

Real Analysis

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1 Introduction

The following is intended for anyone who stumbles over these notes. This is intended to be my personal notes in real analysis. I will hopefully be attending MAT2400 Real Analysis at the University of Oslo in the spring of 2026. However, as I am not a program student at this institution, but just someone who takes individual courses there of my own volition, I do not and cannot attend lectures and therefore I have to learn the material on my own. As far as I understand, while this course is called Real Analysis, it is a bit different than a first course from what I understand. The earlier exams give a hint of functional analysis and also include topics such as fourier analysis, measure- and integration theory among other things. Thus the different supplementary course material I will use to aid myself in learning the content of this course will most likely be a bit scattered and all over the place, which these notes will undoubtedly reflect. I will do my best to keep things organized for my own sake, but keep this in mind if you are someone who intends to use these notes to learn Real Analysis.

2 Basic Banach Space theory

The following section of notes is derived from the first video in the lecture series MIT 18.102 Introduction to Functional Analysis, Spring 2021 (found on youtube).

Definition 2.1 (Vector Space). A vector space V over a field \mathbb{F} is a nonempty set of elements called "vectors" together with a binary operation $+$ on V and a binary function \cdot which maps elements of V, \mathbb{F} to V satisfying:

1. Associativity of vector addition:

$$u + (v + w) = (u + v) + w, \forall u, v, w \in V$$

2. Commutativity of vector addition:

$$u + v = v + w, \forall u, v \in V$$

3. Identity element:

$$\exists 0 \in V : v + 0 = 0 + v = v, \forall v \in V$$

4. Each $v \in V$ has an inverse $-v$ under the vector-addition operation.

5. Scalar multiplication is compatible with field multiplication:

$$a(bv) = (ab)v$$

where $a, b \in \mathbb{F}$ and $v \in V$.

6. The multiplicative identity $1 \in \mathbb{F}$ satisfies:

$$1v = v, \forall v \in V$$

7. Distributivity of scalar multiplication with respect to vector addition:

$$a(u + v) = au + av$$

where $a \in \mathbb{F}$ and $u, v \in V$.

8. Distributivity of scalar multiplication with respect to field addition:

$$(a + b)v = av + bv$$

where $a, b \in \mathbb{F}$ and $v \in V$.

When proving that something is a vector space, most of these follow naturally from showing closure under addition and scalar multiplication and those two properties, are generally enough to show that it is indeed a vector space.

A subspace U of V is a set $U \subseteq V$ which is also a vector space. It is enough to show that $U \subseteq V$ and that it is closed under the two operations.

Some typical examples of vector spaces are \mathbb{F}^n where \mathbb{F} is the reals or the complex numbers. We also have spaces like the space of real polynomials of degree $\leq n$, i.e. $\mathcal{P}_n = \{\sum_{i=0}^n \alpha_i x^i : \alpha_i \in \mathbb{R}\}$, which is itself a subspace of the space of continuous real-valued functions $C(\mathbb{R})$.

So \mathbb{R}^2 and $C(\mathbb{R})$ are both vector spaces over \mathbb{R} , but they have one really big difference, that being the dimension.

Definition 2.2. Let V be a vector space. A set $\{v_1, \dots, v_n\} \subseteq V$ is linearly independent if

$$\sum_{i=1}^n \alpha_i v_i = 0 \Leftrightarrow \alpha_1 = \dots = \alpha_n = 0 \in \mathbb{F}$$

Note: the right-to-left direction of this implication is always true.

The two spaces discussed above are different in dimension, \mathbb{R}^2 being 2-dimensional and the other being infinite-dimensional. One definition of finite-dimensional is that every linearly independent set in the space is finite. I however like the definition using bases more. Both of these definitions are equivalent.

We won't give a rigorous definition of a basis, but in short a basis of V is a linearly independent set of vectors which spans V , i.e. every vector in V can be expressed as a linear combination of basis-vectors. If the basis is finite then V is finite dimensional. Moreover, if the basis is finite then the dimension of V is the number of basis-vectors. Note that if a finite dimensional space V has a basis with n elements then every basis of V has n elements. A space is infinite-dimensional if no finite set of linearly independent vectors spans the space.

2.1 Norms and Metrics

Definition 2.3 (Norm). Let V be a vector space. A norm $\|\cdot\|$ is a function from $V \rightarrow [0, \infty)$ satisfying:

1. $\|v\| = 0$ if and only if $v = 0$.
2. $\|\alpha v\| = |\alpha| \cdot \|v\|$ where α is an element of the ground field.
3. $\|v + w\| \leq \|v\| + \|w\|$.

The tuple $(V, \|\cdot\|)$ is called a normed space.

Example 2.1. $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$ defines a norm on \mathbb{R}^n . In fact it constitutes a norm on \mathbb{C}^2 as well. Formally, if we take $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ the p -norm of a vector $v \in \mathbb{F}^n$ ($p \in [1, \infty]$) is

$$\|v\|_p := \begin{cases} \left(\sum_{i=1}^n |v_i|^p \right)^{1/p} & p < \infty \\ \max_{i=1, \dots, n} |v_i| & p = \infty \end{cases}$$

Norms give us a notion of the "length" of a vector. Now all we need to do analysis on spaces is a notion of distance. Intuitively, norms already give us a notion of distance from 0.

Definition 2.4. Let X be a set. A metric is a function $d: X^2 \rightarrow [0, \infty)$ satisfying:

1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$.

The metric gives us a notion of distance. In a typical first course in analysis where we work on the reals, $d(a, b) = |a - b|$ is the metric we deal with.

Proposition 2.1. *Let V be a normed space with norm $\|\cdot\|$. Then we can define the distance (a metric) between two vectors by*

$$d(x, y) := \|x - y\|$$

In other words you can define a metric in terms of the norm in any normed space. This metric is usually referred to as the metric induced by the norm.

We won't provide a proof of this as it's fairly intuitive. Now we can get a sense of convergence and continuity in vector spaces by saying that a sequence $\{a_n\}_{n \in \mathbb{N}}$ converges to a value a if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : n \geq N \Rightarrow \|a_n - a\| < \varepsilon$$

and a linear transformation (look up definition if necessary) $T \in \mathcal{L}(U, V)$ is (uniformly) continuous if

$$\forall \varepsilon > 0, \exists \delta > 0 : \|x - y\|_U < \delta \Rightarrow \|Tx - Ty\|_V < \varepsilon$$

for every $x, y \in U$. Notice that if you replace $\|x - y\|$ with $d(x, y)$ it looks like the standard definitions in terms of metric spaces.

2.2 Banach Spaces

Definition 2.5 (Banach Space). A normed space V is a Banach Space if it is complete with respect to the metric induced by the norm, meaning that every Cauchy sequence converges to a value in the space.

Example 2.2. \mathbb{R}^n or \mathbb{C}^n form Banach Spaces with respect to the ℓ^p norms (see Example 2.1).

Theorem 2.1. If X is a complete metric space, then $C_\infty(X)$ is a Banach Space.

Recall that $C_\infty(X)$ is the space of bounded continuous functions on X .

Proof. We show that every Cauchy sequence in $C_\infty(X)$ converges to an element of $C_\infty(X)$.

Let $\{u_n\}_{n=1}^\infty \subseteq C_\infty(X)$ be a Cauchy sequence with respect to the supremum norm. Then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\|u_n - u_m\|_\infty < \varepsilon \quad \text{for all } n, m \geq N.$$

Equivalently,

$$|u_n(x) - u_m(x)| < \varepsilon \quad \text{for all } x \in X \text{ and all } n, m \geq N.$$

Fix $x \in X$. Then $\{u_n(x)\}_{n=1}^\infty$ is a Cauchy sequence in \mathbb{R} (or \mathbb{C}), since

$$|u_n(x) - u_m(x)| \leq \|u_n - u_m\|_\infty.$$

Because \mathbb{R} (or \mathbb{C}) is complete, the limit

$$u(x) := \lim_{n \rightarrow \infty} u_n(x)$$

exists. This defines a function $u : X \rightarrow \mathbb{R}$.

We now show that $u_n \rightarrow u$ uniformly on X . Let $\varepsilon > 0$ and choose N such that $\|u_n - u_m\|_\infty < \varepsilon$ for all $n, m \geq N$. Fix $n \geq N$ and $x \in X$. Taking the limit $m \rightarrow \infty$ gives

$$|u_n(x) - u(x)| = \lim_{m \rightarrow \infty} |u_n(x) - u_m(x)| \leq \varepsilon.$$

Since $x \in X$ was arbitrary, it follows that

$$\|u_n - u\|_\infty \leq \varepsilon \quad \text{for all } n \geq N.$$

Thus $u_n \rightarrow u$ uniformly on X .

Since each u_n is bounded and the convergence is uniform, the limit function u is bounded. Moreover, since each u_n is continuous and uniform limits of continuous functions are continuous, u is continuous on X .

Therefore $u \in C_\infty(X)$ and $\{u_n\}$ converges to u in the supremum norm. Hence $C_\infty(X)$ is complete, and thus a Banach space. \square

3 A step back into Calculus Theory

The following section is largely in accordance with chapter 2 of Lindstrøm's Spaces: An Introduction To Real Analysis.

Definition 3.1 (Limit of a sequence). Let (x_n) be a sequence of real numbers and $x \in \mathbb{R}$.

We say

$$x_n \rightarrow x$$

if for every $\varepsilon > 0$, there exists a natural number N such that

$$|x_n - x| < \varepsilon, \forall n \geq N$$

This is among the most important definitions in a first course in Real-Analysis. Intuitively what it captures is the sense of being able to get arbitrarily close to a value. We will all be familiar with examples from calculus like $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

Theorem 3.1. If a sequence converges, then its limit is unique.

Proof. Suppose, for a contradiction, that we have some sequence (x_n) with

$$x_n \rightarrow x \text{ and } x_n \rightarrow y$$

of course with $x \neq y$.

x_n converges so we can choose any $\varepsilon > 0$ for the condition in definition 3.1 to hold.

Consider the case of $\varepsilon = \frac{|x-y|}{2}$. $\varepsilon > 0$ since $x \neq y$ and is therefore a value for which the condition must hold. Namely,

$$|x_n - x| < \varepsilon \text{ and } |x_n - y| < \varepsilon$$

leading to

$$|x - x_n| + |x_n - y| < 2\varepsilon = |x - y|$$

which, via triangle inequality, yields

$$|x - y| = |x - x_n + x_n - y| \leq |x - x_n| + |x_n - y| < |x - y|$$

in particular

$$|x - y| < |x - y|$$

which is not possible, a contradiction.

□

Theorem 3.2 (Algebra of Limits). If $x_n \rightarrow x$ and $y_n \rightarrow y$, then:

1. $x_n + y_n \rightarrow x + y$
2. $x_n y_n \rightarrow xy$
3. If $y_n \neq 0$ and $y \neq 0$, then $\frac{x_n}{y_n} \rightarrow \frac{x}{y}$.

Proof of (1): Sum of Limits. Let (x_n) and (y_n) be convergent sequences with $x_n \rightarrow x$ and $y_n \rightarrow y$.

Let $\varepsilon > 0$. Since both sequences converge to their respective limits we have some $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned} |x_n - x| &< \varepsilon/2, \forall n \geq N_1 \\ |y_n - y| &< \varepsilon/2, \forall n \geq N_2 \end{aligned}$$

Let $N = \max\{N_1, N_2\}$, then $n \geq N$ satisfies $n \geq N_1$ and $N \geq N_2$ so

$$|x_n - x| + |y_n - y| < \varepsilon$$

Notice that via the triangle inequality we get

$$|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y| < \varepsilon$$

completing the proof. \square

Theorem 3.3. A sequence x_n converges to x if and only if

$$\limsup x_n = \liminf x_n = x$$

We will take this theorem without proof as it requires some background about monotone sequences and completeness which are covered later.

Definition 3.2 (Limit of a function). Let $f: D \subset \mathbb{R} \rightarrow \mathbb{R}$, and let a be a limit point of D .

We say

$$\lim_{x \rightarrow a} f(x) = L$$

if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Later, when we get to metric spaces again, this condition becomes

$$d_X(x, a) < \delta \Rightarrow d_Y(f(x), f(a)) < \varepsilon$$

for continuity.

Theorem 3.4 (Sequential criterion).

$$\lim_{x \rightarrow a} f(x) = L \Leftrightarrow \text{for every sequence } x_n \rightarrow a \text{ with } x_n \neq a, f(x_n) \rightarrow L$$

Proof sketch. We only prove \Rightarrow , for now.

Assume $\lim_{x \rightarrow a} f(x) = L$.

Let $x_n \rightarrow a$ with $x_n \neq a$.

Given $\varepsilon > 0$, choose $\delta > 0$ from the definition of the limit.

Since $x_n \rightarrow a$ there exists $N \in \mathbb{N}$ such that with $n \geq N$ we have

$$|x_n - a| < \delta$$

Hence for $n \geq N$,

$$|f(x_n) - L| < \varepsilon$$

Thus $f(x_n) \rightarrow L$.

□

4 Completeness

Definition 4.1 (Upper Bound). A set $A \subset \mathbb{R}$ is bounded above if there exists $M \in \mathbb{R}$ such that

$$a \leq M, \forall a \in A$$

Such an M is called an upper bound.

Definition 4.2 (Supremum). Let $A \subset \mathbb{R}$. A number $s \in \mathbb{R}$ is the supremum of A if it is an upper bound of A and for any upper bound u of A we have

$$s \leq u$$

and we use the notation

$$s = \sup A$$

The completeness axiom says that every nonempty subset of \mathbb{R} that is bounded above has a supremum in \mathbb{R} , this being the distinguishing quality separating \mathbb{R} from \mathbb{Q} .

Sidenote: You can also characterize the reals as the *unique* ordered field containing \mathbb{Q} which has the least upper bound property. More precisely, every \mathbb{F} with this property satisfies $\mathbb{F} \cong \mathbb{R}$.

Proposition 4.1. If A has a supremum then for every $\varepsilon > 0$, there exists $a \in A$ such that

$$s - \varepsilon < a \leq s$$

Proof. Let $\varepsilon > 0$. If no $a \in A$ existed with the desired property, then $s - \varepsilon$ would be a supremum, contradicting minimality of s . \square

This proposition becomes useful later in many arguments.

The infimum $\inf A$ is also something we need and classically we define it as being the greatest lower bound, but for the purposes of keeping things simple, we just let the infimum of a set A , $\inf A$, be defined as

$$\inf A = -\sup(-A)$$

Theorem 4.1 (Monotone convergence). *Let (x_n) be a monotone increasing sequence (a sequence with which $x_n \leq x_{n+1}$ for every $n \in \mathbb{N}$) which is bounded above. Then (x_n) converges, and*

$$\lim x_n = \sup\{x_n : n \in \mathbb{N}\}$$

Proof. Let

$$A = \{x_n : n \in \mathbb{N}\}$$

Since A is bounded above, A has some supremum s . We claim $x_n \rightarrow s$. Let $\varepsilon > 0$.

By the supremum property, there exists N such that

$$s - \varepsilon < x_N \leq s$$

Since the sequence is increasing, for all $n \geq N$,

$$s - \varepsilon < x_n \leq s$$

Thus

$$|x_n - s| < \varepsilon$$

□

Lemma 4.1. *Every cauchy sequence in \mathbb{R} is bounded.*

Proof. Let (x_n) be cauchy.

Choose $\varepsilon = 1$. Then there exists N such that

$$|x_n - x_m| < 1, \forall n, m \geq N$$

Fix $m = N$. Then for all $n \geq N$:

$$|x_n| \leq |x_N| + |x_n - x_N| < |x_N| + 1$$

Thus all terms are bounded.

□

Theorem 4.2. *Every cauchy sequence in \mathbb{R} converges.*

Proof idea. Let (x_n) be cauchy.

(x_n) is bounded. Use the cauchy property to construct nested intervals. Show the intersection of these to be nonempty via completeness of \mathbb{R} . (x_n) will converge to a point in this intersection.

A bit more formally, we choose indices n_1, n_2, \dots such that

$$|x_n - x_m| < 2^{-k}, \forall n, m \geq n_k$$

and define the intervals

$$I_k = [x_{n_k} - 2^{-k}, x_{n_k} + 2^{-k}]$$

Then each I_k is closed and bounded with $I_{k+1} \subset I_k$. Completeness will give us

$$\bigcap_{k \in \mathbb{N}} I_k \neq \emptyset$$

There will be an x in this set which is the limit point. □

5 Open and Closed Sets in Metric Spaces

Let X be a set and $A \subseteq X$. Intuitively, a point x is either **inside** of A , **outside** of A , or **on the boundary** of A . We know what it means to not be in A ; $a \notin A \Leftrightarrow a \in A^c$. We also know what $a \in A$ means, but set-theoretically we don't have a notion of being on the "boundary" of A . We can make sense of this notion in a metric space.

Recall that the open ball is the set $B(x;r) = \{z \in X : d(x,z) < r\}$. Likewise we can define the closed ball as follows.

Definition 5.1 (Closed Ball). *The closed ball centered at $x \in X$ with radius $r \geq 0$ is $B(x;r) = \{z \in X : d(x,z) \leq r\}$.*

Let (X,d) be a metric space with $A \subseteq X$.

- $x \in X$ is an interior point of A if $B(x;r) \subseteq A$ for some $r > 0$.
- $y \in X$ is an interior point of A if $B(y;r) \subseteq A^c$ for some $r > 0$.
- $z \in X$ is a boundary point of A if it is neither of the above, i.e. $B(z;r) \cap A \neq \emptyset$ and $B(z;r) \cap A^c \neq \emptyset$ for every $r > 0$.

Notably, every point in X is one of these three.

- $A^0 = \{\text{all interior points of } A\}$
- $\partial A = \{\text{all boundary points of } A\}$
- $\overline{A} = A \cup \partial A = ((A^c)^0)^c$

Proposition 5.1. *For any $A \subset X$, we have $\partial A = \partial(A^c)$.*

Theorem 5.1. *Useful characterizations of openness:*

- $A \subseteq X$ is open if $A = A^0$
- $A \subseteq X$ is open if A contains none of its boundary points.
- $A \subseteq X$ is open if $A \cap \partial A = \emptyset$
- $A \subseteq X$ is open if $\forall x \in A$ there is some $r > 0$ s.t. $B(x;r) \subseteq A$

Theorem 5.2. *Useful characterizations of openness:*

- $B \subseteq X$ is closed if $B = \overline{B}$
- $B \subseteq X$ is closed if B contains all of its boundary points.
- $A \subseteq X$ is closed if $\partial B \subseteq B$

Problem 5.1. Consider $X = \mathbb{R}$ with the canonical metric $|\cdot|$. Show that (a, b) is open and $[a, b]$ is closed for $a \leq b \in \mathbb{R}$, and $(a, b]$ is neither open nor closed for $a < b$.

Proof. Let $x \in (a, b)$, i.e. $a < x < b$. Take

$$r := \min\{x - a, x - b\} > 0$$

Take any $y \in (x - r, x + r)$. Then

$$y > x - r \geq a, y < x + r \leq b$$

Therefore

$$(x - r, x + r) \subset (a, b)$$

In other words, every $x \in (a, b)$ admits an open ball contained entirely in the interval, so (a, b) is open.

Notice that $\partial(a, b) = \{a, b\}$ and that $[a, b] = \partial(a, b) \cup (a, b)$ hence $[a, b]$ is the closure of (a, b) and thus is closed.

It is clear that $(a, b]$ is neither open nor closed as $\partial(a, b] = \{a, b\} \not\subset (a, b]$ and $(a, b] \cap \partial(a, b] \neq \emptyset$. \square

6 Week 4 problem set

Proposition 6.1 (Proposition 3.1.4 in Spaces). *If (X, d) is a metric space with $x, y, z \in X$, then*

$$|d(x, y) - d(y, z)| \leq d(x, z)$$

Problem 6.1. *Prove Proposition 6.1.*

Proof. Let (X, d) be a metric space and recall the standard triangle inequality:

$$d(x, z) \leq d(x, y) + d(y, z)$$

Now observe the following rearrangements:

$$\begin{aligned} d(x, y) &\leq d(y, z) + d(x, z) \\ d(x, y) - d(y, z) &\leq d(x, z) \end{aligned}$$

and

$$\begin{aligned} d(x, y) &\leq d(y, z) + d(x, z) \\ d(y, z) &\leq d(x, y) + d(x, z) \\ d(y, z) - d(x, z) &\leq d(x, y) \\ -d(x, z) &\leq d(x, y) - d(y, z) \end{aligned}$$

Together:

$$|d(x, y) - d(y, z)| \leq d(x, z)$$

□

Problem 6.2 (3.1.6 Spaces). *Let $(V, \|\cdot\|)$ be a normed space. Show that it induces a norm, i.e.*

$$d(x, y) := \|x - y\|$$

is a norm.

Proof. Let $(V, \|\cdot\|)$ be a normed space.

Non-negativity and Identity of Indiscernibles:

$$d(x, y) = \|x - y\| \geq 0$$

since $x - y \in V$ and $\|\cdot\|$ is non-negative.

$$d(x, x) = \|x - x\| = \|0\| = 0$$

Assume $d(x, y) = 0$.

$$d(x, y) = \|x - y\| = 0$$

Then $x - y = 0 \Rightarrow x = y$.

Symmetry:

Recall for a norm we have $\|\alpha x\| = |\alpha|\|x\|$, hence

$$d(x, y) = \|1\| \|x - y\| = \|-1\| \|x - y\| = \|y - x\| = d(y, x)$$

Triangle Inequality:

$$\begin{aligned} d(x, z) &= \|x - z\| = \|x - (y + z - y)\| \\ &\leq \|x - y\| + \|y - z\| \\ &= \|x - y\| + \|z - y\| \\ &= d(x, y) + d(y, z) \end{aligned}$$

□

Problem 6.3 (3.1.7 Spaces). Show that if x_1, x_2, \dots, x_n are points in a metric space, then

$$d(x_1, x_n) \leq \sum_{i=1}^{n-1} d(x_i, x_{i+1})$$

Proof. We proceed by induction on $n \in \mathbb{N}$.

Base Case ($n = 1$).

Clearly,

$$d(x_1, x_1) \leq d(x_1, x_1)$$

Hypothesis. Now assume, for any $k \in \mathbb{N}$,

$$d(x_1, x_k) \leq \sum_{i=1}^{k-1} d(x_i, x_{i+1})$$

Induction. Let $n = k + 1$.

$$\begin{aligned} d(x_1, x_n) &= d(x_1, x_{k+1}) \leq d(x_1, x_k) + d(x_k, x_{k+1}) \\ &\leq \left(\sum_{i=1}^{k-1} d(x_i, x_{i+1}) \right) + d(x_k, x_{k+1}) \\ &= \sum_{i=1}^{n-1} d(x_i, x_{i+1}) \end{aligned}$$

as desired.

□

Problem 6.4 (3.2.1 Spaces). Assume that (X, d) is a discrete metric space. Show that the sequence $\{x_n\}$ converges to a if and only if there is an $N \in \mathbb{N}$ such that $x_n = a$ for all $n \geq N$.

Proof. Let (X, d) be a discrete metric space, i.e. a metric space where

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

\Leftarrow .

Assume there is some $N \in \mathbb{N}$ such that $\forall n \geq N$,

$$x_n = a$$

Then it is clear that for any $\varepsilon > 0$,

$$x_n = a \Rightarrow d(x_n, a) = 0 < \varepsilon$$

so $\{x_n\} \rightarrow a$.

\Rightarrow .

Assume that $\{x_n\} \rightarrow a$. Then for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \geq N$, then

$$d(x_n, a) < \varepsilon$$

In particular, this must hold for $\varepsilon = 0.5$. Notice that this requires $x_n = a$ since otherwise we would have $1 < 0.5$, a contradiction. \square

Problem 6.5 (3.2.5 Spaces). Let (X, d) be a metric space and fix some $a \in X$. Show that the function $f: X \rightarrow \mathbb{R}$ defined as

$$f(x) = d(x, a)$$

is continuous. Note: \mathbb{R} is finite dimensional so we are free to choose any metric. $d_{\mathbb{R}}(x, y) = |x - y|$ is fine.

Proof. Let (X, d) be a metric space and fix $a \in X$. Let f be defined as above.

Consider a family of sequences in X such that every sequence $\{x_n\}$ converges to $a \in X$. Let $\{x_n\}$ be some arbitrary sequence in this family.

Then, $\forall \varepsilon > 0$ we have some natural number N , s.t. for any $n \geq N$,

$$d(x_n, a) < \varepsilon$$

Notice that

$$d(x_n, a) = |d(x_n, a) - 0| < \varepsilon$$

where $\varepsilon > 0$ was arbitrary. Hence $f(\{x_n\})$ converges to $0 = d(a, a) = f(a)$.

Thus f is continuous. \square

Problem 6.6. Let (X, d) be a metric space. Prove that every finite subset of X is closed.

Proof. Recall that proposition 3.3.13 from Spaces tells us that for any finite collection F_1, \dots, F_n of closed sets, their union

$$\bigcup_{i \in [n]} F_i$$

is closed.

Consider any singleton $\{x\} \subseteq X$. It is clear that $\{x\}$ is closed since $\{x\}^c$ is open. Observe that the open ball $B(y; \varepsilon) \subseteq \{x\}^c$ for any $y \in X \setminus \{x\}$ with $\varepsilon = d(x, y)$.

Let $\{x_1, \dots, x_n\} \subseteq X$ be any finite subset. As shown above, $\{x_i\}$ is closed, and by proposition 3.3.13

$$\{x_1, \dots, x_n\} = \bigcup_{i \in [n]} \{x_i\}$$

is closed. \square

Problem 6.7 (3.3.13 Spaces). A metric space (X, d) is disconnected if it is the union of two non-empty, disjoint and open subsets. If it is not disconnected it is connected.

a) Let $X = (-1, 1) \setminus \{0\}$ and let d be the usual metric on X . Show that (X, d) is disconnected.

b) Let $X = \mathbb{Q}$ and let d be the usual metric again. Show that (X, d) is disconnected.

c) Assume that (X, d) is a connected metric space and that $f: X \rightarrow Y$ is continuous and surjective. Show that Y is connected.

Proof of (a). It is clear that $(-1, 0)$ and $(0, 1)$ are open with respect to $d(x, y) = |x - y|$ as, we can find an open ball around any point in either. Since

$$X = (-1, 0) \cup (0, 1)$$

we conclude that X is disconnected. \square

Proof of (b). Recall that $\sqrt{2} \notin \mathbb{Q}$ so the open subsets

$$(-\infty, \sqrt{2}) \cap \mathbb{Q} \text{ and } (\sqrt{2}, \infty) \cap \mathbb{Q}$$

Both of these sets are open in \mathbb{R} and thus they are also open in \mathbb{Q} . \square

Proof of (c). Suppose, for a contradiction that we have some continuous surjection $f: X \rightarrow Y$ where X is connected and Y is not.

From surjection we can glean that for any $y \in Y$ there is some $x \in X$ such that $f(x) = y$. From continuity at every point we have that there is some $\delta > 0$ for any $\varepsilon > 0$ such that

$$f[B_X(x; \delta)] \subseteq B_Y(f(x); \varepsilon)$$

By assumption

$$Y = O_1 \cup O_2$$

for some nonempty, disjoint, open O_1 and O_2 . So for any point in O_1 or O_2 we have an open neighbourhood surrounding it. Consider some family open sets in O_1 , $\{\omega_n\}_{n \in \mathbb{N}}$ such that

$$\bigcup_{n \in \mathbb{N}} \omega_n = O_1$$

It is clear that $\omega_i \not\subseteq O_2$ for any ω_i . Suppose ω_i is centered around $y_i \in Y$. By surjectivity we have that there are some $x_i \in X$ such that the image open ball around x_i goes to the open ball around y_i . It is not hard to see that we are characterizing an open set in X which is mapped to O_1 . Repeat this process for O_2 and we will see that since there is no open ball which falls in both sets, and correspondingly via continuity there then is no ball in X which falls in the pre-images of both O_1 and O_2 , hence we have partitioned X into two disjoint non-empty subsets. So X is disconnected, a catastrophic contradiction. \square

Another attempt at a proof of (c). Suppose, for a contradiction that we have a continuous surjection $f: X \rightarrow Y$ where X is connected and Y is not.

Let $A \subseteq X$ be a subset such that $f|_A: A \rightarrow Y$ is a bijection. Since it is the restriction of f to A is still continuous f constitutes a homeomorphism from $A \rightarrow Y$. Then, the inverse $g: Y \rightarrow A$ exists, which is still continuous of course.

Recall that a function g from a metric space $Y \rightarrow A$ is continuous if and only if for any $\varepsilon > 0$ we have some $\delta > 0$ such that

$$g[B_Y(y; \delta)] \subseteq B_A(g(y); \varepsilon)$$

for any $y \in Y$. By assumption, there exist O_1 and O_2 such that

$$Y = O_1 \cup O_2$$

with $O_1, O_2 \neq \emptyset$ and disjoint. Furthermore, O_1 and O_2 are open. We now claim that A is partitioned into two disjoint, nonempty, open sets

$$g[O_1] \text{ and } g[O_2]$$

It is not hard to see that they are both open as the open ball around any point in O_1 or O_2 is sent to some open ball around a point in their

image via continuity. Furthermore it is clear that they are nonempty since g is a bijection and O_1, O_2 are nonempty.

Lastly, assume (for contradiction) that there is some $y \in Y$ such that $g(y) \in g[O_1]$ and $g(y) \in g[O_2]$, i.e. they are not disjoint. Then we would have $f|_A(g(y)) \in O_1$ and $f|_A(g(y)) \in O_2$. Recalling that $f|_A \circ g = id_Y$ we have that $y \in O_1$ and $y \in O_2$, but it is established that no such point exists, contradicting the assumption that $g[O_1] \cap g[O_2] \neq \emptyset$.

Thus we have shown that A is disconnected, but then X is also disconnected, a catastrophic contradiction to our original assumption. \square

Problem 6.8 (3.3.13 Spaces (again)). *A space X is path connected if there is a continuous $r : [0, 1] \rightarrow X$ for every pair $x, y \in X$ such that $r(0) = x$ and $r(1) = y$.*

d) Let d be the usual metric on \mathbb{R}^n :

$$d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$$

Show that (\mathbb{R}^n, d) is path-connected.

e) Show that every path-connected metric space is connected.

Proof of (d). Consider two distinct points $x, y \in \mathbb{R}^n$. Then

$$x = (x_1, \dots, x_n), \quad y = (y_1, \dots, y_n)$$

Consider the vector

$$(y - x) \cdot z = (y_1 - x_1, \dots, y_n - x_n) \cdot z, \quad z \in [0, 1]$$

It is clear that

$$x + (y - x) \cdot 0 = x \quad \text{and} \quad x + (y - x) \cdot 1 = y$$

So define $r : [0, 1] \rightarrow X$ by

$$r(z) := x + (y - x) \cdot z$$

Let $z \in [0, 1]$ and fix any $z_0 \in \mathbb{Z}$, let $\varepsilon > 0$ and choose $\delta := \frac{\varepsilon}{\|x - y\|} > 0$. Assume

$$|z - z_0| < \delta$$

Then,

$$\begin{aligned}
||r(z) - r(z_0)|| &= ||(x + (y - x) \cdot z) - (x + (y - x) \cdot z_0)|| \\
&= \left(\sum_{i=1}^n ((x_i - zx_i + zy_i) - (x_i - z_0x_i + z_0y_i))^2 \right)^{1/2} \\
&= \left(\sum_{i=1}^n ((z_0 - z)x_i + (z - z_0)y_i)^2 \right)^{1/2} \\
&= \left(\sum_{i=1}^n ((z - z_0)(y_i - x_i))^2 \right)^{1/2} \\
&= |z - z_0| \left(\sum_{i=1}^n (y_i - x_i)^2 \right)^{1/2} \\
&= |z - z_0| \cdot ||x - y|| < \delta \cdot ||x - y|| = \frac{\varepsilon}{||x - y||} ||x - y|| \\
&= \varepsilon
\end{aligned}$$

□

7 Week 6 problem set

Problem 7.1 (Problem relating to Section 3.3). Let (X, d_X) and (Y, d_Y) be metric spaces, let $f: X \rightarrow Y$ be a continuous function, let $y \in Y$ be some given point, and consider the problem of finding an $x \in X$ such that $f(x) = y$ (that is, we wish to solve the above equation). Prove that the set of solutions of this equation is a closed subset of X .

Hint: Phrase the question in terms of finding f^{-1} of a closed set.

Proof. Apply the assumptions of the problem description.

Let $S_y \subseteq X$ denote the set of solutions to $f(x) = y \in Y$. Notice that S_y is the set $f^{-1}[\{y\}]$. Since $\{y\}$ is a singleton it is a closed set. In the topological sense, if f is continuous then the preimage of every closed set in Y is closed in X . This would complete the proof, but let us give an analytical way of doing it instead.

Once again let $S_y \subset X$ denote the set of $x \in X$ such that $f(x) = y \in Y$. Let $\{x_n\}$ be some sequence in S_y such that $x_n \rightarrow x$ in X . Now we need to show that $x \in S_y$, i.e. $f(x) = y$. Since $x_n \in S_y$ we have that (for every $n \in \mathbb{N}$)

$$f(x_n) = y$$

f is continuous so via sequential continuity we get

$$f(x_n) = f(x)$$

i.e.

$$f(x) = y$$

Thus $x \in S_y$. Since $\{x_n\}$ was an arbitrarily chosen convergent sequence in S_y and we found that $x_n \rightarrow x \in S_y$ we conclude that S_y contains all its limit points. In other words S_y is closed. \square

Problem 7.2. Prove that (\mathbb{Z}, d) , where $d(x, y) = |x - y|$, is complete. Hint: What does it mean for a sequence in (\mathbb{Z}, d) to converge or to be Cauchy?

Proof. Let $\{x_n\}$ be a Cauchy sequence in \mathbb{Z} . Then, for every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ s.t.

$$|x_n - x_m| < \varepsilon$$

for every $n, m \geq N$. Notice that we then require $x_n = x_m$ for all $n, m \geq N$ since if that weren't the case we would have $0 < \varepsilon := 0.5 < 1 \leq |x_n - x_m|$. Thus every Cauchy sequence in \mathbb{Z} will necessarily settle at some $x \in \mathbb{Z}$ and be constant. In other words, $\{x_n\} \rightarrow x \in \mathbb{Z}$, as desired. \square

Problem 7.3 (Problem 3.3.3 Spaces). Assume that F is a nonempty, closed and bounded subset of \mathbb{R} with the usual metric. Show that $\inf F \in F$ and $\sup F \in F$. Give an example of a bounded, but not closed set such that it contains its supremum and infimum.

Proof. Suppose $\emptyset \neq F \subseteq \mathbb{R}$ with F closed and bounded.

Recall that since F is bounded there is some finite $r > 0$, $x \in F$ such that

$$F \subseteq (x - r, x + r)$$

Recall that for any nonempty subset of \mathbb{R} there exists a least upper bound and greatest lower bound, i.e. $\inf F \in \mathbb{R}$ and $\sup F \in \mathbb{R}$.

Let $\{x_n\}$ be a sequence in F such that $\{x_n\} \rightarrow \sup F$. By closedness $\sup F \in F$. Apply this same reasoning to the infimum to get the desired result. Notice that we required boundedness as, if F wasn't bounded above and below we might have had $\inf F = -\infty$ or $\sup F = \infty$. In that case we couldn't have constructed sequences converging to these points.

Let $F = [0, 1] \setminus \{\frac{1}{2}\}$. Clearly not closed since any sequence converging to $1/2$ contradicts it being closed. It is bounded above and below and $\inf F = 0 \in F$ and $\sup F = 1 \in F$. \square

Problem 7.4 (Problem 3.3.11 Spaces). Prove Proposition 3.3.12. Find an infinite collection of open sets whose intersection is closed.

Proposition 7.1 (Proposition 3.3.12 from Spaces). Let (X, d) be a metric space.

a) If \mathcal{G} is a finite or infinite collection of open sets, then

$$\bigcup_{G \in \mathcal{G}} G$$

is open.

b) If G_1, \dots, G_n is a finite collection of open sets then

$$\bigcap_{i=1}^n G_i$$

is open.

Proof of (a). Consider a (potentially infinite) family $\mathcal{G} = \{G_1, G_2, \dots\}$ of open sets. Define

$$\Gamma := \bigcup_{G \in \mathcal{G}} G$$

Take any arbitrary $x \in \Gamma$. Then there is some G_i such that $x \in G_i$. As G_i is open there is some $r \in \mathbb{R}^+$ with

$$B(x; r) \subseteq G_i \subseteq \Gamma$$

Hence Γ is open. \square

Proof of (b). Let $\mathcal{G} = \{G_i\}_{i \in [n]}$ be a finite family of open sets. Define

$$\Gamma := \bigcap_{i \in [n]} G_i$$

Suppose $x \in \Gamma$. Then $\bigwedge_{i=1}^n x \in G_i$ and for each G_i there is some $r_i > 0$ such that

$$B(x; r_i) \subseteq G_i$$

Then

$$B(x; \min\{r_i : 1 \leq i \leq n\}) = \bigcap_{i=1}^n B(x; r_i) \subseteq \bigcap_{i \in [n]} G_i = \Gamma$$

so Γ is open. \square

Example 7.1 (Counter example for infinite intersection.). Consider the metric space $(\mathbb{R}, |\cdot|)$. Consider the following infinite family of open sets

$$\{(-1/n, 1/n)\}_{n \in \mathbb{N}}$$

Clearly,

$$\bigcap_{n \in \mathbb{N}} (-1/n, 1/n) = \{0\}$$

which is closed in \mathbb{R} .

Problem 7.5. prob Consider $X = \mathbb{R}$ with the canonical metric $d(x, y) = |x - y|$. The support of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is the set of points x where $f(x) \neq 0$. Prove that the support of a continuous function is always open. Note: In the literature, the "support" of f is usually defined to be the closure of the above set, which is of course closed, not open.

Proof. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and let

$$\Omega := \{x \in \mathbb{R} : f(x) \neq 0\}.$$

Suppose, for a contradiction, that Ω is not open. Then there exists $x \in \Omega$ such that for every $r > 0$ there is some $y \in (x - r, x + r)$ with $f(y) = 0$.

Since f is continuous at x , for $\varepsilon := |f(x)| > 0$ there exists $\delta > 0$ such that

$$|f(y) - f(x)| < \varepsilon \quad \text{whenever} \quad |y - x| < \delta.$$

By assumption, there exists $y \in (x - \delta, x + \delta)$ with $f(y) = 0$. Then

$$|f(x)| = |f(x) - f(y)| < |f(x)|,$$

a contradiction. Hence Ω is open. \square

Problem 7.6. Show that the equation $\cos t = 2t$ has a unique solution. Hint: Formulate the problem as finding the fixed point of a function f .

Proof. Banach's fixed-point theorem tells us that for some complete metric space X , a contraction $F : X \rightarrow X$ will have a unique $t \in X$ such that

$$t = F(t)$$

Let $F : [0, \pi/2] \rightarrow [0, \pi/2]$ be defined by

$$F(t) := \frac{1}{2} \cos t$$

Clearly, $F(t) = t$ if and only if $\cos t = 2t$. Furthermore, as a closed bounded subset of a complete metric space, $([0, \pi/2], |\cdot|_{[0, \pi/2]})$ is complete. Now all we need is to show that F constitutes a contraction on \mathbb{R} . Notice that

$$\frac{1}{2} \cos(t) \in [0, 1/2], \quad t \in [0, \pi/2]$$

so F certainly constitutes a contraction. Thus by Banach's fixed point theorem there is a unique point $t \in [0, \pi/2]$ such that $t = F(t)$. t constitutes the unique solution to

$$\cos(t) = 2t$$

\square