Lecture Notes: Real Analysis — Set Theory Revision and Least Upper Bound Property (Course: MIT 18.100A)

Thobias K. Høivik

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Cardinalities

Theorem 1 (Cantor's Theorem)**.** *If A is a set, then $|A| < |P(A)|$.*

Proof. Let A be a set. Define $f : A \to P(A)$ by $f(x) = \{x\}$. We can see that f is injective:

$$f(x) = y(x)$$
$$\{x\} = \{y\}$$
$$x = y$$

thus f is injective which implies

$$\boxed{|A| \leq |P(A)|}$$

Now we shall show that $|A| \neq |P(A)|$. By contradiction suppose $|A| = |P(A)|$. Then $\exists$ bijective function $g : A \to P(A)$. Define a subset $B \subset A$ by $B = \{x \in A : x \notin g(x)\}$. Then by the definition of the powerset we have $B \in P(A)$. Since g is surjective, $\exists b \in A$ such that $g(b) = B$.
Case 1:

$$b \in g(b) = B \Rightarrow b \in B$$
$$\Rightarrow b \notin g(b)$$

Case 2:

$$b \notin g(b) \Rightarrow b \in B$$
$$b \in g(b)$$

Thus we have arrived at a contradiction $b \in g(b) \Leftrightarrow b \notin g(b)$ by our assumption and so it cannot be true. Thus

$$\boxed{|A| \neq |P(A)|}$$

$$|A| \leq |P(A)| \wedge |A| \neq |P(A)|$$
$$\Rightarrow \boxed{|A| < |P(A)|}$$

□

The Reals and Order Relations

We will now restate the definition of the real numbers, slightly differently from before, as a theorem rather than a definition.

Theorem 2 (Real Numbers)**.** *There exists a unique ordered field containing $\mathbb{Q}$ with the least upper bound property, which we denote by $\mathbb{R}$.*

We shall not go through the process of proving this theorem at this time, but rather explore the properties of $\mathbb{R}$.

Ordered Sets

Definition 1 (Ordered Set)**.** *An ordered set is a set S with a relation $<$ such that*

1. $\forall x, y \in S : x = y \vee x < y \vee y < x$
2. *If $x < y$ and $y < z$ then $x < z$, i.e the relation is transitive.*

Examples:
For $m, n \in \mathbb{Z}$, $m < n$ if $m - n \in \mathbb{Z}$.
For $q, r \in \mathbb{Q}$, $q < r$ if $\exists m, n \in \mathbb{N}$ s.t $r - q = \frac{m}{n}$.
Non-Example:
Let S be a subset $S = P(\mathbb{N})$. We define the relation R, where $(A, B) \in R$ if $A \subset B$. Clearly this is transitive, but property one is not satisfied because, for example $\{0\} \neq \{1\}$, and neither $(\{0\}, \{1\}) \notin R$ nor $(\{1\}, \{0\}) \notin R$.

Bounds

Recall the concepts of bounds, supremum and infimum.
Examples:
Let $S = \mathbb{Z}, E = \{-1, 0, 2\} \subset S$. Examples of upper- and lower bounds would be $2, 3, 4, \ldots$, and $\ldots, -2, -1$, respectively. The supremum $sup(E)$ would be 2 and the infimum $inf(E)$ would be -1.
Let $S = \mathbb{Q}$ and $E = \{q \in \mathbb{Q} : 0 \leq q \leq 1\}$. Upper bounds: $1, \frac{3}{2}, \frac{5}{3}, \ldots$, and lower bounds: $0, -1, -\frac{2}{3}, \ldots$. Supremum $sup(E) = 1$ and infimum $inf(E) = 0$.

Least Upper Bound Property

Definition 2. *An ordered set S has the least upper bound property if every nonempty $E \subset S$ which is bounded above has a supremum in S.*

A trivial example of this is $S = \{0\}$ where every nonzero subset is the set itself and so every supremum lies in S.
Example:
$S = \{-1, -2, -3, \ldots\}$ with the regular ordering relation found in the integers. If $E \subset S$, E nonempty then $-E = \{-x : x \in E\} \subset \mathbb{N}$. By the well-ordering property of the natural

numbers, there exists $m \in -E$ such that $\forall x \in E : m \leq -x \Rightarrow -m \in E$ and $\forall x \in E : x \leq -m \Rightarrow -m$ is the supremum of E.

Proposition 1. $\mathbb{Q}$ *does not have the least upper bound property.*

Proof. Consider the set

$$E = \{q \in \mathbb{Q} \mid q > 0 \text{ and } q^2 < 2\}.$$

The set E is nonempty since, for example, $q = 1$ belongs to E. Moreover, E is bounded above in $\mathbb{Q}$ by any rational number greater than $\sqrt{2}$, such as $r = 2$.
We claim that E has no least upper bound in $\mathbb{Q}$. First, observe that for all $q \in E$, we have $q < \sqrt{2}$, so any least upper bound of E in $\mathbb{R}$ must be $\sup(E) = \sqrt{2}$. However, since $\sqrt{2} \notin \mathbb{Q}$, $\sup(E)$ cannot be a rational number.
Suppose for contradiction that E has a least upper bound $s \in \mathbb{Q}$. Then $s \geq q$ for all $q \in E$, and for any $\mathcal{E} > 0$, there exists some $q \in E$ with $s - \epsilon < q$. If $s = \sup(E)$, then we must have $s^2 = 2$. However, no rational number satisfies this equation, contradicting our assumption that s is rational. Thus, E has no least upper bound in $\mathbb{Q}$, proving that $\mathbb{Q}$ does not satisfy the least upper bound property. $\square$