

# Lecture Notes: Abstract Algebra - Isomorphisms

## (Course By: Alvaro Lozano-Robledo)

Thobias K. Høivik

March 8, 2025

**Definition 1.** Two groups  $(G, \star_G)$  and  $(H, \star_H)$  are said to be **isomorphic**  $\Leftrightarrow G \cong H$  if there exists a bijective function

$$\phi : G \rightarrow H$$

that preserves the group operation, i.e.,  $\forall a, b \in G$ ,

$$\phi(a \star_G b) = \phi(a) \star_H \phi(b).$$

A function

$$\psi : G \rightarrow H$$

that preserves the group operation, but is not necessarily bijective is called a **homomorphism**. In other words an **isomorphism** is a bijective **homomorphism**.

### Example

Show  $(\mathbb{Z}/_4\mathbb{Z}, +) \cong (\mu_4, \times)$  where  $\mu_4 = \{1, -1, i, -i\}$  are the 4-th roots of unity in  $\mathbb{C}^x$ .

*Proof.* Let  $\phi : \mathbb{Z}/_4\mathbb{Z} \rightarrow \mu_4$ ,  $\phi(n) = i^n$ .

$$\begin{aligned}\phi(1) &= i \\ \phi(2) &= i^2 = -1 \\ \phi(3) &= i^3 = -i \\ \phi(0) &= i^0 = 1\end{aligned}$$

thus it is injective and surjective  $\Leftrightarrow$  it is bijective, and

$$\phi(m+n) = i^{m+n} = i^m \times i^n$$

so  $\phi$  is a structure preserving bijection, making it an isomorphism. □

## Another Example

Let

$$F = C^\infty(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f^{(n)} \text{ exists for all } n \in \mathbb{Z}_{\geq 0}\}$$

be the space of infinitely differentiable functions. Define addition as  $(f+g)(x) = f(x) + g(x)$  for all  $f, g \in F$ . Show that  $F \cong F$  via the mapping

$$\phi : F \rightarrow F, \quad \phi(f)(x) = \int_0^x f(t)dt.$$

*Proof.* We set out to prove that  $\phi$  is a bijective homomorphism. Let  $f, g \in F$ ,

### 1: Homomorphism

$$\begin{aligned} \phi(f+g)(x) &= \int_0^x f(t) + g(t)dt \\ &= \int_0^x f(t)dt + \int_0^x g(t)dt = \phi(f)(x) + \phi(g)(x) \end{aligned}$$

### 2: Injectivity

$$\begin{aligned} \phi(f)(x) &= \phi(g)(x) \\ \int_0^x f(t)dt &= \int_0^x g(t)dt \\ \frac{d}{dx} \int_0^x f(t)dt &= \frac{d}{dx} \int_0^x g(t)dt \end{aligned}$$

By the fundamental theorem of calculus:

$$f(x) = g(x)$$

### 3: Surjectivity

$$\begin{aligned} \phi(f)(x) &= g(x) \\ \int_0^x f(t)dt &= g(x) \end{aligned}$$

By the fundamental theorem of calculus:

$$\begin{aligned} \frac{d}{dx} \int_0^x f(t)dt &= f(x) \\ \Rightarrow f(x) &= \frac{d}{dx} g(x) \\ \forall g \in F : g' \in F, \quad \therefore f \in F \end{aligned}$$

Thus we have shown  $\phi$  to be a bijective homomorphism, completing the proof. □

**Lemma 1.** Let  $\phi : G \rightarrow H$  be an isomorphism of groups. Then  $\phi(e_H) = e_H$ .

*Proof.* Let  $\phi(e_G) = h \in H$ . Also

$$\phi(e_G) = \phi(e_G \star_G e_G) = \phi(e_G) \star_H \phi(e_G) = h \star_H h$$

So  $h = h \star_H h$  and so

$$h \star_H h^{-1} = h^{-1} \star_H (h \star_H h) = h$$

Hence  $h = e_H$ . □

## Example

Prove that  $\mathbb{Z}/_4\mathbb{Z} \not\cong \mathbb{Z}/_2\mathbb{Z} \times \mathbb{Z}/_2\mathbb{Z}$ .

*Proof.* Suppose there is an isomorphism  $\phi : \mathbb{Z}/_4\mathbb{Z} \rightarrow \mathbb{Z}/_2\mathbb{Z} \times \mathbb{Z}/_2\mathbb{Z}$ . Then

$$\exists g \in \mathbb{Z}/_2\mathbb{Z} \times \mathbb{Z}/_2\mathbb{Z} \text{ s.t } \phi(g) = 1 \bmod 4$$

If so,

$$\phi(g + g) = \phi(g) + \phi(g) = 1 + 1 = 2 \bmod 4,$$

but

$$g + g = 2g = (a, b) + (a, b) = (2a, 2b) \equiv (0, 0) \bmod 2$$

$$\phi(2g) = \phi(0) = 0 \not\equiv 2 \bmod 4$$

Thus we get

$$\phi(g + g) = 2 \bmod 4 \wedge \phi(g + g) = 0 \Rightarrow \perp,$$

a contradiction. So an isomorphism cannot exist between these groups, hence they are not isomorphic. □