Some Practice Problems from Fraleigh

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March 9, 2025

These are some practice problems, primarily to work on proof writing.

Problem 44 — Page 67

Let G be a cyclic group with generator a, and let G' be a group isomorphic to G. If $\varphi : G \to G'$ is an isomorphism, show that, for every $x \in G$, $\varphi(x)$ is completely determined by the value $\varphi(a)$. That is, if $\varphi : G \to G'$ and $\psi : G \to G'$ are two isomorphisms such that $\varphi(a) = \psi(a)$, then $\varphi(x) = \psi(x)$ for all $x \in G$.

Proof. Let $G =< a >= \{e, a, a^2, \dots\}$ be a cyclic group generated by a isomorphic to G' via the arbitrary isomorphisms:

$$\varphi : G \to G' \land \psi : G \to G'$$

where

$$\varphi(a) = \psi(a)$$

Given that G is cyclic with a as generator we deduce

$$\forall x \in G \quad \exists n \in \mathbb{Z} : x = a^n$$

$$\therefore \varphi(x) = \varphi(a^n) = \varphi(aa \dots a)$$

which, by the structure preserving property of group isomorphisms yields

$$\varphi(aa \dots a) = \varphi(a^n) = \varphi(a)^n = \varphi(a)\varphi(a) \dots \varphi(a)$$

which means that $\varphi(x)$ can be written in terms of $\varphi(a)$ alone. Furthermore, recall

$$\varphi(a) = \psi(a)$$

meaning

$$\begin{aligned}
\varphi(a)\varphi(a) &= \psi(a)\psi(a) \\
\varphi(a)\varphi(a) \dots \varphi(a) &= \psi(a)\psi(a) \dots \psi(a) \\
\varphi(a)^n &= \psi(a)^n \\
\varphi(a^n) &= \psi(a^n) \\
\varphi(x) &= \psi(x)
\end{aligned}$$

$\square$

Problem 45 — Page 67

Let a and b be elements of a group G. Show that if ab has finite order n, then ba also has order n.

Proof. Take a and b to be arbitrary elements of G, with $|ab| = n \neq \infty$, then

$$\begin{gathered} |ab| = n \Rightarrow (ab)^n = e \\ (ab)^n = e \\ (ab)(ab)\ldots(ab) = e \end{gathered}$$

by associativity of group operations:

$$\begin{gathered} a(ba)(ba)\ldots(ba)b = e \\ a(ba)^{n-1}b = e \\ (ba)^{n-1}b = a^{-1} \\ (ba)^{n-1} = a^{-1}b^{-1} \\ (ba)^{n-1}b = a^{-1}b^{-1}b \\ (ba)^{n-1}b = a^{-1} \\ (ba)^{n-1}ba = a^{-1}a \\ (ba)^{n-1}(ba) = e \\ (ba)^n = e \end{gathered}$$

thus $|ba| = n$.

□