Real Analysis

Thobias Høivik

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1 Introduction

The following is intended for anyone who stumbles over these notes. This is intended to be my personal notes in real analysis. I will hopefully be attending MAT2400 Real Analysis at the University of Oslo in the spring of 2026. However, as I am not a program student at this institution, but just someone who takes individual courses there of my own volition, I do not and cannot attend lectures and therefore I have to learn the material on my own. As far as I understand, while this course is called Real Analysis, it is a bit different than a first course from what I understand. The earlier exams give a hint of functional analysis and also include topics such as fourier analysis, meassure- and integration theory among other things. Thus the different supplementary coursematerial I will use to aid myself in learning the content of this course will most likely be a bit scattered and all over the place, which these notes will undoubtedly reflect. I will do my best to keep things organized for my own sake, but keep this in mind if you are someone who intends to use these notes to learn Real Analysis.

2 Basic Banch Space theory

The following section of notes is derived from the first video in the lecture series MIT 18.102 Introduction to Functional Analysis, Spring 2021 (found on youtube).

Definition 2.1 (Vector Space). *A vector space V over a field $\mathbb{F}$ is a nonempty set of elements called "vectors" together with a binary operation $+$ on V and a binary function $\cdot$ which maps elements of $V, \mathbb{F}$ to V satisfying:*

1. *Associativity of vector addition:*
$$u + (v + w) = (u + v) + w, \forall u, v, w \in V$$
2. *Commutativity of vector addition:*
$$u + v = v + w, \forall u, v \in V$$
3. *Identity element:*
$$\exists 0 \in V : v + 0 = 0 + v = v, \forall v \in V$$
4. *Each $v \in V$ has an inverse $-v$ under the vector-addition operation.*
5. *Scalar multiplication is compatible with field multiplication:*
$$a(bv) = (ab)v$$
where $a, b \in \mathbb{F}$ and $v \in V$.
6. *The multiplicative identity $1 \in \mathbb{F}$ satisfies:*
$$1v = v, \forall v \in V$$
7. *Distributivity of scalar multiplication with respect to vector addition:*
$$a(u + v) = au + av$$
where $a \in \mathbb{F}$ and $u, v \in V$.
8. *Distributivity of scalar multiplication with respect to field addition:*
$$(a + b)v = av + bv$$
where $a, b \in \mathbb{F}$ and $v \in V$.

When proving that something is a vector space, most of these follow naturally from showing closure under addition and scalar multiplication and those two properties, are generally enough to show that it is indeed a vector space.
A subspace U of V is a set $U \subseteq V$ which is also a vector space. It is enough to show that $U \subseteq V$ and that it is closed under the two operations.

Some typical examples of vector spaces are $\mathbb{F}^n$ where $\mathbb{F}$ is the reals or the complex numbers. We also have spaces like the space of real polynomials of degree $\leq n$, i.e. $\mathcal{P}_n = \{\sum_{i=0}^n \alpha_i x^i : \alpha_i \in \mathbb{R}\}$, which is itself a subspace of the space of continuous real-valued functions $C(\mathbb{R})$.

So $\mathbb{R}^2$ and $C(\mathbb{R})$ are both vector spaces over $\mathbb{R}$, but they have one really big difference, that being the dimension.

Definition 2.2. *Let V be a vector space. A set $\{v_1, \dots, v_n\} \subseteq V$ is linearly independent if*

$$\sum_{i=1}^{n} \alpha_i v_i = 0 \Leftrightarrow \alpha_1 = \dots = \alpha_n = 0 \in \mathbb{F}$$

Note: the right-to-left direction of this implication is always true.

The two spaces discussed above are different in dimension, $\mathbb{R}^2$ being 2-dimensional and the other being infinite-dimensional. One definition of finite-dimensional is that every linearly independent set in the space is finite. I however like the definition using bases more. Both of these definitions are equivalent.

We won't give a rigorous definition of a basis, but in short a basis of V is a linearly independent set of vectors which spans V, i.e. every vector in V can be expressed as a linear combination of basis-vectors. If the basis is finite then V is finite dimensional. Moreover, if the basis is finite then the dimension of V is the number of basis-vectors. Note that if a finite dimensional space V has a basis with n elements then every basis of V has n elements. A space is infinite-dimensional if no finite set of linearly independent vectors spans the space.

2.1 Norms and Metrics

Definition 2.3 (Norm). *Let V be a vector space. A norm $||\cdot||$ is a function from $V \to [0, \infty)$ satisfying:*

1. $||v|| = 0$ *if and only if* $v = 0$.
2. $||\alpha v|| = |\alpha| \cdot ||v||$ *where α is an element of the ground field.*
3. $||v + w|| \leq ||v|| + ||w||$.

The tuple $(V, ||\cdot||)$ is called a normed space.

Example 2.1. $||x||_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ *defines a norm on $\mathbb{R}^n$. In fact it constitutes a norm on $\mathbb{C}^2$ as well. Formally, if we take $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ the p-norm of a vector $v \in \mathbb{F}^n$ ($p \in [1, \infty]$) is*

$$||v||_p := \begin{cases} \left(\sum_{i=1}^{n} |v_i|^p \right)^{1/p} & p < \infty \\ \max_{i=1,\ldots,n} |v_i| & p = \infty \end{cases}$$

Norms give us a notion of the "length" of a vector. Now all we need to do analysis on spaces is a notion of distance. Intuitively, norms already give us a notion of distance from 0.

Definition 2.4. *Let X be a set.*
A metric is a function $d : X^2 \to [0, \infty)$ satisfying:

1. $d(x, y) = 0$ *if and only if* $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$.

The metric gives us a notion of distance. In a typical first course in analysis where we work on the reals, $d(a, b) = |a - b|$ is the metric we deal with.

Proposition 2.1. *Let V be a normed space with norm $||\cdot||$. Then we can define the distance (a metric) between two vectors by*

$$d(x,y) := ||x-y||$$

In other words you can define a metric in terms of the norm in any normed space. This metric is usually refered to as the metric induced by the norm.

We won't provide a proof of this as it's fairly intuitive. Now we can get a sense of convergence and continuity in vector spaces by saying that a sequence $\{a_n\}_{n\in\mathbb{N}}$ converges to a value a if

$$\forall\varepsilon > 0, \exists N \in \mathbb{N} : n \geq N \Rightarrow ||a_n - a|| < \varepsilon$$

and a linear transformation (look up definition if necessary) $T \in \mathcal{L}(U,V)$ is (uniformly) continuous if

$$\forall\varepsilon > 0, \exists\delta > 0 : ||x-y||_U < \delta \Rightarrow ||Tx - Ty||_V < \varepsilon$$

for every $x, y \in U$. Notice that if you replace $||x-y||$ with $d(x,y)$ it looks like the standard definitions in terms of metric spaces.

2.2 Banach Spaces

Definition 2.5 (Banach Space). *A normed space V is a Banach Space if it is complete with respect to the metric induced by the norm, meaning that every cauchy sequence converges to a value in the space.*

Example 2.2. *$\mathbb{R}^n$ or $\mathbb{C}^n$ form Banach Spaces with respect to the ℓ^p norms (see Example 2.1).*

Theorem 2.1. *If X is a complete metric space, then $C_\infty(X)$ is a Banach Space.*

Recall that $C_\infty(X)$ is the space of bounded continuous functions on X.

Proof. We show that every Cauchy sequence in $C_\infty(X)$ converges to an element of $C_\infty(X)$.

Let $\{u_n\}_{n=1}^\infty \subseteq C_\infty(X)$ be a Cauchy sequence with respect to the supremum norm. Then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|u_n - u_m\|_\infty < \varepsilon \quad \text{for all } n, m \geq N.$$

Equivalently,

$$|u_n(x) - u_m(x)| < \varepsilon \quad \text{for all } x \in X \text{ and all } n, m \geq N.$$

Fix $x \in X$. Then $\{u_n(x)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$ (or $\mathbb{C}$), since

$$|u_n(x) - u_m(x)| \leq \|u_n - u_m\|_\infty.$$

Because $\mathbb{R}$ (or $\mathbb{C}$) is complete, the limit

$$u(x) := \lim_{n\to\infty} u_n(x)$$

exists. This defines a function $u : X \to \mathbb{R}$.

We now show that $u_n \to u$ uniformly on X. Let $\varepsilon > 0$ and choose N such that $\|u_n - u_m\|_\infty < \varepsilon$ for all $n, m \geq N$. Fix $n \geq N$ and $x \in X$. Taking the limit $m \to \infty$ gives

$$|u_n(x) - u(x)| = \lim_{m\to\infty} |u_n(x) - u_m(x)| \leq \varepsilon.$$

Since $x \in X$ was arbitrary, it follows that

$$\|u_n - u\|_\infty \leq \varepsilon \quad \text{for all } n \geq N.$$

Thus $u_n \to u$ uniformly on X.

Since each u_n is bounded and the convergence is uniform, the limit function u is bounded. Moreover, since each u_n is continuous and uniform limits of continuous functions are continuous, u is continuous on X.

Therefore $u \in C_\infty(X)$ and $\{u_n\}$ converges to u in the supremum norm. Hence $C_\infty(X)$ is complete, and thus a Banach space. $\square$

3 A step back into Calculus Theory

The following section is largely in accordance with chapter 2 of Lindstrøm's Spaces: An Introduction To Real Analysis.

Definition 3.1 (Limit of a sequence). *Let (x_n) be a sequence of real numbers and $x \in \mathbb{R}$.*
We say

$$x_n \to x$$

if for every $\varepsilon > 0$, there exists a natural number N such that

$$|x_n - x| < \varepsilon, \ \forall n \geq N$$

This is among the most important definitions in a first course in Real-Analysis. Intuitively what it captures is the sense of being able to get arbitrarily close to a value. We will all be familiar with examples from calculus like $\lim_{n\to\infty} \frac{1}{n} = 0$.

Theorem 3.1. *If a sequence converges, then its limit is unique.*

Proof. Suppose, for a contradiction, that we have some sequence (x_n) with

$$x_n \to x \text{ and } x_n \to y$$

of course with $x \neq y$.

x_n converges so we can choose any $\varepsilon > 0$ for the condition in definition 3.1 to hold.

Consider the case of $\varepsilon = \frac{|x-y|}{2}$. $\varepsilon > 0$ since $x \neq y$ and is therefore a value for which the condition must hold. Namely,

$$|x_n - x| < \varepsilon \text{ and } |x_n - y| < \varepsilon$$

leading to

$$|x - x_n| + |x_n - y| < 2\varepsilon = |x - y|$$

which, via triangle inequality, yields

$$|x - y| = |x - x_n + x_n - y| \leq |x - x_n| + |x_n - y| < |x - y|$$

in particular

$$|x - y| < |x - y|$$

which is not possible, a contradiction.

□

Theorem 3.2 (Algebra of Limits). *If $x_n \to x$ and $y_n \to y$, then:*

1. $x_n + y_n \to x + y$
2. $x_n y_n \to xy$
3. *If $y_n \neq 0$ and $y \neq 0$, then $\frac{x_n}{y_n} \to \frac{x}{y}$.*

Proof of (1): Sum of Limits. Let (x_n) and (y_n) be convergent sequences with $x_n \to x$ and $y_n \to y$.

Let $\varepsilon > 0$. Since both sequences converge to their respective limits we have some $N_1, N_2 \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon/2, \ \forall n \geq N_1$$
$$|y_n - y| < \varepsilon/2, \ \forall n \geq N_2$$

Let $N = \max\{N_1, N_2\}$, then $n \geq N$ satisfies $n \geq N_1$ and $N \geq N_2$ so

$$|x_n - x| + |y_n - y| < \varepsilon$$

Notice that via the triangle inequality we get

$$|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y| < \varepsilon$$

completing the proof. □

Theorem 3.3. *A sequence x_n converges to x if and only if*

$$\limsup x_n = \liminf x_n = x$$

We will take this theorem without proof as it requires some background about monotone sequences and completeness which are covered later.

Definition 3.2 (Limit of a function). *Let $f : D \subset \mathbb{R} \to \mathbb{R}$, and let a be a limit point of D.*
We say

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Later, when we get to metric spaces again, this condition becomes

$$d_X(x, a) < \delta \Rightarrow d_Y(f(x), f(a)) < \varepsilon$$

for continuity.

Theorem 3.4 (Sequential criterion).

$$\lim_{x \to a} f(x) = L \Leftrightarrow \textit{for every sequence } x_n \to a \textit{ with } x_n \neq a, f(x_n) \to L$$

Proof sketch. We only prove $\Rightarrow$, for now.

Assume $\lim_{x \to a} f(x) = L$.

Let $x_n \to a$ with $x_n \neq a$.

Given $\varepsilon > 0$, choose $\delta > 0$ from the definition of the limit.

Since $x_n \to a$ there exists $N \in \mathbb{N}$ such that with $n \geq N$ we have

$$|x_n - a| < \delta$$

Hence for $n \geq N$,

$$|f(x_n) - L| < \varepsilon$$

Thus $f(x_n) \to L$. □

4 Completeness

Definition 4.1 (Upper Bound). *A set $A \subset \mathbb{R}$ is bounded above if there exists $M \in \mathbb{R}$ such that*

$$a \leq M, \forall a \in A$$

Such an M is called an upper bound.

Definition 4.2 (Supremum). *Let $A \subset \mathbb{R}$.*
A number $s \in \mathbb{R}$ is the supremum of A if it is an upper bound of A and for any upper bound u of A we have

$$s \leq u$$

and we use the notation

$$s = \sup A$$

The completeness axiom says that every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$, this being the distinguishing quality separating $\mathbb{R}$ from $\mathbb{Q}$.

Sidenote: You can also characterize the reals as the *unique* ordered field containing $\mathbb{Q}$ which has the least upper bound property. More precisely, every $\mathbb{F}$ with this property satisfies $\mathbb{F} \cong \mathbb{R}$.

Proposition 4.1. *If A has a supremum then for every $\varepsilon > 0$, there exists $a \in A$ such that*

$$s - \varepsilon < a \leq s$$

Proof. Let $\varepsilon > 0$. If no $a \in A$ existed with the desired property, then $s - \varepsilon$ would be a supremum, contradicting minimality of s. $\square$

This proposition becomes useful later in many arguments.

The infimum $\inf A$ is also something we need and classically we define it as being the greatest lower bound, but for the purposes of keeping things simple, we just let the infimum of a set A, $\inf A$, be defined as

$$\inf A = -\sup(-A)$$

Theorem 4.1 (Monotone convergence). *Let (x_n) be a monotone increasing sequence (a sequence with which $x_n \leq x_{n+1}$ for every $n \in \mathbb{N}$) which is bounded above.*
Then (x_n) converges, and

$$\lim x_n = \sup\{x_n : n \in \mathbb{N}\}$$

Proof. Let

$$A = \{x_n : n \in \mathbb{N}\}$$

Since A is bounded above, A has some supremum s. We claim $x_n \to s$.

Let $\varepsilon > 0$.

By the supremum property, there exists N such that

$$s - \varepsilon < x_N < \leq s$$

Since the sequence is increasing, for all $n \geq N$,

$$s - \varepsilon < x_n \leq s$$

Thus

$$|x_n - s| < \varepsilon$$

□

Lemma 4.1. *Every cauchy sequence in $\mathbb{R}$ is bounded.*

Proof. Let (x_n) be cauchy.

Choose $\varepsilon = 1$. Then there exists N such that

$$|x_n - x_m| < 1,\ \forall n, m \geq N$$

Fix $m = N$. Then for all $n \geq N$:

$$|x_n| \leq |x_N| + |x_n - x_N| < |x_N| + 1$$

Thus all terms are bounded. □

Theorem 4.2. *Every cauchy sequence in $\mathbb{R}$ converges.*

Proof idea. Let (x_n) be cauchy.

(x_n) is bounded. Use the cauchy property to construct nested intervals. Show the intersection of these to be nonempty via completeness of $\mathbb{R}$. (x_n) will converge to a point in this intersection.

A bit more formally, we choose indices $n_1, n_2, \dots$ such that

$$|x_n - x_m| < 2^{-k},\ \forall n, m \geq n_k$$

and define the intervals

$$I_k = [x_{n_k} - 2^{-k}, x_{n_k} + 2^{-k}]$$

Then each I_k is closed and bounded with $I_{k+1} \subset I_k$. Completeness wil give us

$$\bigcap_{k \in \mathbb{N}} I_k \neq \emptyset$$

There will be an x in this set which is the limit point. □

5 Open and Closed Sets in Metric Spaces

Let X be a set and $A \subseteq X$. Intuitively, a point x is either **inside of** A, **outside of** A, or **on the boundary** of A. We know what it means to not be in A; $a \notin A \Leftrightarrow a \in A^c$. We also know what $a \in A$ means, but set-theoretically we don't have a notion of being on the "boundary" of A. We can make sense of this notion in a metric space.

Recall that the open ball is the set $B(x;r) = \{z \in X : d(x,z) < r\}$. Likewise we can define the closed ball as follows.

Definition 5.1 (Closed Ball). *The closed ball centered at $x \in X$ with radius $r \geq 0$ in $B(x;r) = \{z \in X : d(x,z) \leq r\}$.*

Let (X, d) be a metric space with $A \subseteq X$.

- $x \in X$ is an interior point of A if $B(x;r) \subseteq A$ for some $r > 0$.
- $y \in X$ is an interior point of A if $B(x;r) \subseteq A^c$ for some $r > 0$.
- $z \in X$ is a boundary point of A if it is neither of the above, i.e. $B(z;r) \cap A \neq \emptyset$ and $B(z;r) \cap A^c \neq \emptyset$ for every $r > 0$.

Notably, every point in X is one of these three.

- $A^0 = \{$all interior points of $A\}$
- $\partial A = \{$all boundary points of $A\}$
- $\overline{A} = A \cup \partial A = ((A^c)^0)^c$

Proposition 5.1. *For any $A \subset X$, we have $\partial A = \partial(A^c)$.*

Theorem 5.1. *Useful characterizations of openness:*
$A \subseteq X$ is open if $A = A^0$
$A \subseteq X$ is open if A contains none of its boundary points.
$A \subseteq X$ is open if $A \cap \partial A = \emptyset$
$A \subseteq X$ is open if $\forall x \in A$ there is some $r > 0$ s.t. $B(x;r) \subseteq A$

Theorem 5.2. *Useful characterizations of openness:*
$B \subseteq X$ is closed if $B = \overline{B}$
$B \subseteq X$ is closed if A contains all of its boundary points.
$A \subseteq X$ is closed if $\partial B \subseteq B$

Problem 5.1. *Consider $X = \mathbb{R}$ with the canonical metric $|\cdot|$. Show that (a, b) is open and $[a, b]$ is open for $a \leq b \in \mathbb{R}$, and $(a, b]$ is neither open nor closed for $a < b$.*

Proof. Let $x \in (a, b)$, i.e. $a < x < b$. Take

$$r := \text{min}\{x - a, x - b\} > 0$$

Take any $y \in (x - r, x + y)$. Then

$$y > x - r \geq a,\ y < x + r \leq b$$

Therefore

$$(x - r, x + r) \subset (a, b)$$

In other words, every $x \in (a, b)$ admits and open ball contained entirely in the interval, so (a, b) is open.

Notice that $\partial(a, b) = \{a, b\}$ and that $[a, b] = \partial(a, b) \cup (a, b)$ hence $[a, b]$ is the closure of (a, b) and thus is closed.

It is clear that $(a, b]$ is neither open nor closed as $\partial(a, b] = \{a, b\} \not\subseteq (a, b]$ and $(a, b] \cap \partial(a, b] \neq \emptyset$. $\square$

6 Week 4 problem set

Proposition 6.1 (Proposition 3.1.4 in Spaces). *If (X, d) is a metric space with $x, y, z \in X$, then*

$$|d(x, y) - d(y, z)| \leq d(x, z)$$

Problem 6.1. *Prove Proposition 6.1.*

Proof. Let (X, d) be a metric space and recall the standard triangle inequality:

$$d(x, z) \leq d(x, y) + d(y, z)$$

Now observe the following rearangements:

$$\begin{aligned} d(x, y) &\leq d(y, z) + d(x, z) \\ d(x, y) - d(y, z) &\leq d(x, z) \\ &\text{and} \\ d(x, y) &\leq d(y, z) + d(x, z) \\ d(y, z) &\leq d(x, y) + d(x, z) \\ d(y, z) - d(x, z) &\leq d(x, y) \\ -d(x, z) &\leq d(x, y) - d(y, z) \end{aligned}$$

Together:

$$|d(x, y) - d(y, z)| \leq d(x, z)$$

□

Problem 6.2 (3.1.6 Spaces). *Let $(V, ||\cdot||)$ be a normed space. Show that it induces a norm, i.e.*

$$d(x, y) := ||x - y||$$

is a norm.

Proof. Let $(V, ||\cdot||)$ be a normed space.

Non-negativity and Identity of Indiscernibles:

$$d(x, y) = ||x - y|| \geq 0$$

since $x - y \in V$ and $||\cdot||$ is non-negative.

$$d(x, x) = ||x - x|| = ||0|| = 0$$

Assume $d(x, y) = 0$.

$$d(x, y) = ||x - y|| = 0$$

Then $x - y = 0 \Rightarrow x = y$.

Symmetry:

Recall for a norm we have $||\alpha x|| = |\alpha|||x||$, hence

$$d(x, y) = |1|||x - y|| = || - 1(x - y)|| = ||y - x|| = d(y, x)$$

Triangle Inequality:

$$\begin{aligned} d(x, z) = ||x - z|| &= ||x - (y + y) - z|| \\ &\leq ||x - y|| + ||y - z|| \\ &= ||x - y|| + ||z - y|| \\ &= d(x, y) + d(y, z) \end{aligned}$$

□

Problem 6.3 (3.1.7 Spaces). *Show that if $x_1, x_2, \ldots, x_n$ are points in a metric space, then*

$$d(x_1, x_n) \leq \sum_{i=1}^{n-1} d(x_i, x_{i+1})$$

Proof. We proceed by induction on $n \in \mathbb{N}$.

Base Case ($n = 1$).

Clearly,

$$d(x_1, x_1) \leq d(x_1, x_1)$$

Hypothesis. Now assume, for any $k \in \mathbb{N}$,

$$d(x_1, x_k) \leq \sum_{i=1}^{k-1} d(x_i, x_{i+1})$$

Induction. Let $n = k + 1$.

$$\begin{aligned} d(x_1, x_n) = d(x_1, x_{k+1}) &\leq d(x_1, x_k) + d(x_k, x_{k+1}) \\ &\leq \left(\sum_{i=1}^{k-1} d(x_i, x_{i+1}) \right) + d(x_k, x_{k+1}) \\ &= \sum_{i=1}^{n-1} d(x_i, x_{i+1}) \end{aligned}$$

as desired. □

Problem 6.4 (3.2.1 Spaces). *Assume that (X, d) is a discrete metric space. Show that the sequence $\{x_n\}$ converges to a if and only if there is an $N \in \mathbb{N}$ such that $x_n = a$ for all $n \geq N$.*

Proof. Let (X, d) be a discrete metric space, i.e. a metric space where

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

$\Leftarrow$.

Assume there is some $N \in \mathbb{N}$ such that $\forall n \geq N$,

$$x_n = a$$

Then it is clear that for any $\varepsilon > 0$,

$$x_n = 0 \Rightarrow d(x_n, a) = 0 < \varepsilon$$

so $\{x_n\} \to a$.

$\Rightarrow$.

Assume that $\{x_n\} \to a$. Then for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \geq N$, then

$$d(x_n, a) < \varepsilon$$

In particular, this must hold for $\varepsilon = 0.5$. Notice that this requires $x_n = a$ since otherwise we would have $1 < 0.5$, a contradiction. $\square$

Problem 6.5 (3.2.5 Spaces). *Let (X, d) be a metric space and fix some $a \in X$. Show that the function $f : X \to \mathbb{R}$ defined as*

$$f(x) = d(x, a)$$

is continuous. Note: $\mathbb{R}$ is finite dimensional so we are free to choose any metric. $d\mathbb{R}(x, y) = |x - y|$ is fine.

Proof. Let (X, d) be a metric space and fix $a \in X$. Let f be defined as above.

Consider a family of sequences in X such that every sequence $\{x_n\}$ converges to $a \in X$. Let $\{x_n\}$ be some arbitrary sequence in this family.

Then, $\forall \varepsilon > 0$ we have some natural number $\mathbb{N}$, s.t. for any $n \geq N$,

$$d(x_n, a) < \varepsilon$$

Notice that

$$d(x_n, a) = |d(x_n, a) - 0| < \varepsilon$$

where $\varepsilon > 0$ was arbitrary. Hence $f(\{x_n\})$ converges to $0 = d(a,a) = f(a)$.

Thus f is continuous. □

Problem 6.6. *Let (X,d) be a metric space. Prove that every finite subset of X is closed.*

Proof. Recall that proposition $3.3.13$ from Spaces tells us that for any finite collection $F_1, \dots, F_n$ of closed sets, their union

$$\bigcup_{i\in[n]} F_i$$

is closed.

Consider any singleton $\{x\} \subseteq X$. It is clear that $\{x\}$ is closed since $\{x\}^c$ is open. Observe that the open ball $B(y;\varepsilon) \subseteq \{x\}^c$ for any $y \in X \setminus \{x\}$ with $\varepsilon = d(x,y)$.

Let $\{x_1, \dots, x_n\} \subseteq X$ be any finite subset. As shown above, $\{x_i\}$ is closed, and by proposition 3.3.13

$$\{x_1, \dots, x_n\} = \bigcup_{i\in[n]} \{x_i\}$$

is closed. □

Problem 6.7 (3.3.13 Spaces). *A metric space (X,d) is disconnected if it is the union of two non-empty, disjoint and open subsets. If it is not disconnected it is connected.*
a) Let $X = (-1,1) \setminus \{0\}$ and let d be the usual metric on X. Show that (X,d) is disconnected.
b) Let $X = \mathbb{Q}$ and let d be the usual metric again. Show that (X,d) is disconnected.
c) Assume that (X,d) is a connected metric space and that $f : X \to Y$ is continuous and surjective. Show that Y is connected.

Proof of (a). It is clear that $(-1,0)$ and $(0,1)$ are open with respect to $d(x,y) = |x-y|$ as, we can find an open ball around any point in either. Since

$$X = (-1,0) \cup (0,1)$$

we conclude that X is disconnected. □

Proof of (b). Recall that $\sqrt{2} \notin \mathbb{Q}$ so the open subsets

$$(-\infty, \sqrt{2}) \cap \mathbb{Q} \text{ and } (\sqrt{2}, \infty) \cap \mathbb{Q}$$

Both of these sets are open in $\mathbb{R}$ and thus they are also open in $\mathbb{Q}$. □

Proof of (c). Suppose, for a contradiction that we have some continuous surjection $f : X \to Y$ where X is connected and Y is not.

From surjection we can gleam that for any $y \in Y$ there is some $x \in X$ such that $f(x) = y$. From continuity at every point we have that there is some $\delta > 0$ for any $\varepsilon > 0$ such that

$$f\left[B_X(x;\delta)\right] \subseteq B_Y(f(x);\varepsilon)$$

By assumption

$$Y = O_1 \cup O_2$$

for some nonempty, disjoint, open O_1 and O_2. So for any point in O_1 or O_2 we have an open neighbourhood surrounding it. Consider some family open sets in O_1, $\{\omega_n\}_{n\in\mathbb{N}}$ such that

$$\bigcup_{n\in\mathbb{N}} \omega_n = O_1$$

It is clear that $\omega_i \notin O_2$ for any ω_i. Suppose ω_i is centered around $y_i \in Y$. By surjectivity we have that there are some $x_i \in X$ such that the image open ball around x_i goes to the open ball around y_i. It is not hard to see that we are characterizing an open set in X which is mapped to O_1. Repeat this process for O_2 and we will see that since there is no open ball which falls in both sets, and correspondingly via continuity there then is no ball in X which falls in the pre-images of both O_1 and O_2, hence we have partitioned X into two disjoint non-empty subsets. So X is disconnected, a catastrophic contradiction. $\square$

Another attempt at a proof of (c). Suppose, for a contradiction that we have a continuous surjection $f : X \to Y$ where X is connected and Y is not.

Let $A \subseteq X$ be a subset such that $f\,|_A : A \to Y$ is a bijection. Since it is the restriction of f to A is still continuous f constitutes a homeomorphism from $A \to Y$. Then, the inverse $g : Y \to A$ exists, which is still continuous of course.

Recall that a function g from a metric space $Y \to A$ is continuous if and only if for any $\varepsilon > 0$ we have some $\delta > 0$ such that

$$g\left[B_Y(y;\delta)\right] \subseteq B_A(g(y);\varepsilon)$$

for any $y \in Y$. By assumption, there exist O_1 and O_2 such that

$$Y = O_1 \cup O_2$$

with $O_1, O_2 \neq \emptyset$ and disjoint. Furthermore, O_1 and O_2 are open. We now claim that A is partitioned into two disjoint, nonempty, open sets

$$g[O_1] \text{ and } g[O_2]$$

It is not hard to see that they are both open as the open ball around any point in O_1 or O_2 is sent to some open ball around a point in their image via continuity. Furthermore it is clear that they are nonempty since g is a bijection and O_1, O_2 are nonempty.

Lastly, assume (for contradiction) that there is some $y \in Y$ such that $g(y) \in g[O_1]$ and $g(y) \in g[O_2]$, i.e. they are not disjoint. Then we would have $f|_A (g(y)) \in O_1$ and $f|_A (g(y)) \in O_2$. Recalling that $f|_A \circ g = id_Y$ we have that $y \in O_1$ and $y \in O_2$, but it is established that no such point exists, contradicting the assumption that $g[O_1] \cap g[O_2] \neq \emptyset$.

Thus we have shown that A is disconnected, but then X is also disconnected, a catastrophic contradiction to our original assumption. $\square$

Problem 6.8 (3.3.13 Spaces (again)). *A space X is path connected if there is a continuous $r : [0,1] \to X$ for every pair $x, y \in X$ such that $r(0) = x$ and $r(1) = y$.*
d) Let d be the usual metric on $\mathbb{R}^n$:

$$d(x,y) = ||x - y|| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Show that $(\mathbb{R}^n, d)$ is path-connected.
e) Show that every path-connected metric space is connected.

Proof of (d). Consider two distinct points $x, y \in \mathbb{R}^n$. Then

$$x = (x_1, \dots, x_n), \;\; y = (y_1, \dots, y_n)$$

Consider the vector

$$(y - x) \cdot z = (y_1 - x_1, \dots, y_n - x_n) \cdot z, \;\; z \in [0,1]$$

It is clear that

$$x + (y - x) \cdot 0 = x \text{ and } x + (y - x) \cdot 1 = y$$

So define $r : [0,1] \to X$ by

$$r(z) := x + (y - x) \cdot z$$

Let $z \in [0,1]$ and fix any $z_0 \in \mathbb{Z}$, let $\varepsilon > 0$ and choose $\delta := \frac{\varepsilon}{||x-y||} > 0$. Assume

$$|z - z_0| < \delta$$

Then,

$$\begin{aligned}
||r(z) - r(z_0)|| &= ||(x + (y - x) \cdot z) - (x + (y - x) \cdot z_0)|| \\
&= \left(\sum_{i=1}^{n} \left((x_i - zx_i + zy_i) - (x_i - z_0 x_i + z_0 y_i)\right)^2 \right)^{1/2} \\
&= \left(\sum_{i=1}^{n} \left((z_0 - z)x_i + (z - z_0)y_i\right)^2 \right)^{1/2} \\
&= \left(\sum_{i=1}^{n} \left((z - z_0)(y_i - x_i)\right)^2 \right)^{1/2} \\
&= |z - z_0| \left(\sum_{i=1}^{n} (y_i - x_i)^2 \right)^{1/2} \\
&= |z - z_0| \cdot ||x - y|| < \delta \cdot ||x - y|| = \frac{\varepsilon}{||x - y||} ||x - y|| \\
&= \varepsilon
\end{aligned}$$

$\square$

7 Week 6 problem set

Problem 7.1 (Problem relating to Section 3.3). *Let (X, d_X) and (Y, d_Y) be metric spaces, let $f : X \to Y$ be a continuous function, let $y \in Y$ be some given point, and consider the problem of finding an $x \in X$ such that $f(x) = y$ (that is, we wish to solve the above equation). Prove that the set of solutions of this equation is a closed subset of X.*
Hint: *Phrase the question in terms of finding f^{-1} of a closed set.*

Proof. Apply the assumptions of the problem description.

Let $S_y \subseteq X$ denote the set of solutions to $f(x) = y \in Y$. Notice that S_y is the set $f^{-1}[\{y\}]$. Since $\{y\}$ is a singleton it is a closed set. In the topological sense, if f is continuous then the preimage of every closed set in Y is closed in X. This would complete the proof, but let us give an analytical way of doing it instead.

Once again let $S_y \subset X$ denote the set of $x \in X$ such that $f(x) = y \in Y$. Let $\{x_n\}$ be some sequence in S_y such that $x_n \to x$ in X. Now we need to show that $x \in S_n$, i.e. $f(x) = y$. Since $x_n \in S_y$ we have that (for every $n \in \mathbb{N}$)

$$f(x_n) = y$$

f is continuous so via sequential continuity we get

$$f(x_n) = f(x)$$

i.e.

$$f(x) = y$$

Thus $x \in S_y$. Since $\{x_n\}$ was an arbitrarily chosen convergent sequence in S_y and we found that $x_n \to x \in S_y$ we conclude that S_y contains all it's limit points. In other words S_y is closed. $\square$

Problem 7.2. *Prove that $(\mathbb{Z}, d)$, where $d(x, y) = |x - y|$, is complete.*
Hint: *What does it mean for a sequence in $(\mathbb{Z}, d)$ to converge or to be Cauchy?*

Proof. Let $\{x_n\}$ be a cauchy sequence in $\mathbb{Z}$. Then, for every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ s.t.

$$|x_n - x_m| < \varepsilon$$

for every $n, m \geq N$. Notice that we then require $x_n = x_m$ for all $n, m \geq N$ since if that weren't the case we would have $0 < \varepsilon := 0.5 < 1 \leq |x_n - x_m|$. Thus every cauchy sequence in $\mathbb{Z}$ will necessarily settle at some $x \in \mathbb{Z}$ and be constant. In other words, $\{x_n\} \to x \in \mathbb{Z}$, as desired. $\square$

Problem 7.3 (Problem 3.3.3 Spaces). *Assume that F is a nonempty, closed and bounded subset of $\mathbb{R}$ with the usual metric. Show that* $\inf F \in F$ *and* $\sup F \in F$*. Give an example of a bounded, but not closed set such that it contains it's supremum and infinmum.*

Proof. Suppose $\emptyset \neq F \subseteq \mathbb{R}$ with F closed and bounded.

Recall that since F is bounded there is some finite $r > 0$, $x \in F$ such that

$$F \subseteq (x - r, x + r)$$

Recall that for any nonempty subset of $\mathbb{R}$ there exists a least upper bound and greates lower bound, i.e. $\inf F \in \mathbb{R}$ and $\sup F \in \mathbb{R}$.

Let $\{x_n\}$ be a sequence in F such that $\{x_n\} \to \sup F$. By closedness $\sup F \in F$. Apply this same reasoning to the infimum to get the desired result. Notice that we required boundedness as, if F wasn't bounded above and below we might have had $\inf F = -\infty$ or $\sup F = \infty$. In that case we couldn't have constructed sequences converging to these points.

Let $F = [0, 1] \setminus \{\frac{1}{2}\}$. Clearly not closed since any sequence converging to $1/2$ contradicts it being closed. It is bounded above and below and $\inf F = 0 \in F$ and $\sup F = 1 \in F$. □

Problem 7.4 (Problem 3.3.11 Spaces). *Prove Proposition 3.3.12. Find an infinite collection of open sets whose intersection is closed.*

Proposition 7.1 (Proposition 3.3.12 from Spaces). *Let (X, d) be a metric space.*
a) If $\mathcal{G}$ is a finite or infinite collection of open sets, then

$$\bigcup_{G \in \mathcal{G}} G$$

is open.
b) If $G_1, \ldots, G_n$ is a finite collection of open sets then

$$\bigcap_{i=1}^{n} G_i$$

is open.

Proof of (a). Consider a (potentially infinite) family $\mathcal{G} = \{G_1, G_2, \ldots\}$ of open sets. Define

$$\Gamma := \bigcup_{G \in \mathcal{G}} G$$

Take any arbitrary $x \in \Gamma$. Then there is some G_i such that $x \in G_i$. As G_i is open there is some $r \in \mathbb{R}^+$ with

$$B(x; r) \subseteq G_i \subseteq \Gamma$$

Hence Γ is open. □

Proof of (b). Let $\mathcal{G} = \{G_i\}_{i \in [n]}$ be a finite family of open sets. Define

$$\Gamma := \bigcap_{i \in [n]} G_i$$

Suppose $x \in \Gamma$. Then $\bigwedge_{i=1}^{n} x \in G_i$ and for each G_i there is some $r_i > 0$ such that

$$B(x; r_i) \subseteq G_i$$

Then

$$B(x; \mathtt{min}\{r_i : 1 \leq i \leq n\}) = \bigcap_{i=1}^{n} B(x; r_i) \subseteq \bigcap_{i \in [n]} G_i = \Gamma$$

so Γ is open. □

Example 7.1 (Counter example for infinite intersection.). *Consider the metric space $(\mathbb{R}, |\cdot|)$. Consider the following infinite family of open sets*

$$\{(-1/n, 1/n)\}_{n \in \mathbb{N}}$$

Clearly,

$$\bigcap_{n \in \mathbb{N}} (-1/n, 1/n) = \{0\}$$

which is closed in $\mathbb{R}$.

Problem 7.5. *prob Consider $X = \mathbb{R}$ with the canonical metric $d(x, y) = |x - y|$. The support of a function $f : \mathbb{R} \to \mathbb{R}$ is the set of points x where $f(x) \neq 0$. Prove that the support of a continuous function is always open.* Note: *In the literature, the "support" of f is usually defined to be the closure of the above set, which is of course closed, not open.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let

$$\Omega := \{x \in \mathbb{R} : f(x) \neq 0\}.$$

Suppose, for a contradiction, that Ω is not open. Then there exists $x \in \Omega$ such that for every $r > 0$ there is some $y \in (x - r, x + r)$ with $f(y) = 0$.

Since f is continuous at x, for $\varepsilon := |f(x)| > 0$ there exists $\delta > 0$ such that

$$|f(y) - f(x)| < \varepsilon \quad \text{whenever } |y - x| < \delta.$$

By assumption, there exists $y \in (x-\delta, x+\delta)$ with $f(y) = 0$. Then

$$|f(x)| = |f(x) - f(y)| < |f(x)|,$$

a contradiction. Hence Ω is open. □

Problem 7.6. *Show that the equation* $\cos t = 2t$ *has a unique solution.* Hint: *Formulate the problem as finding the fixed point of a function* f.

Proof. Banach's fixed-point theorem tells us that for some complete metric space X, a contraction $F : X \to X$ will have a unique $t \in X$ such that

$$t = F(t)$$

Let $F : [0, \pi/2] \to [0, \pi/2]$ be defined by

$$F(t) := \frac{1}{2} \cos t$$

Clearly, $F(t) = t$ if and only if $\cos t = 2t$. Furthermore, as a closed bounded subset of a compelte emtric space, $([0, \pi/2], |\cdot|_{[0,\pi/2]})$ is complete. Now all we need is to show that F constitutes a contraction on $\mathbb{R}$. Notice that

$$\frac{1}{2} \cos(t) \in [0, 1/2],\ t \in [0, \pi/2]$$

so F ceirtanly constitutes a contraction. Thus by Banach's fixed point theorem there is a unique point $t \in [0, \pi/2]$ such that $t = F(t)$. t constitutes the unique solution to

$$\cos(t) = 2t$$

□

8 3.4 & 3.5

Proposition 8.1. *Let (X, d) be a metric space. Let $f : X \to X$ be a contraction. Then f is continuous.*

Proof. Let $f : X \to X$ be a contraction. In other words, for some $\omega \in (0, 1)$, let f satisfy

$$d(f(x), f(y)) \leq \omega d(x, y),\ \forall x, y \in X$$

Let $\varepsilon > 0$. Define $\delta = \frac{\varepsilon}{\omega}$. Suppose $d(x, y) < \delta$. Then

$$d(f(x), f(y)) \leq \omega d(x, y) < \omega \cdot \delta = \varepsilon$$

$\square$

Recall that completeness is a very usefull property as if a sequence seems to converge then it does converge, and to a point in the space. However, sometimes it is hard to show that a sequence is cauchy. It would be nice to substitute this requirement to get some other way of showing convergence of sequence in a space. We could require that every sequence converges, but that is a ridiculously strict requirement and false most of the time. So instead, we require all sequences to have convergent subseqneces.

Recall the definition of a subsequence:

Definition 8.1 (Subsequence). *Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence. A subsequence is a sequence of the form $\{x_{n(k)}\}_{k \in \mathbb{N}}$, where $n(k) \in \mathbb{N}$ and*

$$n(1) < n(2) < n(3) < \cdots$$

Useful note: $n(k) \geq k$.

Proposition 8.2. *Let (X, d) be a metric space. If $\{x_n\}_n$ converges, then all possible subsequences converge, and the limit is the same.*

Proof. Let $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that $d(x_n, x) < \varepsilon$. Then $n(k) \geq n \geq N$ so also

$$d(x_{n(k)}, x) < \varepsilon$$

$\square$

Definition 8.2 (Compactness). *A metric space (X, d) is said to be* compact *if every sequence $\{x_n\}_n$ has a convergent subsequence $\{x_n(k)\}_k$.*

Definition 8.3. *A subset $K \subseteq X$ of a metric space (X, d) is compact if (K, d) is compact.*

Theorem 8.1. *Let (X, d) be a metric space. If $K \subseteq X$ is finite then (K, d) is compact.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $K = \{k_1, k_2, \dots, k_m\}$. As $\{x_n\}_n$ is infinite, there must be at least one element k_i where $x_j = k_i$ for infinitely many $j \in \mathbb{N}$. Then we can pick out indices such that $\{x_n(p)\}_{p\in\mathbb{N}}$ is a constant sequence for some $k_i \in K$ and thus convergent. $\square$

Theorem 8.2. *Let (X, d) is compact. Then (X, d) is complete.*

Proof. Let $\{x_n\}_n$ be cauchy. As (X, d) is compact there exists some convergent $\{x_{n(k)}\}_k$ which converges to $x \in X$. For $\varepsilon > 0$, let $N_1 \in \mathbb{N}$ be s.t.

$$d(x_{n(k)}, x) < \varepsilon,\ n \geq N_1$$

and let $N_2 \in \mathbb{N}$ s.t.

$$d(x_n, x_m) < \varepsilon,\ n, m \geq N_2$$

Let $N = \max\{n(N_1), N_2\}$. If $m \geq N$ then

$$d(x_m, x) \leq \underbrace{d(x_m, x_{n(m)})}_{<\varepsilon} + \underbrace{d(x_{n(m)}, x)}_{<\varepsilon} < 2\varepsilon$$

Hence $\{x_n\}_n$ converges to $x \in X$. $\square$

Theorem 8.3. *If $K \subseteq X$ is compact then it is closed and bounded.*

Proof. Let $K \subseteq X$ be compact.

Closed.

Let $\{x_n\}_n$ be a sequence in K converging to $x \in X$. By compactness we can find a subsequence $\{x_{n(k)}\}_k$ converging to some $y \in K$. Then $x = y \in K$, as the subsequence converges to the same value.

Bounded.

Assume K is not bounded. Fix $\overline{x} \in X$. Then for every $n \in \mathbb{N}$, there is some $x_n \in K$ with

$$d(x_n, \overline{x}) \geq n$$

Let $\{x_{n(k)}\}_k$ be a convergent subsequence of $\{x_n\}_n$. Then

$$n(k) \leq d(x_{n(k)}, \overline{x}) \underset{k\to\infty}{\to} d(x, \overline{x})$$

In other words, a value which tends to infinity is less than a fixed value. □

Proposition 8.3. *Let $X = \mathbb{R}^n$, $d(x, y) = ||x - y||$ (all norms are equivalent in $\mathbb{R}^n$). Let $K \subseteq \mathbb{R}^n$. Then*

$$K \text{ is compact} \Leftrightarrow K \text{ is closed and bounded}$$

Proof. The l2r direction is already covered so we tackle the other way.

Let K be closed and bounded. Let $\{x_n\}_n$ be a sequence in K. Then $\{x_n\}_n$ is bounded, so by Bolzano-Weierstrass, there is a convergent subsequence $\{x_{n(k)}\}_k$ converging to some $x \in \mathbb{R}^n$. Since K is closed $x \in K$. We conclude that K is compact. □

Theorem 8.4. *Let (X, d_X) and (Y, d_Y) be metric spaces with $f : X \to Y$ continuous. If $K \subseteq X$ is compact then $f[K]$ is, also.*

Theorem 8.5. *If (X, d) is compact and $f : X \to \mathbb{R}$ continuous. Then f attains a minimum and maximum.*

Definition 8.4. *Let (X, d) be a metric space. A set $K \subseteq X$ is* totally bounded *if $\forall \varepsilon > 0$, there exists finitely many points $x_1, \dots, x_n \in K$ such that*

$$K \subseteq \bigcup_{i=1}^{n} B(x_i; \varepsilon)$$

In other words, we say K is totally bounded if we can find finitely many open balls which cover it for any radius ε. This sounds a lot like the topological definition of compactness (Every open cover has a finite subcover). What we have here is a stronger property than meere boundedness.

Lemma 8.1. *Every compact set is totally bounded.*

Proof. Let $K \subseteq X$ be a compact subspace of a metric space (X, d), and assume it is not totally bounded. So K compact with

$$K \nsubseteq \bigcup_{i=1}^{n} B(x_i; \varepsilon)$$

for any choice of $x_1, \dots, x_n$ for some ε. Pick any $x_1 \in K$. Then $K \nsubseteq B(x_1; \varepsilon)$, so there is some $x_2 \in K \setminus B(x_1; \varepsilon)$. Iteratively pick

$x_1, \dots, x_n$, pick any $x_{n+1} \in K \setminus \bigcup_{i=1}^n B(x_i; \varepsilon)$. Note: $d(x_n, x_m) \geq \varepsilon$ for any $x \neq m$. Then $\{x_n\}_n$ is a sequence in K, so has convergent subsequence with limit $x \in K$. But then $\varepsilon \leq d(x_n, x) + d(x, x_m) \underset{x,m\to\infty}{\to} 0$. $\square$

Definition 8.5. *Let (X, d) be a metric space. A set $B \subseteq X$ is separable if there exists a dense, countable set $D \subseteq B$.*

Example 8.1.

- *$\mathbb{R}$ is separable ($\mathbb{Q} \subseteq \mathbb{R}$)*
- *Any subset of $\mathbb{R}$ is separable*
- *$B_1, B_2, \dots$ separable, then*

$$\bigcup_{i\in\mathbb{N}} B_i$$

  is, also
- *$\ell^p(\mathbb{R})$ is separable for all $p < \infty$, but not $p = \infty$*

Theorem 8.6. *Let (X, d) be a complete metric space, $K \subseteq X$.*

$$K \text{ compact} \Leftrightarrow K \text{ totally bounded}$$

Theorem 8.7. *Every compact set is separable.*

9 3.6 - Compactness IV

Definition 9.1. *Let (X, d) be a metric space and let $K \subseteq X$. An* open covering *of K is a family of open sets $\mathcal{F}$ satisfying*

$$K \subseteq \bigcup_{\mathcal{O} \in \mathcal{F}} \mathcal{O}$$

Definition 9.2. *A set $K \subseteq X$ has the open covering property if for any open covering $\mathcal{F}$ of K, there exists $\mathcal{O}_1, \ldots, \mathcal{O}_n \in \mathcal{F}$ such that $\{\mathcal{O}_1, \ldots, \mathcal{O}_n\}$ is an open covering of K.*

Theorem 9.1. *A set $K \subseteq X$ is compact if and only if it has the open covering property.*

This is a very powerful classification of compactness and is the one you typically first encounter in a topology course. Notice that we don't need to assume completeness of X, closedness of K (as that comes for free), etc.

10 4.1 - Models of Continuity

Definition 10.1. *A function* $f : X \to Y$ *is*

- *continuous at* x *if* $\forall \varepsilon > 0, \exists \delta > 0$ *such that*
$$d_Y(f(x), f(y)) < \varepsilon$$
for every $y \in X$ *satisfying* $d_X(x, y) < \delta$.
- *continuous if it is continuous at all* $x \in X$.
- *uniformly continuous if* $\forall \varepsilon > 0, \exists \delta > 0$ *s.t.*
$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y))$$
$\forall x, y \in X$.
- *Lipschitz continuous if* $\exists C > 0$ *s.t.*
$$d_Y(f(x), f(y)) < C d_x(x, y), \quad \forall x, y \in X$$

with each subsequent property implying the one before it.

Definition 10.2. *Denote*
$$C(X, Y) = \{\text{all continuous } f : X \to Y\}$$
$$C(X) = C(X, \mathbb{R})$$

Lemma 10.1. *If* $f : A \to \mathbb{R}$ *(*$A \subseteq \mathbb{R}$*) satisfies* $|f'(x)| \leq M, \forall x \in A$*, then* f *is Lipschitz with constant* M.

Lemma 10.2. *If* $f : [a, b] \to \mathbb{R}$ *is continuously differentiable, then it is Lipschitz.*

Theorem 10.1. *Let* $f : X \to Y$ *be continuous and* (X, d_X) *compact. Then* f *is uniformly continuous.*

11 Problems from week of 9/2 to 15/2

Problem 11.1 (Problem 5, Spaces p.67). *Let (X, d) be a metric space. For a subset $A \subseteq X$, let ∂A denote the set of all boundary points of A. Recall that the closure of A is $\overline{A} = A \cup \partial A$.*

a) A subset A of X is called precompact *if $\overline{A}$ is compact. Show that A is precompact if and only if all sequences in A have a convergent subsequence.*

b) Show that a subset of $\mathbb{R}^m$ is precompact if and only if it is bounded.

Proof of (a). Notice that what we're really saying here is that A is precompact if and only if A is compact.

By an earlier theorem we have that if $A \subseteq X$ is compact then it is closed and bounded. Namely A is closed so $A = \overline{A}$. Thus it is clear to see that A is precompact if and only if it is compact. □

Proof of (b). By an earlier proposition we have that a $A \subseteq \mathbb{R}^m$ is compact if and only if it is closed and bounded. As we saw in (a) we know that A is precompact if and only if it is compact. □

Problem 11.2 (Problem 6, Spaces p.67). *Assume that (X, d) is a metric space and that $f : X \to [0, \infty)$ is continuous. Assume that for each $\varepsilon > 0$, there is a compact set $K_\varepsilon \subseteq X$ such taht $f(x) < \varepsilon$ when $x \notin K_\varepsilon$, Show that f has a maximum point.*

Proof. If $f \equiv 0$, then f trivially attains its maximum. Assume therefore that there exists $x_0 \in X$ such that $f(x_0) > 0$.

Let

$$\varepsilon := \frac{1}{2} f(x_0) > 0.$$

By assumption, there exists a compact set $K_\varepsilon \subseteq X$ such that

$$f(x) < \varepsilon \quad \text{for all } x \notin K_\varepsilon.$$

Since f is continuous and K_ε is compact, the restriction

$$f\,\big|_{K_\varepsilon} : K_\varepsilon \to [0, \infty)$$

attains a maximum. Let $x^* \in K_\varepsilon$ be such that

$$f(x^*) = \max_{x \in K_\varepsilon} f(x) =: M.$$

Note that

$$M \geq f(x_0) > \varepsilon.$$

Now let $x \in X$ be arbitrary. If $x \in K_\varepsilon$, then $f(x) \leq M$. If $x \notin K_\varepsilon$, then

$$f(x) < \varepsilon < M.$$

Hence in all cases,

$$f(x) \leq M = f(x^*).$$

Therefore f attains its maximum at x^*. □

Problem 11.3 (Problem 7 Spaces p.67). *Let (X, d) be a compact metric space, and assume that $f : X \to \mathbb{R}$ is continuous when $\mathbb{R}$ is given the usual metric. Show that if $f(x) > 0$ for every $x \in X$, then there is a positive, real number a such that $f(x) > a$ for every $x \in X$.*

Proof. Notice that what the claim gives us is that $f[X]$ is bounded below by some *positive* real number a. By assumption $f[X]$ is already bounded below by 0. We give a constructive argument for the existence of $a > 0$ as follows:

As f is continuous, X compact, we conclude that $f[X] \subseteq \mathbb{R}$ is compact. By the Heine-Borel theorem $f[X]$ is closed and bounded. Thus $\inf f[X]$ exists and furthermore $0 < \inf f[X] \in f[X]$. (We conclude that $\inf f[X]$ is strictly greater than 0 as $f(x) > 0$ by assumption). Let $a = \frac{1}{2} \inf f[X]$.

Clearly,

$$0 < a < \inf f[X]$$

so

$$0 < a < f(x), \, \forall x \in X$$

□

Problem 11.4 (Problem 8 Spaces p.68). *Assume that $f : X \to Y$ is a continuous function between metric spaces, and let K be a compact subset of Y. Show that $f^{-1}(K)$ is closed.*

Proof. Let $K \subseteq Y$ compact, $f : X \to Y$ continuous, and denote

$$\Omega := f^{-1}[K] = \{x \in X \mid f(x) \in K\}$$

Suppose Ω is open. Then there is some sequence $\{x_n\}_{n\in\mathbb{N}}$ in Ω which converges to a limit point $x \notin \Omega$. By sequential continuity we have that

$$f(x_n) \to f(x)$$

$f(x_n) \in K$. Since K is compact (and thus closed), we conclude that

$$f(x) \in K$$

so

$$x \in f^{-1}[K] = \Omega$$

In other words, $x \in \Omega$ and $x \notin \Omega$.

(No this did not need to be a proof by contradiction) □

Problem 11.5 (Problem 9 Spaces p.68). *Show that a totally bounded subset of a metric space is always bounded. Find an example of a bounded set in a metric space that is not totally bounded.*

Proof. Let (X, d) be a metric space with $K \subseteq X$ totally bounded. Recall that what we mean by totally bounded is that for every $\varepsilon > 0$, there exists finitely many $x_1, \dots, x_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^{n} B(x_i; \varepsilon)$$

K is bounded if

$$K \subseteq B(x; r)$$

for some $x \in K$ and $r > 0$.

Pick $x := x_i$ for any $i \in [n]$, and let

$$r := \sup_{1 \le j \le n} d_X(x, x_j) + \varepsilon$$

Consider any $y \in B(x_j; \varepsilon)$. Then

$$d_X(y, x_j) < \varepsilon$$

and, by triangle inequality

$$d(y, x) \le d(y, x_j) + d(x_j, x)$$

Hence,

$$d(y, x) < \varepsilon + d(x_j, x)$$

Since $d(x_j, x) \le \sup_{1 \le j \le n} d(x, x_j)$, we get

$$d(y, x) < \varepsilon + \sup_{1 \le j \le n} d(x, x_j)$$

As r is defined,

$$\varepsilon + \sup_{j} d(x, x_j) = r$$

Thus

$$d(y, x) < r.$$

As $y \in B(x_j; \varepsilon)$ was arbitrarily chosen we may conclude that

$$K \subseteq B(x; r)$$

□

Problem 11.6 (Problem 11 Spaces p.68). *A metric space (X, d) is locally compact if there for each $a \in X$ is some $r > 0$ such that the closed ball $\overline{B}(a; r)$ is compact.*
a) Show that $\mathbb{R}^n$ is locally compact.
b) Show that if $X = \mathbb{R} \setminus \{0\}$, and $d : X \to \mathbb{R}$ is the metric defined by $d(x, y) = |x - y|$, then (X, d) is locally compact, but not complete.

Proof of (a). Let $a \in \mathbb{R}^n$ and take any $r > 0$. Clearly

$$\overline{B}(a; r)$$

is closed and furthermore

$$\overline{B}(a; r) \subseteq \overline{B}(a; r)$$

so bounded.

By the Heine-Borel theorem $\overline{B}(a; r)$ is compact. As $a \in \mathbb{R}^n$ was chosen arbitrarily we conclude that $\mathbb{R}^n$ is locally compact. □

Proof of (b). Pick any $a \in X$. If $a > 0$ we define the closed ball

$$a \pm \frac{|a|}{2}$$

which is clearly closed and bounded (thus compact).

The space is not complete because

$$\left\{\frac{1}{n}\right\}_{n \in \mathbb{N}} \underset{n \to \infty}{\to} 0 \notin X$$

i.e. we have a cauchy sequence which does not converge to a point in the space. □

Problem 11.7 (Problem 15 Spaces p.68). *Assume that C and K are disjoint, compact subsets of a metric space (X, d), and define*

$$a = \text{inf}\left\{d(x, y) \mid x \in C, y \in K\right\}$$

Show that a is strictly positive and that there are point $x_0 \in C$, $y_0 \in K$ such that $d(x_0, y_0) = a$. Show by an example that the result does not hold if we only assume that one of the sets is compact and the other is closed.

Proof. As the metric is non-negative we can clearly see that $a \geq 0$. Assume, in search of contradiction, that $a = 0$. Then, for every $n \in \mathbb{N}$ there exists $x_n \in C$, $y_n \in K$ such that

$$d(x_n, y_n) < \frac{1}{n}$$

C is compact so you can take a convergent subsequence

$$\{x_n(k)\}_k \to x_0 \in C$$

Then, as

$$d(x_{n(k)}, y_{n(k)}) \to 0$$

we see

$$\{y_{n(k)}\}_k \to x_0 \in K$$

and thus conclude

$$x_0 \in C \cap K \neq \emptyset$$

contradiction our assumption. Hence $a > 0$.

Now consider two sequences $\{x_n\}_n \subseteq C$ and $\{y_n\}_n \subseteq K$ such that

$$d(x_n, y_n) \underset{n\to\infty}{\to} a$$

By sequential continuity (metric is continuous) we get that

$$d(x_n, y_n) \to d(x, y)$$

and by uniqueness of limit point we conclude that

$$d(x, y) = a$$

Now, as C and K are compact we can find convergent subsequences which must then again converge to x and y in their respective sets. In other words,

$$x \in C \text{ and } y \in K$$

Therefore you can find points in C and K such that the minimum distance is attained.

(Perhaps this could have been shown by arguing that as d is continuous and thus the set of distances between C and K are compact in $\mathbb{R}$ ($\Rightarrow$ closed and bounded), the extreme-value theorem applies?) $\square$

Problem 11.8 (Problem 2 Spaces p.80). *Prove that* $f : (0, 1) \to \mathbb{R}$ *given by* $f(x) = \frac{1}{x}$ *is not uniformly continuous.*

Proof. Recall that a function $f : X \to Y$ is uniformly continuous if

$$\forall \varepsilon > 0,\ \exists \delta > 0 : d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)),\ \forall x, y \in X$$

Crucially there must be a δ for every ε which works at every point similtaneously. Informally, the problem with $\frac{1}{x}$ is that it grows without bound when $x \to 0$.

Consider what happens as $x \to 0$. Let $x_n = \frac{1}{n}$ and $y = \frac{1}{n+1}$. Then

$$d_X(x_n, y_n) \underset{n\to\infty}{\to} 0$$

but

$$d_Y(f(x_n), f(y_n)) = |n - (n+1)| = 1$$

So we can fix $\varepsilon_0 = 1/2$ and for any $\delta > 0$ we can pick n large enough so that

$$d_X(x_n, y_n) < \delta$$

but

$$d_Y(f(x_n), f(y_n)) = 1 > \varepsilon_0$$

□

Problem 11.9 (Problem 4 Spaces p.80). *Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function and assume that the derivative f' is bounded. Show that f is uniformly continuous.*

Proof. We know that f is a real-valued function such that there exists $M \in \mathbb{N}$ for which

$$\left|\frac{d}{dx} f(x)\right| = \left|\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\right| \leq M, \; \forall x \in \mathbb{R}$$

Suppose f is not uniformly continuous. Then it is ceirtainly not Lipschitz continuous. I.e. for every constant C there are points x_C, y_C such that

$$|f(x_C) - f(y_C)| > C|x_C - y_C|$$

Specifically:

$$\begin{aligned} |f(x_M) - f(y_M)| &> M|x_M - y_M| \\ \left|\frac{f(x_M) - f(y_M)}{x_M - y_M}\right| &> M \\ M \geq \left|\frac{f(x_M) - f(y_M)}{x_M - y_M}\right| &> M \end{aligned}$$

Our assumption that f is not Lipschitz continuous has led to a contradiction, so f is Lipschitz and hence uniformly continuous. □

Problem 11.10 (A reach problem from section 4.6). *Assume that $X \subset \mathbb{R}^n$ is not compact. Show that there is an unbounded continuous function $f : X \to \mathbb{R}$.*

Proof. Recall that from the Heine-Borel theorem a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. Therefore we may conclude that X is either open or unbounded. In the case where X is unbounded we construct the function $f : X \to \mathbb{R}$ with

$$f(x) = ||x||$$

which very clearly is continuous and not bounded.

Now suppose X is open. Fix some $y \in \partial X \setminus X$. Define $f : X \to \mathbb{R}$ as

$$f(x) = \frac{1}{||x - y||} = \frac{1}{d_X(x, y)}$$

□

12 Spaces 4.2 - Models of convergence

Definition 12.1. *Let $(X, d_X), (Y, d_Y)$ be metric spaces.*
Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions with domain X and co-domain Y. Let $f : X \to Y$ be another function.
We say that $\{f_n\}_{n\in\mathbb{N}}$ converges to f pointwise *if $\forall \varepsilon > 0$ and $x \in X$ there exists $n \in \mathbb{N}$ such that*

$$d_Y(f_n(x), f(x)) < \varepsilon, \; \forall n \geq N$$

and will often write $f_n \underset{n\to\infty}{\to} f$ pointwise.
We say that $\{f_n\}_{n\in\mathbb{N}}$ converges to f uniformly *if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that*

$$d_Y(f_(x), f(x)) < \varepsilon, \; \forall n \geq N \text{ and } \forall x \in X$$

and we often write $f_n \underset{n\to\infty}{\to} f$ uniformly.
The difference being that for uniform convergence, the choice of N doens't depend on $x \in X$.

Proposition 12.1. *Let $\{f_n\}_n$ be a sequence of $f_n : X \to Y$, and let $f : X \to Y$. Then the following are equivalent.*

1. *$f_n \to f$ pointwise.*
2. *$f_n(x) \to f(x)$ for every $x \in X$.*

Definition 12.2. *For $f, g : X \to Y$, define the* supremum metric

$$\rho(f, g) = \sup_{x\in X} d_Y(f(x), g(x))$$

also denoted

$$d_\infty(f, g), \;\; ||f - g||_\infty \text{ or } ||f - g||_{L^\infty}$$

Proposition 12.2. *Let $\{f_n\}_n$ be a sequence of functions from $X \to Y$, and let $f : X \to Y$. Then the following are equivalent.*

1. *$f_n \to f$ uniformly.*
2. *$\rho(f_n, f) \underset{n\to\infty}{\to} 0$.*

Example 12.1. *Let $f_n, f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$, $f_n(x) = \min(n, x^2)$. Show that f_n converges to f pointwise, but not uniformly.*

Proof. For any $x \in \mathbb{R}$, we have $f_n(x) = f(x)$ for every $n \geq x^2$, so $f_n(x) \to f(x)$. Next $\rho(f_n, f) = \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} |\min(n, x^2) - x^2| = \infty$. $\square$

Theorem 12.1. *Let $\{f_n\}_n$ be a sequence of $f_n : X \to Y$ and suppose $f_n \to f$ uniformly for some $f : X \to Y$.*
Then, f is continuous.

13 Some problems for week of Feb 16th

Problem 13.1. *Let $f_n : \mathbb{R} \to \mathbb{R}$ and assume that the sequence $\{f_n\}_n$ of continuous functions converges uniformly to $f : \mathbb{R} \to \mathbb{R}$ on all intervals $[-k, k]$, $k \in \mathbb{N}$. Show that f is continuous.*

Proof. Let $x_0 \in \mathbb{R}$ be arbitrary. We prove that f is continuous at x_0.

Choose $k \in \mathbb{N}$ such that $k > |x_0| + 1$. Then a neighbourhood of x_0 is contained in $[-k, k]$. By assumption, $f_n \to f$ uniformly on $[-k, k]$.

Let $\varepsilon > 0$ be given. Since the convergence is uniform on $[-k, k]$, there exists $N \in \mathbb{N}$ such that for all $x \in [-k, k]$ and all $n \geq N$,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}.$$

Fix such an N. Because f_N is continuous at x_0, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \quad \Rightarrow \quad |f_N(x) - f_N(x_0)| < \frac{\varepsilon}{3}.$$

We may assume $\delta \leq 1$, so that $|x - x_0| < \delta$ implies $x \in [-k, k]$.

For such x we estimate:

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus,

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, f is continuous at x_0. Because x_0 was arbitrary, f is continuous on $\mathbb{R}$. $\square$

Problem 13.2 (Problem 9, Spaces p.85). *Let (X, d) be a metric space and assume that the sequence f_n of continuous functions on X converges uniformly to f. Show that if $\{x_n\}_n$ is a sequence in X converging to x, then $f_n(x_n) \to f(x)$. Find an example which shows that this is not necessarily the case if the sequence of functions converges pointwise.*

Proof. Recall that, as $f_n \to f$ uniformly, we can for every ε, find $N \in \mathbb{N}$ such that

$$d(f_n(x), f(x)) < \varepsilon, \forall n \geq N \text{ and } \forall x \in X$$

As f_n is continuous we can use the property of sequential continuity. I.e.

$$f_n(x_{n'}) \underset{n' \to \infty}{\to} f_n(x)$$

Thus

$$d(f_n(x_{n'}), f_n(x)) \to 0,\ n' \to \infty$$

Let $\varepsilon > 0$. Pick $N \in \mathbb{N}$ such that

$$d(f_n(x_n), f_n(x)) < \varepsilon/2$$

and

$$d(f_n(x), f(x)) < \varepsilon/2,\ \forall x \in X$$

Then

$$d(f_n(x_n), f(x)) \leq d(f_n(x_n), f_n(x)) + d(f_n(x), f(x)) < \varepsilon$$

$\square$

Problem 13.3 (Problem 4.5.1, Spaces p.100). *Let $f, g : [0,1] \to \mathbb{R}$ be given by $f(x) = x$, $g(x) = x^2$. Find $\rho(f,g)$.*

Solution.

$$\rho(f,g) = \sup_{0 \leq x \leq 1} |g(x) - f(x)|$$

As f, g continuous we know that there exists extremal values in $[0,1]$. $|f(0) - g(0)| = 0$ and $|f(1) - f(1)| = 0$ so the endpoints are not it. We know that there must be some point in $[0,1]$ where a maxima or minima of $g(x) - f(x)$ is attained.

$$\begin{aligned} g(x) - f(x) &= x^2 - x \\ &= x(x-1) \\ \frac{d}{dx}(g-f)(x) &= 2x - 1 \\ 2x - 1 &= 0 \\ x &= \frac{1}{2} \end{aligned}$$

$(g - f)(1/2) = -1/4$ so

$$\rho(f,g) = \frac{1}{4}$$

$\square$

Problem 13.4 (Problem 4.5.7, Spaces p.101). *For $f \in B(\mathbb{R}, \mathbb{R})$ and $r \in \mathbb{R}$, we define a function f_r by $f_r(x) = f(x+r)$.*
a) Show that if f is uniformly continuous, then $\lim_{r \to 0} \rho(f_r, f) = 0$.
b) Show that the function g defined by $g(x) = \cos(\pi x^2)$ is not uniformly continuous on $\mathbb{R}$.

c) Is it true that $\lim_{r\to 0}\rho(f_r, f) \to 0$ *for all* $f \in B(\mathbb{R}, \mathbb{R})$*?*

Proof of (a). Suppose $f \in B(\mathbb{R}, \mathbb{R})$ is uniformly continuous. Namely, $\forall\varepsilon > 0,\ \exists\delta > 0$ for which

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon,\ \forall x, y \in \mathbb{R}$$

We are asked to show that

$$\lim_{r\to 0}\rho(f_r, f) = 0$$

In other words

$$\sup_{x\in\mathbb{R}}|f(x+r) - f(x)| \underset{r\to 0}{\to} 0$$

Notice then that $f(x+r)$ converges to $f(x)$ for every x. Let $\varepsilon > 0$ and choose δ from uniform continuity. If $|r| < \varepsilon$, then for every $x \in \mathbb{R}$,

$$|(x+r) - r| = |r| < \delta$$

Therefore

$$|f(x+r) - f(x)| < \varepsilon,\ \forall x \in \mathbb{R}$$

Hence

$$\rho(f_r, f) \leq \varepsilon \text{ whenever } |r| < \delta$$

so

$$\lim_{r\to 0}\rho(f_r, f) = 0$$

$\square$

Problem 13.5. *The space* $C_b(X, Y)$ *is always "larger" than* Y*, in the sense that* Y *can be embedded in* $C_b(X, Y)$*:*
Indeed, show that the map $i : Y \to C_b(X, Y)$ *which maps* $y \in Y$ *to the constant function* $f(x) \equiv y$*, is an embedding (cf. Definition 3.1.3).*
Show that $i(Y)$ *is precisely the subset of constant functions, and that this set is a closed subset of* $C_b(X, Y)$*.*
Conclude that $C_b(X, Y)$ *is complete if and only if* Y *is complete.*

Proof sketch. Define $i : Y \to C_b(X, Y)$ by

$$y \mapsto f(x) \equiv y$$

We begin by showing the necessary condition of an embedding before arguing that it is injective (as the latter will follow clearly from the former).

Let $y_1, y_2 \in Y$ and denote constant functions which map to them by f_{y_1}, f_{y_2} respectively.

$$\begin{aligned}\rho(f_{y_1}, f_{y_2}) &= \sup_{x \in X} d_Y(f_{y_1}(x), f_{y_2}(x)) \\ &= \sup\{d_Y(y_1, y_2), d_Y(y_1, y_2) \ldots\} \\ &= d_Y(y_1, y_2)\end{aligned}$$

It is clear that for distinct y_1, y_2 their images are distinct since otherwise the supremum distance between f_{y_1}, f_{y_2} would be 0, but then it would follow that $d_Y(y_1, y_2) = 0$. Therefore we conclude that i is an injection.

From a similar argument we can also see that there necessarily exists one and only one constant function f_y for every $y \in Y$, so $i[Y]$ must be exactly the set of all of these, and therefore we could create a one-to-one mapping from y's and constant functions. $\square$