Lecture Notes: Real Analysis — First Lecture (Course By: The Bright Side of Mathematics)

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Introduction to Real Analysis

Definition 1 (Axioms of The Reals)**.** *A non-empty set* $\mathbb{R}$ *together with operations* $+, \times$ *and ordering* $\leq$ *is called the real numbers if it satisfies:*

- *(A)* $(\mathbb{R}, +)$ *is an abelian group with additive identity 0.*
- *(M)* $(\mathbb{R}, \cdot)$ *is an abelian group with multiplicative identity 1.*
- *(D) Distributive law:* $x \cdot (y + z) = x \cdot y + x \cdot z$*.*
- *(O)* $\leq$ *is a total order, compatible with* $+$ *and* $\cdot$*, Archimedean property .*
- *(C) Every Cauchy sequence is a convergent sequence.*

Notice that properties A, M and D makes $\mathbb{R}$ a field.

Definition 2 (The Absolute Value Function)**.** *Let* $x \in \mathbb{R}$*. Then the absolute value of* x *is:*

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Sequences and Limits

Definition 3 (Sequences)**.** *A sequence of real numbers is a map* $a_n : \mathbb{N} \to \mathbb{R}$ *or* $a_n : \mathbb{N}_0 \to \mathbb{R}$ *if you have* $0 \in \mathbb{N}$*. (Truly, a number theorist's worst nightmare)*
We will more often use the notations $(a_1, a_2, a_3, \ldots)$ *or* $(a_n)_{n \in \mathbb{N}}$ *or* $(a_n)_{n=1}^{\infty}$ *or* (a_n)*.*

Examples:

1. $$(a_n)_{n \in \mathbb{N}} = ((-1)^n)_{n \in \mathbb{N}} = (-1, 1, -1, 1, \ldots)$$

2. $$(a_n)_{n \in \mathbb{N}} = \left(\frac{1}{n}\right)_{n \in \mathbb{N}} = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$$

3. $$(a_n)_{n \in \mathbb{N}} = (2^n)_{n \in \mathbb{N}} = (2, 4, 8, 16, 32, \ldots)$$

Convergent Series

Definition 4 (Convergent Series)**.** *A sequence* $(a_n)_{a\in\mathbb{N}}$ *is called convergent to* $a \in \mathbb{R}$ *if*

$$\mathcal{E} > 0, \quad \exists N \in \mathbb{N}, \quad \forall n \geq N : |a_n - a| < \mathcal{E}$$

Example: $(a_n) = (\frac{1}{n})$ (change in notation) is convergent to $0 \in \mathbb{R}$.

Proof. Let $\mathcal{E} > 0$. Choose $N \in \mathbb{N}$ such that $N \cdot \mathcal{E} > 1$ or, in other words, let $N > \frac{1}{\mathcal{E}}$ which must exist because of the Archimedian property. Then for $n \geq N$, we have:

$$|a_n - 0| = |\frac{1}{n} - 0| = |\frac{1}{n}| \leq \frac{1}{N} < \mathcal{E}$$

$\square$