Lecture Notes: Abstract Algebra - Isomorphisms (Course By: Alvaro Lozano-Robledo)

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Definition 1. *Two groups* $(G, \star_G)$ *and* $(H, \star_H)$ *are said to be* ***isomorphic*** $\Leftrightarrow G \cong H$ *if there exists a bijective function*

$$\phi : G \to H$$

that preserves the group operation, i.e., $\forall a, b \in G$*,*

$$\phi(a \star_G b) = \phi(a) \star_H \phi(b).$$

A function

$$\psi : G \to H$$

that preserves the group operation, but is not necessarily bijective is called a ***homomorphism****. In other words an* ***isomorphism*** *is a bijective* ***homomorphism****.*

Example

Show $(\mathbb{Z}/_4\mathbb{Z}, +) \cong (\mu_4, \times)$ where $\mu_4 = \{1, -1, i, -i\}$ are the 4-th roots of unity in $\mathbb{C}^x$.

Proof. Let $\phi : \mathbb{Z}/_4\mathbb{Z} \to \mu_4$, $\phi(n) = i^n$.

$$\begin{gathered}\phi(1) = i \\ \phi(2) = i^2 = -1 \\ \phi(3) = i^3 = -i \\ \phi(0) = i^0 = 1\end{gathered}$$

thus it is injective and surjective $\Leftrightarrow$ it is bijective, and

$$\phi(m + n) = i^{m+n} = i^m \times i^n$$

so ϕ is a structure preserving bijection, making it an isomorphism. $\square$

Another Example

Let

$$F = C^{\infty}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f^{(n)} \text{ exists for all } n \in \mathbb{Z}_{\geq 0}\}$$

be the space of infinitely differentiable functions. Define addition as $(f+g)(x) = f(x)+g(x)$ for all $f, g \in F$. Show that $F \cong F$ via the mapping

$$\phi : F \to F, \quad \phi(f)(x) = \int_0^x f(t)dt.$$

Proof. We set out to prove that ϕ is a bijective homomorphism. Let $f, g \in F$,

1: Homomorphism

$$\phi(f+g)(x) = \int_0^x f(t) + g(t)dt$$
$$= \int_0^x f(t)dt + \int_0^x g(t)dt = \phi(f)(x) + \phi(g)(x)$$

2: Injectivity

$$\phi(f)(x) = \phi(g)(x)$$
$$\int_0^x f(t)dt = \int_0^x g(t)dt$$
$$\frac{d}{dx}\int_0^x f(t)dt = \frac{d}{dx}\int_0^x g(t)dt$$

By the fundemental theorem of calculus:

$$f(x) = g(x)$$

3: Surjectivity

$$\phi(f)(x) = g(x)$$
$$\int_0^x f(t)dt = g(x)$$

By the fundemental theorem of calculus:

$$\frac{d}{dx}\int_0^x f(t)dt = f(x)$$
$$\Rightarrow f(x) = \frac{d}{dx}g(x)$$
$$\forall g \in F : g' \in F, \quad \therefore f \in F$$

Thus we have shown ϕ to be a bijective homomorphism, completing the proof.

□

Lemma 1. *Let $\phi : G \to H$ be an isomorphism of groups. Then $\phi(e_G) = e_H$.*

Proof. Let $\phi(e_G) = h \in H$. Also

$$\phi(e_G) = \phi(e_G \star_G e_G) = \phi(e_G) \star_H \phi(e_G) = h \star_H h$$

So $h = h \star_H h$ and so

$$h \star_H h^-1 = h^-1 \star_H (h \star_H h) = h$$

Hence $h = e_H$.

$\square$

Example

Prove that $\mathbb{Z}/_4\mathbb{Z} \ncong \mathbb{Z}/_2\mathbb{Z} \times \mathbb{Z}/_2\mathbb{Z}$.

Proof. Suppose there is an isomorphism $\phi : \mathbb{Z}/_4\mathbb{Z} \to \mathbb{Z}/_2\mathbb{Z} \times \mathbb{Z}/_2\mathbb{Z}$. Then

$$\exists g \in \mathbb{Z}/_2\mathbb{Z} \times \mathbb{Z}/_2\mathbb{Z} \text{ s.t } \phi(g) = 1 \bmod 4$$

If so,

$$\phi(g + g) = \phi(g) + \phi(g) = 1 + 1 = 2 \bmod 4,$$

but

$$g + g = 2g = (a, b) + (a, b) = (2a, 2b) \equiv (0, 0) \bmod 2$$
$$\phi(2g) = \phi(0) = 0 \not\equiv 2 \bmod 4$$

Thus we get

$$\phi(g + g) = 2 \bmod 4 \wedge \phi(g + g) = 0 \Rightarrow \perp,$$

a contradiction. So an isomorphism cannot exist between these groups, hence they are not isomorphic. $\square$