MAT-INF3600 Assignment

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Problem 1

Solution for (a). Let $A = \{0, 1\}$, $c^{\mathfrak{A}}$ be any element in A and $R^{\mathfrak{A}}(x, y)$ if $x = y$. Let $f^{\mathfrak{A}}(x) = x$ be the identity function on A.

Then (i) is satisfied and (ii), $(\forall x)[R(x, f(x)]$ is satisfied. $\qquad\square$

Solution for (b). Let $A = \{0, 1\}$, $c^{\mathfrak{A}}$ be any element in A and $R^{\mathfrak{A}}(x, y)$ if $x < y$ with $f^{\mathfrak{A}}(x) = x$ as before. Then, we get a 2 element universe, but $R(x, f(x))$ is not satisfied for any x since 0 is not less than itself and 1 is not less than itself. $\qquad\square$

Solution for (c). Now we must construct a 5 element universe with an injective function.

Let $A = \{1, \ldots, 5\}$ with $c^{\mathfrak{A}} = 1$ (or any other choice), $R = \emptyset$ (again, arbitrary) and $f(5) = 1$, $f(x) = x + 1$ otherwise. $\qquad\square$

Solution and proof of (d). Let $A = \mathbb{N}$ (0 included) with $c^{\mathfrak{A}} = 0$, $R = \emptyset$ (choice is arbitrary) and $f^{\mathfrak{A}}(n) = n + 1$ be the successor function.

Then $\forall x[f(x) \neq c]$ since 0 is not the successor of any natural number. Furthermore the second condition of f being injective is satisfied since

$$
\begin{aligned}
f(x) &= f(y) \\
\rightarrow x + 1 &= y + 1 \\
\rightarrow x &= y
\end{aligned}
$$

Hence $\mathfrak{A} \vDash \Gamma$.

Now to prove that any model of Γ has an infinite universe.

Suppose we have some model of Γ with a finite universe $A = \{c^{\mathfrak{A}}, x_1, x_2, \ldots, x_n\}$. We require $f : A \to A \setminus \{c^{\mathfrak{A}}\}$ and for it to be injective. Since A is finite we have an injective map from a set of size $n + 1$ to a set of size n which is not possible by the pigeonhole principle, thus we arrive at a contradiction.

To vizualize this more clearly we can attempt to construct an injection $f : A \to A$.

$$
\begin{aligned}
f^{\mathfrak{A}}(c^{\mathfrak{A}}) &= x_{i_1} \text{ where } x_{i_1} \neq c^{\mathfrak{A}} \\
f^{\mathfrak{A}}(x_1) &= x_{i_2} \text{ where } x_{i_2} \neq x_{i_1}, \text{ and } x_{i_2} \neq c \\
&\vdots \\
f^{\mathfrak{A}}(x_{n-1}) &= x_{i_n} \text{ where } x_{i_n} \neq x_{i_{n-1}}, \ldots, x_{i_1}, \text{ and } x_{i_n} \neq c^{\mathfrak{A}}
\end{aligned}
$$

But now we arrive at $f^{\mathfrak{A}}(x_n)$ which cannot go to $c^{\mathfrak{A}}$ as that violates $f(x) \neq c$ and $f^{\mathfrak{A}}(x_n)$ cannot go to any x_i as that would violate injectivity. So we cannot construct a well-defined injection that satisfies $f(x) \neq c$ for all x given a finite universe.

Hence any model of Γ necessarily has an infinite universe. $\qquad\square$

Problem 2

Proof. Let $n \geq 1$ and let $\theta_1, \ldots, \theta_n$ be sentences. Let Σ be a set of formulas. We will prove, by induction, that $\Sigma \cup \{\theta_1, \ldots, \theta_n\} \vdash \phi$ if and only if $\Sigma \vdash \theta_1 \wedge \cdots \wedge \theta_n \to \phi$.

Base case $n = 1$.

$\Sigma \cup \theta \vdash \phi$ if any only if $\Sigma \vdash \theta \to \phi$, by **Theorem 2.7.4 (The Deduction Theorem)**.

Assume that for $n = k$. Now look at $n = k + 1$,

$$\Sigma \cup \{\theta_1, \ldots, \theta_k, \theta_{k+1}\} \vdash \phi$$
$$\Sigma \cup \{\theta_1, \ldots, \theta_n\} \vdash \theta_{k+1} \to \phi \text{ by the regular Deduction Theorem}$$
$$\Sigma \vdash \left[\bigwedge_{i=1}^{k} \theta_i\right] \to (\theta_{k+1} \to \phi) \text{ by assumption}$$
$$\alpha \to (\beta \to \gamma) \text{ equivalent to } (\alpha \wedge \beta) \to \gamma$$
$$\text{Hence } \Sigma \vdash \left[\bigwedge_{i=1}^{k+1=n} \theta_i\right] \to \phi$$

Now to show the implication in the other direction:

$$\Sigma \vdash \left[\bigwedge_{i=1}^{k+1=n} \theta_i\right] \to \phi$$
$$\Sigma \vdash \left[\bigwedge_{i=1}^{k} \theta_i\right] \to (\theta_{k+1} \to \phi)$$
$$\Sigma \cup \{\theta_1, \ldots, \theta_k\} \vdash \theta_{k+1} \to \phi \text{ by assumption}$$
$$\Sigma \cup \{\theta_1, \ldots, \theta_{k+1}\} \vdash \phi \text{ by the regular Deduction Theorem}$$

This completes the proof. $\square$

Problem 3

Proof of (a).

$$1. \forall x[Rx \rightarrow Sx]$$
$$2. \forall x[Rx \rightarrow Sx] \rightarrow (Rt \rightarrow St) \qquad (Q1)$$
$$3. (Rt \rightarrow St) \rightarrow (\neg St \rightarrow \neg Rt) \qquad (PC)$$
$$4. \forall x[Rx \rightarrow Sx] \rightarrow (\neg St \rightarrow \neg Rt) \qquad 2,3 \quad (PC)$$
$$5. \forall x[Rx \rightarrow Sx] \rightarrow \forall y[\neg Sy \rightarrow \neg Ry] \qquad 4, \quad (QR)$$
$$6. \forall y[\neg Sy \rightarrow \neg Ry] \qquad 1,5 \quad (PC)$$

Thus we have a deduction of $\forall y[\neg Sy \rightarrow \neg Ry]$ from $\forall x[Rx \rightarrow Sx]$. □

Proof of (b). First we recognize that $\phi \nvdash \psi$ if and only if $\{\phi, \neg\psi\}$ is satisfiable.

Let $\mathfrak{A}$ be a structure with universe $A = \{a, b\}$, $R^{\mathfrak{A}} = \{a\}$, $S^{\mathfrak{A}} = \{a, b\}$.

Then if $t = a$, Ra is true and Sa is true so $Rx \rightarrow Sx$ is true. If $t = b$, Rb is false so the implication is true regardless. Therefore $\mathfrak{A} \models \forall x[Rx \rightarrow Sx]$.

Now to check the other formula. If $t = b$ we have Sb, but we do not have Rb. Hence the implication does not hold for all terms and $\mathfrak{A} \nvDash \forall y[Sy \rightarrow Ry]$. □

Problem 4

Proof of (a). We will appeal to Lemma 1 by showing that the negation leads to the given contradiction.

$$1. \exists x(R(x)) \land \forall x(\neg R(x) \land \neg S(x))$$

$2. \exists x(R(x))$	1	(PC)
$3. \forall x(\neg R(x) \land \neg S(x))$	1	(PC)
$4. \forall x(\neg R(x) \land \neg S(x)) \to (\neg R(x) \land \neg S(x))$		$(Q1)$
$5. \neg R(x) \land \neg S(x)$	$3,4$	(PC)
$6. \neg R(x)$	5	(PC)
$7. \neg R(x) \to (R(x) \to \neg(\forall x[x = x]))$		(PC)
$8. R(x) \to \neg(\forall x[x = x])$	$6,7$	(PC)
$9. \exists x(R(x)) \to \neg(\forall x[x = x])$	8	(QR)
$10. \neg(\forall x[x = x])$	$2,9$	(PC)
$11. \forall x(x = x)$		$(E1)$
$12. \bot$	$10,11$	(PC)

Therefore

$$\vdash \exists x(R(x)) \to [\exists x(R(x) \lor S(x))]$$

$\square$

Proof of (b). Once again, we will appeal to Lemma 1 by showing that

$$\forall x(R(x) \land \exists x \neg R(x)) \vdash \bot$$

$$1. \forall x(R(x) \land \exists x \neg R(x))$$

$2. \forall x(R(x) \land \exists x \neg R(x)) \to R(x) \land \exists x \neg (R(x))$		$(Q1)$
$3. R(x) \land \exists x \neg (R(x))$	$1,2$	(PC)
$4. R(x)$	3	(PC)
$5. \exists \neg (R(x))$	3	(PC)
$6. R(x) \to (\neg R(x) \to \neg[\forall x(x = x)])$		(PC)
$7. \neg R(x) \to \neg[\forall x(x = x)]$	$4,6$	(PC)
$8. \exists x(\neg R(x)) \to \neg[\forall x(x = x)]$	7	(QR)
$9. \neg[\forall x(x = x)]$	$5,8$	(PC)
$10. \forall x(x = x)$		$(E1)$
$11. \bot$	$9,10$	(PC)

$\square$

Problem 5

Proof. We proceed by structural induction on t.

Base case $(t = 0)$.

Then there is $p = 0$, $T \vdash 0 = 0$.

Assume that for any variable-free $\mathfrak{L}$-term t of height n, we can find some prime term p such that

$$T \vdash t = p$$

Inductive step.

Now we look at variable-free terms of height $n + 1$.

Case 1 (f). Suppose $t = f(t_1, t_2)$ is variable-free. Then t_1, t_2 are variable free $\mathfrak{L}$-terms. By our hypothesis there exist prime terms p_1, p_2 such that

$$T \vdash p_1 = t_1, \quad T \vdash p_2 = t_2$$

By $E2$ we have

$$p_1 = t_1 \wedge p_2 = t_2 \rightarrow f(t_1, t_2) = f(p_1, p_2)$$

By the inductive definition of prime terms, $f(p_1, p_2)$ is a prime term and we have

$$T \vdash t = p$$

where $t = f(t_1, t_2)$ and $p = f(p_1, p_2)$.

Case 2 (v). Suppose $t = v(t')$ is variable-free. Then t' is variable-free. By our hypothesis there exists a prime term p' such that

$$T \vdash p' = t'$$

p' is prime so it is the constant 0 or of the form $f(p'_1, p'_2)$ where p'_1, p'_2 are prime. If p' is 0 we have (by $E2$)

$$v(t') = v(0) = 0$$

which is a prime term. Otherwise if $p' = f(p'_1, p'_2)$ then

$$v(t') = v(f(p'_1, p'_2)) = p'_1$$

by T_3 (and $E2$), which is prime.

In other words there exists, in both cases, prime p such that

$$T \vdash t = p$$

Case 3 (h). Suppose $t = h(t')$ is variable-free. Then t' is variable-free. By our hypothesis there exists a prime term p' such that

$$T \vdash p' = t'$$

p' is prime so it is the constant 0 or of the form $f(p'_1, p'_2)$ where p'_1, p'_2 are prime. If p' is 0 we have (by $E2$)

$$h(t') = h(0) = 0$$

which is a prime term. Otherwise if $p' = f(p'_1, p'_2)$ then

$$h(t') = h(f(p'_1, p'_2)) = p'_2$$

by T_4 (and $E2$), which is prime.

In other words there exists, in both cases, prime p such that

$$T \vdash t = p$$

$\square$