Lecture Notes: Real Analysis — Uncountability of Real Numbers (Course: MIT 18.100A)

Thobias K. Høivik

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Proving Uncountability

Theorem 1 (Triangle Inequality)**.** $\forall x, y \in \mathbb{R} : |x + y| \leq |x| + |y|$

Proof. By the definition of absolute value, for any $x, y \in \mathbb{R}$, we have:

$$-|x| \leq x \leq |x|$$

$$-|y| \leq y \leq |y|$$

Adding these inequalities together:

$$-(|x| + |y|) \leq x + y \leq |x| + |y|$$

By the definition of absolute value, this implies:

$$|x + y| \leq |x| + |y|$$

which proves the theorem. □

Definition 1. *Let $x \in (0, 1]$ and let $d_{-j} \in \{0, 1, \ldots, 9\}$ for $j \in \mathbb{N}$. We say x is represented by the digits $\{d_{-j} : j \in \mathbb{N}\}$, $x = 0.d_{-1}d_{-2}\ldots$ if $x = \sup\{d_{-1}10^{-1} + \cdots + d_{-n}10^{-n} | n \in \mathbb{N}\}$.*

Example:

$$\begin{aligned} 0.25000 &= \sup\{\frac{2}{10}, \frac{2}{10} + \frac{5}{100}, \frac{2}{10} + \frac{5}{100} + \frac{0}{1000}, \ldots\} \\ &= \sup\{\frac{2}{10}, \frac{25}{100}\} = \frac{1}{4} \end{aligned}$$

Theorem 2.

1. *For every set of digits $\{d_{-j} : j \in \mathbb{N}\}$ with $d_j \in \{0, 1, 2, \ldots, 9\}$, there exists a unique $x \in [0, 1]$ such that $x = 0.d_{-1}d_{-2}d_{-3}\ldots$*

2. *$\forall x \in (0, 1], \exists! \{d_{-j} : j \in \mathbb{N}\}$ such that $x = 0.d_{-1}d_{-2}\ldots$ and $0.d_{-1}\ldots d_{-n} < x \leq 0.d_{-1}\ldots d_{-n} + 10^{-n}$.*

Theorem 3 (Cantor)**.** $(0,1]$ *is uncountable.*

Proof. Assume, by contradiction, that $|(0,1]| = |\mathbb{N}|$. Then $\exists f : \mathbb{N} \to (0,1]$ which is bijective. $\forall n$, write $f(n) = 0.d^n_{-1}d^n_{-2}\dots$ satisfying $f(n) \leq 0.d^n_{-1}d^n_{-2}\dots + 10^{-n}$.
Let

$$e_{-j} = \begin{cases} 1 \text{ if } d^j_{-j} \neq 1 \\ 2 \text{ if } d^j_{-j} = 1 \end{cases}$$

By 1) of the previous theorem $\exists! y \in (0,1]$ such that $y = 0.e_{-1}e_{-2}\dots$. Since all e_{-j} are either 1 or 2, they are non-zero. $\forall n \in \mathbb{N} : 0.e_{-1}e_{-2}\dots e_{-n} < y \leq 0.e_{-1}\dots e_{-n} + 10^{-n}$. y is then the unique decimal representation of this number from 2) in the previous theorem. Since f is surjective, $\exists m \in \mathbb{N}$ such that $y = f(m)$.
Then

$$d^m_{-m} = e_{-m} = \begin{cases} 1 \text{ if } d^m_{-m} \neq 1 \\ 2 \text{ if } d^m_{-m} = 1 \end{cases} \neq d^m_{-m}$$

Thus we have arrived at a contradiction so our assumption that $(0,1]$ is countable is false. ☐

Sequences and Series

Definition 2. *A sequence of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$. We denote $f(n)$ with X_n and the sequence by $\{X_n\}_{n=1}^{\infty}$ or $\{X_n\}$ or otherwise $X_1, X_2, X_3, \dots$*
We may also write a_n.

Example: $\{\frac{1}{n}\} = 1, \frac{1}{2}, \frac{1}{4}, \dots$

Definition 3. *A sequence X_n is bounded if $\exists b \geq 0$ such that $\forall n \in \mathbb{N} : |X_n| \leq B$.*

Example: $\{\frac{1}{n}\}$ is bounded by 1 because it is never larger than 1 for any natural number n.
Non-Example: $\{n\}$ is not bounded as it will just grow and grow for increasing n.

Proof. Let $B \geq 0$. By Archimedes Property $\exists n \in \mathbb{N} : n > B$. Therefore $\{n\}$ is unbounded. ☐

Definition 4. *A sequence $\{X_n\}$ converges to $X \in \mathbb{R}$ if $\forall \mathcal{E} > 0, \exists M \in \mathbb{N} : \forall n \geq M$, $|X_n - X| < \mathcal{E}$. If a sequence converges we say it's convergent, otherwise it is divergent.*

Theorem 4. *If $x, y \in \mathbb{R}$ and $\forall \mathcal{E} > 0, |x - y| < \mathcal{E} \Rightarrow x = y$.*

Proof. Suppose $x, y \in \mathbb{R}$ and $\forall \mathcal{E} > 0, |x-y| < \mathcal{E}$, $x \neq y$. Then $|x - y| > 0$. Then by $|x - y| < \frac{|x-y|}{2} \Rightarrow \frac{1}{2}|x-y| < 0 \Rightarrow |x-y| < 0$ which is not possible. ☐

Theorem 5. *If $\{X_n\}$ converges to x and y, then $x = y$.*

Proof. Let (a_n) be a sequence such that:

$$a_n \to x \quad \text{and} \quad a_n \to y$$

By the definition of convergence, for any $\epsilon > 0$, there exists a positive integer N_1 such that for all $n \geq N_1$:

$$|a_n - x| < \epsilon$$

Similarly, since $a_n \to y$, for the same $\epsilon > 0$, there exists a positive integer N_2 such that for all $n \geq N_2$:

$$|a_n - y| < \epsilon$$

Let $N = \max(N_1, N_2)$. Then, for all $n \geq N$, we have both:

$$|a_n - x| < \epsilon \quad \text{and} \quad |a_n - y| < \epsilon$$

Now, using the triangle inequality, we obtain:

$$|x - y| = |(x - a_n) + (a_n - y)| \leq |x - a_n| + |a_n - y|$$

For $n \geq N$, we know:

$$|x - y| \leq |x - a_n| + |a_n - y| < \epsilon + \epsilon = 2\epsilon$$

Since ϵ is arbitrary, we can make ϵ as small as we want. Thus, for any $\epsilon > 0$, we have:

$$|x - y| \leq 2\epsilon$$

Taking the limit as $\epsilon \to 0$, we get:

$$|x - y| = 0$$

Therefore, $x = y$.

$\square$