Lecture Notes: Real Analysis — Supremum and Infimum (From: Wrath of Math on Youtube)

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Definitions

Definition 1 (Upper Bound)**.** *Let S be a subset of a field F. An element $b \in F$ is called an upper bound of S if for all $x \in S$, $x \leq b$.*

Definition 2 (Lower Bound)**.** *Let S be a subset of a field F. An element $a \in F$ is called a lower bound of S if for all $x \in S$, $x \geq a$.*

Definition 3 (Supremum)**.** *Let F be an ordered field and $S \subseteq F$ be nonempty. The* ***supremum*** *of S, if it exists, is some $b_0 \in F$ such that*

1. *b_0 is an upper bound of S*
2. *if b is any other upper bound of S then $b_0 \leq b$*

If it exists, the supremum of S is denoted $sup(S)$.

Definition 4 (Infimum)**.** *Let F be an ordered field and $S \subseteq F$ be nonempty. The* ***infimum*** *of S, if it exists, is some $b_0 \in F$ such that*

1. *b_0 is a lower bound of S*
2. *if b is any other lower bound of S then $b_0 \geq b$*

If it exists, the infimum of S is denoted $inf(S)$.

We may also call the supremum and infimum the **least upper bound** and the **greates lower bound**, respectively. As always when we say field we are usually refering to $\mathbb{R}$ in real analysis as that is the premier field of interest in the subject. If we take S to be a subset of the reals we may visualize S as an interval on the real number line. The supremum could then be thought of as the closest you could get to the interval from the right without being in it and the same is true for the infimum, but on the other side.

In Practice

Observe the concept of Supremums and Infimums in these examples:
Let $S = \{1, 2, 3, 5, 8\} \subset \mathbb{R}$. The supremum of S is:

$$\sup(S) = 8$$

because 8 is the greatest element in the set and there are no elements in S greater than 8.
Let $S = \{1, 2, 3, 5, 8\} \cup (10, 12) \subset \mathbb{R}$. The supremum of S is:

$$\sup(S) = 12$$

because 12 is the least upper bound of the set, even though 12 is not an element of the set (since S contains the interval $(10, 12)$ but not 12).
Let $S = (0, 1) \subset \mathbb{R}$. The supremum of S is:

$$\sup(S) = 1$$

because 1 is the least upper bound of the set, even though 1 is not an element of the set.
Let $S = [0, 1] \subset \mathbb{R}$. The supremum of S is:

$$\sup(S) = 1$$

because 1 is the greatest element of the set, and there is no element in S greater than 1.