Mat210 Advanced Discrete Mathematics Notes

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1 Pre-Semester Start – Cardinality

The following chapter contains notes based on what I think the course will cover in the first week (week 33). According to the syllabus, cardinality is mentioned early, so this section will review some basics.

Definition 1.1: Cardinality

Let A and B be sets. We say A and B have the same *cardinality*, written $|A| = |B|$, if there exists a bijection $f : A \to B$. If no such bijection exists, the sets have different cardinalities.

Example 1.1

Let $A = \{1,2\}$, $B = \{3,4\}$. While this is a trivial example, we can show that there are as many elements in A as in B by constructing a function $f : A \to B$ and showing that f is a bijection.

Proof that $|A| = |B|$. Let $f : A \to B$ be defined by

$$f(n) = n + 2.$$

Let $x, y \in A$ and suppose $f(x) = f(y)$. Then

$$\begin{aligned} f(x) &= f(y) \\ x + 2 &= y + 2 \\ x &= y. \end{aligned}$$

Thus, f is injective.
Now let $b \in B$. Then $b - 2 \in A$, since $B = \{3,4\}$ and subtracting 2 yields values in $A = \{1,2\}$. So for every $b \in B$, there exists $a = b - 2 \in A$ such that $f(a) = b$. Hence, f is surjective.
Since f is both injective and surjective, it is a bijection, and therefore $|A| = |B|$. $\square$

Definition 1.2: Finite and Infinite Sets

A set A is *finite* if there exists a natural number $n \in \mathbb{N}$ such that $|A| = |\{1,2,\ldots,n\}|$. Otherwise, A is *infinite*.

Definition 1.3: Countably Infinite

A set A is *countably infinite* if there exists a bijection $f : \mathbb{N} \to A$. A set is *countable* if it is finite or countably infinite.

Definition 1.4: Uncountable Set

A set A is *uncountable* if it is not countable; that is, there does not exist a bijection from $\mathbb{N}$ to A.

Example 1.2

The set $\mathbb{R}$ is famously uncountable, as is rigorously demonstrated in any introductory analysis course (e.g., via Cantor's diagonal argument).

Definition 1.5: Power Set

Let A be a set. The *power set* of A, denoted $\mathscr{P}(A)$, is the set of all subsets of A.

Theorem 1.1: Cantor's Theorem

For any set A, we have $|\mathscr{P}(A)| > |A|$. In particular, there is no surjection from A onto $\mathscr{P}(A)$.

Proof. It suffices to show that there cannot exist a surjective function $f : A \to \mathscr{P}(A)$. Suppose, for contradiction, that such a surjective function f exists. Define the set

$$B = \{a \in A \mid a \notin f(a)\}.$$

Then $B \subseteq A$, so $B \in \mathscr{P}(A)$. Since f is surjective, there exists $b \in A$ such that $f(b) = B$. We now ask: is $b \in B$?

- If $b \in B$, then by the definition of B, $b \notin f(b) = B$, a contradiction.
- If $b \notin B$, then by the definition of B, $b \in f(b) = B$, again a contradiction.

In either case, we reach a contradiction. Therefore, our assumption that f is surjective must be false. Hence, there is no surjection from A onto $\mathscr{P}(A)$, and so

$$|\mathscr{P}(A)| > |A|.$$

□

After showing that the power set is strictly larger, we usually demonstrate that

$$|\mathscr{P}(A)| = 2^{|A|} > |A|$$

even for infinite sets. However, for infinite cardinals, exponentiation behaves differently than for finite numbers. For example, $2^{\aleph_0} = \mathfrak{c} = |\mathbb{R}|$.

Problem 1.1

Prove that $|\mathbb{N}| = |\mathbb{Z}|$, assuming $0 \in \mathbb{N}$.

Proof of Problem 1.1. We will construct a bijection $f : \mathbb{N} \to \mathbb{Z}$.

Define:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

We first show that f is injective. Suppose $f(x) = f(y)$.

Case 1: Both x and y are even. Then:

$$\frac{x}{2} = \frac{y}{2} \Rightarrow x = y.$$

Case 2: Both x and y are odd. Then:

$$-\frac{x+1}{2} = -\frac{y+1}{2} \Rightarrow x+1 = y+1 \Rightarrow x = y.$$

Case 3: One is even, one is odd. Then $f(x) \in \mathbb{Z}_{\geq 0}$, $f(y) \in \mathbb{Z}_{<0}$, so $f(x) \neq f(y)$. Hence, f is injective.

Now we show that f is surjective. Let $z \in \mathbb{Z}$. We find $n \in \mathbb{N}$ such that $f(n) = z$:

Case 1: $z \geq 0$. Then let $n = 2z$. Since $z \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{N}$, and $f(n) = z$.

Case 2: $z < 0$. Then let $n = -2z - 1$. Since $z \in \mathbb{Z}_{<0}$, $n \in \mathbb{N}$, and:

$$f(n) = -\frac{n+1}{2} = -\frac{(-2z-1+1)}{2} = -\frac{-2z}{2} = z.$$

In both cases, such an $n \in \mathbb{N}$ exists, so f is surjective.

Thus, f is a bijection and $|\mathbb{N}| = |\mathbb{Z}|$. $\square$