Algebraic and Enumerative Combinatorics: Overview

Thobias K. Høivik

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1 Introduction to Combinatorics

Combinatorics is the branch of mathematics concerned with counting, arrangement, and structure of sets, particularly finite sets. The two main subfields of combinatorics are:

- **Enumerative Combinatorics**: Counting the number of ways to arrange or select objects under various conditions.
- **Algebraic Combinatorics**: Applying algebraic techniques to combinatorial structures, often focusing on generating functions, group actions, and symmetric functions.

2 Basic Counting Principles

One of the fundamental problems in combinatorics is counting how many ways we can arrange or choose objects. Some basic counting principles include:

2.1 The Addition Rule

If we have two disjoint sets A and B, the total number of elements in $A \cup B$ is:

$$|A \cup B| = |A| + |B|.$$

This is known as the addition rule, and it generalizes to the union of any finite number of disjoint sets.

2.2 The Multiplication Rule

If we are selecting an element from set A and an element from set B, and the choice of an element from A does not affect the choice from B, the total number of possibilities is:

$$|A \times B| = |A| \times |B|.$$

This is known as the multiplication rule.

3 Permutations and Combinations

A permutation is an arrangement of objects in a specific order, while a combination is a selection of objects without regard to the order.

3.1 Permutations

The number of ways to arrange n objects in order is given by:

$$P(n) = n! = n \times (n-1) \times (n-2) \times \cdots \times 1.$$

If only r objects are selected from a set of n objects, the number of possible arrangements (permutations) is:

$$P(n,r) = \frac{n!}{(n-r)!}.$$

3.2 Combinations

The number of ways to choose r objects from a set of n objects, regardless of the order, is given by:

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

4 Introduction to Generating Functions

Generating functions are a powerful tool in combinatorics that allow us to encode sequences of numbers into a formal power series.

4.1 The Ordinary Generating Function (OGF)

The ordinary generating function of a sequence $\{a_n\}$ is defined as:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

This function helps in solving recurrence relations and counting problems.

4.2 The Exponential Generating Function (EGF)

The exponential generating function for a sequence $\{a_n\}$ is defined as:

$$A(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}.$$

EGFs are particularly useful for counting labeled structures, like trees or graphs.

5 Applications of Combinatorics

In algebraic and enumerative combinatorics, one often works with specific problems like:

- Counting the number of distinct objects under symmetries (e.g., counting distinct colorings or graphs).
- Solving recurrences using generating functions.
- Applying group theory to count invariant objects.

6 A fun problem

The lecturer introduced a problem at the start of the lecture which we were to find 4 formulas for. Given a $2 \times n$ rectangle, $n \in \mathbb{N}$, how many ways can we tile it with 2×1 rectangles.

1. Explicit formula
2. Recursive formula
3. Formula for generating function
4. Asymptotic formula

6.1 Explicit formula

Here we were simply given the formula:

$$\begin{aligned} T_n &= \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor} \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \end{aligned}$$

Which reveals to us that the question we were posed is the same thing as asking how many ways can we sum up to n using only 1 and 2.

6.2 Recursive formula

$$T_n = T_{n-1} + T_{n-2},\, T_0 = 0 \wedge T_1 = 0$$

Observe that this turns out to be the formula for the n^{th} fibbonacci number. This means that the number of ways to tile the rectangle is the same as numbers of ways to sum up to n using only 1s and 2s, which is the same as getting the n^{th} fibbonacci number. These are all the same question we're asking, just made more and more familiar.