq c]$  since  $f^{\mathfrak{A}}(c^{\mathfrak{A}}) = 0 + 1$ ,  $f^{\mathfrak{A}}(1) = 1 + 1$ , and so on. By the peano axioms, 0 is not the successor of any natural number. Furthermore the second condition of  $f$  being injective is satisfied since

$$\begin{aligned} f(x) &= f(y) \\ x + 1 &= y + 1 \\ x &= y \end{aligned}$$

Hence  $\mathfrak{A} \models \Gamma$ .

Now to prove that any model of  $\Gamma$  has an infinite universe.

Suppose we have some model of  $\Gamma$  with a finite universe  $A = \{c^{\mathfrak{A}}, x_1, x_2, \dots, x_n\}$ . We require  $f : A \rightarrow A \setminus \{c^{\mathfrak{A}}\}$  and for it to be injective. Since  $A$  is finite we have an injective map from a set of size  $n + 1$  to a set of size  $n$  which is not possible by the pigeonhole principle, thus we arrive at a contradiction.

To visualize this more clearly we can attempt to construct an injection  $f : A \rightarrow A$ .

$$\begin{aligned} f^{\mathfrak{A}}(c^{\mathfrak{A}}) &= x_{i_1} \text{ where } x_{i_1} \neq c^{\mathfrak{A}} \\ f^{\mathfrak{A}}(x_1) &= x_{i_2} \text{ where } x_{i_2} \neq x_{i_1}, \text{ and } x_{i_2} \neq c \\ &\vdots \\ f^{\mathfrak{A}}(x_{n-1}) &= x_{i_n} \text{ where } x_{i_n} \neq x_{i_{n-1}}, \dots, x_{i_1}, \text{ and } x_{i_n} \neq c^{\mathfrak{A}} \end{aligned}$$

But now we arrive at  $f^{\mathfrak{A}}(x_n)$  which cannot go to  $c^{\mathfrak{A}}$  as that violates  $f(x) \neq c$  and  $f^{\mathfrak{A}}(x_n)$  cannot go to any  $x_i$  as that would violate injectivity. So we cannot construct a well-defined injection that satisfies  $f(x) \neq c$  for all  $x$  given a finite universe.

Hence any model of  $\Gamma$  necessarily has an infinite universe.  $\square$

## Problem 2

*Proof.* Let  $n \geq 1$  and let  $\theta_1, \dots, \theta_n$  be sentences. Let  $\Sigma$  be a set of formulas. We will prove, by induction, that  $\Sigma \cup \{\theta_1, \dots, \theta_n\} \vdash \phi$  if and only if  $\Sigma \vdash \theta_1 \wedge \dots \wedge \theta_n \rightarrow \phi$ .

*Base case  $n = 1$ .*

$\Sigma \cup \theta \vdash \phi$  if and only if  $\Sigma \vdash \theta \rightarrow \phi$ , by **Theorem 2.7.4 (The Deduction Theorem)**.

Assume that for  $n = k$ . Now look at  $n = k + 1$ ,

$$\begin{aligned} \Sigma \cup \{\theta_1, \dots, \theta_k, \theta_{k+1}\} &\vdash \phi \\ \Sigma \cup \{\theta_1, \dots, \theta_k\} &\vdash \theta_{k+1} \rightarrow \phi \text{ by the regular Deduction Theorem} \\ \Sigma &\vdash \left[ \bigwedge_{i=1}^k \theta_i \right] \rightarrow (\theta_{k+1} \rightarrow \phi) \text{ by assumption} \\ \alpha \rightarrow (\beta \rightarrow \gamma) &\text{ equivalent to } (\alpha \wedge \beta) \rightarrow \gamma \\ \text{Hence } \Sigma &\vdash \left[ \bigwedge_{i=1}^{k+1=n} \theta_i \right] \rightarrow \phi \end{aligned}$$

Now to show the implication in the other direction:

$$\begin{aligned} \Sigma &\vdash \left[ \bigwedge_{i=1}^{k+1=n} \theta_i \right] \rightarrow \phi \\ \Sigma &\vdash \left[ \bigwedge_{i=1}^k \theta_i \right] \rightarrow (\theta_{k+1} \rightarrow \phi) \\ \Sigma \cup \{\theta_1, \dots, \theta_k\} &\vdash \theta_{k+1} \rightarrow \phi \text{ by assumption} \\ \Sigma \cup \{\theta_1, \dots, \theta_{k+1}\} &\vdash \phi \text{ by the regular Deduction Theorem} \end{aligned}$$

This completes the proof. □

### Problem 3

*Proof of (a).*

1.	$\forall x[Rx \rightarrow Sx]$	
2.	$(\forall x[Rx \rightarrow Sx]) \rightarrow (Ry \rightarrow Sy)$	(Q1)
3.	$Ry \rightarrow Sy$	1,2 (PC)
4.	$\neg Sy \rightarrow \neg Ry$	3 (PC)
5.	$(\neg Sy \rightarrow \neg Ry) \rightarrow (\neg Sy \rightarrow \neg Ry)$	(PC)
6.	$(\neg Sy \rightarrow \neg Ry) \rightarrow \forall y(\neg Sy \rightarrow \neg Ry)$	(QR)
7.	$\forall y(\neg Sy \rightarrow \neg Ry)$	4,6 (PC)

Thus we have a deduction of  $\forall y[\neg Sy \rightarrow \neg Ry]$  from  $\forall x[Rx \rightarrow Sx]$ . □

*Proof of (b).* First we recognize that  $\phi \not\models \psi$  if and only if  $\{\phi, \neg\psi\}$  is satisfiable.

Let  $\mathfrak{A}$  be a structure with universe  $A = \{a, b\}$ ,  $R^{\mathfrak{A}} = \{a\}$ ,  $S^{\mathfrak{A}} = \{a, b\}$ .

Then if  $t = a$ ,  $Ra$  is true and  $Sa$  is true so  $Rx \rightarrow Sx$  is true. If  $t = b$ ,  $Rb$  is false so the implication is true regardless. Therefore  $\mathfrak{A} \models \forall x[Rx \rightarrow Sx]$ .

Now to check the other formula. If  $t = b$  we have  $Sb$ , but we do not have  $Rb$ . Hence the implication does not hold for all terms and  $\mathfrak{A} \not\models \forall y[Sy \rightarrow Ry]$ . □