MAT-INF3600 Assignment

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Problem 1

Solution for (a). Let $A = \{0, 1\}$, $c^{\mathfrak{A}}$ be any element in A and $R^{\mathfrak{A}}(x, y)$ if $x = y$. Let $f^{\mathfrak{A}}(x) = x$ be the identity function on A.

Then (i) is satisfied and (ii), $(\forall x)[R(x, f(x)]$ is satisfied. $\square$

Solution for (b). Let $A = \{0, 1\}$, $c^{\mathfrak{A}}$ be any element in A and $R^{\mathfrak{A}}(x, y)$ if $x < y$ with $f^{\mathfrak{A}}(x) = x$ as before. Then, we get a 2 element universe, but $R(x, f(x))$ is not satisfied for any x since 0 is not less than itself and 1 is not less than itself. $\square$

Solution for (c). Now we must construct a 5 element universe with an injective function.

Let $A = \{1, \ldots, 5\}$ with $c^{\mathfrak{A}} = 1$ (or any other choice), $R = \emptyset$ (again, arbitrary) and $f(5) = 1$, $f(x) = x + 1$ otherwise. $\square$

Solution and proof of (d). Let $A = \mathbb{N}$ (0 included) with $c^{\mathfrak{A}} = 0$, $R = \emptyset$ (choice is arbitrary) and $f^{\mathfrak{A}}(n) = n + 1$ be the successor function.

Then $\forall x[f(x) \neq c]$ since $f^{\mathfrak{A}}(c^{\mathfrak{A}}) = 0 + 1$, $f^{\mathfrak{A}}(1) = 1 + 1$, and so on. By the peano axioms, 0 is not the successor of any natural number. Furthermore the second condition of f being injective is satisfied since

$$\begin{aligned} f(x) &= f(y) \\ x + 1 &= y + 1 \\ x &= y \end{aligned}$$

Hence $\mathfrak{A} \vDash \Gamma$.

Now to prove that any model of Γ has an infinite universe.

Suppose we have some model of Γ with a finite universe $A = \{c^{\mathfrak{A}}, x_1, x_2, \ldots, x_n\}$. We require $f : A \to A \setminus \{c^{\mathfrak{A}}\}$ and for it to be injective. Since A is finite we have an injective map from a set of size $n + 1$ to a set of size n which is not possible by the pigeonhole principle, thus we arrive at a contradiction.

To vizualize this more clearly we can attempt to construct an injection $f : A \to A$.

$$\begin{aligned} f^{\mathfrak{A}}(c^{\mathfrak{A}}) &= x_{i_1} \text{ where } x_{i_1} \neq c^{\mathfrak{A}} \\ f^{\mathfrak{A}}(x_1) &= x_{i_2} \text{ where } x_{i_2} \neq x_{i_1}, \text{ and } x_{i_2} \neq c \\ &\vdots \\ f^{\mathfrak{A}}(x_{n-1}) &= x_{i_n} \text{ where } x_{i_n} \neq x_{i_{n-1}}, \ldots, x_{i_1}, \text{ and } x_{i_n} \neq c^{\mathfrak{A}} \end{aligned}$$

But now we arrive at $f^{\mathfrak{A}}(x_n)$ which cannot go to $c^{\mathfrak{A}}$ as that violates $f(x) \neq c$ and $f^{\mathfrak{A}}(x_n)$ cannot go to any x_i as that would violate injectivity. So we cannot construct a well-defined injection that satisfies $f(x) \neq c$ for all x given a finite universe.

Hence any model of Γ necessarily has an infinite universe. $\square$

Problem 2

Proof. Let $n \geq 1$ and let $\theta_1, \ldots, \theta_n$ be sentences. Let Σ be a set of formulas. We will prove, by induction, that $\Sigma \cup \{\theta_1, \ldots, \theta_n\} \vdash \phi$ if and only if $\Sigma \vdash \theta_1 \wedge \cdots \wedge \theta_n \to \phi$.

Base case $n = 1$.

$\Sigma \cup \theta \vdash \phi$ if any only if $\Sigma \vdash \theta \to \phi$, by **Theorem 2.7.4 (The Deduction Theorem)**.

Assume that for $n = k$. Now look at $n = k + 1$,

$$\begin{aligned}
&\Sigma \cup \{\theta_1, \ldots, \theta_k, \theta_{k+1}\} \vdash \phi \\
&\Sigma \cup \{\theta_1, \ldots, \theta_n\} \vdash \theta_{k+1} \to \phi \text{ by the regular Deduction Theorem} \\
&\Sigma \vdash \left[\bigwedge_{i=1}^{k} \theta_i\right] \to (\theta_{k+1} \to \phi) \text{ by assumption} \\
&\alpha \to (\beta \to \gamma) \text{ equivalent to } (\alpha \wedge \beta) \to \gamma \\
&\text{Hence } \Sigma \vdash \left[\bigwedge_{i=1}^{k+1=n} \theta_i\right] \to \phi
\end{aligned}$$

Now to show the implication in the other direction:

$$\begin{aligned}
&\Sigma \vdash \left[\bigwedge_{i=1}^{k+1=n} \theta_i\right] \to \phi \\
&\Sigma \vdash \left[\bigwedge_{i=1}^{k} \theta_i\right] \to (\theta_{k+1} \to \phi) \\
&\Sigma \cup \{\theta_1, \ldots, \theta_k\} \vdash \theta_{k+1} \to \phi \text{ by assumption} \\
&\Sigma \cup \{\theta_1, \ldots, \theta_{k+1}\} \vdash \phi \text{ by the regular Deduction Theorem}
\end{aligned}$$

This completes the proof. $\square$

Problem 3

Proof of (a).

$1. \forall x[Rx \to Sx]$		
$2. \forall x[Rx \to Sx] \to (Rt \to St)$		$(Q1)$
$3. (Rt \to St) \to (\neg St \to \neg Rt)$		(PC)
$4. \forall x[Rx \to Sx] \to (\neg St \to \neg Rt)$	2, 3	(PC)
$5. \forall x[Rx \to Sx] \to \forall y[\neg Sy \to \neg Ry]$	4,	(QR)
$6. \forall y[\neg Sy \to \neg Ry]$	1, 5	(PC)

Thus we have a deduction of $\forall y[\neg Sy \to \neg Ry]$ from $\forall x[Rx \to Sx]$. □

Proof of (b). First we recognize that $\phi \nvDash \psi$ if and only if $\{\phi, \neg\psi\}$ is satisfiable.

Let $\mathfrak{A}$ be a structure with universe $A = \{a, b\}$, $R^{\mathfrak{A}} = \{a\}$, $S^{\mathfrak{A}} = \{a, b\}$.

Then if $t = a$, Ra is true and Sa is true so $Rx \to Sx$ is true. If $t = b$, Rb is false so the implication is true regardless. Therefore $\mathfrak{A} \vDash \forall x[Rx \to Sx]$.

Now to check the other formula. If $t = b$ we have Sb, but we do not have Rb. Hence the implication does not hold for all terms and $\mathfrak{A} \nvDash \forall y[Sy \to Ry]$. □

Problem 4

Proof of (a). We will appeal to Lemma 1 by showing that the negation leads to the given contradiction.

$1. \exists x(R(x)) \wedge \forall x(\neg R(x) \wedge \neg S(x))$		
$2. \exists x(R(x))$	1	(PC)
$3. \forall x(\neg R(x) \wedge \neg S(x))$	1	(PC)
$4. \forall x(\neg R(x) \wedge \neg S(x)) \rightarrow (\neg R(x) \wedge \neg S(x))$		$(Q1)$
$5. \neg R(x) \wedge \neg S(x)$	3,4	(PC)
$6. \neg R(x)$	5	(PC)
$7. \neg R(x) \rightarrow (R(x) \rightarrow \neg(\forall x[x = x])$		(PC)
$8. R(x) \rightarrow \neg(\forall x[x = x])$	6,7	(PC)
$9. \exists x(R(x)) \rightarrow \neg(\forall x[x = x])$	8	(QR)
$10. \neg(\forall x[x = x])$	2,9	(PC)
$11. \forall x(x = x)$		$(E1)$
$12. \perp$	10,11	(PC)

Therefore

$$\vdash \exists x(R(x)) \rightarrow [\exists x(R(x) \vee S(x)]$$

□

Proof of (b). Once again, we will appeal to Lemma 1 by showing that

$$\forall x(R(x) \wedge \exists x \neg R(x)) \vdash \perp$$

$1. \forall x(R(x) \wedge \exists x \neg R(x))$		
$2. \forall x(R(x) \wedge \exists x \neg R(x)) \rightarrow R(x) \wedge \exists x \neg(R(x))$		$(Q1)$
$3. R(x) \wedge \exists x \neg(R(x))$	1,2	(PC)
$4. R(x)$	3	(PC)
$5. \exists \neg(R(x))$	3	(PC)
$6. R(x) \rightarrow (\neg R(x) \rightarrow \neg[\forall x(x = x)])$		(PC)
$7. \neg R(x) \rightarrow \neg[\forall x(x = x)]$	4,6	(PC)
$8. \exists x(\neg R(x)) \rightarrow \neg[\forall x(x = x)]$	7	(QR)
$9. \neg[\forall x(x = x)]$	5,8	(PC)
$10. \forall x(x = x)$		$(E1)$
$11. \perp$	9,10	(PC)

□

Problem 5

Proof. We proceed by structural induction on t. First notice that since we require t to be variable-free, then t is either 0, $v(t_1)$, $h(t_1)$ or $f(t_1, t_2)$ where t_1, t_2 are themselves variable-free $\mathfrak{L}$-terms.

Base case $(t = 0)$.

Then there is obviously $p = 0$, $T \vdash 0 = 0$.

Assume, for any variable-free $\mathfrak{L}$-term t' we can find some p' such that $T \vdash t' = p'$.

Inductive step.

Case 1 (v).

For t to be variable-free, it must be the case that $t = v(t')$

$\square$