Lecture Notes: Real Analysis — Characterizing the Reals (Course: MIT 18.100A)

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Abstract Algebra revision

Recall the definition of the field from abstract algebra:

Definition 1 (Field)**.** *A field $(F, +, \cdot)$ is a set with two operations denoted as $+$ and $\cdot$ satisfying the following conditions,*

1. *Closure of addition; If* $x, y \in F \Rightarrow x + y \in F$
2. *Commutativity of addition;* $x + y = y + x, \forall x, y \in F$
3. *Associativity of addition;* $\forall x, y, z \in F : (x + y) + z = x + (y + z)$
4. *Additive identity;* $\forall x \in F : x + 0 = 0 + x = x$
5. *Additive inverses;* $\forall x \in F, \exists(-x) : x + (-x) = (-x) + x = 0$
6. *Closure of multiplication; If* $x, y \in F \Rightarrow x \cdot y \in F$
7. *Commutativity of multiplication;* $x \cdot y = y \cdot x, \forall x, y \in F$
8. *Associativity of multiplication;* $\forall x, y, z \in F : (xy)z = x(yz)$
9. *Multiplicative identity;* $\forall x \in F : 1 \cdot x = x \cdot 1 = x$
10. *Multiplicative inverses:* $\forall x \in F, (x \neq 0) \Rightarrow \exists x^{-1} \in F : x \cdot x^{-1} = x^{-1} \cdot x = 1$
11. *(Left & right) distributive property* $x(y+z) = xy+xz \wedge (x+y)z = xz+yz \quad \forall x, y, z \in F$

Examples:
A clear example of this would be $\mathbb{Q}$. $\mathbb{Z}$, however, may only have sufficient structure for an integral domain or even a euclidean domain if we want to be a bit more specific.

Theorem 1. *If F is a field then $\forall x \in F$,*

$$0 \cdot x = 0$$

Proof. If $x \in F$,

$$0 = 0x + (-0x) = (0 + 0)x + (-0x) = 0x + 0x + (-0x)$$
$$= 0x + 0 = 0x$$

□

Definition 2 (Ordered Field)**.** *An ordered field is a field F which is also an ordered set such that*

1. $\forall x, y, z \in F$, *if* $x < y \Rightarrow x + z < y + z$
2. *If* $x > 0$ *and* $y > 0$ *then* $xy > 0$

If $x > 0$ *we say* x *is positive and for* $x \geq 0$ *we say* x *is non-negative.*

Example:
Here $\mathbb{Q}$ is an example, yet again.

Theorem 2. *If F is an ordered field, then if $x > 0 \Rightarrow -x < 0$*

Proof. If $0 < x \Rightarrow -x + 0 < -x + x \Rightarrow -x < 0$ □

The same can be said for $x < 0 \Rightarrow -x > 0$ and it is equally trivial to prove.

Field with the Least Upper Bound property

Theorem 3. *Let F be an ordered field with the least upper bound property. Then if $A \subset F, A \neq \emptyset$ and bounded below then $inf(A)$ exists in F.*

Proof. We are given that for any $\emptyset \neq A \subset F \Rightarrow \exists sup(A)$. Consider then the set $-A = \{-x : x \in A\}$. Clearly $-A$ is nonempty and since F is a field $x \in F \Rightarrow -x \in F$ so $-A \subset F$. Thus $-A$ is a nonempty subset of F. We assume there exists a supremum of $-A$ since F has the least upper bound property. If $b \geq x, \forall x \in -A$ is an arbitrary upper bound of $-A$, the supremum $sup(A)$ is the bound $b_0 \leq b$, observe what happens when we take their inverses.

$$\begin{gathered}
b \geq x \\
b + (-b) \geq x + (-b) \\
0 \geq x + (-b) \\
(-x) + 0 \geq (-x) + x + (-b) \\
-x \geq -b
\end{gathered}$$

Thus the additive inverse of any arbitrary upper bound of $-A$ is a lower bound of the set containing the additive inverses of $-A$

$$\begin{gathered}
b_0 \leq b \\
b_0 + (-b_0) \leq b + (-b_0) \\
0 \leq b + (-b_0) \\
(-b) + 0 \leq (-b) + b + (-b_0) \\
-b \leq -b_0
\end{gathered}$$

Thus the inverse of the supremum of $-A$ is the infimum of the set of A's inverses. Recall that $-A = \{-x : x \in A\}$ so $A = \{x : -x \in -A\}$. Thus, a subset Abounded below of an ordered field has $\inf(A) = -\sup(-A)$ and so a ordered field with the least upper bound property has the greates lower bound property, also.

□

The reals again

Theorem 4. *There exists a unique (up to isomorphisms) ordered field with the least upper bound property containing $\mathbb{Q}$. This is the field denoted by $\mathbb{R}$.*

This is not trivial to prove and will, once again, not be done here.

Theorem 5. *There exists unique $r \in \mathbb{R}$ such that $r > 0 \land r^2 = 2$.*

Proof. Let $E = \{x \in \mathbb{R} : x > 0 \land x^2 < 2\}$. Then E is bounded above by 2, so $\sup E$ exists by the least upper bounde property. Let $r = \sup E$. We will show that $r^2 = 2$. Assume for contradiction that $r^2 < 2$ or $r^2 > 2$.
Case 1: $r^2 < 2$ Since r is the least upper bound, for any $\mathcal{E} > 0$, there exists $x \in E$ such that $r - \mathcal{E} < x$. Choose $\mathcal{E} > 0$ small enough such that $(r + \mathcal{E})^2 < 2$. Then $r + \mathcal{E} \in E$, contradicting that r is an upper bound. Hence, $r^2 \geq 2$.
Case 2: $r^2 > 2$ Choose $\mathcal{E} > 0$ small enough such that $(r - \mathcal{E})^2 > 2$. Then $r - \mathcal{E}$ is an upper bound of E, contradicting the assumption that $r = \sup E$. Hence, $r^2 \leq 2$.
Since both cases lead to contradictions, we conclude that $r^2 = 2$.
Uniqueness: Suppose there exists another $s > 0$ such that $s^2 = 2$. If $s > r$, then s is an upper bound smaller than $\sup E$, contradicting the definition of r. If $s < r$, then r is not the least upper bound, also a contradiction. Hence, $r = s$, proving uniqueness. □