Lecture Notes: Abstract Algebra (Course by: Alvaro Lozano-Robledo)

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Hamilton's Quaternions

Definition

Let Q8

$$= \{\pm 1, \pm i, \pm j, \pm k. | i^2 = j^2 = k = 2 = -1, ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j\}$$

Then, $< Q8, \times >$ is called Hamilton's Quaternions or the Quaternion group.
Q1) Is $< Q8, \times >$ a group?
Closure: We can see by the given definition of the set above that when an element of Q8 is multiplied with another element of Q8, we get another element of Q8, so it is indeed closed.
Associative: We know multiplication is associative and we can experimentally test every combination of elements in Q8, multiply them, and find that Q8 is indeed associative. Note also that Q8 can be represented as matrices,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad -I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad k = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$$-i = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \quad -j = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad -k = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

These matrices satisfy the defining relations:

$$i^2 = j^2 = k^2 = ijk = -I.$$

and observe also that Q8 then must be a subset of $GL(2, \mathbb{C})$.
Identity: 1 is the identity since $q \times 1 = 1 \times q = q$.
Inverses: 4 of the elements in Q8 are just negative versions of the others since $(\pm 1)^{-1} = \pm 1, (\pm i)^{-1} = \mp i$, etc. Thus each element of Q8 has an inverse, and all group axioms are satisfied for Q8 under multiplication.

Q2) What is the order of Q8? $|Q8| = 8 = 2^3$, Q8 is a 2-group. If the order of G is p^n for p prime, then G is a p-group, making Q8 a p-group.
Q3) Is Q8 abelian? No it is not abelian because $ij = k \neq ji = -k$.
Q4) What are the orders of each element of Q8?

- ord(1) = 1
- ord(-1) = 2
- ord(i) = ord(-i) = 4
- ord(j) = ord(-j) = 4
- ord(k) = ord(-k) = 4

Here we get to see a corollary of Lagrange's theorem as the order of elements in Q8 divide the order of Q8.
Q5) What does the Caley table look like?

$\times$	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

Q6) What are the elements of $< i >$? $< i >= \{e, i, i^2, \cdots\} = \{1, i, -1, -i\} =< -i >$.
Q7) What do the other cyclic subgroups of Q8 look like? Well, all elements in Q8 not equal to ± 1 will look the same as $< i >$, while $< 1 >= \{1\}, < -1 >= \{-1, 1\}$.
Q8) What would other subgroups of Q8 look like?

$$\begin{gathered} H \leq G \ \wedge \ i, j \in H \text{ then} \\ < i >\subseteq H \ \wedge \ < j >\subseteq H \\ i \times j = h \in H, -1 \in H, 1 \in H, i \times j = k \in H \\ \text{so } \pm 1, \pm i, \pm j, \pm k \in H = Q8 \end{gathered}$$

This means that the subgroups of Q8 are $Q8, < 1 >, < -1 >, < i >, < j >, < k >$.