Lecture Notes: Real Analysis — The Archimedian Property, Density of the Rationals, and Absolute Value (Course: MIT 18.100A)

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$\mathbb{R}$ is the unique ordered field with the least upper bound property containing $\mathbb{Q}$.
Simple fact: If $x, y \in \mathbb{R} \wedge x < y \Rightarrow \exists r \in \mathbb{R}$ such that $x < r < y$, for instance $r = \frac{x+y}{2}$. Then, we ask the question: $x, y \in \mathbb{R} \wedge x < y$, then, does there exist $r \in \mathbb{Q}$ such that $x < r < y$?

Theorem 1 (The Archimedian Property)**.** *If $x, y \in \mathbb{R}$ and $x > 0$ then $\exists n \in \mathbb{N}$ such that $nx > y$.*

Theorem 2 (Density of the Rationals (the answer))**.** *If $x, y \in \mathbb{R}$ and $x < y$ then $\exists r \in \mathbb{Q}$ such that $x < r < y$*

Proof (Archimedian Property). We prove this by contradiction.

1. **Assume the negation**: Suppose the claim is false. Then there exists $x, y \in \mathbb{R}$ with $x > 0$ such that for every $n \in \mathbb{N}$,
$$nx \leq y.$$
This means that the set
$$S = \{nx \mid n \in \mathbb{N}\}$$
is bounded above by y.

2. **Apply the Least Upper Bound (LUB) property**: Since $\mathbb{R}$ has the *Least Upper Bound Property*, the set S has a least upper bound, say b_0, i.e.,
$$b_0 = \sup S.$$
By the definition of supremum, for every $\varepsilon > 0$, there exists some $n \in \mathbb{N}$ such that
$$b_0 - \varepsilon < nx \leq b_0.$$

3. **Derive a contradiction**: Consider $b_0 - x$. Since b_0 is the least upper bound, it must be that
$$b_0 - x < nx$$

for some $n \in \mathbb{N}$, implying

$$(n+1)x = nx + x > b_0.$$

But this contradicts the assumption that b_0 is an upper bound for S, because $(n+1)x \in S$ and should not exceed b_0.

Thus, the assumption was false, proving that for any $x > 0$ and any $y \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that

$$nx > y.$$

□

Proof (Density of the Rationals). Suppose $x, y \in \mathbb{R}, x < y$. Then, there are three cases:

1. $x < 0 < y$
2. $0 \leq x < y$
3. $x < y \leq 0$

Case 1: Choose $r = 0 \in \mathbb{Q}$
Case 2: Suppose $0 \leq x < y$. By the archimedian property $\exists n \in \mathbb{N}$ such that $n(y-x) > 1$. Then, by the archimedian property again $\exists l \in \mathbb{N}$ such that $l > nx$.
Thus, $S = \{k \in \mathbb{N} : k > nx\} \neq \emptyset$. The well-ordering property of $\mathbb{N}$, S has least element m. Since $m \in S \Rightarrow nx < m$. Since m is the least element of S, $m - 1 \notin S \Rightarrow m - 1 \leq nx \Rightarrow m \leq nx + 1$. Then, $nx < m \leq nx + 1 < ny \Rightarrow x < \frac{m}{n} < y$, so $r = \frac{m}{n} \in \mathbb{Q}$.
Case 3: Suppose $x < y \leq 0$. Then, $0 \leq -y < x$. By case 2, $\exists r' \in \mathbb{Q}$ such that $-y < r' < -x$. Then, multiplying by -1 gives $x < -r' < y$ so $r = r'$. □

Theorem 3. *Assume $S \subset \mathbb{R}$ is nonempty and bounded above, i.e $\exists \sup S$. Then $x = \sup S \Leftrightarrow$*

1. *x is an upper bound of S*
2. *$\forall \mathcal{E} > 0, \exists y \in S$ s.t $x - \mathcal{E} < y \leq x$*

Proof. Suppose $S \subset \mathbb{R}$ nonempty and bounded above. First we prove the left-to-right implication: Assume x is the supremum of S. Then x is an upper bound of S. Let $\mathcal{E} > 0$. Then, $x - \mathcal{E} \in \mathbb{R}$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists y \in \mathbb{Q}$ such that $x - \mathcal{E} < y < x \Rightarrow x - \mathcal{E} < y \leq x$.
Now for the right-to-left implication:
Suppose x is an upper bound of S and that $\forall \mathcal{E} > 0, \exists y \in S$ s.t $x - \mathcal{E} < y \leq x$, but x is not the supremum of S. Then $\exists z \in \mathbb{R}$ s.t $x - \mathcal{E} < y \leq z < x$. Since $z < x \Rightarrow x - z > 0$. Choose $\mathcal{E} = x - z$. Then, $x - (x - z) < y \leq z < x \Rightarrow z < y \leq z < x$, which contradicts our assumption that $z < x$ so our assumption must be false and properties 1 and 2 must imply that x is the supremum of S.
Thus, we have shown that x is the supremum of S if and only if x is an upper bound of S and for any value arbitrarily close to x there exists some value y in between them.

□

Remark: There is an analogous statement for the infimum of a set S, which works in the same way.

Theorem 4. $\sup\{1 - \frac{1}{n} : n \in \mathbb{N}\} = 1$

Proof. Note $1 - \frac{1}{n} < 1, \forall n \in \mathbb{N}$ so 1 is an upper bound. Let $\mathcal{E} > 0$. Then by the archimedian principle $\exists n \in \mathbb{N}$ such that $\frac{1}{\mathcal{E}} < n$. Then, $1 - \mathcal{E} < 1 - \frac{1}{n} < 1$. Thus $\sup\{1 - \frac{1}{n} : n \in \mathbb{N}\} = 1$. □

Definition 1. *For* $x \in \mathbb{R}, A \subset \mathbb{R}$ *we define* $x + A = \{x + a : a \in A\}$, $x \cdot A = \{x \cdot a : a \in A\}$.

Theorem 5. *If* $x \in \mathbb{R}$ *and* A *is bounded above then* $x + A$ *is bounded above and* $\sup(x + A) = x + \sup A$.

Proof. Suppose $x \in \mathbb{R}$ and A is bounded above with $\sup A = b_0$. Then $\forall a \in A : a \leq b$, where b is any arbitrary upper bound of A. $b_0 \leq b \Rightarrow a \leq b_0 \Rightarrow x + a \leq x + b_0$. So, $x + b_0$ is an upper bound, but we can more concretely show that it is in fact the l.u.b. Suppose $x + b_0$ is not in fact the least upper bound. Then, $\exists c < x + b_0$ a smaller upper bound. However, subtracting x $c - x < b_0$ which means that b_0 is not the least upper bound of A anymore with this assumption, and thus it must be false and $x + b_0$ is in fact the l.u.b. Thus $\sup(x + A) = x + \sup A$. □

Note that in the proof of **Theorem 5** we could also (and it may have been more rigorous to) use the epsilon definition to prove we had indeed found the l.u.b.

Theorem 6. *If* $x \in \mathbb{R}$ *and* A *is bounded above then* $x \cdot A$ *is bounded above and* $\sup(x \cdot A) = x \cdot \sup A$.

This is proved in a similar fashion as in **Theorem 5**, one just needs to find some epsilon that will make it magically work.

Theorem 7. *If* $A, B \subset \mathbb{R}$ *with* A *bounded above and* B *bounded below and* $\forall x \in A, \forall y \in B$ *if* $x \leq y$ *then* $\sup A \leq \inf B$.

Proof. Let $y \in B$. Then $\forall x \in A : x \leq y \Rightarrow y$ is an upper bound of A and therefore $\sup A \leq y$. Thus, $\sup A$ is a lower bound of B $\Rightarrow \sup A \leq \inf B$.

□

Definition 2 (Absolute Value)**.** *If* $x \in \mathbb{R}$

$$|x| = \begin{cases} x \ \textit{if} \ x \geq 0 \\ -x \ \textit{if} \ x < 0 \end{cases}$$