MAT-INF3600 Exam Practice

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1 Problem set

Problem 1.1. *Let $\mathfrak{L}$ be the language with a binary relation symbol R. Consider the sentence ϕ:*

$$\forall x \neg R(x,x) \wedge \forall x,y,z[R(x,y) \wedge R(y,z) \rightarrow R(x,z)]$$

Prove or disprove: If a structure $\mathcal{M}$ satisfies ϕ with a finite domain M, then there must exist $a \in M$ such that $(a,b) \notin R, \forall b \in M$.

Proof. The claim is true. Assume $\mathcal{M} \models \phi$ with $|M| = |\{m_1, \dots, m_n\}| < \infty$. Suppose, for a contradiction, that there is some m_j for every m_i such that $(m_i, m_j) \in R$.

Then we have $(m_{i_1}, m_{i_2}) \in R$. R is irreflexive so m_{i_2} must relate to some element other than m_{i_1} (as otherwise we would get $(m_{i_1}, m_{i_1}) \in R$ through transitivity), thus $(m_{i_2}, m_{i_3}) \in R$. Extrapolating we get $(m_{i_{n-1}}, m_{i_n}) \in R$, but now m_{i_n} must relate to some element, but this would lead to $(m_{i_n}, m_{i_n}) \in R$ through transitivity so $\mathcal{M} \not\models \phi$, a contradiction. □

Problem 1.2. *Provide a formal derivation of:*

$$(\neg P \rightarrow \neg Q) \vdash (Q \rightarrow P)$$

Proof. The deduction, using L & K's system, is trivial.

Let $\Sigma = \{\neg P \rightarrow \neg Q\}$.

$$\begin{aligned} &1. \neg P \rightarrow \neg Q && (\Sigma) \\ &2. Q \rightarrow P && 1, (PC) \end{aligned}$$

We could also use the deduction theorem where we have $\phi \vdash \psi$ if and only if $\vdash \phi \rightarrow \psi$ for formula ψ and sentence ϕ. □

Problem 1.3. *Let Σ be a set of first-order sentences. Suppose that for every natural number n, there exists a model $\mathcal{M}_n$ of Σ such that $\mathcal{M}_n$ has n elements. Prove Σ has an infinite model.*

Proof. Assume $\mathcal{M}_n \vDash \Sigma, |M| = n$ for every $n \in \mathbb{N}$.

Consider the expanded set $\Sigma' \supseteq \Sigma$ defined as

$$\Sigma' := \Sigma \cup \{\exists x_1, \dots, \exists x_n \bigwedge_{i<j} x_i \neq x_j : n \in \mathbb{N}\}$$

This set is essentially Σ as well as "there are at least n elements" (for every n). Consider the set

$$\Delta \subset \Sigma'$$

such that $\phi_k \in \Delta$ where ϕ_k is the statement "there are at least k elements". It is then straightforward that every finite subset of Σ' is satisfiable so Σ' is as well, and in particular, it's model is infinite. $\Sigma' \supseteq \Sigma$ so this model also satisfies Σ. $\square$

2 Problem set

Problem 2.1. *Show that the class of all finite structures (in a language with at least one relation) is not first-order axiomatizable.*

Sketch proof. Classifying all finite structures includes structures that are arbitrarily large. It is straightforward to show that if some set of sentences has arbitrarily large finite models then it also has an infinite model.

As seen in problem 1.3, defining ”at least n elements” for every natural n and having that sentence satisfied for every n means that it is satisfied by an infinite structure. □

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Solution of problem 1.

$$\begin{array}{lr}
1. \forall x[Q(Sx) \to Q(x)] & \Sigma_1 \\
2. \forall x[Q(Sx) \to Q(x)] \to [Q(S0) \to Q(0)] & (Q1) \\
3. Q(S0) \to Q(0) & 1,2 \quad (PC) \\
4. Q(S0) & \Sigma_1 \\
5. Q(0) & 3,4 \quad (PC)
\end{array}$$

Hence we have $\Sigma_1 \vdash Q(0)$. □

Solution of problem 2.

$$\begin{array}{lr}
1. \forall x[Q(Sx) \to Q(x)] & \Sigma_1 \\
\vdots & \\
5. Q(0) & 4,5 \quad (PC) \\
6. 0 = x \to (Q(0) \to Q(x)) & E3 \\
7. Q(0) \to (0 = x \to Q(x)) & 1 \quad (PC) \\
8. 0 = x \to Q(x) & 5,7 \quad (PC)
\end{array}$$

Hence we have $\Sigma_1 \vdash 0 = x \to Q(x)$. □

Proof of problem 3. Let $\mathfrak{A} \vDash \Sigma_{17}$, then we have $\mathfrak{A} \vDash Q(\overline{n})$ for $n \in \{1, \dots, 17\}$. Notice that then $\mathfrak{A} \vDash \Sigma_{16}$. Thus we have $\Sigma_{17} \vDash \Sigma_{16}$.

Let $\mathfrak{B}$ be a structure with universe $[16]$, that is, the natural numbers from 0 to 16. Let $S^{\mathfrak{B}}$ be the successor function for $n = 0, \dots, 15$ while $S^{\mathfrak{B}}(16) = 16$. Furthermore let $Q^{\mathfrak{B}} = \{0, \dots, 16\}$. We see that $\mathfrak{B} \vDash \Sigma_{16}$, but $\mathfrak{B} \nvDash \Sigma_{17}$. Therefore $\Sigma_{16} \nvDash \Sigma_{17}$. □

Solution of problem 4. Let $\phi := \forall x[Qx]$. Then we can create a model for Σ which satisfies ϕ in the same manner as done above with universe $\mathbb{N}$, but we can also have a model $\mathfrak{A}$ where the universe is $\mathbb{N} \cup \{\omega\}$ such that $S^{\mathfrak{A}}(\omega) = \omega$ and $\omega \notin Q^{\mathfrak{A}}$. □

Solution of problem 5. A set of formulas Σ is consistent if it is not inconsistent. Meaning $\Sigma \nvdash \perp$, equivalently Σ has a model.

Γ_1 is consistent. Take a model $\mathfrak{A}$ with universe $\{c\}$, where the constant is interpreted as c, $f^{\mathfrak{A}}(c, c) = c$.

Γ_2 is not consistent.

$$\begin{array}{lr}
1. \forall x[\neg x = c] & \Gamma_2 \\
2. \forall x[\neg x = c] \to f(c, c) = c & (Q_1) \\
3. f(c, c) = c & 1, 2 \quad (PC) \\
4. \neg f(c, c) = c & \Gamma_2 \\
5. f(c, c) = c \land \neg f(c, c) = c & 3, 4 \quad (PC)
\end{array}$$

From here we get such nonsense as $\Gamma_2 \vdash x \neq x$.

Γ_3 is inconsistent.

$$\begin{array}{lr}
1. \forall x[\neg x = c] & \Gamma_3 \\
2. \forall x[\neg x = c] \to \neg c = c & (Q_1) \\
3. \neg c = c & 1, 2 \quad (PC)
\end{array}$$

but $\Gamma_3 \vdash c = c$ using E_1. Nonsense.

Γ_4 is fine.

Let $\mathfrak{B}$ be a structure with universe $\{a, c\}$, $c^{\mathfrak{B}} = c$, $f^{\mathfrak{B}}(c, c) = c$. Now it doesn't matter how we define the rest we are already done as $a \neq c$ so there exists an element different from c. □