Mega Super Awesome Ultimate 3D Notes

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Table of Contents

- Linear Algebra
- Real Analysis
- Hilbert Spaces
- Topology
- Probability

Linear Algebra

Definition: Vector Space

A **vector space** over a field $\mathbb{F}$ is a set V together with two operations (vector addition and scalar multiplication) satisfying the usual axioms (associativity, distributivity, identity elements, inverses, etc.).

Theorem: Rank-Nullity Theorem

For a linear transformation $T : V \to W$ between finite-dimensional vector spaces,

$$\dim(\ker T) + \dim(\operatorname{im} T) = \dim(V)$$

Proof

Choose a basis for $\ker T$ and extend it to a basis of V. The images of the additional basis vectors form a basis of $\operatorname{im} T$. Counting dimensions gives the formula.

Real Analysis

Definition: Uniform Convergence

A sequence of functions $\{f_n\}$ converges **uniformly** to f on a set A if

$$\forall \epsilon > 0, \exists N \text{ such that } n > N \implies |f_n(x) - f(x)| < \epsilon \ \forall x \in A$$

Theorem: Weierstrass M-Test

Let $\{f_n\}$ be a sequence of functions on A, and suppose $|f_n(x)| \leq M_n$ for all $x \in A$.
If $\sum M_n$ converges, then $\sum f_n$ converges **uniformly** on A.

Hilbert Spaces

Definition: Inner Product Space

A **Hilbert space** is a complete inner product space $(H, \langle \cdot, \cdot \rangle)$ where the norm is induced by the inner product: $\|x\| = \sqrt{\langle x, x \rangle}$.

Lemma: Cauchy-Schwarz Inequality

For all $x, y \in H$,

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

Proof

Consider the vector $x - \frac{\langle x, y \rangle}{\|y\|^2} y$. Its norm squared is nonnegative:

$$\|x - \frac{\langle x, y \rangle}{\|y\|^2} y\|^2 \geq 0$$

Expanding and simplifying gives the inequality.

Topology

Definition: Open Set

A set U in a topological space (X, τ) is **open** if $U \in \tau$.

Proposition: Union of Open Sets

The union of any collection of open sets is open.

Proof

Follows directly from the definition of a topology: $\bigcup_\alpha U_\alpha \in \tau$.

Probability

Definition: Expected Value

For a discrete random variable X taking values x_i with probabilities p_i,

$$\mathbb{E}[X] = \sum_i x_i p_i$$

Theorem: Law of Large Numbers (Weak)

Let $X_1, X_2, \ldots$ be i.i.d. with mean μ and variance $\sigma^2 < \infty$. Then

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \to \mu \text{ in probability as } n \to \infty.$$

Proof (Sketch)

By Chebyshev's inequality:

$$\mathbb{P}(|\overline{X}_n - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2} \to 0$$
