Lecture Notes: Professor Dave's Differential Equations
Exact First-Order Differential Equations

Thobias K. Høivik

March 28, 2025

Suppose $f : \mathbb{R}^2 \to \mathbb{R}$. The function can be plotted as a surface in $\mathbb{R}^3$ where $Z = f(x, y)$ is the height of the shape above the x-y-plane. If we take a horizonal slice of the surface at some height, then the intersection of a plane at that height with the shape will give a countour line. $Z = f(x, y)$ remains constant along the countour line. Suppose we move in the direction h on the countour line so that we are taken from $(x, y, f(x, y))$ to $(x + dx, y + dy, f(x, y))$. Then, clearly, the directional derivative

$$\nabla f \cdot h = \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix} \cdot \begin{bmatrix} dx \\ dy \end{bmatrix} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

since there is no change in f. In other words, a step h on the contour line will have no change in height Z. From here if we name the partial derivatives as functions we can get

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$M(x, y)dx + N(x, y)dy = 0$$

If we get a differential equation in this form we can try to find the multivariable function it could have come from. Then, any contour line of $f(x)$ will then be a solution to the differential equation.

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}$$

First we must check if the function f exists.

$$\frac{\partial f}{\partial x} = M, \frac{\partial f}{\partial y} = N$$

$$\Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial M}{\partial y},$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial N}{\partial x}$$

By Clairaut's theorem:

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

Thus

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

must be satisfied. We call this the condition for exactness, when faced with a differential equation in the form above. F is called the potential function for the differential equation.

Example:

Consider the DE:

$$\frac{dy}{dx} = \frac{x^2 - 4y^2}{8xy + y^4}$$

$$\Rightarrow (4y^2 - x^2)dx + (8xy + y^4)dy = 0$$

Now check for exactness:

$$\frac{\partial}{\partial y}\left[4y^2 - x^2\right] = 8y$$

$$\frac{\partial}{\partial x}\left[8xy + y^4\right] = 8y$$

So we know we have an exact differential equation. We know then that

$$\exists F : \mathbb{R}^2 \to R : F(x,y) = \int (4y^2 - x^2)dx = \int (8xy + y^4)dy$$

$$\Rightarrow 4xy^2 - \frac{x^3}{3} + f(y) = 4xy^2 + \frac{y^5}{5} + g(x)$$

$$-\frac{x^3}{3} + f(y) = \frac{y^5}{5} + g(x)$$

Notice that if $f(y) = \frac{y^5}{5}$ and $g(x) = -\frac{x^3}{3}$ everything works out. Thus

$$F(x,y) = 4xy^2 - \frac{x^3}{3} + \frac{y^5}{5}$$

All contour lines of F will give a solution to the differential equation so an implpcit general solution would be

$$4xy^2 - \frac{x^3}{3} + \frac{y^5}{5} = C$$