Killing an ant with an ICBM

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The following is a proof that the n-th root of 2 is irrational for $n \geq 2$, inspired by a similar proof that $\sqrt[3]{2}$ is irrational I saw on YT shorts.

Proof. Let $n \geq 2$. Then

$$\sqrt[n]{2} \text{ is irrational}$$

Case 1 ($n = 2$).

We proceed in the usual way. Assume, for a contradiction, that the square-root of 2 is rational.

$$\begin{aligned}
\sqrt{2} &= \frac{a}{b} \\
\text{where } a, b &\in \mathbb{Z}, \text{ and } gcd(a,b) = 1 \\
2 &= \frac{a^2}{b^2} \\
2b^2 &= a^2 \\
\Rightarrow a &= 2k, k \in \mathbb{Z} \\
2b^2 &= 4k^2 \\
b^2 &= 2k^2 \\
\Rightarrow b &= 2j, j \in \mathbb{Z}
\end{aligned}$$

Now we see that a and b share a factor larger than 1, contradicting our assumption. Hence $\sqrt{2}$ cannot be a rational number.

Case 2 ($n > 2$).

Proceed as above, assuming the n-th root is rational.

$$\begin{aligned}
\sqrt[n]{2} &= \frac{a}{b}, a, b \in \mathbb{Z} \\
2 &= \frac{a^n}{b^n} \\
2b^n &= a^n \\
b^n + b^n &= a^n
\end{aligned}$$

Which has no non-zero integer solutions (which we would require since $b \neq 0$) by Fermat's Last Theorem [1], a contradiction. Thus $\sqrt[n]{2}$ cannot be rational.

$\square$

References

[1] Wiles, A. (1995). Modular elliptic curves and Fermat's Last Theorem. *Annals of Mathematics*, 141(3), 443–551.