Lecture Notes: Real Analysis — Uncountability of Real Numbers (Course: MIT 18.100A)

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April 7, 2025

Proving Uncountability

Theorem 1 (Triangle Inequality). $\forall x, y \in \mathbb{R} : |x + y| \leq |x| + |y|$

Proof. By the definition of absolute value, for any $x, y \in \mathbb{R}$, we have:

$$-|x| \leq x \leq |x|$$

$$-|y| \leq y \leq |y|$$

Adding these inequalities together:

$$-(|x| + |y|) \leq x + y \leq |x| + |y|$$

By the definition of absolute value, this implies:

$$|x + y| \leq |x| + |y|$$

which proves the theorem. $\square$

Definition 1. *Let $x \in (0, 1]$ and let $d_{-j} \in \{0, 1, \ldots, 9\}$ for $j \in \mathbb{N}$. We say x is represented by the digits $\{d_{-j} : j \in \mathbb{N}\}$, $x = 0.d_{-1}d_{-2}\ldots$ if $x = \sup\{d_{-1}10^{-1} + \cdots + d_{-n}10^{-n} | n \in \mathbb{N}\}$.*

Example:

$$0.25000 = \sup\{\frac{2}{10}, \frac{2}{10} + \frac{5}{100}, \frac{2}{10} + \frac{5}{100} + \frac{0}{1000}, \ldots\}$$

$$= \sup\{\frac{2}{10}, \frac{25}{100}\} = \frac{1}{4}$$

Theorem 2.

1. *For every set of digits $\{d_{-j} : j \in \mathbb{N}\}$ with $d_j \in \{0, 1, 2, \ldots, 9\}$, there exists a unique $x \in [0, 1]$ such that $x = 0.d_{-1}d_{-2}d_{-3}\ldots$*

2. *$\forall x \in (0, 1], \exists! \{d_{-j} : j \in \mathbb{N}\}$ such that $x = 0.d_{-1}d_{-2}\ldots$ and $0.d_{-1}\ldots d_{-n} < x \leq 0.d_{-1}\ldots d_{-n} + 10^{-n}$.*

Theorem 3 (Cantor). $(0, 1]$ *is uncountable.*

Proof. Assume, by contradiction, that $|(0, 1]| = |\mathbb{N}|$. Then $\exists f : \mathbb{N} \to (0, 1]$ which is bijective.
$\forall n$, write $f(n) = 0.d^n_{-1}d^n_{-2}\dots$ satisfying $f(n) \leq 0.d^n_{-1}d^n_{-2}\cdots + 10^{-n}$.
Let

$$e_{-j} = \begin{cases} 1 \text{ if } d^j_{-j} \neq 1 \\ 2 \text{ if } d^j_{-j} = 1 \end{cases}$$

By 1) of the previous theorem $\exists! y \in (0, 1]$ such that $y = 0.e_{-1}e_{-2}\dots$. Since all e_{-j} are either 1 or 2, they are non-zero. $\forall n \in \mathbb{N} : 0.e_{-1}e_{-2}\dots e_{-n} < y \leq 0.e_{-1}\dots e_{-n} + 10^{-n}$. y is then the unique decimal representation of this number from 2) in the previous theorem. Since f is surjective, $\exists m \in \mathbb{N}$ such that $y = f(m)$.
Then

$$d^m_{-m} = e_{-m} = \begin{cases} 1 \text{ if } d^m_{-m} \neq 1 \\ 2 \text{ if } d^m_{-m} = 1 \end{cases} \neq d^m_{-m}$$

Thus we have arrived at a contradiction so our assumption that $(0, 1]$ is countable is false. $\square$

Sequences and Series

Definition 2. *A sequence of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$. We denote $f(n)$ with X_n and the sequence by $\{X_n\}^\infty_{n=1}$ or $\{X_n\}$ or otherwise $X_1, X_2, X_3, \dots$*
We may also write a_n.

Example: $\{\frac{1}{n}\} = 1, \frac{1}{2}, \frac{1}{4}, \dots$

Definition 3. *A sequence X_n is bounded if $\exists b \geq 0$ such that $\forall n \in \mathbb{N} : |X_n| \leq B$.*

Example: $\{\frac{1}{n}\}$ is bounded by 1 because it is never larger than 1 for any natural number n.
Non-Example: $\{n\}$ is not bounded as it will just grow and grow for increasing n.

Proof. Let $B \geq 0$. By Archimedes Property $\exists n \in \mathbb{N} : n > B$. Therefore $\{n\}$ is unbounded. $\square$

Definition 4. *A sequence $\{X_n\}$ converges to $X \in \mathbb{R}$ if $\forall \mathcal{E} > 0, \exists M \in \mathbb{N} : \forall n \geq M$, $|X_n - X| < \mathcal{E}$. If a sequence converges we say it's convergent, otherwise it is divergent.*

Theorem 4. *If $x, y \in \mathbb{R}$ and $\forall \mathcal{E} > 0, |x - y| < \mathcal{E} \Rightarrow x = y$.*

Proof. Suppose $x, y \in \mathbb{R}$ and $\forall \mathcal{E} > 0, |x - y| < \mathcal{E}$, $x \neq y$. Then $|x - y| > 0$. Then by $|x - y| < \frac{|x-y|}{2} \Rightarrow \frac{1}{2}|x - y| < 0 \Rightarrow |x - y| < 0$ which is not possible. $\square$

Theorem 5. *If $\{X_n\}$ converges to x and y, then $x = y$.*

Proof. Suppose, by contradiction, that $\{X_n\}$ converges to x and y where $x \neq y$. Then $\forall \mathcal{E} > 0, \exists N_x \in \mathbb{N}$ such that $\forall n_x \geq N_x : |X_{n_x} - x| < \mathcal{E}$ and $\forall \mathcal{E} > 0, \exists N_y \in \mathbb{N}$ such that $\forall n_y \geq N_y : |X_{n_y} - y| < \mathcal{E}$. Then if we choose $N = \max\{N_x, N_y\}$ we get that $\forall \mathcal{E} > 0, \exists N \in \mathbb{N}$ such that $\forall n > N : |X_n - x| < \mathcal{E} \wedge |X_n - y| < \mathcal{E}$.

Then

$$|X_n - x| - |X_n - y| < \mathcal{E} - \mathcal{E} = 0$$
$$|X_n - x| \le |X_n| + |-x| = |X_n| + |x|$$
via the triangle inequality
$$|X_n - x| - |X_n - y| \le (|X_n| + |x|) - (|X_n| + |y|) < 0$$
$$\Rightarrow |x| - |y| \le |x - y| < 0$$
via the reverse triangle inequality

Which is not possible as the absolute value returns non-negative numbers. Thus our assumption that $x \ne y$ was false, meaning that a sequence converges to one number only.

$\square$