Oblig 1

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Problem (1). *Let (X,d) be a metric space and $K \subseteq X$ a subset. A function $f : K \to \mathbb{R}$ is* lower semicontinuous *if for all $x \in K$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$y \in K,\ d(x,y) < \delta \Rightarrow f(x) < f(y) + \varepsilon$$

The goal of this problem is to show that if $f : K \to \mathbb{R}$ is lower semicontinuous *and K is compact, then f attains a minimum, i.e. there is some $\overline{x} \in K$ such that*

$$f(\overline{x}) \leq f(x) \qquad \forall x \in K.$$

You may proceed as follows:
(a) Let $m :=$ inf*$\{f(x) : x \in K\}$. Argue that there exists a* minimizing sequence *for f, i.e. a sequence $(x_n)_{n\in\mathbb{N}}$ in K such that $f(x_n) \to m$ as $n \to \infty$.*
Assume from now that K is compact.
(b) Show that there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and some $\overline{x} \in K$ such that $x_{n_k} \to \overline{x}$ as $k \to \infty$.
(c) Show that $m \leq f(\overline{x}) \leq m + \varepsilon$ for any $\varepsilon > 0$, and conclude that $\overline{x}$ is a minimum of f.

Proof of (a). Let $f : K \to \mathbb{R}$ be some function and let $m := \inf\{f(x) : x \in K\}$.

By the definition of the infimum we have that, for every $\varepsilon > 0$ there exists $x \in K$ such that

$$f(x) \leq m + \varepsilon$$

In particular for $\varepsilon := \frac{1}{n}$ this must be the case. Then, for $n \in \mathbb{N}$ there exists $x_n \in K$ such that

$$f(x_n) \leq m + \frac{1}{n}$$

but m is a lower bound for all $f(x_n)$ hence

$$m \leq f(x_n) \leq m + \frac{1}{n}$$

so

$$f(x_n) \underset{n\to\infty}{\to} m$$

□

Proof of (b). $(x_n)_{n\in\mathbb{N}} \in K$ as described in proof of (a). Since K is compact, $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence which converges to some point $\overline{x} \in K$. □

Proof of (c). By definition of m as the infimum of all values of f we can see that

$$m \leq f(\overline{x})$$

By the virtue of f being lower semicontinuous we may also conclude that for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$d(\overline{x}, y) < \delta \Rightarrow f(\overline{x}) < f(y) + \varepsilon$$

In particular, since $x_{n_k} \to \overline{x}$, we can choose k large enough so that $d(\overline{x}, x_{n_k}) < \delta$. I.e.

$$d(\overline{x}, x_{n_k}) < \delta \Rightarrow f(\overline{x}) < f(x_{n_k}) + \varepsilon$$

As $k \to \infty$ we know that $f(x_{n_k}) \to m$ so taking the limit we get

$$m \leq f(\overline{x}) \leq m + \varepsilon$$

Taking $\varepsilon \to 0$ we conclude that

$$m \leq f(\overline{x}) \leq m$$

so $\overline{x}$ is a minimum of f. □

Problem (2). *Let (X, d) be a metric space and $E \subseteq X$ any subset. Let $f : E \to \mathbb{R}$ be any bounded function (i.e. there is some $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in E$). Define $g : \overline{E} \to \mathbb{R}$ by*

$$g(x) := \liminf_{\substack{y \to x \\ y \in E}} f(y) \qquad \forall x \in \overline{E},$$

that is,

$$g(x) := \lim_{n \to \infty} f_n(x), \qquad \text{where } f_n(x) := \inf\{f(y) : y \in E \cap B(x; 1/n)\}.$$

(Here, $\overline{E} := E \cup \partial E$ is the closure of E.) The function g is called the lower semicontinuous envelope *of f.*
(a) Show that g is well-defined, i.e. that the above limit exists, and that

$$g(x) \leq f(x)$$

for all $x \in E$.
(b) Show that g is lower semicontinuous.
(c) Show that if f is lower semicontinuous on E, then $g(x) = f(x)$ for all $x \in E$.

Proof of (a). Denote $\xi_{x,n} := E \cap B(x; 1/n)$. It is easy to see that $\xi_{x,n+1} \subseteq \xi_{x,n}$ (and that they are nonempty for all $n \in \mathbb{N}$). Notice then that

$$f_n(x) \leq f_{n+1}(x) \leq f_{n+2}(x) \leq \cdots \leq M$$

I.e. we get a monotone increasing sequence which is bounded above, which converges.

It is quite clear that for $x \in E$

$$\lim_{n \to \infty} f_n(x) \leq f(x)$$

as for every n $x \in B(x; \frac{1}{n})$ because $d(x,x) = 0 < 1/n$ and by the definition of the infimum $f_n(x) = \text{inf}\{f(y) : y \in \xi_{x,n}\}$ is a lower bound for the value of every specific $x \in \xi_{x,n}$. $\square$

Proof of (b). We wish to show that for every $x \in \overline{E}$ and $\varepsilon > 0$ we can find $\delta > 0$ such that

$$z \in \overline{E},\ d(x,z) < \delta \Rightarrow g(x) < g(z) + \varepsilon$$

Let $x \in \overline{E}$ and $\varepsilon > 0$. Since $g(x) = \text{lim}_{n\to\infty} f_n(x)$ and the sequence of $f_n(x)$ is non-decreasing, we have $f_n(x) \leq g(x)$ for all $n \in \mathbb{N}$. Thus there exists $N \in \mathbb{N}$ such that

$$g(x) < f_N(x) + \varepsilon$$

Recall that $f_N(x) = \text{inf}\{f(y) : y \in E \cap B(x; 1/N)\}$. Choose $\delta > 0$ such that $B(z;\delta) \subseteq B(x; 1/N)$. For any $M > 1/\delta$ we have $B(z; 1/M) \subseteq B(z;\delta) \subseteq B(x; 1/N)$. As we said in (a), the infimum over a larger set is less than or equal to the infimum over a subset. Therefore

$$f_N(x) \leq f_M(z)$$

As $m \to \infty$ we get

$$f_N(x) \leq g(z)$$

Recalling that

$$g(x) < f_N(x) + \varepsilon$$

we finally conclude that

$$g(x) < g(z) + \varepsilon$$

$\square$

Proof of (c). Let $x \in E \subseteq \overline{E}$. If f is lower semicontinuous then $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t.

$$z \in E,\ d(x,z) < \delta \Rightarrow f(x) < f(y) + \varepsilon \Rightarrow f(x) \leq f(y) + \varepsilon$$

In particular this holds for every $z \in E \cap B(x; 1/N)$ (assume $N \geq 1/\delta$), so it holds for the infimum as well, i.e.

$$d(x,z) < \delta \Rightarrow f(x) \leq f_n(x) + \varepsilon,\ n \geq N$$

Once again, nondecreasing sequence so take the limit to get

$$f(x) \leq g(x) + \varepsilon$$

Together with the results from (a) and (b) we conclude the desired result. $\square$