MAT-INF3600 Assignment

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Problem 1

Solution for (a). Let $A = \{0, 1\}$, $c^{\mathfrak{A}}$ be any element in A and $R^{\mathfrak{A}}(x, y)$ if $x = y$. Let $f^{\mathfrak{A}}(x) = x$ be the identity function on A.

Then (i) is satisfied and (ii), $(\forall x)[R(x, f(x)]$ is satisfied. □

Solution for (b). Let $A = \{0, 1\}$, $c^{\mathfrak{A}}$ be any element in A and $R^{\mathfrak{A}}(x, y)$ if $x < y$ with $f^{\mathfrak{A}}(x) = x$ as before. Then, we get a 2 element universe, but $R(x, f(x))$ is not satisfied for any x since 0 is not less than itself and 1 is not less than itself. □

Solution for (c). Now we must construct a 5 element universe with an injective function.

Let $A = \{1, \ldots, 5\}$ with $c^{\mathfrak{A}} = 1$ (or any other choice), $R = \emptyset$ (again, arbitrary) and $f(5) = 1$, $f(x) = x + 1$ otherwise. □

Solution and proof of (d). Let $A = \mathbb{N}$ (0 included) with $c^{\mathfrak{A}} = 0$, $R = \emptyset$ (choice is arbitrary) and $f^{\mathfrak{A}}(n) = n + 1$ be the successor function.

Then $\forall x[f(x) \neq c]$ since $f^{\mathfrak{A}}(c^{\mathfrak{A}}) = 0 + 1$, $f^{\mathfrak{A}}(1) = 1 + 1$, and so on. By the peano axioms, 0 is not the successor of any natural number. Furthermore the second condition of f being injective is satisfied since

$$
\begin{aligned}
f(x) &= f(y) \\
x + 1 &= y + 1 \\
x &= y
\end{aligned}
$$

Hence $\mathfrak{A} \models \Gamma$.

Now to prove that any model of Γ has an infinite universe.

Suppose we have some model of Γ with a finite universe $A = \{c^{\mathfrak{A}}, x_1, x_2, \ldots, x_n\}$. We require $f : A \to A \setminus \{c^{\mathfrak{A}}\}$ and for it to be injective. Since A is finite we have an injective map from a set of size $n + 1$ to a set of size n which is not possible by the pigeonhole principle, thus we arrive at a contradiction.

To vizualize this more clearly we can attempt to construct an injection $f : A \to A$.

$$
\begin{aligned}
f^{\mathfrak{A}}(c^{\mathfrak{A}}) &= x_{i_1} \text{ where } x_{i_1} \neq c^{\mathfrak{A}} \\
f^{\mathfrak{A}}(x_1) &= x_{i_2} \text{ where } x_{i_2} \neq x_{i_1}, \text{ and } x_{i_2} \neq c \\
&\vdots \\
f^{\mathfrak{A}}(x_{n-1}) &= x_{i_n} \text{ where } x_{i_n} \neq x_{i_{n-1}}, \ldots, x_{i_1}, \text{ and } x_{i_n} \neq c^{\mathfrak{A}}
\end{aligned}
$$

But now we arrive at $f^{\mathfrak{A}}(x_n)$ which cannot go to $c^{\mathfrak{A}}$ as that violates $f(x) \neq c$ and $f^{\mathfrak{A}}(x_n)$ cannot go to any x_i as that would violate injectivity. So we cannot construct a well-defined injection that satisfies $f(x) \neq c$ for all x given a finite universe.

Hence any model of Γ necessarily has an infinite universe. □

Problem 2

Proof. Let $n \geq 1$ and let $\theta_1, \ldots, \theta_n$ be sentences. Let Σ be a set of formulas. We will prove, by induction, that $\Sigma \cup \{\theta_1, \ldots, \theta_n\} \vdash \phi$ if and only if $\Sigma \vdash \theta_1 \wedge \cdots \wedge \theta_n \to \phi$.

Base case $n = 1$.

$\Sigma \cup \theta \vdash \phi$ if any only if $\Sigma \vdash \theta \to \phi$, by **Theorem 2.7.4 (The Deduction Theorem)**.

Assume that for $n = k$. Now look at $n = k + 1$,

$$\begin{aligned}
\Sigma \cup \{\theta_1, \ldots, \theta_k, \theta_{k+1}\} &\vdash \phi \\
\Sigma \cup \{\theta_1, \ldots, \theta_n\} &\vdash \theta_{k+1} \to \phi \text{ by the regular Deduction Theorem} \\
\Sigma &\vdash \left[\bigwedge_{i=1}^{k} \theta_i\right] \to (\theta_{k+1} \to \phi) \text{ by assumption} \\
\alpha \to (\beta \to \gamma) &\text{ equivalent to } (\alpha \wedge \beta) \to \gamma \\
\text{Hence } \Sigma &\vdash \left[\bigwedge_{i=1}^{k+1=n} \theta_i\right] \to \phi
\end{aligned}$$

Now to show the implication in the other direction:

$$\begin{aligned}
\Sigma &\vdash \left[\bigwedge_{i=1}^{k+1=n} \theta_i\right] \to \phi \\
\Sigma &\vdash \left[\bigwedge_{i=1}^{k} \theta_i\right] \to (\theta_{k+1} \to \phi) \\
\Sigma \cup \{\theta_1, \ldots, \theta_k\} &\vdash \theta_{k+1} \to \phi \text{ by assumption} \\
\Sigma \cup \{\theta_1, \ldots, \theta_{k+1}\} &\vdash \phi \text{ by the regular Deduction Theorem}
\end{aligned}$$

This completes the proof. □

Problem 3

Proof of (a).

1.$\forall x[Rx \to Sx]$		
2.$(\forall x[Rx \to Sx]) \to (Ry \to Sy)$		$(Q1)$
3.$Ry \to Sy$	1,2	(PC)
4.$\neg Sy \to \neg Ry$	3	(PC)
5.$(\neg Sy \to \neg Ry) \to (\neg Sy \to \neg Ry)$		(PC)
6.$(\neg Sy \to \neg Ry) \to \forall y(\neg Sy \to \neg Ry)$		(QR)
7.$\forall y(\neg Sy \to \neg Ry)$	4,6	(PC)

Thus we have a deduction of $\forall y[\neg Sy \to \neg Ry]$ from $\forall x[Rx \to Sx]$. □

Proof of (b). First we recognize that $\phi \nvDash \psi$ if and only if $\{\phi, \neg\psi\}$ is satisfiable.

Let $\mathfrak{A}$ be a structure with universe $A = \{a, b\}$, $R^{\mathfrak{A}} = \{a\}$, $S^{\mathfrak{A}} = \{a, b\}$.

Then if $t = a$, Ra is true and Sa is true so $Rx \to Sx$ is true. If $t = b$, Rb is false so the implication is true regardless. Therefore $\mathfrak{A} \vDash \forall x[Rx \to Sx]$.

Now to check the other formula. If $t = b$ we have Sb, but we do not have Rb. Hence the implication does not hold for all terms and $\mathfrak{A} \nvDash \forall y[Sy \to Ry]$. □

Problem 4

Proof of (a).

$$
\begin{array}{lr}
1. R(x) \to (R(x) \vee S(x)) & (PC) \\
2. \forall x(R(x) \to [R(x) \vee S(x)]) & 1,\ \text{Lemma 2} \\
3. [\forall x(R(x) \to [R(x) \vee S(x)])] \to [R(t) \to (R(t) \vee S(t))] & (Q1) \\
4. R(t) \to (R(t) \vee S(t)) & 2,3 \quad (PC) \\
5. (R(t) \vee S(t)) \to [\exists x(R(x) \vee S(x))] & (Q2) \\
6. R(t) \to [\exists x(R(x) \vee S(x))] & 4,5 \quad (PC) \\
7. R(t) \to R(t) & (PC) \\
8. \exists x(R(x)) \to R(t) & 7 \quad (QR) \\
9. \exists x(R(x)) \to [\exists x(R(x) \vee S(x)] & 6,7 \quad (PC)
\end{array}
$$

Therefore

$$\vdash \exists x(R(x)) \to [\exists x(R(x) \vee S(x)]$$

□

Proof of (b). Here we will appeal to Lemma 1 by showing that

$$\forall x(R(x) \wedge \exists x \neg R(x)) \vdash \perp$$

$$
\begin{array}{lr}
1. \forall x(R(x) \wedge \exists x \neg R(x)) & \\
2. \forall x(R(x) \wedge \exists x \neg R(x)) \to R(t) \wedge \exists x \neg R(x) & (Q1) \\
3. R(t) \wedge \exists x \neg R(x) & 1,2 \quad (PC)
\end{array}
$$

□

Note: Fix (a) and complete (b)

Problem 5