Lecture Notes: Design and Analysis of Algorithms
Fast Fourier Transform (Course: MIT-6.046J)

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Fast Fourier Transform

Definition 1. *A polynomial of degree $n-1$ can be written as*

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} = \sum_{k=0}^{n-1} a_k x^k, a_k \in \mathbb{R}$$

$$= \langle a_0, a_1, a_2, \ldots, a_{n-1} \rangle$$
$$= \{(x_k, A(x_k)) \quad | \quad k = 1, 2, \ldots, n-1\}$$
$$= c(x - r_0)(x - r_1) \cdots (x - r_{n-1})$$

Operations on polynomials

1. Evaluation: $A(x)$ & $x_0 \to A(x_0)$?
   Horner's Rule: $A(x) = a_0 + x(a_1 + x(a_2 + \cdots x(a_{n-1}))) \Rightarrow \mathcal{O}(n)$

2. Add: $A(x)$ & $B(x) \to C(x) = A(x) + B(x), \forall x$
   $c_k = a_k + b_k \Rightarrow \mathcal{O}(n)$

3. Multiplication: $A(x)$ & $B(x) \to C(x) = A(x) \cdot B(x), \forall x$

$$c_k = \sum_{k}^{j=0} a_j b_{k-j}$$

$$C(x) = \sum_{k=0}^{n-1} \sum_{j=0}^{k} a_j b_{k-j}$$

$\Rightarrow \mathcal{O}(n^2)$ which is very bad.

Convolution

Suppose we have a vector representing a wave where each entry in the vector represents the amplitude at that frequency. We then want to take a gaussian distribution and shift it along each possible frequency of the wave and take the dot product of the wave and gaussiang distribution to "smooth out" the wave. This is more or less finding c_k for all k. In other words it's finding $C(x)$ which we know is $\mathcal{O}(n^2)$ (very bad).