Experiments in Paper Soccer

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1 Rules of Paper Soccer

Paper soccer (or paper hockey) is an abstract strategy game played on a square grid representing a soccer or hockey field. (Wikipedia, needs citation)

1.1 Rules

etc.

2 Formalizing the game

Definition 2.1: Board

Let the *pitch* be an $m \times n$ grid of lattice points

$$V = \{(i, j) \mid 0 \leq i \leq m, 0 \leq j \leq n\} \subset \mathbb{Z}^2,$$

where m and n are odd integers. Let the goal vertices be

$$g_1 = (\lceil m/2 \rceil, n) \quad \text{and} \quad g_2 = (\lceil m/2 \rceil, 0),$$

each corresponding to the opponent's goal line. Two vertices $p, q \in V$ are said to be *adjacent* if $\|p - q\|_2 = 1$ or $\|p - q\|_2 = \sqrt{2}$. Let

$$E(V) = \big\{\{p, q\} \subseteq V \mid p, q \text{ are adjacent}\big\}$$

be the set of all possible edges between adjacent vertices.

Here we make a simplifying assumption by constraining the game to a rectangular grid, whereas the goals typically lie "outside" the grid (usually 1×2 on either end of the pitch).

Definition 2.2: Ball

Let $b_0 = (\lceil m/2 \rceil, \lceil n/2 \rceil)$ denote the initial position of the ball. After t plies (see Definition below), let b_t be the current position of the ball. When the specific index is unimportant, we write simply b.

Definition 2.3: Game state

A state is (G_s, b, p) where

- G_s is an undirected graph on V whose edges are the set of drawn segments so far.
- $b \in V$ is the current ball vertex.
- $p \in \{1, 2\}$ is the player whose turn it is to move.

Usually we can determine whose turn it is to move by counting the number of moves played. In the cases where it won't lead to confusion we may denote a game state by G, G', G_i, etc.

Definition 2.4: Ply

A ply consists of choosing an adjacent vertex u (orthogonal or diagonal) such that $\{b, u\} \notin E(G_s)$ and $\{b, u\}$ is within V. After drawing $\{b, u\}$, we set $E(G_s) := E(G_s) \cup \big\{\{b, u\}\big\}$ and $b := u$.

Definition 2.5: Bounce

If the new ball vertex u satisfies $\deg_{G_s}(u) \geq 2$ after adding the edge $\{b, u\}$, then the same player continues, and another ply is forced. A player's *turn* is the maximal sequence of plies performed by the same player until the ball lands on a vertex v with $\deg_{G_s}(v) = 1$.

Definition 2.6: Winning conditions

If the ball lands on g_1, player 2 wins. If it lands on g_2, player 1 wins.
If no legal ply exists for the player to move, that player loses. (Equivalently, the other player wins.) If all edges have been used, we may optionally declare a draw; however, this cannot occur before all possible plies are exhausted.

Definition 2.7: Weak equivalence

Two game states $G_1 = (G_{s_1}, b_1, p_1)$ and $G_2 = (G_{s_2}, b_2, p_2)$ are *weakly equivalent*, written $G_1 \simeq G_2$, if they share the same drawn edges:

$$E(G_{s_1}) = E(G_{s_2}).$$

Definition 2.8: Strong equivalence

Two game states are *strongly equivalent*, written $G_1 \cong G_2$, if they are weakly equivalent and the ball occupies the same vertex:

$$E(G_{s_1}) = E(G_{s_2}) \quad \text{and} \quad b_1 = b_2.$$

3 Brain storm

Definition 3.1: Gamestate

Graph? Discrete, finite vector space? How will moves be represented? Is composition of moves commutative? (Probably not)

Definition 3.2: Weak equivalence of gamestates

Two gamestates are weakly equivalent if the same edges are used.
When two gamestates G_1 and G_2 are equivalent we will write $G_1 \simeq G_2$ Do these exist? (Probably)

Definition 3.3: Strong equivalence of gamestates

Two gamestates are strongly equivalent if they are weakly equivalent, and the ball is in the same place.
We will write $G_1 \cong G_2$.
(These ceirtanly exist, via experimentation with games of depth 5).

Lemma 3.1

You cannot reach two equivalent game states in a different number of moves (perhaps this is only true for strong equivalence? maybe move chaining will prevent this from being true).

Theorem 3.1

You cannot have two equivalent game states with different players to move.
(This will follow from above Lemma if it is true)

Theorem 3.2

There does not exist any equivalent (weak or strong) gamestates at depth < 3. In other words you cannot reach two like structures with less than 3 moves.

Outline of proof. If the depth is 1 there are 8 positions (move north, west, north-east,etc.). If depth is 2 there should be 64 positions, none of which are equal (no move chaining should make this true). □

4 Immediate results

Notation.

If G_1 and G_2 are gamestates, we may write

$$E_{G_1} \quad \text{and} \quad E_{G_2}$$

to denote their respective edges.

Theorem 4.1: Equivalence relations

The weak- ($\simeq$) and strong ($\cong$) equivalences are equivalence relations.

Proof. We verify the three defining properties.

(Reflexivity.) For any game state G_a, we have $E_{G_a} = E_{G_a}$ and $b_{G_a} = b_{G_a}$, so $G_a \cong G_a$ and hence $G_a \simeq G_a$.

(Symmetry.) If $G_a \cong G_b$, then by definition $b_{G_a} = b_{G_b}$ and $E_{G_a} = E_{G_b}$. Thus also $b_{G_b} = b_{G_a}$ and $E_{G_b} = E_{G_a}$, so $G_b \cong G_a$ and therefore $G_b \simeq G_a$.

(Transitivity.) If $G_a \cong G_b$ and $G_b \cong G_c$, then $b_{G_a} = b_{G_b} = b_{G_c}$ and $E_{G_a} = E_{G_b} = E_{G_c}$, hence $G_a \cong G_c$ and consequently $G_a \simeq G_c$. □

Definition 4.1: Equivalence classes

The *weak equivalence class* of a game state G is $[G]_\simeq = \{H \mid H \simeq G\}$. Likewise, the *strong equivalence class* of G is $[G]_\cong = \{H \mid H \cong G\}$.

Theorem 4.2: Boundedness of Paper Soccer

The game is bounded. I.e. it will end in a finite number of plies.

Proof. Every ply strictly increases $|E(G_s)|$ by 1. If $E(G_s) = E(V)$ (every edge is colored), there are not any plies left, and the player whose turn it is loses. Thus the game ends in at most $|E(V)|$ plies. □

Lemma 4.1

Let G_1, G_2 be game states such that $G_1 \simeq G_2$. Then they must have been reached in the same number of plies.

Proof. Since $G_1 \simeq G_2$, we have $E_{G_1} = E_{G_2}$. Each ply adds exactly one new edge, so the number of plies is determined uniquely by $|E_G|$. Hence G_1 and G_2 were reached after the same number of plies. □

Theorem 4.3

Let G_1, G_2 be game states such that $G_1 \cong G_2$. Then they must have been reached in the same number of plies.

Proof. Since $G_1 \cong G_2$ implies $G_1 \simeq G_2$, the claim follows immediately from the previous lemma. □

Lemma 4.2

No turn can have more than one ply during the first 3 turns.

Proof. On the first move we have 8 possible moves since any given vertex has 8 adjacent vertices. One of these is chosen

□