Lecture Notes: Combinatorics

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1 Counting Principles

Multiplication principle

If there are a ways of performing task A, and b ways of performing task B, **regardless of the outcome of A**, there are ab ways of performing A then B

Addition Principle

If there are a ways of performing task A, and b ways of performing task B, then there are $a + b$ ways of performing either A or B

Example

Consider a map with 4 sectors A, B, C, D, all adjacent except B and D. We have q colors and we want to color the adjacent sectors different colors. How many ways are there to do this? How many **proper** colorings? Note that while this looks like the four color question, it isn't quite the same.

1. Color A (q posibilities)
2. Color B ($q - 1$ posibilities)
3. Color C ($q - 2$ posibilities)
4. Color D ($q - 2$ posibilities)

By the multiplication principle there are $q(q - 1)(q - 2)^2$ posibilities as long as $q \geq 3$ since we cannot satisfy the condition of the color of $C \neq D$ otherwise.

1. Color B (q posibilities)
2. Color C ($q - 1$ posibilities, any but B)
3. Color D ($q - 1$ posibilities, any but C)
4. Color A ($q - 2$ or $q - 3$ posibilities, any but B, C, D)

Case 1: B,C,D have 3 different colors $\Rightarrow$ # of colorings $= q(q-1)(q-2)(q-3)$.
Case 2: B,C,D don't have 3 different colors. $\Rightarrow$ # of colorings $= q(q-1)(q-2)$.
The answer then becomes $q(q-1)(q-2)(q-3)+q(q-1)(q-2) = q(q-1)(q-2)\times(q-3+1) = q(q-1)(q-2)\times(q-2) = q(q-1)(q-2)^2$

2 Permutations

Let S_n be the set of permutations of $[n] = 1, 2, 3, ..., n$.
Proposition: $|S_n| = n!$
Proof: To choose a permutation of $\pi_1, \pi_2, \pi_3, ..., \pi_n$,

1	Choose π_1 (n possibilities)
2	Choose π_2 ($n-1$ possibilities)
3	Choose π_3 ($n-2$ possibilities)
$\vdots$	$\vdots$
n	Choose π_n (1 possibility)

We have to make a choice for each element and each choice is independent of all the other choices, $\therefore S_n = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 1 = n!$ $\square$

3 Subsets

Let S be a finite set of size n.
Let $2^s = \{$ subsets of set S $\}$, $\binom{S}{k} = \{k$ -subsets of S$\}$

Proposition: $|2^s| = 2^n$
Proof: $S = \{a_1, a_2, a_3, ..., a_n\}$

$a_1 \in S$?	2 posibilities
$a_2 \in S$?	2 posibilities
$a_3 \in S$?	2 posibilities
$\vdots$	$\vdots$
$a_n \in S$?	2 posibilities

Thus we have to make independent binary choices, meaning 2 posibilities, n times.
$\therefore |2^s| = 2 \times 2 \times 2 \times \cdots \times 2 = 2^n$ $\square$
Notice this gives a bijection between the $\{$subsets of $S\}$ and the $\{$sqeuences $(e_1, e_2, ..., e_n), e = 0 \vee e = 1\}$. So now we know how many posibilities there are when we have to make n binary independent choices.

Definition: $\binom{n}{k} = |\binom{S}{k}|$ for $|S| = n$
Proposition: $\binom{n}{k} =$
To choose a k-subset $\{b_1, ..., b_k\} \subseteq S$

Choose b_1	n posibilities
Choose b_2	$n-1$ posibilities
Choose b_3	$n-2$ posibilities
$\vdots$	$\vdots$
Choose b_k	$n-k+1$ posibilies

Yielding $n(n-1)(n-2)...(n-k+1) = \frac{n!}{k!(n-k)!}$. Note $k!$ making it the unordered choosing of k elements from n elements. If it were ordered it would be $\frac{n!}{(n-k)!}$.

4 Subsets & Generating Functions

The multivariant generating function for subets of $[n]$.

$$\sum_{A \subseteq [n]} \prod_{i \in A} x_i = (1+x_1)(1+x_2)...(1+x_n)$$

Example: $n = 2 \rightarrow x_1x_2 + x_1 + x_2 + 1, \ \leftrightarrow \{1,2\}, \{1\}, \{2\}, \emptyset$
$x_1x_2 + x_1 + x_2 + 1 = (x_1+1)(x_2+1)$

Plug in $x_1 = x_2 = x_3 = ... = x_n$

$$\sum_{A \subseteq [n]} x^{|A|} = (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

We get the binomial theorem for free!

5 Compositions

A composition of n is a way of expressing n as an ordered sum of of positive integers.
Example: The compositions of 3 are $1+1+1 = 2+1 = 1+2 = 3$.