

Real Analysis

Thobias Høivik

Spring 2026

Contents

| | | |
|----------|----------------------------------|----------|
| 1 | Introduction | 3 |
| 2 | Basic Banach Space theory | 4 |
| 2.1 | Norms and Metrics | 6 |
| 2.2 | Banach Spaces | 8 |

1 Introduction

The following is intended for anyone who stumbles over these notes. This is intended to be my personal notes in real analysis. I will hopefully be attending MAT2400 Real Analysis at the University of Oslo in the spring of 2026. However, as I am not a program student at this institution, but just someone who takes individual courses there of my own volition, I do not and cannot attend lectures and therefore I have to learn the material on my own. As far as I understand, while this course is called Real Analysis, it is a bit different than a first course from what I understand. The earlier exams give a hint of functional analysis and also include topics such as fourier analysis, measure- and integration theory among other things. Thus the different supplementary course material I will use to aid myself in learning the content of this course will most likely be a bit scattered and all over the place, which these notes will undoubtedly reflect. I will do my best to keep things organized for my own sake, but keep this in mind if you are someone who intends to use these notes to learn Real Analysis.

2 Basic Banach Space theory

The following section of notes is derived from the first video in the lecture series MIT 18.102 Introduction to Functional Analysis, Spring 2021 (found on youtube).

Definition 2.1 (Vector Space). A vector space V over a field \mathbb{F} is a nonempty set of elements called "vectors" together with a binary operation $+$ on V and a binary function \cdot which maps elements of V, \mathbb{F} to V satisfying:

1. Associativity of vector addition:

$$u + (v + w) = (u + v) + w, \forall u, v, w \in V$$

2. Commutativity of vector addition:

$$u + v = v + w, \forall u, v \in V$$

3. Identity element:

$$\exists 0 \in V : v + 0 = 0 + v = v, \forall v \in V$$

4. Each $v \in V$ has an inverse $-v$ under the vector-addition operation.

5. Scalar multiplication is compatible with field multiplication:

$$a(bv) = (ab)v$$

where $a, b \in \mathbb{F}$ and $v \in V$.

6. The multiplicative identity $1 \in \mathbb{F}$ satisfies:

$$1v = v, \forall v \in V$$

7. Distributivity of scalar multiplication with respect to vector addition:

$$a(u + v) = au + av$$

where $a \in \mathbb{F}$ and $u, v \in V$.

8. Distributivity of scalar multiplication with respect to field addition:

$$(a + b)v = av + bv$$

where $a, b \in \mathbb{F}$ and $v \in V$.

When proving that something is a vector space, most of these follow naturally from showing closure under addition and scalar multiplication and those two properties, are generally enough to show that it is indeed a vector space.

A subspace U of V is a set $U \subseteq V$ which is also a vector space. It is enough to show that $U \subseteq V$ and that it is closed under the two operations.

Some typical examples of vector spaces are \mathbb{F}^n where \mathbb{F} is the reals or the complex numbers. We also have spaces like the space of real polynomials of degree $\leq n$, i.e. $\mathcal{P}_n = \{\sum_{i=0}^n \alpha_i x^i : \alpha_i \in \mathbb{R}\}$, which is itself a subspace of the space of continuous real-valued functions $C(\mathbb{R})$.

So \mathbb{R}^2 and $C(\mathbb{R})$ are both vector spaces over \mathbb{R} , but they have one really big difference, that being the dimension.

Definition 2.2. Let V be a vector space. A set $\{v_1, \dots, v_n\} \subseteq V$ is linearly independent if

$$\sum_{i=1}^n \alpha_i v_i = 0 \Leftrightarrow \alpha_1 = \dots = \alpha_n = 0 \in \mathbb{F}$$

Note: the right-to-left direction of this implication is always true.

The two spaces discussed above are different in dimension, \mathbb{R}^2 being 2-dimensional and the other being infinite-dimensional. One definition of finite-dimensional is that every linearly independent set in the space is finite. I however like the definition using bases more. Both of these definitions are equivalent.

We won't give a rigorous definition of a basis, but in short a basis of V is a linearly independent set of vectors which spans V , i.e. every vector in V can be expressed as a linear combination of basis-vectors. If the basis is finite then V is finite dimensional. Moreover, if the basis is finite then the dimension of V is the number of basis-vectors. Note that if a finite dimensional space V has a basis with n elements then every basis of V has n elements. A space is infinite-dimensional if no finite set of linearly independent vectors spans the space.

2.1 Norms and Metrics

Definition 2.3 (Norm). Let V be a vector space. A norm $\|\cdot\|$ is a function from $V \rightarrow [0, \infty)$ satisfying:

1. $\|v\| = 0$ if and only if $v = 0$.
2. $\|\alpha v\| = |\alpha| \cdot \|v\|$ where α is an element of the ground field.
3. $\|v + w\| \leq \|v\| + \|w\|$.

The tuple $(V, \|\cdot\|)$ is called a normed space.

Example 2.1. $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$ defines a norm on \mathbb{R}^n . In fact it constitutes a norm on \mathbb{C}^2 as well. Formally, if we take $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ the p -norm of a vector $v \in \mathbb{F}^n$ ($p \in [1, \infty]$) is

$$\|v\|_p := \begin{cases} \left(\sum_{i=1}^n |v_i|^p \right)^{1/p} & p < \infty \\ \max_{i=1, \dots, n} |v_i| & p = \infty \end{cases}$$

Norms give us a notion of the "length" of a vector. Now all we need to do analysis on spaces is a notion of distance. Intuitively, norms already give us a notion of distance from 0.

Definition 2.4. Let X be a set. A metric is a function $d: X^2 \rightarrow [0, \infty)$ satisfying:

1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$.

The metric gives us a notion of distance. In a typical first course in analysis where we work on the reals, $d(a, b) = |a - b|$ is the metric we deal with.

Proposition 2.1. *Let V be a normed space with norm $\|\cdot\|$. Then we can define the distance (a metric) between two vectors by*

$$d(x, y) := \|x - y\|$$

In other words you can define a metric in terms of the norm in any normed space. This metric is usually referred to as the metric induced by the norm.

We won't provide a proof of this as it's fairly intuitive. Now we can get a sense of convergence and continuity in vector spaces by saying that a sequence $\{a_n\}_{n \in \mathbb{N}}$ converges to a value a if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : n \geq N \Rightarrow \|a_n - a\| < \varepsilon$$

and a linear transformation (look up definition if necessary) $T \in \mathcal{L}(U, V)$ is (uniformly) continuous if

$$\forall \varepsilon > 0, \exists \delta > 0 : \|x - y\|_U < \delta \Rightarrow \|Tx - Ty\|_V < \varepsilon$$

for every $x, y \in U$. Notice that if you replace $\|x - y\|$ with $d(x, y)$ it looks like the standard definitions in terms of metric spaces.

2.2 Banach Spaces

Definition 2.5 (Banach Space). A normed space V is a Banach Space if it is complete with respect to the metric induced by the norm, meaning that every Cauchy sequence converges to a value in the space.

Example 2.2. \mathbb{R}^n or \mathbb{C}^n form Banach Spaces with respect to the ℓ^p norms (see Example 2.1).

Theorem 2.1. If X is a complete metric space, then $C_\infty(X)$ is a Banach Space.

Recall that $C_\infty(X)$ is the space of bounded continuous functions on X .

Proof. We show that every Cauchy sequence in $C_\infty(X)$ converges to an element of $C_\infty(X)$.

Let $\{u_n\}_{n=1}^\infty \subseteq C_\infty(X)$ be a Cauchy sequence with respect to the supremum norm. Then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|u_n - u_m\|_\infty < \varepsilon \quad \text{for all } n, m \geq N.$$

Equivalently,

$$|u_n(x) - u_m(x)| < \varepsilon \quad \text{for all } x \in X \text{ and all } n, m \geq N.$$

Fix $x \in X$. Then $\{u_n(x)\}_{n=1}^\infty$ is a Cauchy sequence in \mathbb{R} (or \mathbb{C}), since

$$|u_n(x) - u_m(x)| \leq \|u_n - u_m\|_\infty.$$

Because \mathbb{R} (or \mathbb{C}) is complete, the limit

$$u(x) := \lim_{n \rightarrow \infty} u_n(x)$$

exists. This defines a function $u : X \rightarrow \mathbb{R}$.

We now show that $u_n \rightarrow u$ uniformly on X . Let $\varepsilon > 0$ and choose N such that $\|u_n - u_m\|_\infty < \varepsilon$ for all $n, m \geq N$. Fix $n \geq N$ and $x \in X$. Taking the limit $m \rightarrow \infty$ gives

$$|u_n(x) - u(x)| = \lim_{m \rightarrow \infty} |u_n(x) - u_m(x)| \leq \varepsilon.$$

Since $x \in X$ was arbitrary, it follows that

$$\|u_n - u\|_\infty \leq \varepsilon \quad \text{for all } n \geq N.$$

Thus $u_n \rightarrow u$ uniformly on X .

Since each u_n is bounded and the convergence is uniform, the limit function u is bounded. Moreover, since each u_n is continuous and uniform limits of continuous functions are continuous, u is continuous on X .

Therefore $u \in C_\infty(X)$ and $\{u_n\}$ converges to u in the supremum norm. Hence $C_\infty(X)$ is complete, and thus a Banach space. \square