Mathematical Logic

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Contents

1 Introduction

These notes are for (introductory) mathematical logic. As of writing this introduction, I have not yet been accepted to take the mathematical logic course (University of Oslo). These notes will be based on the book used in the aforementioned course, as well as resources I can find online (e.g. publicly avaliable courses like UCLA's Math 220A). Therefore the structure of these notes might be a bit scattered and disorganized.

1.1 High level overview / rambling

First-order logic is the standard way to formalize mathematics. For instance Peano arithmetic formalizes number theory, Zermelo-Fraenkel set theory formalizes set theory. In symbolic logic, with formal languages, we don't consider interpretations of symbols. Using structures we can have a notion of validity or satisfaction to logical expressions. With these, we have notions of syntax and semantics.

1.2 Gödel's Completeness Theorem

This is a fundamental result in mathematical logic which states (informally) that a formula with no free variables can be formally deduced from a given set of axioms if and only if it is valid in every structure satisfying these axioms.

2 Languages and Structures

2.1 Languages

What does $\forall x(x > 0 \Rightarrow \exists y(y \cdot y = x))$ mean? While we might recognize that this is a statement that holds in an ordered field where every positive element is a square , it is just a sequence of symbols.

Definition 2.1: First order language

A (first-order) language is a set of symbols $\mathfrak{L}$ composed of two disjoint subsets:

1. The first part (common to all languages) consists of "(" and ")" together with the following logical symbols: the set of variables $\mathcal{V} = \{v_n : n \in \mathbb{N}\}$, the equality symbol "=", connectives "$\neg$", "$\wedge$", the existential quantifier "$\exists$"

2. The second part, called the signature of $\mathfrak{L}$ denoted $\sigma^{\mathfrak{L}}$, consists of the non-logical symbols of $\mathfrak{L}$. It consists of:

   - a set of constant symbols $C^{\mathfrak{L}}$
   - a sequence of sets $\mathscr{F}_n^{\mathfrak{L}}, n \in \mathbb{N}_{\geq 1}$, where elements of this set are called the n-ary function symbols
   - a sequence of sets $\mathscr{R}_n^{\mathfrak{L}}, n \in \mathbb{N}_{\geq 1}$, where elements of this set are called n-ary relation symbols (or, n-ary predicates)

The language $\mathfrak{L}$ is given by the disjoint union of these sets.

Note that the existential quantifier and logical or may be swapped for their negations: $\forall, \vee$ and nothing will change. The choice is a matter of taste. Also note that while we give practical names to the sets in $\mathfrak{L}$, we should attempt not to imbue any sort of interpretation on languages.

Now, a few *remarks*:

1. A language is always infinite. Furthermore, the logical part is countable while the non-logical can have arbitrary cardinality.

2. Abusing the notation: $\mathfrak{L}$ and $\sigma^{\mathfrak{L}}$ may be identified.

Example 2.1

$\mathfrak{L}_{\emptyset}$ the empty language (still contains logical part).
$\mathfrak{L}_{ring} = \{\underline{0}, \underline{1}, +, -, \cdot\}$ the ring language.
$\mathfrak{L}_{ord} = \{<\}$ the order language.
$\mathfrak{L}_{o.ring} := \mathfrak{L}_{ring} \cup \mathfrak{L}_{ord}$ the ordered ring language.
$\mathfrak{L}_{set} = \{\in\}$ the language of set theory.
$\mathfrak{L}_{grp} = \{\cdot, ^{-1}\}, 1$ the language of groups.
$\mathfrak{L}_{graph} = \{E\}$ the language of graphs.
$\mathfrak{L}_{ar} = \{\underline{0}, S, +, \cdot, <\}$ the language of arithmetic.

2.2 Structures

Definition 2.2: Structures

Let $\mathfrak{L}$ be a first-order language. An $\mathfrak{L}$-structure, $\mathfrak{A}$, constists of a non-empty set A (called the base set or universe of $\mathfrak{A}$ together with an element $c^{\mathfrak{A}} \in A$ for each $c \in c^{\mathfrak{L}}$, a function $f^{\mathfrak{A}} : A^n \longrightarrow A$ for each $f \in F_n^{\mathfrak{L}}$, and a subset $R^{\mathfrak{A}} \subseteq A^n$ for each $R \in R_n^{\mathfrak{L}}$. We write $\mathfrak{A} = \left(A, \left(Z^{\mathfrak{A}}\right)_{Z \in \sigma^{\mathfrak{L}}}\right)$. $Z^{\mathfrak{A}}$ is called the interpretation of the symbol $Z \in \sigma^{\mathfrak{L}}$ in the structure $\mathfrak{A}$

Note that there are different ways of formally defining this and the specifics of this can safely be taken for granted later on.

Example 2.2

1. $\mathcal{N} = (\mathbb{N}, 0, \underset{x \mapsto x+1}{S}, +, \cdot, <)$ is an $\mathfrak{L}_{ar}$-structure
2. $C = (\mathbb{C}, 0, 1, +, -, \cdot)$ is an $\mathfrak{L}_{ring}$-structure
3. $R = (\mathbb{R}, 0, 1, +, -, \cdot, <)$ the ordered field of real numbers, is an $\mathfrak{L}_{o.ring}$-structures

Definition 2.3: isomorphism of structures

We say that two $\mathfrak{L}$-structures $\mathfrak{A}, \mathfrak{B}$ are isomorphic, $\mathfrak{A} \cong \mathfrak{B}$, if there exists an isomorphism $F : A \underset{\text{bijection}}{\longrightarrow} B$ between the base sets which commutes with the interpretations of the symbols $\sigma^{\mathfrak{L}}$, that is:

1. $F(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$ for every constant symbol $c \in C^{\mathfrak{L}}$
2. $F(f^{\mathfrak{A}}(a_1, \ldots, a_n)) = f^{\mathfrak{B}}(F(a_1), \ldots, F(a_n))$ for every function symbol $f \in \mathscr{F}_n^{\mathfrak{L}}$, and every tuple $(a_1, \ldots, a_n) \in A^n$
3. $(a_1, \ldots, a_n) \in R^{\mathfrak{A}} \Leftrightarrow (F(a_1), \ldots, F(a_n)) \in R^{\mathfrak{B}}$ for every predicate $R \in \mathscr{R}_n^{\mathfrak{L}}$ and every tuple $(a_1, \ldots, a_n) \in A^n$

3 Terms and Formulas

3.1 Terms

Definition 3.1

A word w over a set (alphabet) E is a finite string $w = a_0 a_1 \dots a_{k-1}$ with $a_i \in E$ for every i. We call k the length of w, and we denote E^* the set of all words over E.

Definition 3.2

Let $\mathfrak{L}$ be a language. The set $\mathcal{T}^{\mathfrak{L}}$ of $\mathfrak{L}$-terms is the smallest subset D of $\mathfrak{L}^*$ containing the variables and the constants of $\mathfrak{L}$, such that if $f \in F_n^{\mathfrak{L}}$ and $t_1, \dots, t_n \in D$, then $f t_1 \dots t_n \in D$.

$$\Rightarrow \mathcal{T}^{\mathfrak{L}} = \bigcup_{n \in \mathbb{N}} \mathcal{T}_n^{\mathfrak{L}}$$

where $\mathcal{T}_0^{\mathfrak{L}} = C^{\mathfrak{L}} \cup \mathcal{V}^{\mathfrak{L}}$ and, inductively

$$\mathcal{T}_{n+1}^{\mathfrak{L}} = \mathcal{T}_n^{\mathfrak{L}} \cup \{f t_1 \dots t_k : k \in \mathbb{N}_{\geq 1}, f \in \mathcal{F}_k^{\mathfrak{L}} \text{ and } t_1, \dots, t_k \in \mathcal{T}_n^{\mathfrak{L}}\}$$

.

Proposition:

Any term $t \in \mathcal{T}^{\mathfrak{L}}$ satisfies one and only one of the following:

1. t is a variable
2. t is a constant symbol
3. there exists a unique integer $n \geq 1$, a unique n-ary function symbol $f \in \mathcal{F}_n^{\mathfrak{L}}$ and a unique sequence $(t_1, \dots, t_n)$ of terms such that $t = f t_1 \dots t_n$

Proof. We proceed by structural induction on the term $t \in \mathcal{T}^{\mathfrak{L}}$, as defined by the inductive construction of terms in a first-order language.

Base cases.

If t is a *variable* (i.e., $t \in \mathcal{V}$), then it satisfies condition (1).

If t is a *constant symbol* (i.e., a 0-ary function symbol), then it satisfies condition (2).

These two cases are mutually exclusive since the sets of variables and constant symbols are disjoint.

Inductive step.

Assume the statement holds for terms $t_1, \dots, t_n \in \mathcal{T}^{\mathfrak{L}}$.

Let $f \in \mathcal{F}_n^{\mathfrak{L}}$ be an n-ary function symbol for some $n \geq 1$, and define a new term

$$t = f t_1 \dots t_n.$$

Then t is neither a variable nor a constant symbol. Furthermore, the structure of the term guarantees that there exists a *unique* integer n, a unique function symbol f, and a unique sequence $(t_1, \dots, t_n)$ of terms such that

$$t = f t_1 \dots t_n,$$

due to the syntactic rules of term formation in first-order logic. Thus, condition (3) holds uniquely.

Exclusivity.

We now verify that the three cases are mutually exclusive:

- A term cannot be both a variable and a constant symbol (by definition of the syntax).
- A term cannot be a variable or constant symbol and simultaneously of the form $f t_1 \dots t_n$ for any $n \geq 1$.
- Finally, due to the unique structure of terms, a term of the form $f t_1 \dots t_n$ cannot be written in any other way.

Therefore, every term falls under exactly one of the three cases, completing the proof.

□

We introduce some *notation* here, for practical purposes: we shall often write $f(t_1, \dots, t_n)$ instead of $f t_1 \dots t_n$. When f is binary we might write $t_1 f t_2$ instead of $f t_1 t_2$. For example: $(x+y) \cdot z$ means $\cdot + x y z$.

Definition 3.3

The height of a term t, denoted $ht(t)$, is defined as the smallest natural number k such that $t \in \mathcal{T}_k^{\mathfrak{L}}$.

From this definition and the unique reading property for terms, it follows that $ht(f(t_1, \dots, t_n)) = 1 + \max\{ht(t_i) : 1 \leq i \leq n\}$.

3.2 Formulas

Definition 3.4: Atomic formula

An atomic $\mathfrak{L}$-formula is one of the following

- a word of the form $t_1 = t_2$, where t_1, t_2 are terms of the language
- a word of the form $R t_1 \dots t_n$, where $R \in \mathcal{R}_n^{\mathfrak{L}}$ and all $t_1, \dots, t_n$ are terms of the language

Then the set $Fml^{\mathfrak{L}}$ of $\mathfrak{L}$-formulas is the smallest subset D of $\mathfrak{L}^*$ that contains all atomic $\mathfrak{L}$-formulas such that if $x \in \mathcal{V}^{\mathfrak{L}}$ and $\phi, \psi \in D$, then $\neg\phi$, $(\phi \wedge \psi)$ and $\exists x \phi$ are all in D.

$$\Rightarrow Fml^{\mathfrak{L}} = \bigcup_{n \in \mathbb{N}} Fml_n^{\mathfrak{L}}$$

where $Fml_0^{\mathfrak{L}}$ is the set of atomic $\mathfrak{L}$-formulas, and inductively

$$Fml_{n+1}^{\mathfrak{L}} := Fml_n^{\mathfrak{L}} \cup \{\neg\phi : \phi \in Fml_n^{\mathfrak{L}}\} \cup \{(\phi \wedge \psi) : \phi, \psi \in Fml_n^{\mathfrak{L}}\} \cup \{\exists x \phi : \phi \in Fml_n^{\mathfrak{L}}, x \in \mathcal{V}^{\mathfrak{L}}\}$$

Note that the inclusion of $Fml_n^{\mathfrak{L}}$ in the union in the inductive definition is technically redundant, but we defined it like this for practicallity in other definitions.

The same can be said for the definition of $\mathcal{T}_n^{\mathfrak{L}}$.

3.3 Exercises from 1.6.1

Problem 3.1: Task 4

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are two $\mathfrak{L}$-structures. We say that they are *isomorphic*, $\mathfrak{A} \cong \mathfrak{B}$, if there exists a bijection $i : A \to A$ such that for each constant symbol c of $C^{\mathfrak{L}}$, $i(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$, for each n-ary function symbol f in $\mathscr{F}_n^{\mathfrak{L}}$, and for each $(a_1, \dots, a_n) \in A^n$, $i(f^{\mathfrak{A}}(a_1, \dots, a_n)) = f^{\mathfrak{B}}(i(a_1), \dots, i(a_n))$, and for each n-ary relation/predicate symbol R in $\mathscr{R}_n^{\mathfrak{L}}$, and for each $(a_1, \dots, a_n) \in A^n$, we have $(a_1, \dots, a_n) \in R^{\mathfrak{A}} \Leftrightarrow (i(a_1), \dots, i(a_n)) \in R^{\mathfrak{B}}$.

A Show that $\cong$ is an equivalence relation.

B Find two different structures for a particular language and prove that they are not isomorphic.

Proof of first. Let $\mathfrak{L}$ be an arbitrary language, and $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be $\mathfrak{L}$-structures.

First we show that $\mathfrak{A} \cong \mathfrak{A}$, i.e. $\cong$ is reflexive. Let $i : A \to A$ be the identity map $i(a) = a$. It is trivial to show that i is bijective, being its own inverse. Then we have $i(c^{\mathfrak{A}}) = c^{\mathfrak{A}}$ satisfying the first property of structure isomorphisms. Furthermore, if we take f to be an n-ary function symbol of the language, and $(a_1, \dots, a_n) \in A^n$, we get that $i(f^{\mathfrak{A}}(a_1, \dots, a_n)) = f^{\mathfrak{A}}(a_1, \dots, a_n)$. Now recognize that $i(a_j) = a_j$ for all j, and we find that $i(f^{\mathfrak{A}}(a_1, \dots, a_n)) = f^{\mathfrak{A}}(i(a_1), \dots, i(a_n))$, showing that the second property holds. Lastly, with R being some n-ary relation symbol in the language and $(a_1, \dots, a_n) \in A^n$, we have, through the same identity argument,

$$(a_1, \dots, a_n) \in R^{\mathfrak{A}} \Leftrightarrow (i(a_1), \dots, i(a_n)) \in R^{\mathfrak{A}}.$$

Thus $\mathfrak{A} \cong \mathfrak{A}$.

Second we show that $\cong$ is symmetric. Suppose $i : A \to B$ is an isomorphism, so $\mathfrak{A} \cong \mathfrak{B}$ via i. Since i is bijective, it has an inverse $j := i^{-1} : B \to A$ which is also a bijection.

We now show that j is a structure isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$.

Constants: For any constant symbol c in the language, we know $i(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$. Applying j to both sides gives:

$$j(c^{\mathfrak{B}}) = j(i(c^{\mathfrak{A}})) = c^{\mathfrak{A}}.$$

So the condition on constants is satisfied.

Functions: Let f be an n-ary function symbol and $(b_1, \dots, b_n) \in B^n$. Since j maps from B to A, we define $a_k := j(b_k)$ for each k. Then $i(a_k) = b_k$ for all k, and:

$$\begin{aligned} j(f^{\mathfrak{B}}(b_1, \dots, b_n)) &= j(f^{\mathfrak{B}}(i(a_1), \dots, i(a_n))) \\ &= j(i(f^{\mathfrak{A}}(a_1, \dots, a_n))) = f^{\mathfrak{A}}(a_1, \dots, a_n), \end{aligned}$$

which shows that:

$$f^{\mathfrak{A}}(j(b_1), \dots, j(b_n)) = j(f^{\mathfrak{B}}(b_1, \dots, b_n)),$$

satisfying the function compatibility condition.

Relations: Let R be an n-ary relation symbol and $(b_1, \dots, b_n) \in B^n$. Let $a_k := j(b_k)$ as before. Then, since $i(a_k) = b_k$, we get:

$$(b_1, \dots, b_n) \in R^{\mathfrak{B}} \Leftrightarrow (i(a_1), \dots, i(a_n)) \in R^{\mathfrak{B}} \Leftrightarrow (a_1, \dots, a_n) \in R^{\mathfrak{A}}$$

by the isomorphism definition for i. Applying j to the tuple and reversing the direction of the implication gives the desired equivalence. Hence $\mathfrak{B} \cong \mathfrak{A}$.

Lastly, we show that $\cong$ is transitive. Suppose $\mathfrak{A} \underset{i}{\cong} \mathfrak{B}$ and $\mathfrak{B} \underset{j}{\cong} \mathfrak{C}$. Define $k := j \circ i : A \to C$, which is bijective since it is the composition of two bijections.

We show that k is a structure isomorphism from $\mathfrak{A}$ to $\mathfrak{C}$.

Constants: For any constant symbol c, we have:

$$k(c^{\mathfrak{A}}) = j(i(c^{\mathfrak{A}})) = j(c^{\mathfrak{B}}) = c^{\mathfrak{C}}.$$

Functions: Let f be an n-ary function symbol and $(a_1, \dots, a_n) \in A^n$. Then:

$$\begin{aligned} k(f^{\mathfrak{A}}(a_1, \dots, a_n)) &= j(i(f^{\mathfrak{A}}(a_1, \dots, a_n))) \\ &= j(f^{\mathfrak{B}}(i(a_1), \dots, i(a_n))) \\ &= f^{\mathfrak{C}}(j(i(a_1)), \dots, j(i(a_n))) \\ &= f^{\mathfrak{C}}(k(a_1), \dots, k(a_n)). \end{aligned}$$

Relations: Let R be an n-ary relation symbol and $(a_1, \dots, a_n) \in A^n$. Then:

$$(a_1, \dots, a_n) \in R^{\mathfrak{A}} \Leftrightarrow (i(a_1), \dots, i(a_n)) \in R^{\mathfrak{B}} \Leftrightarrow (j(i(a_1)), \dots, j(i(a_n))) \in R^{\mathfrak{C}} \Leftrightarrow (k(a_1), \dots, k(a_n)) \in R^{\mathfrak{C}}.$$

The reverse direction follows similarly by applying the inverses of j and i. Hence, k preserves all relations.

Thus, $\mathfrak{A} \cong \mathfrak{C}$, and we have shown that $\cong$ is transitive.

Therefore, $\cong$ is an equivalence relation (reflexive, symmetric, transitive). □

Solution to the second (Informal). Let $\mathfrak{L}_{NT} = \{0, S, +, \cdot, E, <\}$ be the language of number theory.

Let $\mathcal{N}, \mathfrak{A}$ be $\mathfrak{L}_{NT}$-structures. More specifically, let $\mathcal{N}$ be the usual structure which models number theory. In $\mathfrak{A}$, let $a +^{\mathfrak{A}} b = a$ for all $a, b \in A$. To show that two structures like these cannot be isomorphic we proceed by contradiction, supposing there does exist an isomorphism $i : \mathbb{N} \to A$.

i is some function that, among other properties, satisfies $i(n+m) = i(n) +^{\mathfrak{A}} i(m)$, much like a group homomorphism, fun fun. Now since $i(n) +^{\mathfrak{A}} i(m) = i(n)$ we have

$$i(n+m) = i(n)$$

Choose $n = 1 = m$ and we get

$$i(2) = i(1)$$

which contradicts injectivity of i.

4 Semantics of first-order formulas

4.1 Unique readability, free variables, notation

Proposition. (Unique readability of formulas) Any $\mathfrak{L}$-formula ϕ satisfies one and only one of the following:

1. ϕ is an atomic formula
2. ϕ is equal to $\neg\psi$ for some unique $\mathfrak{L}$-formula ψ
3. ϕ is equal to $\psi \wedge \mathfrak{X}$ for some unique $\mathfrak{L}$-formulas $\psi, \mathfrak{X}$
4. ϕ is equal to $\exists x\psi$ for some unique variable x and some unique $\mathfrak{L}$-formula ψ

Proof. We proceed by structural induction on the formation of formulas in $\mathfrak{L}$.

Let $Fml_n^{\mathfrak{L}}$ denote the set of formulas of depth at most n:

- Base case: $Fml_0^{\mathfrak{L}}$ consists of atomic formulas.
- Inductive step: $Fml_{n+1}^{\mathfrak{L}}$ consists of formulas obtained by applying negation, conjunction, or existential quantification to formulas in $Fml_n^{\mathfrak{L}}$.

Base case ($n = 0$)**:** Any formula $\phi \in Fml_0^{\mathfrak{L}}$ is, by definition, atomic. Hence, case (1) holds, and the rest do not apply. The decomposition is trivially unique.

Inductive hypothesis: Assume that for all $\psi \in Fml_n^{\mathfrak{L}}$, ψ satisfies exactly one of the four cases of the proposition, with the associated subformulas and variables being unique.

Inductive step: Let $\phi \in Fml_{n+1}^{\mathfrak{L}} \setminus Fml_n^{\mathfrak{L}}$. Then ϕ is constructed as exactly one of the following:

- $\phi = \neg\psi$, where $\psi \in Fml_n^{\mathfrak{L}}$,
- $\phi = \psi \wedge \chi$, where $\psi, \chi \in Fml_n^{\mathfrak{L}}$,
- $\phi = \exists x\,\psi$, where x is a variable and $\psi \in Fml_n^{\mathfrak{L}}$.

Each of these constructions matches exactly one of cases (2), (3), or (4). We now prove uniqueness for each case.

Case 2: Suppose $\phi = \neg\psi_1 = \neg\psi_2$. Then by the syntax of negation, it follows that $\psi_1 = \psi_2$. Hence, ψ is uniquely determined.

Case 3: Suppose $\phi = \psi_1 \wedge \chi_1 = \psi_2 \wedge \chi_2$. Then the only way such a formula can be parsed as a conjunction is with:

$$\psi_1 = \psi_2 \quad \text{and} \quad \chi_1 = \chi_2.$$

Thus, both components are uniquely determined.

Case 4: Suppose $\phi = \exists x_1\,\psi_1 = \exists x_2\,\psi_2$. By the syntax of quantification, it must be that $x_1 = x_2$ and $\psi_1 = \psi_2$. Hence, both the variable and subformula are uniquely determined.

Mutual exclusivity: Finally, we verify that no formula can belong to more than one of the cases (1)–(4).

- Atomic formulas cannot start with $\neg$, $\wedge$, or $\exists$.
- Formulas of the form $\neg\psi$ cannot be written as conjunctions or quantifiers.
- Conjunctions and quantifiers have distinct syntactic structures and cannot be confused.

Hence, each formula ϕ satisfies exactly one of the four cases, and the decomposition is unique in each.

□

Definition 4.1

Let $v_k \in \mathcal{V}^{\mathfrak{L}}$ be a variable. We define: a free occurence of v_k in ϕ:

- If ϕ is atomic (i.e. $t_1 = t_2$ for some terms), all occurences of v_k in ϕ are free.
- If ϕ is of the form $\neg\psi$, then the free occurences of v_k in ϕ are those which are free in ψ.
- If ϕ is of the form $(\psi \wedge \chi)$, then the free occurences of v_k in ϕ are those that are free in ψ and those that are free in χ
- If ϕ is of the form $\exists v_l \psi$ and $l \neq k$, then the free occurences of v_k in ϕ are those which are free in ψ
- If ϕ is of the form $\exists v_k \psi$, then no occurence of v_k in ϕ is free

Occurences of v_k in ϕ that are not free are called bound.
The free variables of ϕ are those having at least one reoccurence in ϕ. Denote $Free(\phi)$ the set of free variables of ϕ.
A sentence is a formula with no free variables.

Example 4.1

If ϕ is the formula $(\exists v_0 v_0 < v_1 \wedge v_0 = v_1)$ (note that there is no use of "(" or ")" in the formula itself).
Then:

- The first two occurences of v_0 are bound
- The third v_0 is free
- All occurences of v_1 are free

Notation. We will use the following abbreviations:

- $(\phi \vee \psi)$ for $\neg(\neg\phi \wedge \neg\psi)$
- $(\phi \Rightarrow \psi)$ for $\neg(\phi \wedge \neg\psi)$
- $(\phi \Leftrightarrow \psi)$ for $((\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi))$
- $\forall x\phi$ for $\neg\exists x\neg\phi$

Note that these do not constitute new symbols in the meaning of formulas in the formation rules, this is simply for convenience. We, formally, stick to the minimum amount of symbols for the full power of first-order logic.

We will also write $\exists x_1,\dots,x_n$ instead of $\exists x_1,\dots,\exists x_n$ and do so for the universal quantifier, also. Furthermore we write $R(t_1,\dots,t_n)$ instead of $Rt_1,\dots,t_n$ as well as write $t_1 R t_2$ instead of $Rt_1 t_2$ when R is a binary relation. We will write $(\phi_0 \wedge \dots \wedge \phi_n)$ or sometimes $\bigwedge_{i=0}^n \phi_i$ instead of $(\dots(\phi_0 \wedge \phi_1) \wedge \phi_2) \wedge \dots \phi_n)$, as well as similar abbreviations for disjunction. We will add parentheses, or omit them for readability; with the convention that symbols from $\{\exists, \neg, \forall\}$ bind strongest, followed by $\wedge$, then $\vee$, then the symbols from $\{\Rightarrow, \Leftrightarrow\}$.

Example 4.2

$\forall x \phi \wedge \psi \Rightarrow \chi$ is read as $((\forall x \phi \wedge \psi) \Rightarrow \chi)$, that is $\neg(\forall x \phi \wedge \psi) \wedge \neg\chi)$, and so finally we get

$$\neg((\neg\exists x \neg\phi \wedge \psi) \wedge \neg\chi)$$

4.2 Semantics

Definition 4.2

Let $\mathfrak{A}$ be an $\mathfrak{L}$-structure.

1. An assignment (with value in $\mathfrak{A}$) is a function $\alpha : \mathcal{V}^{\mathfrak{L}} \to A$ from the set of variables to the base set or universe of $\mathfrak{A}$.

2. If α is an assignment and t is an $\mathfrak{L}$-term, we define $t^{\mathfrak{A}}[\alpha]$ by induction on the height of t $ht(t)$:

    - $v_i^{\mathfrak{A}}[\alpha] := \alpha(v_i)$ (for $v_i \in \mathcal{V}^{\mathfrak{A}}$) and $c^{\mathfrak{A}}[\alpha] := c^{\mathfrak{A}}$ (for constant symbol $c \in C^{\mathfrak{L}}$)
    - $f(t_1,\dots,t_n)^{\mathfrak{A}}[\alpha] := f^{\mathfrak{A}}(t_1^{\mathfrak{A}}[\alpha],\dots,t_n^{\mathfrak{A}}[\alpha])$

Lemma 4.1

If two assignments α, β concide on all variables occuring in t, then $t[\alpha] = t[\beta]$.

Notation. If t is a term, we might denote it by $t(x_1,\dots,x_n)$ if the variables x_i are all distinct and all variables having at least one occurence in t all belong to the set $\{x_1,\dots,x_n\}$. If $t(x_1,\dots,x_n)$ and a tuple of elements $a_1,\dots,a_n \in A$, we define $t^{\mathfrak{A}}[a_1,\dots,a_n]$ by $t^{\mathfrak{A}}[\alpha]$ with α defined to be $\alpha(x_i) = a_i$ for all i.

Definition 4.3: Satisfaction of a formula/"Tarski's definition of truth"

Let $\mathfrak{A}$ be an $\mathfrak{L}$-structure. By induction on the height $ht(\phi)$ of a formula we define $\mathfrak{A} \models \phi[\alpha]$ (read as "ϕ is satisfied in $\mathfrak{A}$ in α"):

- $\mathfrak{A} \models t_1 = t_2$ if and only if $t_1^{\mathfrak{A}}[\alpha] = t_2^{\mathfrak{A}}[\alpha]$

- $\mathfrak{A} \models Rt_1 \dots t_n[\alpha]$ if and only if $(t_1[\alpha],\dots,t_n[\alpha] \in R^{\mathfrak{A}}$

- $\mathfrak{A} \models \neg\psi[\alpha]$ if and only if $\not\models \psi[\alpha]$

- $\mathfrak{A} \models (\psi \wedge \chi)[\alpha]$ if and only if $\mathfrak{A} \models \psi[\alpha]$ and $\mathfrak{A} \models \chi[\alpha]$

- $\mathfrak{A} \models \exists x \psi[\alpha]$ if and only if there exists $a \in A$ with $\mathfrak{A} \models \psi[\alpha_{a/x}]$, where $\alpha_{a/x}$ denotes the assignment defined by $\alpha_{a/x}(x) = a$ and $\alpha_{a/x}(y) = \alpha(y)$ for all y different from x

Proposition. If two assignments α, β coincide on $Free(\phi)$, then one has:

$$\mathfrak{A} \vDash \phi[\alpha] \text{ if and only if } \mathfrak{A} \vDash \phi[\beta]$$

Proof. By induction on $ht(\phi)$. The case of atomic formulas follows from the earlier lemma about terms.

Inductive step:

Consider the case ϕ is equal to $\exists x \psi$. If we have that $\mathfrak{A} \vDash \phi[\alpha]$, there exists $a \in A$ such that $\mathfrak{A} \vDash \psi[\alpha_{a/x}]$. Any variable $y \neq x$ that is free in ψ is also free in ϕ. Hence we have $\mathfrak{A} \vDash \psi[\beta_{a/x}]$. By the inductive hypothesis, we also have $\mathfrak{A} \vDash \phi[\beta]$. □

Notation. A formula ϕ will be denoted by $\phi(x_1, \ldots, x_n)$ if the variables x_i are all distinct and free in ϕ and they belong to the set $\{x_1, \ldots, x_n\}$. If a formula $\phi(x_1, \ldots, x_n)$ and elements $a_1, \ldots, a_n \in A$ are given, we define $\mathfrak{A} \vDash \phi[a_1, \ldots, a_n]$ by $\mathfrak{A} \vDash \phi[\alpha]$ where α is an assignment such that $\alpha(x_i) = a_i$ (well defined by the previous proposition). Thus the formula $\phi(x_1, \ldots, x_n)$ defines an n-ary relation on the underlying base set A given by $\{(a_1, \ldots, a_n) \in A^n : \mathfrak{A} \vDash \phi[a_1, \ldots, a_n]\}$. In particular, when ϕ is a sentence, then the relation $\mathfrak{A} \vDash \phi$ is interpreted as "ϕ is satisfied or true in $\mathfrak{A}$" or "$\mathfrak{A}$ is a model of ϕ".