Mat210 Advanced Discrete Mathematics Notes

Thobias Høivik

Fall 2025

Contents

1 Pre-Semester Start – Cardinality

The following chapter contains notes based on what I think the course will cover in the first week (week 33). According to the syllabus, cardinality is mentioned early, so this section will review some basics.

Definition 1.1: Cardinality

Let A and B be sets. We say A and B have the same *cardinality*, written $|A| = |B|$, if there exists a bijection $f : A \to B$. If no such bijection exists, the sets have different cardinalities.

Example 1.1

Let $A = \{1,2\}$, $B = \{3,4\}$. While this is a trivial example, we can show that there are as many elements in A as in B by constructing a function $f : A \to B$ and showing that f is a bijection.

Proof that $|A| = |B|$. Let $f : A \to B$ be defined by

$$f(n) = n + 2.$$

Let $x, y \in A$ and suppose $f(x) = f(y)$. Then

$$\begin{aligned} f(x) &= f(y) \\ x + 2 &= y + 2 \\ x &= y. \end{aligned}$$

Thus, f is injective.
Now let $b \in B$. Then $b - 2 \in A$, since $B = \{3,4\}$ and subtracting 2 yields values in $A = \{1,2\}$. So for every $b \in B$, there exists $a = b - 2 \in A$ such that $f(a) = b$. Hence, f is surjective.
Since f is both injective and surjective, it is a bijection, and therefore $|A| = |B|$. □

Definition 1.2: Finite and Infinite Sets

A set A is *finite* if there exists a natural number $n \in \mathbb{N}$ such that $|A| = |\{1,2,\ldots,n\}|$. Otherwise, A is *infinite*.

Definition 1.3: Countably Infinite

A set A is *countably infinite* if there exists a bijection $f : \mathbb{N} \to A$. A set is *countable* if it is finite or countably infinite.

Definition 1.4: Uncountable Set

A set A is *uncountable* if it is not countable; that is, there does not exist a bijection from $\mathbb{N}$ to A.

Example 1.2

The set $\mathbb{R}$ is famously uncountable, as is rigorously demonstrated in any introductory analysis course (e.g., via Cantor's diagonal argument).

Definition 1.5: Power Set

Let A be a set. The *power set* of A, denoted $\mathscr{P}(A)$, is the set of all subsets of A.

Theorem 1.1: Cantor's Theorem

For any set A, we have $|\mathscr{P}(A)| > |A|$. In particular, there is no surjection from A onto $\mathscr{P}(A)$.

Proof. It suffices to show that there cannot exist a surjective function $f : A \to \mathscr{P}(A)$. Suppose, for contradiction, that such a surjective function f exists. Define the set

$$B = \{a \in A \mid a \notin f(a)\}.$$

Then $B \subseteq A$, so $B \in \mathscr{P}(A)$. Since f is surjective, there exists $b \in A$ such that $f(b) = B$. We now ask: is $b \in B$?

- If $b \in B$, then by the definition of B, $b \notin f(b) = B$, a contradiction.
- If $b \notin B$, then by the definition of B, $b \in f(b) = B$, again a contradiction.

In either case, we reach a contradiction. Therefore, our assumption that f is surjective must be false. Hence, there is no surjection from A onto $\mathscr{P}(A)$, and so

$$|\mathscr{P}(A)| > |A|.$$

□

After showing that the power set is strictly larger, we usually demonstrate that

$$|\mathscr{P}(A)| = 2^{|A|} > |A|$$

even for infinite sets. However, for infinite cardinals, exponentiation behaves differently than for finite numbers. For example, $2^{\aleph_0} = \mathfrak{c} = |\mathbb{R}|$.

Problem 1.1

Prove that $|\mathbb{N}| = |\mathbb{Z}|$, assuming $0 \in \mathbb{N}$.

Proof of Problem 1.1. We will construct a bijection $f : \mathbb{N} \to \mathbb{Z}$.

Define:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

We first show that f is injective. Suppose $f(x) = f(y)$.

Case 1: Both x and y are even. Then:

$$\frac{x}{2} = \frac{y}{2} \Rightarrow x = y.$$

Case 2: Both x and y are odd. Then:

$$-\frac{x+1}{2} = -\frac{y+1}{2} \Rightarrow x + 1 = y + 1 \Rightarrow x = y.$$

Case 3: One is even, one is odd. Then $f(x) \in \mathbb{Z}_{\geq 0}$, $f(y) \in \mathbb{Z}_{<0}$, so $f(x) \neq f(y)$. Hence, f is injective.

Now we show that f is surjective. Let $z \in \mathbb{Z}$. We find $n \in \mathbb{N}$ such that $f(n) = z$:

Case 1: $z \geq 0$. Then let $n = 2z$. Since $z \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{N}$, and $f(n) = z$.

Case 2: $z < 0$. Then let $n = -2z - 1$. Since $z \in \mathbb{Z}_{<0}$, $n \in \mathbb{N}$, and:

$$f(n) = -\frac{n+1}{2} = -\frac{(-2z-1+1)}{2} = -\frac{-2z}{2} = z.$$

In both cases, such an $n \in \mathbb{N}$ exists, so f is surjective.

Thus, f is a bijection and $|\mathbb{N}| = |\mathbb{Z}|$. $\square$

2 Tasks from 7.4

Problem 2.1: Task 17

Show that $\mathbb{Q}$ is dense along the number line by showing that given two rational numbers r_1 and r_2 with $r_1 < r_2$, there exists a rational number x such that $r_1 < x < r_2$.

Proof. Let $r_1, r_2 \in \mathbb{Q}$ such that $r_1 < r_2$. Consider that average of these numbers

$$\begin{aligned} x &= \frac{r_1 + r_2}{2} \\ &= \frac{\frac{a}{b} + \frac{c}{d}}{2} \\ &= \frac{a+c}{2bd} \end{aligned}$$

Clearly, x is a rational number since if $a, b, c, d \in \mathbb{Z}$ then $a + c \in \mathbb{Z}$ and $2bd \in \mathbb{Z}$. Furthermore

$$2r_1 < r_1 + r_2 \Rightarrow r_1 < \frac{r_1 + r_2}{2} = x$$
$$r_1 + r_2 < 2r_2 \Rightarrow \frac{r_1 + r_2}{2} = x < r_2$$

Thus we have that x is a rational number satisfying the desired property. Hence $\mathbb{Q}$ is dense along the number line. □

Problem 2.2: Task 26

Prove that any uncountably infinite set A has a countably infinite subset.

Proof. Let A be a set such that $|A| > \aleph_0$. To construct a countably infinite subset we proceed by induction as follows:

Let $a_0 \in A$ be the first element. Then for our next element choose some element $a_1 \in A \setminus \{a_0\}$. We know $A \setminus \{a_0\}$ is non-empty since A is infinite. If we have n elements in our subset take the subsequent element to be

$$a_{n+1} \in A \setminus \{a_0, a_1, \ldots, a_n\}$$

As mentioned earlier, A take away $\{a_1, \ldots, a_n\}$ leaves a non-empty set and a_{n+1} is an available element of this set, meaning we can introduce it to our subset. Then, by mathematical induction, we get a sequence which is itself a type of subset $\{a_i : i \in \mathbb{N}\}$. Clearly we can construct a bijection

$$f : \mathbb{N} \to \{a_i : i \in \mathbb{N}\}$$

such that $f(i) = a_i$. Note that this procedure of making infinitely many choices, means using a weak form of the Axiom of Choice. □

Problem 2.3: Task 27

Let A and B be sets such that $|A| = \aleph_0$. Prove that if there exists some $g : A \to B$ surjection, then B is countable.

Proof. We will proceed by proving that if there exists some surjection from one set Γ to another set Δ, then $|\Gamma| \geq |\Delta|$. With this it follows that B is countable, assuming the conditions set in the problem description. Suppose $\phi : \Gamma \to \Delta$ is surjective, i.e.

$$\forall \delta \in \Delta, \exists \gamma \in \Gamma \text{ s.t } \phi(\gamma) = \delta$$

Since we assume ϕ is well-defined, $\phi(\gamma)$ goes to one and only one $\delta \in \Delta$. Since ϕ is surjective, for any $\delta \in \Delta$ there must be at least one γ mapped to δ. As stated, no γ can map to more than one δ. Therefore, for each δ to have some γ which maps to it there must be at least as many $\gamma \in \Gamma$ as there are $\delta \in \Delta$. In other words,

$$|\Gamma| \geq |\Delta|$$

With this fact, and given that we have sets A, B where $|A| = \aleph_0$ and a surjection $g : A \to B$ it must be the case that

$$|B| \leq |A| = \aleph_0$$

which is what it means to be countable.

□

Problem 2.4: Task 32

Prove that the cartesian product of $\mathbb{Z}$ with itself, $\mathbb{Z} \times \mathbb{Z}$, is countably infinite.

Proof. To show that $\mathbb{Z}^2$ is countably infinite we must show that it is infinite ($|\mathbb{Z}^2| \geq \aleph_0$), and it is countable ($|\mathbb{Z}^2| \leq \aleph_0$), in other words,

$$|\mathbb{Z}^2| = \aleph_0$$

First we show $\mathbb{Z}^2$ is infinite. This should be obvious since $\mathbb{Z}$ is infinite, but to demonstrate this rigorously consider the function $\pi_1 : \mathbb{Z}^2 \to \mathbb{Z}$, defined as follows:

$$\pi_1(a, b) = a$$

Clearly, π_1 is well-defined, since (a, b) is mapped to a unique $a \in \mathbb{Z}$. Also, π_1 is surjective, since for any $a \in \mathbb{Z}$, there exists an infinite amount of elements in $\mathbb{Z}^2$ such that $\pi_1(a, b) = a$. Thus we have shown that we can project $\mathbb{Z}^2$ onto an infinite set $\mathbb{Z}$. Hence $\mathbb{Z}^2$ is infinite. In other words: $|\mathbb{Z}^2| \geq \aleph_0$.

Now we show that there is a surjection from the naturals to $\mathbb{Z}^2$. First define a bijection $h : \mathbb{Z} \to \mathbb{N}$ by

$$h(n) = \begin{cases} 2n, & n \geq 0, \\ -2n - 1, & n < 0. \end{cases}$$

Let $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the Cantor pairing function

$$\pi(a, b) = \frac{(a+b)(a+b+1)}{2} + b,$$

which is a bijection. Its inverse $\pi^{-1} : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ can be written explicitly: for $n \in \mathbb{N}$ set

$$w = \left\lfloor \frac{\sqrt{8n+1} - 1}{2} \right\rfloor, \qquad t = \frac{w(w+1)}{2}, \qquad b = n - t, \qquad a = w - b,$$

so $\pi^{-1}(n) = (a, b)$.

Now define $s : \mathbb{N} \to \mathbb{Z}^2$ by

$$s(n) = \left(h^{-1}(a), h^{-1}(b)\right) \quad \text{where } (a, b) = \pi^{-1}(n).$$

(Here $h^{-1} : \mathbb{N} \to \mathbb{Z}$ exists because h is a bijection.)

To see s is surjective, take any $(x, y) \in \mathbb{Z}^2$. Let $a = h(x)$ and $b = h(y)$. Put $m = \pi(a, b) \in \mathbb{N}$. Then $\pi^{-1}(m) = (a, b)$, hence

$$s(m) = \left(h^{-1}(a), h^{-1}(b)\right) = (x, y).$$

Thus every element of $\mathbb{Z}^2$ has a preimage under s, so s is surjective.

Consequently $|\mathbb{Z}^2| \leq |\mathbb{N}| = \aleph_0$. (Since $\mathbb{Z}^2$ projects onto $\mathbb{Z}$, we also have $|\mathbb{Z}^2| \geq \aleph_0$, so in fact $|\mathbb{Z}^2| = \aleph_0$.)

□

Problem 2.5: Task 38

Suppose $A_1, A_2, \ldots$ is an infinite sequence of countable sets. Prove that

$$\bigcup_{i=1}^{\infty} A_i$$

is countable.

Proof. We intend to show that the countably infinite union of countable sets is countable.

Let $A_1, A_2, \ldots$ be a sequence of countable sets.

Recall that

$$\bigcup_{i=1}^{\infty} A_i = \{x : x \in A_i, i \in \mathbb{Z}_+\}.$$

Since A_i is countable and $\mathbb{Z}_+$ is countable, there exists a surjection $g_i : \mathbb{Z}_+ \to A_i$. Recall also that $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable. Therefore if we can construct a surjective $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \bigcup_{i=1}^{\infty} A_i$, it follows that $\bigcup_{i=1}^{\infty} A_i$ is countable.

Define $f(n, m) = g_n(m)$, where g_n denotes the surjection from $\mathbb{Z}_+$ to A_n. To check surjectivity, let $x \in \bigcup_{i=1}^{\infty} A_i$. Then there exists some $k \in \mathbb{Z}_+$ such that $x \in A_k$. Since g_k is surjective, there exists $m \in \mathbb{Z}_+$ such that $g_k(m) = x$. Hence $f(k, m) = x$. Therefore f is surjective.

Since $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable and f is surjective, it follows that $\bigcup_{i=1}^{\infty} A_i$ is countable. □