

MAT-INF3600 Assignment

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Problem 1

Solution for (a). Let $A = \{0, 1\}$, $c^{\mathfrak{A}}$ be any element in A and $R^{\mathfrak{A}}(x, y)$ if $x = y$. Let $f^{\mathfrak{A}}(x) = x$ be the identity function on A .

Then (i) is satisfied and (ii), $(\forall x)[R(x, f(x))]$ is satisfied. \square

Solution for (b). Let $A = \{0, 1\}$, $c^{\mathfrak{A}}$ be any element in A and $R^{\mathfrak{A}}(x, y)$ if $x < y$ with $f^{\mathfrak{A}}(x) = x$ as before. Then, we get a 2 element universe, but $R(x, f(x))$ is not satisfied for any x since 0 is not less than itself and 1 is not less than itself. \square

Solution for (c). Now we must construct a 5 element universe with an injective function.

Let $A = \{1, \dots, 5\}$ with $c^{\mathfrak{A}} = 1$ (or any other choice), $R = \emptyset$ (again, arbitrary) and $f(5) = 1$, $f(x) = x + 1$ otherwise. \square

Solution and proof of (d). Let $A = \mathbb{N}$ (0 included) with $c^{\mathfrak{A}} = 0$, $R = \emptyset$ (choice is arbitrary) and $f^{\mathfrak{A}}(n) = n + 1$ be the successor function.

Then $\forall x[f(x) \neq c]$ since $f^{\mathfrak{A}}(c^{\mathfrak{A}}) = 0 + 1$, $f^{\mathfrak{A}}(1) = 1 + 1$, and so on. By the peano axioms, 0 is not the successor of any natural number. Furthermore the second condition of f being injective is satisfied since

$$\begin{aligned} f(x) &= f(y) \\ x + 1 &= y + 1 \\ x &= y \end{aligned}$$

Hence $\mathfrak{A} \models \Gamma$.

Now to prove that any model of Γ has an infinite universe.

Suppose we have some model of Γ with a finite universe $A = \{c^{\mathfrak{A}}, x_1, x_2, \dots, x_n\}$. We require $f : A \rightarrow A \setminus \{c^{\mathfrak{A}}\}$ and for it to be injective. Since A is finite we have an injective map from a set of size $n + 1$ to a set of size n which is not possible by the pigeonhole principle, thus we arrive at a contradiction.

To visualize this more clearly we can attempt to construct an injection $f : A \rightarrow A$.

$$\begin{aligned} f^{\mathfrak{A}}(c^{\mathfrak{A}}) &= x_{i_1} \text{ where } x_{i_1} \neq c^{\mathfrak{A}} \\ f^{\mathfrak{A}}(x_1) &= x_{i_2} \text{ where } x_{i_2} \neq x_{i_1}, \text{ and } x_{i_2} \neq c \\ &\vdots \\ f^{\mathfrak{A}}(x_{n-1}) &= x_{i_n} \text{ where } x_{i_n} \neq x_{i_{n-1}}, \dots, x_{i_1}, \text{ and } x_{i_n} \neq c^{\mathfrak{A}} \end{aligned}$$

But now we arrive at $f^{\mathfrak{A}}(x_n)$ which cannot go to $c^{\mathfrak{A}}$ as that violates $f(x) \neq c$ and $f^{\mathfrak{A}}(x_n)$ cannot go to any x_i as that would violate injectivity. So we cannot construct a well-defined injection that satisfies $f(x) \neq c$ for all x given a finite universe.

Hence any model of Γ necessarily has an infinite universe. \square

Problem 2

Proof. Let $n \geq 1$ and let $\theta_1, \dots, \theta_n$ be sentences. Let Σ be a set of formulas. We will prove, by induction, that $\Sigma \cup \{\theta_1, \dots, \theta_n\} \vdash \phi$ if and only if $\Sigma \vdash \theta_1 \wedge \dots \wedge \theta_n \rightarrow \phi$.

Base case $n = 1$.

$\Sigma \cup \theta \vdash \phi$ if and only if $\Sigma \vdash \theta \rightarrow \phi$, by **Theorem 2.7.4 (The Deduction Theorem)**.

Assume that for $n = k$. Now look at $n = k + 1$,

$$\begin{aligned} \Sigma \cup \{\theta_1, \dots, \theta_k, \theta_{k+1}\} &\vdash \phi \\ \Sigma \cup \{\theta_1, \dots, \theta_k\} &\vdash \theta_{k+1} \rightarrow \phi \text{ by the regular Deduction Theorem} \\ \Sigma &\vdash \left[\bigwedge_{i=1}^k \theta_i \right] \rightarrow (\theta_{k+1} \rightarrow \phi) \text{ by assumption} \\ \alpha \rightarrow (\beta \rightarrow \gamma) &\text{ equivalent to } (\alpha \wedge \beta) \rightarrow \gamma \\ \text{Hence } \Sigma &\vdash \left[\bigwedge_{i=1}^{k+1=n} \theta_i \right] \rightarrow \phi \end{aligned}$$

Now to show the implication in the other direction:

$$\begin{aligned} \Sigma &\vdash \left[\bigwedge_{i=1}^{k+1=n} \theta_i \right] \rightarrow \phi \\ \Sigma &\vdash \left[\bigwedge_{i=1}^k \theta_i \right] \rightarrow (\theta_{k+1} \rightarrow \phi) \\ \Sigma \cup \{\theta_1, \dots, \theta_k\} &\vdash \theta_{k+1} \rightarrow \phi \text{ by assumption} \\ \Sigma \cup \{\theta_1, \dots, \theta_{k+1}\} &\vdash \phi \text{ by the regular Deduction Theorem} \end{aligned}$$

This completes the proof. □

Problem 3

Proof of (a).

- | | |
|--|----------|
| 1. $\forall x[Rx \rightarrow Sx]$ | |
| 2. $\forall x[Rx \rightarrow Sx] \rightarrow (Rt \rightarrow St)$ | (Q1) |
| 3. $(Rt \rightarrow St) \rightarrow (\neg St \rightarrow \neg Rt)$ | (PC) |
| 4. $\forall x[Rx \rightarrow Sx] \rightarrow (\neg St \rightarrow \neg Rt)$ | 2,3 (PC) |
| 5. $\forall x[Rx \rightarrow Sx] \rightarrow \forall y[\neg Sy \rightarrow \neg Ry]$ | 4, (QR) |
| 6. $\forall y[\neg Sy \rightarrow \neg Ry]$ | 1,5 (PC) |

Thus we have a deduction of $\forall y[\neg Sy \rightarrow \neg Ry]$ from $\forall x[Rx \rightarrow Sx]$. □

Proof of (b). First we recognize that $\phi \not\models \psi$ if and only if $\{\phi, \neg\psi\}$ is satisfiable.

Let \mathfrak{A} be a structure with universe $A = \{a, b\}$, $R^{\mathfrak{A}} = \{a\}$, $S^{\mathfrak{A}} = \{a, b\}$.

Then if $t = a$, Ra is true and Sa is true so $Rx \rightarrow Sx$ is true. If $t = b$, Rb is false so the implication is true regardless. Therefore $\mathfrak{A} \models \forall x[Rx \rightarrow Sx]$.

Now to check the other formula. If $t = b$ we have Sb , but we do not have Rb . Hence the implication does not hold for all terms and $\mathfrak{A} \not\models \forall y[Sy \rightarrow Ry]$. □

Problem 4

Proof of (a). We will appeal to Lemma 1 by showing that the negation leads to the given contradiction.

1.	$\exists x(R(x)) \wedge \forall x(\neg R(x) \wedge \neg S(x))$	
2.	$\exists x(R(x))$	1 (PC)
3.	$\forall x(\neg R(x) \wedge \neg S(x))$	1 (PC)
4.	$\forall x(\neg R(x) \wedge \neg S(x)) \rightarrow (\neg R(x) \wedge \neg S(x))$	(Q1)
5.	$\neg R(x) \wedge \neg S(x)$	3,4 (PC)
6.	$\neg R(x)$	5 (PC)
7.	$\neg R(x) \rightarrow (R(x) \rightarrow \neg(\forall x[x = x]))$	(PC)
8.	$R(x) \rightarrow \neg(\forall x[x = x])$	6,7 (PC)
9.	$\exists x(R(x)) \rightarrow \neg(\forall x[x = x])$	8 (QR)
10.	$\neg(\forall x[x = x])$	2,9 (PC)
11.	$\forall x(x = x)$	(E1)
12.	\perp	10,11 (PC)

Therefore

$$\vdash \exists x(R(x)) \rightarrow [\exists x(R(x) \vee S(x))]$$

□

Proof of (b). Once again, we will appeal to Lemma 1 by showing that

$$\forall x(R(x) \wedge \exists x\neg R(x)) \vdash \perp$$

1.	$\forall x(R(x) \wedge \exists x\neg R(x))$	
2.	$\forall x(R(x) \wedge \exists x\neg R(x)) \rightarrow R(x) \wedge \exists x\neg(R(x))$	(Q1)
3.	$R(x) \wedge \exists x\neg(R(x))$	1,2 (PC)
4.	$R(x)$	3 (PC)
5.	$\exists x\neg(R(x))$	3 (PC)
6.	$R(x) \rightarrow (\neg R(x) \rightarrow \neg[\forall x(x = x)])$	(PC)
7.	$\neg R(x) \rightarrow \neg[\forall x(x = x)]$	4,6 (PC)
8.	$\exists x(\neg R(x)) \rightarrow \neg[\forall x(x = x)]$	7 (QR)
9.	$\neg[\forall x(x = x)]$	5,8 (PC)
10.	$\forall x(x = x)$	(E1)
11.	\perp	9,10 (PC)

□

Problem 5

Proof. We proceed by structural induction on t . First notice that since we require t to be variable-free, then t is either 0 , $\nu(t_1)$, $h(t_1)$ or $f(t_1, t_2)$ where t_1, t_2 are themselves variable-free \mathcal{L} -terms.

Base case ($t = 0$).

Then there is obviously $p = 0$, $T \vdash 0 = 0$.

Assume, for any variable-free \mathcal{L} -term t' we can find some p' such that $T \vdash t' = p'$.

Inductive step.

Case 1 (ν).

For t to be variable-free, it must be the case that $t = \nu(t')$ for some variable-free t' . In the case that $t = \nu(0) = 0$ we have the same solution as in the base case. If not ... NOTE: Show that $\nu(t')$ must have some prime term because of how it is defined.

Case 2 (h).

Same solution as above.

Case 3 (f).

For t to be variable-free, it must be the case that $t = f(t'_1, t'_2)$ for some variable-free t'_1, t'_2 . Then $\nu(f(t'_1, t'_2)) = t'_1$ and $h(f(t'_1, t'_2)) = t'_2$, both of which have some prime term.

□