# Lecture Notes: Linear Second-Order Differential Equations — Non-Homogeneous (Course by: Professor Dave on YouTube)

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$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

Is non-homogeneous because instead of being equal to 0, it is equal to a function of x. We start by finding the general solution to the equivalnent homogeneous case.

$$a\lambda^2 + b\lambda + c = 0$$

This is known as the complementary solution. Now we must find the **particular integral** (solution compatible with the right side $f(x)$).
Suppose we want to solve

$$y'' - 5y' + 6y = e^x$$

If we pretend that the left hand side equals 0 instead, we get a homogeneous differential equation with general solution

$$y_{CF} = Ae^{2x} + Be^{3x}$$

## Trial Functions

We find a corresponding trial function from the table below and proceed.

| **Form of** $f(x)$ | **Trial Solution** $y_p(x)$ |
|---|---|
| $P_n(x)$ (polynomial of degree $n$) | $A_0 + A_1 x + \cdots + A_n x^n$ |
| $e^{\alpha x}$ | $Ae^{\alpha x}$ |
| $\sin(\beta x),\ \cos(\beta x)$ | $A\cos(\beta x) + B\sin(\beta x)$ |
| $e^{\alpha x} \cdot P_n(x)$ | $(A_0 + A_1 x + \cdots + A_n x^n)e^{\alpha x}$ |
| $e^{\alpha x} \cdot \cos(\beta x)$ or $e^{\alpha x} \cdot \sin(\beta x)$ | $e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x))$ |
| $x^n e^{\alpha x} \sin(px)$ or $x^n e^{\alpha x} \cos(px)$ | $(C_n x^n + C_{n-1}x^{n-1} + \cdots + C_0)(C_s \sin px + C_c \cos px)e^{\alpha x}$ |

Table 1: Choosing Trial Functions for $y'' + ay' + by = f(x)$

We must choose a trial function from this table based on the form of $f(x)$. We have $f(x) = e^x$ so we get the trial function

$$y_{PI}(x) = Ce^x$$

The function must be linearly independent of our complementary function, which this function is. We then substitute in our trial function for $y$ in the original differential equation to get:

$$Ce^x - 5Ce^x + 6Ce^x = e^x \Rightarrow 2Ce^x = e^x \Rightarrow C = \frac{1}{2}$$

$$y_{PI} = \frac{1}{2}e^x$$

Our general solution then becomes

$$y = y_{CF} \ + y_{PI} Ae^{2x} + Be^{3x} + \frac{1}{2}e^x$$

If we find that our trial function is not linearly independent of the complementary solution we multiply the constant term by x.

# Variation of Parameters

Trial functions is not the only way to find the solution to the differential equation. We begin by calculating the **Wronskian**:

$$W(x) = \det \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

In the case of our earlier example we have

$$W(x) = \det \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{5x} - 2e^{5x} = e^{5x}$$

To find the particular integral we use the formula

$$y_{PI} = y_2 \int \frac{y_1 \cdot f(x)}{W(x)} dx - y_1 \int \frac{y_2 \cdot f(x)}{W(x)} dx$$

$$\begin{aligned} y_{PI} &= e^{3x} \int \frac{e^{3x} \cdot e^{x}}{e^{5x}} dx - e^{2x} \int \frac{e^{3x} \cdot e^{x}}{e^{5x}} dx \\ &= e^{3x} \left( -\frac{1}{2} e^{-2x} \right) - e^{2x} \left( -e^{-x} \right) = \frac{1}{2} e^{x} \end{aligned}$$

So we see that we get the same solution as before.