Lecture Notes: Linear First-Order Differential Equations (Course By: Professor Dave Explains on Youtube)

Thobias K. Høivik

March 20, 2025

The general form of a linear first-order differential equation is

$$\frac{dy}{dx} + P(x)y = Q(x)$$

We introduce the integrating factor

$$I(x) = e^{\int P(x)dx} = exp \int P(x)dx$$

If we take the differential equation in it's general form and multiply all terms with the integrating factor we get:

$$e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}P(x)y = e^{\int P(x)dx}Q(x)$$

Consider now the product rule of differentiation

$$(f \cdot g)' = f \cdot g' + g \cdot f'$$

Note how if this resembles the general form when multiplied by the integrating factor. Thus it can be replaced by:

$$\frac{d}{dx}\left(e^{\int P(x)dx}y\right) = e^{\int P(x)dx}Q(x)$$

$$e^{\int P(x)dx}y = \int e^{\int P(x)dx}Q(x)dx$$

$$I(x)y = \int I(x)Q(x)dx$$

Example 1:
Consider the differential equation:

$$\frac{dy}{dx} + \frac{y}{x} = 3x$$

It is not separable, but it is linear with an order of one.

$$I(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$xy = \int x \cdot 3x dx = x^3 + C$$

Giving us the general solution. Dividing by x yields the explicit general solution:

$$\boxed{y = x^2 + \frac{C}{x}}$$

Example 2: Consider the differential equation, valid $\forall x : 0 < x < \frac{\pi}{2}$, with initial condition $y = \frac{1}{2\sqrt{2}}$ when $x = \frac{\pi}{4}$:

$$\frac{dy}{dx} + y \cot x = \sin x$$

$$I(x) = e^{\int \cot(x) dx} = e^{\ln(\sin x)} = \sin x$$

$$y \sin x = \int \sin^2 x = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

Initial conditions:

$$\frac{1}{2\sqrt{2}} \frac{\sqrt{2}}{2} = \frac{pi}{8} - \frac{1}{4} + C$$

$$\Rightarrow C = \frac{4 - \pi}{8}$$

Explicit particular solution:

$$\boxed{y = \frac{1}{\sin x} \left(\frac{x}{2} - \frac{\sin 2x}{4} + \frac{4 - \pi}{8} \right)}$$