Lecture Notes: Abstract Algebra — Internal Direct Products (Course By: Alvaro Lozano-Robledo)

Thobias K. Høivik

March 21, 2025

The goal is to see if a group G is isomorphic to subgroups of G, i.e $G \cong H \times K, H, K \leq G$.
Example:
$U(8) = (\mathbb{Z}/_8\mathbb{Z})^\times = \{1, 3, 5, 7 \bmod 8\}$
$U(8) \cong \mathbb{Z}/_2\mathbb{Z} \times \mathbb{Z}/_2\mathbb{Z}$

Proposition 1. *Let $H = \{1, 3\} =< 3 >, K = \{1, 5\} =< 5 >$. Then $\phi : H \times K \to U(8)$ given by $(h, k) \to h \cdot k$ with the domain- and codomain elements taken mod 8 is a bijection, making $U(8) \cong H \times K$.*

Proof. Let $H = \{1, 3\}, K = \{1, 5\}$. Bijection:

$$
\begin{aligned}
(1, 1) &\to 1 \\
(3, 1) &\to 3 \\
(1, 5) &\to 5 \\
(3, 5) &\to 3 \cdot 5 \equiv 15 \equiv 7 \bmod 8
\end{aligned}
$$

Homomorphism:

$$
\begin{aligned}
\phi((h, k) \cdot (a, b)) &= \phi(ha, kb) \\
&= h \cdot a \cdot k \cdot b = hk \cdot ab \\
&= \phi(h, k) \cdot \phi(a, b)
\end{aligned}
$$

Thus, $U(8) \cong H \times K$ isomorphic to the direct product of two subgroups of $U(8)$. $\square$

Proposition 2. *Let G be a group, and let $H, K \leq G$ subgroups such that:*

1. $G = H \cdot K = \{h \cdot k : h \in H, k \in K\}$
2. $H \cap K = \{e\}$
3. $hk = kh, \forall h \in H, k \in K$

Then G is isomorphic to $H \times K$ via

$$\phi : H \times K \to G$$
$$(h, k) \to h \cdot k$$

If so we say that G is internal direct product of H and K.

Proof. $\phi : H \times K \to G, (h, k) \to h \cdot k$. The map is well defined,

$$h \in H \subseteq G \wedge k \in K \subseteq G \Rightarrow h \star k \in G$$

Injective:

$$\phi(h, k) = \phi(a, b) \Rightarrow h \cdot k = a \cdot b$$
$$\Rightarrow a^{-1}h = bk^{-1} \in H \cap K = \{e\}$$
$$\Rightarrow a = h \wedge b = k, \quad \therefore \boxed{(h, k) = (a, b)}$$

Surjective:
By assumption $HK = G$, so $\exists h \in H \wedge \exists k \in K$ such that $hk = g \in G$.
Homomorphism:

$$\phi(h, k) \cdot \phi(a, b) = hkab$$
$$(hk = kh, \forall h \in H \wedge \forall k \in K)$$
$$\Rightarrow hkab = hakb = \phi(ha, kb) = \phi((h, k) \cdot (a, b))$$

So $H \times K \cong G$. ∎