Lecture Notes: Professor Dave's Differential Equations Linear Second-Order Differential Equations Homogeneous Case

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Second-Order Differential Equations

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x$$

Second-order differential equations are differential equations with a second derivative in them. They are very commonm in physics. For instance:

$$F = ma \Leftrightarrow m\frac{d^2x}{dt^2} = \sum F$$

Linear Homogeneous Second-Order Differential Equations

General form:

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Constant coefficients case:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Characteristic/auxiliary equation:

$$a\lambda^2 + b\lambda + c = 0$$

where we can solve for λ using the quadratic equation. Ansatz solution:

$$e^{\lambda x}$$

Superposition principle:
Given linearly independent solutions: y_1 and y_2 yields a general solution: $y = Ay_1 + By_2$
Now, if we return to our characteristic roots λ_1, λ_2 we have the following possibilities:

1. Values of lambda are real and distinct: Ansatz solutions:
$$e^{\lambda_1 x}, e^{\lambda_2 x}$$
(linearly independent) $\Rightarrow$ general solution:
$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$
2. Values of lambda are complex conjugates; $\lambda_{1,2} = \alpha \pm \beta i$. Then
$$y = e^{\alpha x} \cos \beta x + e^{\alpha x} \sin \beta x$$
yielding, via the superposition principle, the general solution:
$$y = e^{\alpha x}(A \cos \beta X + B \sin \beta x)$$
3. Lambda is a single repeated real root.
$$y = Ae^{\lambda x} + Bxe^{\lambda x} = e^{\lambda x}(A + Bx)$$

Boundary conditions

Initial value problem: values of $y, dy/dx, y'$ at x.
Boundary value problem: two different values of y at two different values of x.

Application of Second-Order Differential Equations

Suppose we have a trolley with mass $m\ kg$ moving on frictionless horizontal rails. Attatched to the side the trolley is a spring with stiffness $k\ N/m$, connecting the trolley to a wall, as well as a dashpot with a damping rate of $c\ (N \cdot s)/m$. We know the following laws hold in this system:

$$F = -kx \text{ /Hooke's law}$$
$$F = -cv = -c\frac{dx}{dt} \text{ /Dashpot}$$
$$\sum F = ma = m\frac{d^2x}{dt^2} \text{ /Newton's second law}$$

The spring- and dashpot forces constitute the net force on the system so we know that

$$\sum F = -kx - c\frac{dx}{dt}$$
$$ma - \sum F = 0$$
$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} - kx = 0$$