Mathematical Logic

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1 Introduction

These notes are for (introductory) mathematical logic. As of writing this introduction, I have not yet been accepted to take the mathematical logic course (University of Oslo). These notes will be based on the book used in the aforementioned course, as well as resources I can find online (e.g. publicly avaliable courses like UCLA's Math 220A). Therefore the structure of these notes might be a bit scattered and disorganized.

1.1 High level overview / rambling

First-order logic is the standard way to formalize mathematics. For instance Peano arithmetic formalizes number theory, Zermelo-Fraenkel set theory formalizes set theory. In symbolic logic, with formal languages, we don't consider interpretations of symbols. Using structures we can have a notion of validity or satisfaction to logical expressions. With these, we have notions of syntax and semantics.

1.2 Gödel's Completeness Theorem

This is a fundamental result in mathematical logic which states (informally) that a formula with no free variables can be formally deduced from a given set of axioms if and only if it is valid in every structure satisfying these axioms.

2 Languages and Structures

2.1 Languages

What does $\forall x(x > 0 \Rightarrow \exists y(y \cdot y = x))$ mean? While we might recognize that this is a statement that holds in an ordered field where every positive element is a square , it is just a sequence of symbols.

Definition 2.1: First order language

A (first-order) language is a set of symbols $\mathfrak{L}$ composed of two disjoint subsets:

A. The first part (common to all languages) consists of "(" and ")" together with the following logical symbols: the set of variables $\mathcal{V} = \{v_n : n \in \mathbb{N}\}$, the equality symbol "=", connectives "¬", "∧", the existential quantifier "∃"

B. The second part, called the signature of $\mathfrak{L}$ denoted $\sigma^{\mathfrak{L}}$, consists of the non-logical symbols of $\mathfrak{L}$. It consists of:

- a set of constant symbols $C^{\mathfrak{L}}$
- a sequence of sets $\mathcal{F}_n^{\mathfrak{L}}, n \in \mathbb{N}_{\geq 1}$, where elements of this set are called the n-ary function symbols
- a sequence of sets $\mathcal{R}_n^{\mathfrak{L}}, n \in \mathbb{N}_{\geq 1}$, where elements of this set are called n-ary relation symbols (or, n-ary predicates)

The language $\mathfrak{L}$ is given by the disjoint union of these sets.

Note that the existential quantifier and logical or may be swapped for their negations: $\forall, \vee$ and nothing will change. The choice is a matter of taste. Also note that while we give practical names to the sets in $\mathfrak{L}$, we should attempt not to imbue any sort of interpretation on languages.

Now, a few *remarks*:

A. A language is always infinite. Furthermore, the logical part is countable while the non-logical can have arbitrary cardinality.

B. Abusing the notation: $\mathfrak{L}$ and $\sigma^{\mathfrak{L}}$ may be identified.

Example 2.1

$\mathfrak{L}_{\emptyset}$ the empty language (still contains logical part).
$\mathfrak{L}_{ring} = \{\underline{0}, \underline{1}, +, -, \cdot\}$ the ring language.
$\mathfrak{L}_{ord} = \{<\}$ the order language.
$\mathfrak{L}_{o.ring} := \mathfrak{L}_{ring} \cup \mathfrak{L}_{ord}$ the ordered ring language.
$\mathfrak{L}_{set} = \{\in\}$ the language of set theory.
$\mathfrak{L}_{grp} = \{\cdot, ^{-1}\}, 1$ the language of groups.
$\mathfrak{L}_{graph} = \{E\}$ the language of graphs.
$\mathfrak{L}_{ar} = \{\underline{0}, S, +, \cdot, <\}$ the language of arithmetic.

2.2 Structures

Definition 2.2: Structures

Let $\mathfrak{L}$ be a first-order language. An $\mathfrak{L}$-structure, $\mathfrak{A}$, constists of a non-empty set A (called the base set or universe of $\mathfrak{A}$ together with an element $c^{\mathfrak{A}} \in A$ for each $c \in c^{\mathfrak{L}}$, a function $f^{\mathfrak{A}} : A^n \to A$ for each $f \in F_n^{\mathfrak{L}}$, and a subset $R^{\mathfrak{A}} \subseteq A^n$ for each $R \in R_n^{\mathfrak{L}}$. We write $\mathfrak{A} = \left(A, \left(Z^{\mathfrak{A}}\right)_{Z\in\sigma^{\mathfrak{L}}}\right)$. $Z^{\mathfrak{A}}$ is called the interpretation of the symbol $Z \in \sigma^{\mathfrak{L}}$ in the structure $\mathfrak{A}$

Note that there are different ways of formally defining this and the specifics of this can safely be taken for granted later on.

Example 2.2

A. $\mathcal{N} = (\mathbb{N}, 0, \underset{x\mapsto x+1}{S}, +, \cdot, <)$ is an $\mathfrak{L}_{ar}$-structure

B. $C = (\mathbb{C}, 0, 1, +, -, \cdot)$ is an $\mathfrak{L}_{ring}$-structure

C. $R = (\mathbb{R}, 0, 1, +, -, \cdot, <)$ the ordered field of real numbers, is an $\mathfrak{L}_{o.ring}$-structures

Definition 2.3: isomorphism of structures

We say that two $\mathfrak{L}$-structures $\mathfrak{A}, \mathfrak{B}$ are isomorphic, $\mathfrak{A} \cong \mathfrak{B}$, if there exists an isomorphism $F : A \underset{\text{bijection}}{\longrightarrow} B$ between the base sets which commutes with the interpretations of the symbols $\sigma^{\mathfrak{L}}$, that is:

A. $F(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$ for every constant symbol $c \in C^{\mathfrak{L}}$

B. $F(f^{\mathfrak{A}}(a_1, \dots, a_n)) = f^{\mathfrak{B}}(F(a_1), \dots, F(a_n))$ for every function symbol $f \in \mathscr{F}_n^{\mathfrak{L}}$, and every tuple $(a_1, \dots, a_n) \in A^n$

C. $(a_1, \dots, a_n) \in R^{\mathfrak{A}} \Leftrightarrow (F(a_1), \dots, F(a_n)) \in R^{\mathfrak{B}}$ for every predicate $R \in \mathscr{R}_n^{\mathfrak{L}}$ and every tuple $(a_1, \dots, a_n) \in A^n$

3 Terms and Formulas

3.1 Terms

Definition 3.1

A word w over a set (alphabet) E is a finite string $w = a_0 a_1 \dots a_{k-1}$ with $a_i \in E$ for every i. We call k the length of w, and we denote E^* the set of all words over E.

Definition 3.2

Let $\mathfrak{L}$ be a language. The set $\mathcal{T}^{\mathfrak{L}}$ of $\mathfrak{L}$-terms is the smallest subset D of $\mathfrak{L}^*$ containing the variables and the constants of $\mathfrak{L}$, such that if $f \in F_n^{\mathfrak{L}}$ and $t_1, \dots, t_n \in D$, then $f t_1 \dots t_n \in D$.
$\Rightarrow \mathcal{T}^{\mathfrak{L}} = \bigcup_{n \in \mathbb{N}} \mathcal{T}_n^{\mathfrak{L}}$, where $\mathcal{T}_0^{\mathfrak{L}} = C^{\mathfrak{L}} \cup \mathcal{V}^{\mathscr{L}}$ and, inductively $\mathcal{T}_{n+1}^{\mathfrak{L}} = \mathcal{T}_n^{\mathfrak{L}} \cup \{f t_1 \dots t_k : k \in \mathbb{N}_{\geq 1}, f \in \mathcal{F}_k^{\mathfrak{L}}$ and $t_1, \dots, t_k \in \mathcal{T}_n^{\mathfrak{L}}\}$.

Proposition:

Any term $t \in \mathcal{T}^{\mathfrak{L}}$ satisfies one and only one of the following:

A. t is a variable

B. t is a constant symbol

C. there exists a unique integer $n \geq 1$, a unique n-ary function symbol $f \in \mathcal{F}_n^{\mathfrak{L}}$ and a unique sequence $(t_1, \dots, t_n)$ of terms such that $t = f t_1 \dots t_n$

Proof. We proceed by structural induction on the term $t \in \mathcal{T}^{\mathfrak{L}}$, as defined by the inductive construction of terms in a first-order language.

Base cases.

If t is a *variable* (i.e., $t \in \mathcal{V}$), then it satisfies condition (1).

If t is a *constant symbol* (i.e., a 0-ary function symbol), then it satisfies condition (2).

These two cases are mutually exclusive since the sets of variables and constant symbols are disjoint.

Inductive step.

Assume the statement holds for terms $t_1, \dots, t_n \in \mathcal{T}^{\mathfrak{L}}$.

Let $f \in \mathcal{F}_n^{\mathfrak{L}}$ be an n-ary function symbol for some $n \geq 1$, and define a new term

$$t = f t_1 \dots t_n.$$

Then t is neither a variable nor a constant symbol. Furthermore, the structure of the term guarantees that there exists a *unique* integer n, a unique function symbol f, and a unique sequence $(t_1, \dots, t_n)$ of terms such that

$$t = f t_1 \dots t_n,$$

due to the syntactic rules of term formation in first-order logic. Thus, condition (3) holds uniquely.

Exclusivity.

We now verify that the three cases are mutually exclusive:

- A term cannot be both a variable and a constant symbol (by definition of the syntax).

- A term cannot be a variable or constant symbol and simultaneously of the form $f t_1 \dots t_n$ for any $n \geq 1$.
- Finally, due to the unique structure of terms, a term of the form $f t_1 \dots t_n$ cannot be written in any other way.

Therefore, every term falls under exactly one of the three cases, completing the proof.

□

We introduce some *notation* here, for practical purposes: we shall often write $f(t_1, \dots, t_n)$ instead of $f t_1 \dots t_n$. When f is binary we might write $t_1 f t_2$ instead of $f t_1 t_2$. For example: $(x+y) \cdot z$ means $\cdot + xyz$.

Definition 3.3

The height of a term t, denoted $ht(t)$, is defined as the smallest natural number k such that $t \in \mathcal{T}_k^{\mathfrak{L}}$.

From this definition and the unique reading property for terms, it follows that $ht(f(t_1, \dots, t_n)) = 1 + \max\{ht(t_i) : 1 \leq i \leq n\}$.

3.2 Formulas

Definition 3.4: Atomic formula

An atomic $\mathfrak{L}$-formula is one of the following

- a word of the form $t_1 = t_2$, where t_1, t_2 are terms of the language
- a word of the form $R t_1 \dots t_n$, where $R \in \mathcal{R}_n^{\mathfrak{L}}$ and all $t_1, \dots, t_n$ are terms of the language

Then the set $Fml^{\mathfrak{L}}$ of $\mathfrak{L}$-formulas is the smallest subset D of $\mathfrak{L}^*$ that contains all atomic $\mathfrak{L}$-formulas such that if $x \in \mathcal{V}^{\mathfrak{L}}$ and $\phi, \psi \in D$, then $\neg\phi, (\phi \wedge \psi)$ and $\exists x \phi$ are all in D.
$\Rightarrow Fml^{\mathfrak{L}} = \bigcup_{n \in \mathbb{N}} Fml_n^{\mathfrak{L}}$, where $Fml_0^{\mathfrak{L}}$ is the set of atomic $\mathfrak{L}$-formulas, and inductively $Fml_{n+1}^{\mathfrak{L}} := Fml_n^{\mathfrak{L}} \cup \{\neg\phi : \phi \in Fml_n^{\mathfrak{L}}\} \cup \{(\phi \wedge \psi : \phi, \psi \in Fml_n^{\mathfrak{L}})\} \cup \{\exists x \phi : \phi \in Fml_n^{\mathfrak{L}}, x \in \mathcal{V}^{\mathfrak{L}}\}$

4 Exercises from 1.6.1

Problem 4.1: Task 4

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are two $\mathfrak{L}$-structures. We say that they are *isomorphic*, $\mathfrak{A} \cong \mathfrak{B}$, if there exists a bijection $i : A \to A$ such that for each constant symbol c of $C^{\mathfrak{L}}$, $i(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$, for each n-ary function symbol f in $\mathscr{F}_n^{\mathfrak{L}}$, and for each $(a_1, \ldots, a_n) \in A^n$, $i(f^{\mathfrak{A}}(a_1, \ldots, a_n)) = f^{\mathfrak{B}}(i(a_1), \ldots, i(a_n))$, and for each n-ary relation/predicate symbol R in $\mathscr{R}_n^{\mathfrak{L}}$, and for each $(a_1, \ldots, a_n) \in A^n$, we have $(a_1, \ldots, a_n) \in R^{\mathfrak{A}} \Leftrightarrow (i(a_1), \ldots, i(a_n)) \in R^{\mathfrak{B}}$.

A Show that $\cong$ is an equivalence relation.

B Find two different structures for a particular language and prove that they are not isomorphic.

Proof of A. Let $\mathfrak{L}$ be an arbitrary language, and $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be $\mathfrak{L}$-structures.

First we show that $\mathfrak{A} \cong \mathfrak{A}$, i.e. $\cong$ is reflexive. Let $i : A \to A$ be the identity map $i(a) = a$. It is trivial to show that i is bijective, being it's own inverse. Then we have $i(c^{\mathfrak{A}}) = c^{\mathfrak{A}}$ satisfying the first property of structure isomorphisms. Furthermore, if we take f to be an n-ary function symbol of the language, and $(a_1, \ldots, a_n) \in A^n$ we get that $i(f^{\mathfrak{A}}(a_1, \ldots, a_n)) = f^{\mathfrak{A}}(a_1, \ldots, a_n)$. Now recognize that $i(a_j) = a_j$ for all j and we find that $i(f^{\mathfrak{A}}(a_1, \ldots, a_n)) = f^{\mathfrak{A}}(i(a_1), \ldots, i(a_n))$, showing that the second property holds. Lastly, with R being some n-ary relation symbol in the language and $(a_1, \ldots, a_n) \in A^n$, we have, through a similar argument, $(a_1, \ldots, a_n) \in R^{\mathfrak{A}} \Leftrightarrow (i(a_1), \ldots, i(a_n)) \in R^{\mathfrak{A}}$. Thus $\mathfrak{A} \cong \mathfrak{A}$.

Second we show the $\cong$ to be symmetric. Supposing $i : A \to B$ is an isomorphism there exists an inverse $j := i^{-1} : B \to A$, since i is bijective. Recognize also that j is bijective since the inverse of a bijection is itself a bijection. Suppose then that $\mathfrak{A} \cong \mathfrak{B}$ via i. Then

$$
\begin{aligned}
i(c^{\mathfrak{A}}) &= c^{\mathfrak{B}} \\
j(i(c^{\mathfrak{A}})) &= j(c^{\mathfrak{B}}) \\
j \circ i &= id \\
\Rightarrow c^{\mathfrak{A}} &= j(c^{\mathfrak{B}})
\end{aligned}
$$

Next we consider some function symbol f and $(a_1, \ldots, a_n \in A^n$ where

$$
\begin{aligned}
i(f^{\mathfrak{A}}(a_1, \ldots, a_n)) &= f^{\mathfrak{B}}(i(a_1), \ldots, i(a_n)) \\
j(i(f^{\mathfrak{A}}(a_1, \ldots, a_n))) &= j(f^{\mathfrak{B}}(i(a_1), \ldots, i(a_n))) \\
\Rightarrow f^{\mathfrak{A}}(a_1, \ldots, a_n) &= j(f^{\mathfrak{B}}(i(a_1), \ldots, i(a_n)))
\end{aligned}
$$

Now since j is the inverse of i, we have

$$
a_k \overset{j}{\underset{i}{\leftrightarrow}} b_k
$$

In other words there is, for each $a_k \in A$ some $b_k \in A$ such that i sends $a_k \to b_k$ and j sends $b_k \to a_k$. Thus

$$
f^{\mathfrak{A}}(j(b_1), \ldots, j(b_n)) = j(f^{\mathfrak{B}}(b_1, \ldots, b_n))
$$

Next we consider some relation symbol and a tuple from A^n where, through a similar argument show that $(a_1, \ldots, a_n) \in R^{\mathfrak{A}} \Leftrightarrow (i(a_1), \ldots, i(a_n)) \in R^{\mathfrak{B}}$. Specifically, by applying j on all of the entries and substituting $j(a_k) = b_k$. Then, by the virtue of j being the i-inverse, it is trivial to show that the implication holds the other way, also.

Lastly we show transitivity of $\cong$. Suppose $\mathfrak{A} \underset{i}{\cong} \mathfrak{B}$ and $\mathfrak{B} \underset{j}{\cong} \mathfrak{C}$. Let $k := j \circ i : A \to C$, remembering that the composition of two bijections is a bijection. Then, for constant symbols $c \in C^{\mathfrak{L}}$, we have $k(c^{\mathfrak{A}}) = j(i(c^{\mathfrak{A}})) = j(c^{\mathfrak{B}}) = c^{\mathfrak{C}}$. Now consider some n-ary function symbol f from $\mathcal{F}_n^{\mathfrak{L}}$ and $(a_1, \ldots, a_n) \in A^n$.

$$\begin{aligned}
k(f^{\mathfrak{A}}(a_1, \ldots, a_n)) &= j(i(f^{\mathfrak{A}}(a_1, \ldots, a_n))) \\
&= j(f^{\mathfrak{B}}(i(a_1), \ldots, i(a_n))) \\
&= f^{\mathfrak{C}}(j(i(a_1)), \ldots, j(i(a_n))) \\
&= f^{\mathfrak{C}}(k(a_1), \ldots, k(a_n))
\end{aligned}$$

Next consider some n-ary relation symbol R and again $(a_1, \ldots, a_n) \in A^n$. We are given, $(a_1, \ldots, a_n) \in R^{\mathfrak{A}} \Leftrightarrow (i(a_1), \ldots, i(a_n)) \in R^{\mathfrak{B}}$ and $(b_1, \ldots, b_n) \in R^{\mathfrak{B}} \Leftrightarrow (j(b_1), \ldots, j(b_n)) \in R^{\mathfrak{C}}$. Recall how there is, via the bijective i, some b_k for a_k.

Then $(i(a_1), \ldots, i(a_n)) \in R^{\mathfrak{B}} \Rightarrow (j(i(a_1)), \ldots, j(i(a_n))) = (k(a_1), \ldots, k(a_n)) \in R^{\mathfrak{C}}$ Through this substitution we have $(a_1, \ldots, a_n) \in R^{\mathfrak{A}} \Rightarrow (k(a_1), \ldots, k(a_n) \in R^{\mathfrak{C}}$. By composing the inverses of i and j we can show the right-to-left implication just as easily by substituting in the same way.

Thus we have shown transitivity of $\cong$, and furthermore $\cong$ is an equivalence relation (reflexive, symmetric, transitive).

$\square$

Solution to B. Let $\mathfrak{L}_{NT} = \{0, S, +, \cdot, E, <\}$ be the language of number theory.

Let $\mathcal{N}, \mathfrak{A}$ be $\mathfrak{L}_{NT}$-structures. More specifically, let $\mathcal{N}$ be the usual structure which models number theory. In $\mathfrak{A}$, let $a +^{\mathfrak{A}} b = a$ for all $a, b \in A$. To show that two structures like these cannot be isomorphic we proceed by contradiction, supposing there does exist an isomorphism $i : \mathbb{N} \to A$.

i is some function that, among other properties, satisfies $i(n+m) = i(n) +^{\mathfrak{A}} i(m)$, much like a group homomorphism. Now since $i(n) +^{\mathfrak{A}} i(m) = i(n)$ we have

$$i(n+m) = i(n)$$

Choose $n = 1 = m$ and we get

$$i(2) = i(1)$$

which contradicts injectivity of i.