Lecture Notes: Abstract Algebra (Course By: Alvaro Lozano-Robledo)

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Comparing Groups

Our long term goal here is to classify/characterize all groups, a tremendous task. We might start by doing this for finit groups or, if we let $n \geq 1$, characterize all groups G of order n.

Example

Consider the group $< Z/4Z, + >$, and it's cayley table

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

vs. $< G, \star >$ with

$\star$	a	b	c	d
a	c	d	b	a
b	d	c	a	b
c	b	a	d	c
d	a	b	c	d

The question then becomes: is $G \cong Z/4Z$?
Answer: Yes it is true if you rearrange

$$\begin{cases} a = 1 \\ b = 3 \\ c = 2 \\ d = 0 \end{cases}$$

so the table may be written as

$\star$	d	a	c	b
d				
a				
c				
b				

to make it match $Z/4Z$.

Example

What if we want to find finite groups $G, |G| = 8$?

1. $Z/8Z = \{0, 1, 2, 3, 4, 5, 6, 7 \bmod 8\}$
2. $Z/4Z \times Z/2Z = \{(a \bmod 4, b \bmod 2)\}$
3. $Z/2Z \times Z/2Z \times Z/2Z = \{(a \bmod 2), (b \bmod 2), (c \bmod 2)\}$,
4. $D_4 = \mathrm{Sym}(\square) = \{id, r, r^2, r^3, s, rs, r^2s, r^3s\}$
5. $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

Q1: Are these different, and **Q2:** are these all the unique groups of order 8? The answer to both of these as it turns out is yes; There are only 5 unique structure on sets with 8 elements which satisfy the group axioms. Any "new" structure of size 8 with the group axioms fulfilled would be one of these 5 groups in disguise.
Now, how would we go about showing that these are indeed different? Well, the first three groups are abelian, while the last two are non-abelian. This means that the first three are distinct from the last two. #1 = $Z/8Z$ is cyclic, but #2 and #3 are not cyclic, making #1 distinct from the other abelian groups. We can continue comparing the qualities of groups to see if they are indeed distinct. For instance for the #4 and #5 we can look at the order of the elements and observe that Q_8 has 1 element with order 2 while D_4 has multiple.