Lecture Notes: Quotient Groups & Normal Subgroups (Course By: Alvaro Lozano-Robledo)

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Normal Subgroups

Definition 1. *Let G be a group and N be a subgroup of G. We say that N is a* ***normal subgroup*** *of G, written as $N \triangleleft G$, if it satisfies the condition*

$$gN = Ng \quad \text{for all } g \in G.$$

Equivalently, N is normal in G if for all $g \in G$ and $n \in N$, we have

$$gng^{-1} \in N.$$

This condition ensures that the left and right cosets of N in G are the same, which allows the construction of the quotient group G/N.

Proposition 1. *Let G be an abelian group, N any subgroup of G. Then $N \triangleleft G$.*

Proof. Let G be an abelian group and N a subgroup of G. The left cosets of N are then $gN = \{gn : n \in N\}$ for some $g \in G$. Since G is abelian and $n \in G \wedge g \in G$ we get

$$gn = ng$$

for any arbitrary $g \in G$. Thus

$$gN = \{gn : n \in N\} = \{ng : n \in N\} = Ng, \quad \forall g \in G$$

□

Quotient Group

Theorem 1. *Let* $G/N = \{gN : g \in G\}$ *respecting* $gN \star g'N = (gg')N$ *is a group with* $[G : N]$ *elements, where* $[G : N]$ *is the index of* N *in* G. $[G : N]$ *is the* n *in* $G = \bigsqcup_{i=1}^{n} g_i N$.

Proof. Assume $N \triangleleft G$. $\#G/N = \#\{gN : g \in G\} =$ the number of distinct cosets of $N = [G : N]$. $(G/N, \star)$ is a group:
$\star$ **well-defined:**

$$aN = a'N$$
$$bN = b'N$$
$$\Rightarrow a' = a \cdot n_1 \wedge b' = b \cdot n_2, n_1, n_2 \in N$$
$$aNbN = a'Nb'N = a'b'N$$
$$aNbN = abN$$
$$a'b'N = a \cdot n_1 b \cdot n_2 N = abN, \because n_1, n_2 \in N$$

$\star$ **associative:**

$$aN \star (bN \star cN) = aN \star (bcN) = abcN$$
$$= (ab)cN = (abN) \star cN = (aN \star bN) \star cN$$

Identity: "e" $= eN = N$

$$aN \star eN = (ae)N = aN$$
$$eN \star aN = (ea)N = aN$$

Inverses:

$$(gN)^{-1} := g^{-1}N$$
$$aN \star a^{-1}N = (aa^{-1})N = eN = N$$
$$a^{-1}N \star aN = (a^{-1}a)N = eN = N$$

Thus $(G/N, \star)$ is a group.

□

Corollary 1. *If* G *is an abelian group and* $N \subseteq G$ *is any subgroup. Then,* G/N *is an abelian group.*

Proof. Since G is abelian $\Rightarrow N \triangleleft G$ is a normal subgroup. By theorem 1 G/N is a group. Now since G is abelian we have

$$\forall aN, bN \in G/N : aN \star bN = (ab)N = (baN) = bN \star aN$$

Thus the quotient group of an abelian group G with any subgroup N is an abelian group. □

Definition 2. *Let* G *be a group and* N *a normal subgroup of* G *(i.e.,* $N \triangleleft G$*). The* ***quotient group*** *or* ***factor group*** G/N *is the set of left cosets of* N *in* G*, defined as*

$$G/N = \{gN \mid g \in G\}.$$

The group operation on G/N *is given by*

$$(gN)(hN) = (gh)N \quad \text{for all } g, h \in G.$$

This operation is well-defined, and G/N *forms a group under this operation.*