Real Analysis

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Spring 2026

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1 Introduction

The following is intended for anyone who stumbles over these notes. This is intended to be my personal notes in real analysis. I will hopefully be attending MAT2400 Real Analysis at the University of Oslo in the spring of 2026. However, as I am not a program student at this institution, but just someone who takes individual courses there of my own volition, I do not and cannot attend lectures and therefore I have to learn the material on my own. As far as I understand, while this course is called Real Analysis, it is a bit different than a first course from what I understand. The earlier exams give a hint of functional analysis and also include topics such as fourier analysis, meassure- and integration theory among other things. Thus the different supplementary coursematerial I will use to aid myself in learning the content of this course will most likely be a bit scattered and all over the place, which these notes will undoubtedly reflect. I will do my best to keep things organized for my own sake, but keep this in mind if you are someone who intends to use these notes to learn Real Analysis.

2 Basic Banch Space theory

The following section of notes is derived from the first video in the lecture series MIT 18.102 Introduction to Functional Analysis, Spring 2021 (found on youtube).

Definition 2.1 (Vector Space). *A vector space V over a field $\mathbb{F}$ is a nonempty set of elements called "vectors" together with a binary operation $+$ on V and a binary function $\cdot$ which maps elements of $V, \mathbb{F}$ to V satisfying:*

1. *Associativity of vector addition:*
$$u + (v + w) = (u + v) + w, \forall u, v, w \in V$$
2. *Commutativity of vector addition:*
$$u + v = v + w, \forall u, v \in V$$
3. *Identity element:*
$$\exists 0 \in V : v + 0 = 0 + v = v, \forall v \in V$$
4. *Each $v \in V$ has an inverse $-v$ under the vector-addition operation.*
5. *Scalar multiplication is compatible with field multiplication:*
$$a(bv) = (ab)v$$
where $a, b \in \mathbb{F}$ and $v \in V$.
6. *The multiplicative identity $1 \in \mathbb{F}$ satisfies:*
$$1v = v, \forall v \in V$$
7. *Distributivity of scalar multiplication with respect to vector addition:*
$$a(u + v) = au + av$$
where $a \in \mathbb{F}$ and $u, v \in V$.
8. *Distributivity of scalar multiplication with respect to field addition:*
$$(a + b)v = av + bv$$
where $a, b \in \mathbb{F}$ and $v \in V$.

When proving that something is a vector space, most of these follow naturally from showing closure under addition and scalar multiplication and those two properties, are generally enough to show that it is indeed a vector space.
A subspace U of V is a set $U \subseteq V$ which is also a vector space. It is enough to show that $U \subseteq V$ and that it is closed under the two operations.

Some typical examples of vector spaces are $\mathbb{F}^n$ where $\mathbb{F}$ is the reals or the complex numbers. We also have spaces like the space of real polynomials of degree $\leq n$, i.e. $\mathcal{P}_n = \{\sum_{i=0}^n \alpha_i x^i : \alpha_i \in \mathbb{R}\}$, which is itself a subspace of the space of continuous real-valued functions $C(\mathbb{R})$.

So $\mathbb{R}^2$ and $C(\mathbb{R})$ are both vector spaces over $\mathbb{R}$, but they have one really big difference, that being the dimension.

Definition 2.2. *Let V be a vector space. A set $\{v_1, \dots, v_n\} \subseteq V$ is linearly independent if*

$$\sum_{i=1}^{n} \alpha_i v_i = 0 \Leftrightarrow \alpha_1 = \cdots = \alpha_n = 0 \in \mathbb{F}$$

Note: the right-to-left direction of this implication is always true.

The two spaces discussed above are different in dimension, $\mathbb{R}^2$ being 2-dimensional and the other being infinite-dimensional. One definition of finite-dimensional is that every linearly independent set in the space is finite. I however like the definition using bases more. Both of these definitions are equivalent.

We won't give a rigorous definition of a basis, but in short a basis of V is a linearly independent set of vectors which spans V, i.e. every vector in V can be expressed as a linear combination of basis-vectors. If the basis is finite then V is finite dimensional. Moreover, if the basis is finite then the dimension of V is the number of basis-vectors. Note that if a finite dimensional space V has a basis with n elements then every basis of V has n elements. A space is infinite-dimensional if no finite set of linearly independent vectors spans the space.

2.1 Norms and Metrics

Definition 2.3 (Norm). *Let V be a vector space. A norm $||\cdot||$ is a function from $V \to [0, \infty)$ satisfying:*

1. $||v|| = 0$ *if and only if* $v = 0$.
2. $||\alpha v|| = |\alpha| \cdot ||v||$ *where* α *is an element of the ground field.*
3. $||v + w|| \leq ||v|| + ||w||$.

The tuple $(V, ||\cdot||)$ is called a normed space.

Example 2.1. $||x||_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ *defines a norm on $\mathbb{R}^n$. In fact it constitutes a norm on $\mathbb{C}^2$ as well. Formally, if we take $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ the p-norm of a vector $v \in \mathbb{F}^n$ ($p \in [1, \infty]$) is*

$$||v||_p := \begin{cases} \left(\sum_{i=1}^{n} |v_i|^p\right)^{1/p} & p < \infty \\ \max_{i=1,\ldots,n} |v_i| & p = \infty \end{cases}$$

Norms give us a notion of the "length" of a vector. Now all we need to do analysis on spaces is a notion of distance. Intuitively, norms already give us a notion of distance from 0.

Definition 2.4. *Let X be a set.*
A metric is a function $d : X^2 \to [0, \infty)$ satisfying:

1. $d(x, y) = 0$ *if and only if* $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$.

The metric gives us a notion of distance. In a typical first course in analysis where we work on the reals, $d(a, b) = |a - b|$ is the metric we deal with.

Proposition 2.1. *Let V be a normed space with norm $||\cdot||$. Then we can define the distance (a metric) between two vectors by*

$$d(x, y) := ||x - y||$$

In other words you can define a metric in terms of the norm in any normed space. This metric is usually refered to as the metric induced by the norm.

We won't provide a proof of this as it's fairly intuitive. Now we can get a sense of convergence and continuity in vector spaces by saying that a sequence $\{a_n\}_{n\in\mathbb{N}}$ converges to a value a if

$$\forall\varepsilon > 0, \exists N \in \mathbb{N} : n \geq N \Rightarrow ||a_n - a|| < \varepsilon$$

and a linear transformation (look up definition if necessary) $T \in \mathcal{L}(U, V)$ is (uniformly) continuous if

$$\forall\varepsilon > 0, \exists\delta > 0 : ||x - y||_U < \delta \Rightarrow ||Tx - Ty||_V < \varepsilon$$

for every $x, y \in U$. Notice that if you replace $||x - y||$ with $d(x, y)$ it looks like the standard definitions in terms of metric spaces.

2.2 Banach Spaces

Definition 2.5 (Banach Space). *A normed space V is a Banach Space if it is complete with respect to the metric induced by the norm, meaning that every cauchy sequence converges to a value in the space.*

Example 2.2. *$\mathbb{R}^n$ or $\mathbb{C}^n$ form Banach Spaces with respect to the ℓ^p norms (see Example 2.1).*

Theorem 2.1. *If X is a complete metric space, then $C_\infty(X)$ is a Banach Space.*

Recall that $C_\infty(X)$ is the space of bounded continuous functions on X.

Proof. We show that every Cauchy sequence in $C_\infty(X)$ converges to an element of $C_\infty(X)$.

Let $\{u_n\}_{n=1}^\infty \subseteq C_\infty(X)$ be a Cauchy sequence with respect to the supremum norm. Then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|u_n - u_m\|_\infty < \varepsilon \quad \text{for all } n, m \geq N.$$

Equivalently,

$$|u_n(x) - u_m(x)| < \varepsilon \quad \text{for all } x \in X \text{ and all } n, m \geq N.$$

Fix $x \in X$. Then $\{u_n(x)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$ (or $\mathbb{C}$), since

$$|u_n(x) - u_m(x)| \leq \|u_n - u_m\|_\infty.$$

Because $\mathbb{R}$ (or $\mathbb{C}$) is complete, the limit

$$u(x) := \lim_{n\to\infty} u_n(x)$$

exists. This defines a function $u : X \to \mathbb{R}$.

We now show that $u_n \to u$ uniformly on X. Let $\varepsilon > 0$ and choose N such that $\|u_n - u_m\|_\infty < \varepsilon$ for all $n, m \geq N$. Fix $n \geq N$ and $x \in X$. Taking the limit $m \to \infty$ gives

$$|u_n(x) - u(x)| = \lim_{m\to\infty} |u_n(x) - u_m(x)| \leq \varepsilon.$$

Since $x \in X$ was arbitrary, it follows that

$$\|u_n - u\|_\infty \leq \varepsilon \quad \text{for all } n \geq N.$$

Thus $u_n \to u$ uniformly on X.

Since each u_n is bounded and the convergence is uniform, the limit function u is bounded. Moreover, since each u_n is continuous and uniform limits of continuous functions are continuous, u is continuous on X.

Therefore $u \in C_\infty(X)$ and $\{u_n\}$ converges to u in the supremum norm. Hence $C_\infty(X)$ is complete, and thus a Banach space. $\square$

3 A step back into Calculus Theory

The following section is largely in accordance with chapter 2 of Lindstrøm's Spaces: An Introduction To Real Analysis.

Definition 3.1 (Limit of a sequence). *Let (x_n) be a sequence of real numbers and $x \in \mathbb{R}$.*
We say

$$x_n \to x$$

if for every $\varepsilon > 0$, there exists a natural number N such that

$$|x_n - x| < \varepsilon, \ \forall n \geq N$$

This is among the most important definitions in a first course in Real-Analysis. Intuitively what it captures is the sense of being able to get arbitrarily close to a value. We will all be familiar with examples from calculus like $\lim_{n\to\infty} \frac{1}{n} = 0$.

Theorem 3.1. *If a sequence converges, then its limit is unique.*

Proof. Suppose, for a contradiction, that we have some sequence (x_n) with

$$x_n \to x \text{ and } x_n \to y$$

of course with $x \neq y$.

x_n converges so we can choose any $\varepsilon > 0$ for the condition in definition 3.1 to hold.

Consider the case of $\varepsilon = \frac{|x-y|}{2}$. $\varepsilon > 0$ since $x \neq y$ and is therefore a value for which the condition must hold. Namely,

$$|x_n - x| < \varepsilon \text{ and } |x_n - y| < \varepsilon$$

leading to

$$|x - x_n| + |x_n - y| < 2\varepsilon = |x - y|$$

which, via triangle inequality, yields

$$|x - y| = |x - x_n + x_n - y| \leq |x - x_n| + |x_n - y| < |x - y|$$

in particular

$$|x - y| < |x - y|$$

which is not possible, a contradiction.

$\square$

Theorem 3.2 (Algebra of Limits). *If $x_n \to x$ and $y_n \to y$, then:*

1. $x_n + y_n \to x + y$
2. $x_n y_n \to xy$
3. *If $y_n \neq 0$ and $y \neq 0$, then $\frac{x_n}{y_n} \to \frac{x}{y}$.*

Proof of (1): Sum of Limits. Let (x_n) and (y_n) be convergent sequences with $x_n \to x$ and $y_n \to y$.

Let $\varepsilon > 0$. Since both sequences converge to their respective limits we have some $N_1, N_2 \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon/2, \ \forall n \geq N_1$$
$$|y_n - y| < \varepsilon/2, \ \forall n \geq N_2$$

Let $N = \max\{N_1, N_2\}$, then $n \geq N$ satisfies $n \geq N_1$ and $N \geq N_2$ so

$$|x_n - x| + |y_n - y| < \varepsilon$$

Notice that via the triangle inequality we get

$$|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y| < \varepsilon$$

completing the proof. $\square$

Theorem 3.3. *A sequence x_n converges to x if and only if*

$$\limsup x_n = \liminf x_n = x$$

We will take this theorem without proof as it requires some background about monotone sequences and completeness which are covered later.

Definition 3.2 (Limit of a function). *Let $f : D \subset \mathbb{R} \to \mathbb{R}$, and let a be a limit point of D.*
We say

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Later, when we get to metric spaces again, this condition becomes

$$d_X(x, a) < \delta \Rightarrow d_Y(f(x), f(a)) < \varepsilon$$

for continuity.

Theorem 3.4 (Sequential criterion).

$\lim_{x \to a} f(x) = L \Leftrightarrow$ *for every sequence* $x_n \to a$ *with* $x_n \neq a, f(x_n) \to L$

Proof sketch. We only prove $\Rightarrow$, for now.

Assume $\lim_{x \to a} f(x) = L$.

Let $x_n \to a$ with $x_n \neq a$.

Given $\varepsilon > 0$, choose $\delta > 0$ from the definition of the limit.

Since $x_n \to a$ there exists $N \in \mathbb{N}$ such that with $n \geq N$ we have

$$|x_n - a| < \delta$$

Hence for $n \geq N$,

$$|f(x_n) - L| < \varepsilon$$

Thus $f(x_n) \to L$. $\square$

4 Completeness

Definition 4.1 (Upper Bound). *A set $A \subset \mathbb{R}$ is bounded above if there exists $M \in \mathbb{R}$ such that*

$$a \leq M, \forall a \in A$$

Such an M is called an upper bound.

Definition 4.2 (Supremum). *Let $A \subset \mathbb{R}$.*
A number $s \in \mathbb{R}$ is the supremum of A if it is an upper bound of A and for any upper bound u of A we have

$$s \leq u$$

and we use the notation

$$s = \sup A$$

The completeness axiom says that every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$, this being the distinguishing quality separating $\mathbb{R}$ from $\mathbb{Q}$.

Sidenote: You can also characterize the reals as the *unique* ordered field containing $\mathbb{Q}$ which has the least upper bound property. More precisely, every $\mathbb{F}$ with this property satisfies $\mathbb{F} \cong \mathbb{R}$.

Proposition 4.1. *If A has a supremum then for every $\varepsilon > 0$, there exists $a \in A$ such that*

$$s - \varepsilon < a \leq s$$

Proof. Let $\varepsilon > 0$. If no $a \in A$ existed with the desired property, then $s - \varepsilon$ would be a supremum, contradicting minimality of s. □

This proposition becomes useful later in many arguments.

The infimum $\inf A$ is also something we need and classically we define it as being the greatest lower bound, but for the purposes of keeping things simple, we just let the infimum of a set A, $\inf A$, be defined as

$$\inf A = -\sup(-A)$$

Theorem 4.1 (Monotone convergence). *Let (x_n) be a monotone increasing sequence (a sequence with which $x_n \leq x_{n+1}$ for every $n \in \mathbb{N}$) which is bounded above.*
Then (x_n) converges, and

$$\lim x_n = \sup\{x_n : n \in \mathbb{N}\}$$

Proof. Let

$$A = \{x_n : n \in \mathbb{N}\}$$

Since A is bounded above, A has some supremum s. We claim $x_n \to s$.

Let $\varepsilon > 0$.

By the supremum property, there exists N such that

$$s - \varepsilon < x_N <\leq s$$

Since the sequence is increasing, for all $n \geq N$,

$$s - \varepsilon < x_n \leq s$$

Thus

$$|x_n - s| < \varepsilon$$

□

Lemma 4.1. *Every cauchy sequence in $\mathbb{R}$ is bounded.*

Proof. Let (x_n) be cauchy.

Choose $\varepsilon = 1$. Then there exists N such that

$$|x_n - x_m| < 1, \; \forall n, m \geq N$$

Fix $m = N$. Then for all $n \geq N$:

$$|x_n| \leq |x_N| + |x_n - x_N| < |x_N| + 1$$

Thus all terms are bounded. □

Theorem 4.2. *Every cauchy sequence in $\mathbb{R}$ converges.*

Proof idea. Let (x_n) be cauchy.

(x_n) is bounded. Use the cauchy property to construct nested intervals. Show the intersection of these to be nonempty via completeness of $\mathbb{R}$. (x_n) will converge to a point in this intersection.

A bit more formally, we choose indices $n_1, n_2, \ldots$ such that

$$|x_n - x_m| < 2^{-k},\ \forall n, m \geq n_k$$

and define the intervals

$$I_k = [x_{n_k} - 2^{-k}, x_{n_k} + 2^{-k}]$$

Then each I_k is closed and bounded with $I_{k+1} \subset I_k$. Completeness wil give us

$$\bigcap_{k \in \mathbb{N}} I_k \neq \emptyset$$

There will be an x in this set which is the limit point. □

5 Open and Closed Sets in Metric Spaces

Let X be a set and $A \subseteq X$. Intuitively, a point x is either **inside of** A, **outside of** A, or **on the boundary** of A. We know what it means to not be in A; $a \notin A \Leftrightarrow a \in A^c$. We also know what $a \in A$ means, but set-theoretically we don't have a notion of being on the "boundary" of A. We can make sense of this notion in a metric space.

Recall that the open ball is the set $B(x;r) = \{z \in X : d(x,z) < r\}$. Likewise we can define the closed ball as follows.

Definition 5.1 (Closed Ball). *The closed ball centered at $x \in X$ with radius $r \geq 0$ in $B(x;r) = \{z \in X : d(x,z) \leq r\}$.*

Let (X,d) be a metric space with $A \subseteq X$.

- $x \in X$ is an interior point of A if $B(x;r) \subseteq A$ for some $r > 0$.
- $y \in X$ is an interior point of A if $B(x;r) \subseteq A^c$ for some $r > 0$.
- $z \in X$ is a boundary point of A if it is neither of the above, i.e. $B(z;r) \cap A \neq \emptyset$ and $B(z;r) \cap A^c \neq \emptyset$ for every $r > 0$.

Notably, every point in X is one of these three.

- $A^0 = \{$all interior points of $A\}$
- $\partial A = \{$all boundary points of $A\}$
- $\overline{A} = A \cup \partial A = ((A^c)^0)^c$

Proposition 5.1. *For any $A \subset X$, we have $\partial A = \partial(A^c)$.*

Theorem 5.1. *Useful characterizations of openness:*
$A \subseteq X$ is open if $A = A^0$
$A \subseteq X$ is open if A contains none of its boundary points.
$A \subseteq X$ is open if $A \cap \partial A = \emptyset$
$A \subseteq X$ is open if $\forall x \in A$ there is some $r > 0$ s.t. $B(x;r) \subseteq A$

Theorem 5.2. *Useful characterizations of openness:*
$B \subseteq X$ is closed if $B = \overline{B}$
$B \subseteq X$ is closed if A contains all of its boundary points.
$A \subseteq X$ is closed if $\partial B \subseteq B$

Problem 5.1. *Consider $X = \mathbb{R}$ with the canonical metric $|\cdot|$. Show that (a,b) is open and $[a,b]$ is open for $a \leq b \in \mathbb{R}$, and $(a,b]$ is neither open nor closed for $a < b$.*

Proof. Let $x \in (a,b)$, i.e. $a < x < b$. Take

$$r := \text{min}\{x - a, x - b\} > 0$$

Take any $y \in (x - r, x + y)$. Then

$$y > x - r \geq a,\ y < x + r \leq b$$

Therefore

$$(x - r, x + r) \subset (a,b)$$

In other words, every $x \in (a,b)$ admits and open ball contained entirely in the interval, so (a,b) is open.

Notice that $\partial(a,b) = \{a,b\}$ and that $[a,b] = \partial(a,b) \cup (a,b)$ hence $[a,b]$ is the closure of (a,b) and thus is closed.

It is clear that $(a,b]$ is neither open nor closed as $\partial(a,b] = \{a,b\} \not\subseteq (a,b]$ and $(a,b] \cap \partial(a,b] \neq \emptyset$. $\square$