Advanced Linear Algebra

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1 Introduction

The first few sections of this collection of notes follows the public advanced linear algebra course from the math at andrews university youtube channel. The lecturer is Dr. Andrew Bosman. These notes will, later on, be for the advanced linear algebra course at the University of Oslo if I'm allowed to take the course. The University of Oslo's course follows the same book as Andrews' does; Advanced Linear and Matrix Algebra by Nathaniel Johnston. Thus these notes will be an amalgamation of the content of those two courses and the book.

2 Vector Spaces and Subspaces

Typically, when vectors are introduced in an introductory course, we look at vectors in $\mathbb{R}^n$. However, vector spaces can be much more abstract and unintuitive. For example, $\mathbb{R}$ itself, can be understood as a vector space over $\mathbb{Q}$ with an uncountably infinite set of bases. So $\mathbb{R}$ is a field, but also a vector space (which, as we'll see in the following definition, is a structure over a field). The same can be said for $\mathbb{C}$, which is a field as well as a vector space over $\mathbb{R}$. We'll also look at vector spaces of functions and all other sorts of abstract spaces.

Definition 2.1: Vector Space

A *vector space* over a field $\mathbb{F}$ is a set V equipped with two operations:

- **Vector addition:** $+ : V \times V \to V$
- **Scalar multiplication:** $\cdot : \mathbb{F} \times V \to V$

These must satisfy 8 axioms such as associativity, distributivity, identity, and existence of additive inverses.

While we will not rigorously define what a field is (that is usually first seen in a first course in abstract algebra), we can aid our intuition by listing a few interesting fields:

- $\mathbb{R}$, as mentioned above
- $\mathbb{Q}$
- $\mathbb{C}$
- $\mathbb{F}_2 = \{0, 1\}$

Example 2.1: Vector Spaces

- $\mathbb{R}^n$, $\mathbb{C}^n$, over $\mathbb{R}$ and $\mathbb{C}$, respectively
- Spaces of functions: $C([a, b])$, $\mathbb{R}^{\mathbb{N}}$
- Polynomial spaces: $\mathbb{R}[x]$

Definition 2.2: Subspace

A subset $U \subseteq V$ is a *subspace* if:

- $\overline{0} \in U$
- $u + v \in U$ for all $u, v \in U$
- $\lambda u \in U$ for all $\lambda \in \mathbb{F}, u \in U$

Note that a subspace has to be a subset which satisfies all of the axioms of a vector space, but checking these properties gives us the rest for free. In fact, checking the last property gives us $0 \in U$ for free, since $0 \in \mathbb{F}$, so that check is also technically redundant.

Example 2.2: Subspaces of $\mathbb{R}^3$

- The zero subspace: $\{0\}$
- Any line or plane through the origin
- The whole space $\mathbb{R}^3$

3 Spans and Linear Independence

Definition 3.1: Span

Given vectors $v_1, \ldots, v_k \in V$, the *span* is:

$$\operatorname{span}(v_1, \ldots, v_k) = \left\{ \sum_{i=1}^{k} \lambda_i v_i : \lambda_i \in \mathbb{F} \right\}.$$

It is the smallest subspace containing all the v_i.

Definition 3.2: Linear Independence

Vectors $v_1, \ldots, v_k$ are *linearly independent* if:

$$\sum_{i=1}^{k} \lambda_i v_i = 0 \quad \Rightarrow \quad \lambda_1 = \cdots = \lambda_k = 0.$$

Otherwise, they are *linearly dependent*.

Example 3.1: Dependence and Independence

- In $\mathbb{R}^3$, any three vectors lying in the same plane are linearly dependent.
- The standard basis vectors e_1, e_2, e_3 in $\mathbb{R}^3$ are linearly independent.

Theorem 3.1: Characterization of Dependence

A set $\{v_1, \ldots, v_k\}$ is linearly dependent if and only if some v_j lies in the span of the others.

4 Bases

Definition 4.1: Basis

A *basis* for a vector space V is a linearly independent set $\{v_1, \dots, v_n\} \subseteq V$ such that

$$\operatorname{span}(v_1, \dots, v_n) = V.$$

Definition 4.2: Dimension

The *dimension* of a vector space V, written $\dim V$, is the number of vectors in any basis for V.

Theorem 4.1: Uniqueness of Representation

If $\{v_1, \dots, v_n\}$ is a basis for V, then every $v \in V$ can be written *uniquely* as

$$v = \lambda_1 v_1 + \cdots + \lambda_n v_n.$$

Example 4.1: Standard Basis

The standard basis of $\mathbb{R}^n$ is $\{e_1, \dots, e_n\}$, where e_i has a 1 in the ith position and 0 elsewhere.

Theorem 4.2: All Bases Have Equal Size

If V has a finite basis, then all bases of V have the same number of vectors.

Sketch of Proof. Any set of more than $\dim V$ vectors is dependent, and any spanning set with fewer than $\dim V$ vectors cannot span. This leads to the conclusion that all bases must contain exactly $\dim V$ vectors. □

5 Some problems for section 2–3

Problem 5.1: Subspace Verification

Let $V = \mathbb{R}^3$ and let $U = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$. Prove that U is a subspace of V.

Proof of 5.1. Let $u, v \in U = \{(x, y, z) \in \mathbb{R}^3 : x+y+z = 0\}$. Trivially, the zero vector is in U since $\overline{0} = (0,0,0)$ and $0+0+0 = 0$. Now denote $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ and let $x \in \mathbb{R}$. Then

$$\begin{aligned} u_1 + u_2 + u_3 &= 0 \\ x \cdot u &= x \cdot u_1 + x \cdot u_2 + x \cdot u_3 \\ &= x \cdot 0 = 0 \\ &\Rightarrow x \cdot u \in U \end{aligned}$$

Next we check vector addition is closed:

$$\begin{aligned} u + v &= (u_1 + v_1, u_2 + v_2, u_3 + v_3) \\ u_1 + v_1 + u_2 + v_2 + u_3 + v_3 &= 0 + 0 \\ &= 0 \\ &\Rightarrow u + v \in U \end{aligned}$$

Thus U is a subspace of $\mathbb{R}^3$ □

Problem 5.2: Span Problem

Let $v_1 = (1,2,1)$, $v_2 = (2,4,2)$, and $v_3 = (0,1,-1)$ in $\mathbb{R}^3$. Determine whether $\text{span}(v_1, v_2, v_3) = \mathbb{R}^3$.

Solution. Note that $v_2 = 2v_1$, so v_1 and v_2 are linearly dependent and span the same line.

We must check whether v_3 lies outside the span of v_1. Suppose $v_3 = av_1$ for some $a \in \mathbb{R}$:

$$(0,1,-1) = a(1,2,1) = (a, 2a, a),$$

which implies $a = 0 \Rightarrow 1 = 0$, contradiction.

So $v_3 \notin \text{span}(v_1)$, and $\{v_1, v_3\}$ is linearly independent.

But since $\mathbb{R}^3$ is 3-dimensional, and $\text{span}(v_1, v_2, v_3) = \text{span}(v_1, v_3)$, a 2-dimensional subspace, the span is not all of $\mathbb{R}^3$.

Therefore,

$$\text{span}(v_1, v_2, v_3) \subsetneq \mathbb{R}^3.$$