Lecture Notes: Professor Dave's Differential Equations Homogeneous Differential Equations and Bernoulli Differential Equations

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Homogeneous Functions

Definition 1. *A function $f(a, b)$ is said to be homogeneous if scaling the arguments by some parameter t results in f being scaled by some exponent of t: $f(ta, tb) = t^n f(a, b)$.*

Example: $f(a, b) = a^2 + ab$ is homogeneous because $f(ta, tb) = (ta)^2 + (ta)(tb) = t^2a^2 + t^2ab = t^2 f(a, b)$.

Definition 2. *The order of the homogeneous function is the constant exponent n which t is raised to: $f(ta, tb) = t^n f(a, b)$.*

So in the example above the order is 2.

Homogeneous Differential Equations

$$f(x, y)dx = g(x, y)dy$$

f, g are homogeneous of the same order. When we encounter homogeneous differential equation like this we can use a substitution $y = ux$ or $u = y/x$. Functions that can be written in terms of $\frac{dy}{dx} = f(\frac{y}{x})$ are also in this form.
Example:

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$$

This function is non separable nor linear nor is it exact. By cross-multiplying we get

$$(x^2 + y^2)dx = (xy)dy$$

Now we have a function in the form

$$f(x, y)dx = g(x, y)dy$$

and since

$$f(tx, ty) = ((tx)^2 + (ty)^2) = t^2(x^2 + y^2)$$

$$g(tx,ty) = (txty)dy = t^2(xy)dy$$

They are both homogeneous functions of order 2. Thus we have a homogeneous differential equation. We could also manipulate the original expression as such:

$$\frac{dy}{dx} = \frac{x^2+y^2}{xy} = \frac{x}{y} + \frac{y}{x} = \left(\frac{y}{x}\right)^{-1} + \frac{y}{x} = f\left(\frac{y}{x}\right)$$

to show that the function is a homogeneous differential equation. We solve this differential equation by using the homogeneous substitution $y = ux$.

$$\frac{dy}{dx} = \frac{x^2+y^2}{xy}$$
$$y = ux$$
$$\frac{dy}{dx} = \frac{d}{dx}ux = x \cdot \frac{du}{dx} + \frac{dx}{dx} \cdot u = x\frac{du}{dx} + u$$
$$x\frac{du}{dx} + u = \frac{x^2 + (ux)^2}{x(ux)} = \frac{1}{u} + u$$
$$x\frac{du}{dx} = \frac{1}{u}$$
$$\frac{du}{dx} = \frac{1}{ux}$$

giving us a separable differential equation with u as the dependent variable, instead of y.

$$(u)du = \frac{1}{x}dx \Rightarrow \int (u)du = \int \frac{1}{x}dx$$
$$\Rightarrow \frac{1}{2}u^2 + C_1 = \ln|x| + C_2$$
$$\frac{1}{2}u^2 = \ln|x| + C = \ln|x| + \ln|e^C| = \ln|x| + \ln|k| = \ln|kx|$$
$$u^2 = 2\ln|kx|$$
$$y = ux \Leftrightarrow u = \frac{y}{x}$$
$$\left(\frac{y}{x}\right)^2 = 2\ln|kx|$$
$$y^2 = 2x^2\ln|kx|$$

Bernoulli Differential Equations

While homogeneous differential equations can be reduced to separable differential equations (via substitution), Bernoulli differential equations can be reduced to linear differential equations (via substituion). Bernoulli differential equations can be written in the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

where P, Q are functions and n is constant (making this a nonlinear DE). The appropriate substitution for these types of differential equations is to let $u = y^{1-n}$.

Example:

$$\frac{dy}{dx} + x^5 y = x^5 y^7$$

$$u = y^{1-n} = y^{-6} \Rightarrow y = u^{-1/6}$$

if we differentiate both sides of the substitution with respect to x we get

$$y = u^{-1/6}$$

$$\frac{dy}{dx} = -\frac{1}{6}u^{-7/6}\frac{du}{dx} \text{ /via chain- and power rule}$$

We can then put this into our equation

$$-\frac{1}{6}u^{-7/6}\frac{du}{dx} + x^5 u^{-1/6} = x^5 u^{-7/6}$$

$$\cdot - 6u^{7/6}$$

$$\frac{du}{dx} - 6x^5 u = -6x^5$$

yielding a linear differential equation.

$$\mu(x) = e^{\int -6x^5\,dx} = e^{-x^6}$$

$$\mu(x)\frac{du}{dx} - 6x^5 u \cdot \mu(x) = -6x^5 \mu(x)$$

$$\frac{d}{dx}(u \cdot \mu(x)) = -6x^5 \mu(x)$$

$$\int \frac{d}{dx}(u \cdot \mu(x))\,dx = \int -6x^5 \mu(x)\,dx$$

$$u \cdot \mu(x) = \int -6x^5 e^{-x^6}\,dx$$

$$\text{Let } w = -x^6 \Rightarrow dw = -6x^5\,dx$$

$$u \cdot e^{-x^6} = \int e^w\,dw = e^w + C = e^{-x^6} + C$$

$$u(x) = 1 + Ce^{x^6}$$

$$y^{-6} = 1 + Ce^{x^6}$$

$$y = \left(1 + Ce^{x^6}\right)^{-1/6}$$