Advanced Linear Algebra

Thobias Høivik

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1 Introduction

The first few sections of this collection of notes follows the public advanced linear algebra course from the math at andrews university youtube channel. The lecturer is Dr. Andrew Bosman. These notes will, later on, be for the advanced linear algebra course at the University of Oslo if I'm allowed to take the course. The University of Oslo's course follows the same book as Andrews' does; Advanced Linear and Matrix Algebra by Nathaniel Johnston. Thus these notes will be an amalgamation of the content of those two courses and the book.

2 Vector Spaces and Subspaces

Typically, when vectors are introduced in an introductory course, we look at vectors in $\mathbb{R}^n$. However, vector spaces can be much more abstract and unintuitive. For example, $\mathbb{R}$ itself, can be understood as a vector space over $\mathbb{Q}$ with an uncountably infinite set of bases. So $\mathbb{R}$ is a field, but also a vector space (which, as we'll see in the following definition, is a structure over a field). The same can be said for $\mathbb{C}$, which is a field as well as a vector space over $\mathbb{R}$. We'll also look at vector spaces of functions and all other sorts of abstract spaces.

Definition 2.1: Vector Space

A *vector space* over a field $\mathbb{F}$ is a set V equipped with two operations:

- **Vector addition:** $+ : V \times V \to V$
- **Scalar multiplication:** $\cdot : \mathbb{F} \times V \to V$

These must satisfy 8 axioms such as associativity, distributivity, identity, and existence of additive inverses.

While we will not rigorously define what a field is (that is usually first seen in a first course in abstract algebra), we can aid our intuition by listing a few interesting fields:

- $\mathbb{R}$, as mentioned above
- $\mathbb{Q}$
- $\mathbb{C}$
- $\mathbb{F}_2 = \{0, 1\}$

Example 2.1: Vector Spaces

- $\mathbb{R}^n$, $\mathbb{C}^n$, over $\mathbb{R}$ and $\mathbb{C}$, respectively
- Spaces of functions: $C([a, b])$, $\mathbb{R}^{\mathbb{N}}$
- Polynomial spaces: $\mathbb{R}[x]$

Definition 2.2: Subspace

A subset $U \subseteq V$ is a *subspace* if:

- $\overline{0} \in U$
- $u + v \in U$ for all $u, v \in U$
- $\lambda u \in U$ for all $\lambda \in \mathbb{F}, u \in U$

Note that a subspace has to be a subset which satisfies all of the axioms of a vector space, but checking these properties gives us the rest for free. In fact, checking the last property gives us $0 \in U$ for free, since $0 \in \mathbb{F}$, so that check is also technically redundant.

Example 2.2: Subspaces of $\mathbb{R}^3$

- The zero subspace: $\{0\}$
- Any line or plane through the origin
- The whole space $\mathbb{R}^3$

3 Spans and Linear Independence

Definition 3.1: Span

Given vectors $v_1, \dots, v_k \in V$, the *span* is:

$$\operatorname{span}(v_1, \dots, v_k) = \left\{ \sum_{i=1}^{k} \lambda_i v_i : \lambda_i \in \mathbb{F} \right\}.$$

It is the smallest subspace containing all the v_i.

Definition 3.2: Linear Independence

Vectors $v_1, \dots, v_k$ are *linearly independent* if:

$$\sum_{i=1}^{k} \lambda_i v_i = 0 \quad \Rightarrow \quad \lambda_1 = \cdots = \lambda_k = 0.$$

Otherwise, they are *linearly dependent*.

Example 3.1: Dependence and Independence

- In $\mathbb{R}^3$, any three vectors lying in the same plane are linearly dependent.
- The standard basis vectors e_1, e_2, e_3 in $\mathbb{R}^3$ are linearly independent.

Theorem 3.1: Characterization of Dependence

A set $\{v_1, \dots, v_k\}$ is linearly dependent if and only if some v_j lies in the span of the others.

4 Bases

Definition 4.1: Basis

A *basis* for a vector space V is a linearly independent set $\{v_1, \ldots, v_n\} \subseteq V$ such that

$$\mathrm{span}(v_1, \ldots, v_n) = V.$$

Definition 4.2: Dimension

The *dimension* of a vector space V, written $\dim V$, is the number of vectors in any basis for V.

Theorem 4.1: Uniqueness of Representation

If $\{v_1, \ldots, v_n\}$ is a basis for V, then every $v \in V$ can be written *uniquely* as

$$v = \lambda_1 v_1 + \cdots + \lambda_n v_n.$$

Example 4.1: Standard Basis

The standard basis of $\mathbb{R}^n$ is $\{e_1, \ldots, e_n\}$, where e_i has a 1 in the ith position and 0 elsewhere.

Theorem 4.2: All Bases Have Equal Size

If V has a finite basis, then all bases of V have the same number of vectors.

Sketch of Proof. Any set of more than $\dim V$ vectors is dependent, and any spanning set with fewer than $\dim V$ vectors cannot span. This leads to the conclusion that all bases must contain exactly $\dim V$ vectors. $\square$

5 Some problems for section 2–3

Problem 5.1: Subspace Verification

Let $V = \mathbb{R}^3$ and let $U = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$. Prove that U is a subspace of V.

Proof of 5.1. Let $u, v \in U = \{(x, y, z) \in \mathbb{R}^3 : x+y+z = 0\}$. Trivially, the zero vector is in U since $\overline{0} = (0,0,0)$ and $0+0+0 = 0$. Now denote $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ and let $x \in \mathbb{R}$. Then

$$\begin{aligned} u_1 + u_2 + u_3 &= 0 \\ x \cdot u &= x \cdot u_1 + x \cdot u_2 + x \cdot u_3 \\ &= x \cdot 0 = 0 \\ &\Rightarrow x \cdot u \in U \end{aligned}$$

Next we check vector addition is closed:

$$\begin{aligned} u + v &= (u_1 + v_1, u_2 + v_2, u_3 + v_3) \\ u_1 + v_1 + u_2 + v_2 + u_3 + v_3 &= 0 + 0 \\ &= 0 \\ &\Rightarrow u + v \in U \end{aligned}$$

Thus U is a subspace of $\mathbb{R}^3$ □

Problem 5.2: Span Problem

Let $v_1 = (1,2,1)$, $v_2 = (2,4,2)$, and $v_3 = (0,1,-1)$ in $\mathbb{R}^3$. Determine whether $\operatorname{span}(v_1, v_2, v_3) = \mathbb{R}^3$.

Solution. Note that $v_2 = 2v_1$, so v_1 and v_2 are linearly dependent and span the same line.

We must check whether v_3 lies outside the span of v_1. Suppose $v_3 = av_1$ for some $a \in \mathbb{R}$:

$$(0,1,-1) = a(1,2,1) = (a, 2a, a),$$

which implies $a = 0 \Rightarrow 1 = 0$, contradiction.

So $v_3 \notin \operatorname{span}(v_1)$, and $\{v_1, v_3\}$ is linearly independent.

But since $\mathbb{R}^3$ is 3-dimensional, and $\operatorname{span}(v_1, v_2, v_3) = \operatorname{span}(v_1, v_3)$, a 2-dimensional subspace, the span is not all of $\mathbb{R}^3$.

Therefore,

$$\operatorname{span}(v_1, v_2, v_3) \subsetneq \mathbb{R}^3.$$

6 Dimensions of a Vector Space

Suppose we have the following vectors in $\mathbb{R}^4$:

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Are these linearly independent in $\mathbb{R}^4$? In other words, does there not exist non-trivial $c_1, c_2, c_3 \in \mathbb{R}$ such that the sum of these three vectors is $\vec{0}$?

$$\begin{aligned} 2c_1 + 3c_3 &= 0 \\ c_1 + c_2 &= 0 \\ c2 &= 0 \\ -3c_1 + c_3 &= 0 \end{aligned}$$

As we can quickly see there is no solution to this system, save for the trivial one. Therefore these three vectors are linearly independent. How about we look at a different space.

Let P^3 be the space of polynomials of degree at most 3. Consider the following polynomials:

$$\begin{aligned} 2 + x - 3x^3 \\ x + x^2 \\ 3 + x^3 \end{aligned}$$

Somehow, the question of these polynomials being linearly independent, is the same as the one for the vectors in $\mathbb{R}^4$. Let us see if we can make this problem again in some different vector space.

Let $M_2(\mathbb{R})$ be the space of real-valued 2×2 matrices, focusing on the following elements:

$$\begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

All of these are equivalent so it would be helpful to find a way to reduce problems of linear independence to an equivalent problem in the familiar $\mathbb{R}^n$.

Definition 6.1

Let $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a finite basis for a vector space V over a field $\mathbb{F}$. If a vector $\vec{v} \in V$ can be written as

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$

for scalars $c_1, \dots, c_n \in \mathbb{F}$, then the tuple

$$[\vec{v}]_B = (c_1, \dots, c_n) \in \mathbb{F}^n$$

is called the **coordinate vector** of $\vec{v}$ with respect to the basis B, and the scalars $c_1, \dots, c_n$ are called the **coordinates** or **coefficients/coefficients** of $\vec{v}$ in the basis B.

Example 6.1

Consider the example from earlier with the vector space of polynomials of degree at most 4. The natural basis $B = \{1, x, x^2, x^3\}$. Then we have that

$$\begin{aligned} \left[2 + x - 3x^3\right]_B &= (2, 1, 0, -3) \\ \left[x + x^2\right]_B &= (0, 1, 1, 0) \\ \left[3 + x^3\right]_B &= (3, 0, 0, 1) \end{aligned}$$

Theorem 6.1

If $B = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a finite basis for some vector space V over a field $\mathbb{F}$, then any basis for V will have n vectors.

Proof. Assume, seeking contradiction, there exists some basis B' which does not have n elements.

Case 1: B' has more than n elements.

Suppose $B' = \{\vec{w}_1, \ldots, \vec{w}_m\} \subseteq V$ has $m > n$ elements. We show B' is linearly dependent in V. We want to find a non-trivial solution to

$$\sum_{i=1}^{m} a_i \vec{w}_i = \vec{0}$$

Let us express these as coefficient vectors

$$\sum_{i=1}^{m} a_i [\vec{w}_i]_B = [\vec{0}]_B$$

$$[\vec{w}_i]_B = (c_{i1}, \ldots, c_{in}) \in \mathbb{F}^n$$

Then the the expression becomes

$$\sum_{i=1}^{m} a_i \begin{bmatrix} c_{i1} \\ \vdots \\ c_{in} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Here we get n equations with $m > n$ variables. By a standard result from linear algebra, any homogeneous system of linear equations with more unknowns than equations has at least one non-trivial solution, regardless of the field. Hence, the vectors $\vec{w}_1, \ldots, \vec{w}_m$ are linearly dependent.

Case 2: B' has less than n elements.

Suppose there is some other basis $B' = \{\vec{w}_1, \ldots, \vec{w}_m\} \subseteq V$ where $m < n$. If it were true that B' constitutes a basis, i.e. spans V and is linearly independent, then since $m < n$ we must have that B is linearly dependent, as shown in *case 1*. In other words if B' was a basis with $m < n$ elements then B cannot be a basis for V since we would, through the same method as above, get that there exists a non-trivial solution for the sum of n coefficients multiplied by $\vec{v}_1, \ldots, \vec{v}_n$ equaling the zero vector. This contradicts our assumption that B is a basis for V and therefore B' cannot be a basis for V.

□

Definition 6.2

V has a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$, we say that the dimension $dim(V) = n$. Note by the theorem above that n is unique since we cannot have a basis with $m \neq n$ elements. This means that V is a finite-dimensional vector space. If there does not exist a finite basis we say V is infinite-dimensional, $dim(V) = \infty$.

7 Tasks from MAT1125 book chapter 2

The following section covers some tasks from the main book used in my course and will therefore be in Norwegian.

Problem 7.1: Oppgave 2.1.3

Vis at følgende funksjoner er lineære.

(a) $T : \mathbb{R}^n \to \mathbb{R}^m$ gitt ved $Tx := Ax$ for en matrise $A \in M_{m\times n}(\mathbb{R})$.

(b) La $a, b \in \mathbb{R}$, $a < b$, og definer $T : C^0([a,b],\mathbb{R}) \to \mathbb{R}$ ved

$$Tf := \int_a^b f(t)dt \qquad \forall f \in C^0([a,b],\mathbb{R})$$

(c) La $g \in C^0(\mathbb{R},\mathbb{R}) \to \mathbb{R}$ være en gitt funksjon, og definer $T : C^0([0,1],\mathbb{R}) \to \mathbb{R}$ ved

$$Tf := \int_0^1 f(t)g(t)dt \qquad \forall f \in C^0([0,1],\mathbb{R})$$

(d) $T : \mathscr{P} \to C^0([a,b],\mathbb{R})$ gitt ved

$$Tp := p\big|_{[a,b]}$$

alstå restriksjonen av p til intervallet $[a,b]$.

Bevis av (a). La $T : \mathbb{R}^n \to \mathbb{M}^m$ definert slik som i oppgavebeskrivelsen.

Vi ønsker å vise at funksjonen er lineær (bevarer vektor-addisjon og skalar-multiplikasjon).

La $x, y \in \mathbb{R}^n$, la $r \in \mathbb{R}$, og la $A \in M_{m\times n}(\mathbb{R})$. Da er x og y av formen:

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad x_y, y_i \in \mathbb{R} \quad (1 \le i \le n)$$

mens

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

som betyr at

$$\begin{aligned} Tx = Ax &= \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ &= \begin{pmatrix} \sum_{i=1}^n a_{1i}x_i \\ \vdots \\ \sum_{i=1}^n a_{mi}x_i \end{pmatrix} \in \mathbb{R}^m \end{aligned}$$

så T er vell-definert.

For så å sjekke lineæritet av vektor addisjon:

$$
\begin{aligned}
T(x+y) &= \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_i \end{pmatrix} \\
&= \begin{pmatrix} \sum_{i=1}^{n} a_{1i}x_i + \sum_{i=1}^{n} a_{1i}y_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi}x_i + \sum_{i=1}^{n} a_{mi}y_i \end{pmatrix} \\
&= \begin{pmatrix} \sum_{i=1}^{n} a_{1i}x_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi}x_i \end{pmatrix} + \begin{pmatrix} \sum_{i=1}^{n} a_{1i}y_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi}y_i \end{pmatrix} \\
&= \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_i \end{pmatrix} \\
&= Tx + Ty
\end{aligned}
$$

Deretter sjekker vi lineæritet av skalar-multiplikasjon:

$$
\begin{aligned}
T(rx) &= \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} rx_1 \\ \vdots \\ rx_n \end{pmatrix} \\
&= \begin{pmatrix} \sum_{i=1}^{n} a_{1i}rx_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi}rx_i \end{pmatrix} = \begin{pmatrix} r\sum_{i=1}^{n} a_{1i}x_i \\ \vdots \\ r\sum_{i=1}^{n} a_{mi}x_i \end{pmatrix} = r\begin{pmatrix} \sum_{i=1}^{n} a_{1i}x_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi}x_i \end{pmatrix} \\
&= rTx
\end{aligned}
$$

Med det har vi vist at T er en lineær. □

Bevis av (b). La $T : C^0([a,b],\mathbb{R}) \to \mathbb{R}$ være definert som i oppgavebeskrivelsen.

Vi ønsker å vise at T er lineær.

T er vell-definert siden å ta integralet fra a til b av en funksjon, kontinuerlig på det intervalet, som har $\mathbb{R}$ som co-domene, gir et reelt tal.

Vi sjekker så lineæritet av vektor-addisjon. La $f, g \in C^0([a,b],\mathbb{R})$. Da har vi

$$
\begin{aligned}
T(f+g) &= \int_a^b (f+g)(t)dt \\
&= \int_a^b f(t)+g(t)dt \\
&= \int_a^b f(t)dt + \int_a^b g(t)dt \\
&= Tf + Tg
\end{aligned}
$$

Deretter sjekker vi lineæritet av skalar-multiplikasjon. La $f \in C^0([a,b],\mathbb{R})$ og $r \in \mathbb{R}$. Da har vi

$$\begin{aligned} T(rf) &= \int_a^b (rf)(t)dt \\ &= \int_a^b rf(t)dt \\ &= r\int_a^b f(t)dt \\ &= rTf \end{aligned}$$

som ønsket.

Med dette har vi vist at T er lineær fra $C^0([a,b],\mathbb{R}) \to \mathbb{R}$. $\square$

Bevis av (c). La $g \in C^0([0,1],\mathbb{R})$ være gitt og $T : C^0([0,1],\mathbb{R}$ definert som i oppgavebeskrivelsen.

Vi ønsker å vise at T er lineær.

Først merk at produktet av to funksjoner kontinuerlige på $[0,1]$ blir en funksjon kontinuerlig på $[0,1]$ (kan vises med å bruke definisjonen, med grenseverdier). Så integralet fra $[0,1]$ av denne funksjonen er vell-definert.

La $f,h \in C^0([0,1],\mathbb{R})$. Da har vi

$$\begin{aligned} T(f+h) &= \int_0^1 (f+h)(t)g(t)dt \\ &= \int_0^1 (f(t)+h(t))g(t)dt \\ &= \int_0^1 f(t)g(t)+h(t)g(t)dt \\ &= \int_0^1 f(t)g(t)dt + \int_0^1 h(t)g(t)dt \\ &= Tf + Th \end{aligned}$$

La $f \in C^0([0,1],\mathbb{R})$ og $r \in \mathbb{R}$. Da har vi

$$\begin{aligned} T(rf) &= \int_0^1 (rf)(t)g(t)dt \\ &= \int_0^1 rf(t)g(t)dt \\ &= r\int_0^1 f(t)g(t)dt \\ &= rTf \end{aligned}$$

som ønsket. $\square$

Bevis av (d). La $T : \mathscr{P} \to C^0([a,b],\mathbb{R})$ definert som i oppgavebeskrivelsen.

Merk at for alle $p \in \mathscr{P}$, p er kontinuerlig. Så restriksjonen av p til $[a,b]$ gir en kontinuerlig funksjon med domene $[a,b]$, så T er vell-definert.

For lineæritet av vektor-addisjon bruker vi faktumet at for to funksjoner p,q med domene $\mathbb{R}$ (som polynomene våre)

$$(p+q)(x) = p(x) + q(x)$$

Siden $[a, b] \subset \mathbb{R}$ må dette også være tilfellet for alle $x \in [a, b]$.

Et lignende argument gjelder for skalar-multiplikasjon. $\square$

Problem 7.2: Oppgave 2.2.5

La $T : \mathscr{P}_8 \to \mathbb{R}$ være den lineære avbildingen

$$T(p) := \int_0^1 p(t)dt \qquad \forall p \in \mathscr{P}_8$$

(a) Vis at $\operatorname{im} T = \mathbb{R}$.
(b) Bruk dimensjonssatsen til å finne dimensjonen til kjernen til T. Vil du si at kjernen til T er "stor" eller "liten"?

Bevis av (a). La $T : \mathscr{P}_8 \to \mathbb{R}$ være definert som i oppgavebeskrivelsen.

Vi ønskjer å vise at $\operatorname{im} T = \mathbb{R}$.

For en hver $\alpha \in \mathbb{R}$, betrakt konstant-polynomet $p(t) = \alpha \in \mathscr{P}_8$. Merk da at

$$T(p) = \int_0^1 \alpha dt = \alpha \int_0^1 1dt = \alpha [t]_0^1 = \alpha \cdot 1 = \alpha$$

Så for hver $\alpha \in \mathbb{R}$ har vi polynomet $p(t) = \alpha$ slik at $T(t) = \alpha$. Så $\operatorname{im} T = \mathbb{R}$.

$\square$

Løsning av (b). Ifølge dimensjonssatsen har vi at viss U, V er vektorrom med $T \in \mathscr{L}(U, V)$ så er

$$\dim(\operatorname{im} T) + \dim(\ker T) = \dim U$$

Vi vet at $\dim \mathscr{P}_8 = 9$ og $\dim(\operatorname{im} T) = \dim \mathbb{R} = 1$ så da vet vi at dimensjonen til kjernen av T er 8, alle polynom utenom konstant-polynomene.

Et underrom med dimensjon $n - 1$ er ganske stort. $\square$

Problem 7.3: Oppgave 2.5.2

Bevis at alle n-dimensjonelle vektorrom over $\mathbb{K}$ er isomorfe med hverandre.

Bevis. La U, V være n-dimensjonelle vektorrom over $\mathbb{K}$.

Vi ønsker å vise at

$$U \cong V$$

Siden U, V er n-dimensjonelle har vi at det finnes basiser $\mathscr{B} = (u_1, \ldots, u_n)$ for U, og $\mathscr{C} = (v_1, \ldots, v_n)$ for V. Siden U, V er isomorfe med $\mathbb{K}^n$, så bør vi ha at komposisjonen av isomorfien fra U til $\mathbb{K}^n$ med isomorfien fra $\mathbb{K}^n$ til V gir en isomorfi fra U til V. $\square$

8 Analysis in vector spaces

Definition 8.1: Norm

Let U be a vector space over either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. A norm on U is a function $\|\cdot\| : U \to \mathbb{R}$ such that

1. $\|u\| \geq 0$ for all $u \in U$, and $\|u| = 0$ if any only if $u = \overline{0}$
2. $\|\alpha u\| = |\alpha||u|$ for all $\alpha \in \mathbb{K}$ and $u \in U$
3. $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in U$

A normed vector space is a pair $(U, \|\cdot\|)$ consisting of a vector space U and a norm $\|\cdot\|$ on U. The "length" of a vector u is $\|u\|$. The "distance" between two vectors u, v is $\|u - v\|$.

The "distance" we are most familiar with in a first course in linear algebra is the euclidian norm

$$\|x\|_{\ell^2} = \sqrt{\sum_{k=1}^{n} |x_k|^2}$$

on $\mathbb{R}^n$ or $\mathbb{C}^n$.

In general for $p \in [1, \infty)$, the ℓ^p-norm on $\mathbb{R}^n$ or $\mathbb{C}^n$ is given by

$$\|x\|_{\ell^p(\mathbb{F}^n)} := \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}$$

where $|x| = \sqrt{x\overline{x}}$ for a complex x, and absolute value if real.

When we set $p = \infty$ we get the so-called "max-norm"

$$\|x\|_{\ell^\infty} := \max_{k=1,\dots,n} |x_k|$$

On the other extreme, when $p = 1$, is often called the "manhattan-metric".

Problem 8.1: Oppgave 3.1.3

La $x, y \in \mathbb{R}^2$, given by

$$x := \begin{pmatrix} 0 \\ 3 \end{pmatrix}, y := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Calculate the length of x and y in the following norms: $\|\cdot\|_{\ell^1}$ $\|\cdot\|_{\ell^2}$ and $\|\cdot\|_{\ell^\infty}$

Solution. Let x, y be given as in the problem description.

ℓ^1)

$$
\begin{aligned}
\|x\| &= \sum_{k=1}^{2} |x_k| \\
&= |0| + |3| = 0 + 3 = 3 \\
\|y\| &= \sum_{k=1}^{2} |y_k| \\
&= |1| + |1| = 1 + 1 = 2
\end{aligned}
$$

ℓ^2)

$$
\begin{aligned}
\|x\| &= \sqrt{\sum_{k=1}^{2} |x_k|} \\
&= \sqrt{|0| + |3|} = \sqrt{|3|} = \sqrt{3} \\
\|y\| &= \sqrt{\sum_{k=1}^{2} |y_k|} \\
&= \sqrt{|1| + |1|} = \sqrt{1+1} = \sqrt{2}
\end{aligned}
$$

ℓ^∞)

$$
\begin{aligned}
\|x\| &= \max_{k=1,2}\{|x_k|\} \\
&= \max\{0, 3\} = 3 \\
\|y\| &= \max_{k=1,2}\{|y_k|\} \\
&= \max\{1\} = 1
\end{aligned}
$$

□

Problem 8.2: Oppgave 3.2.7

Vis at

(a) alle kuler er begrensede

(b) alle åpne kuler $B_r(u)$ er åpne mengder

Bevis av (a). Vi ønsker å vise at det finnes en $r' > 0$ slik at $B_r(u) \subseteq \overline{B}_r(u) \subseteq B_{r'}(0)$.

La $r' = \|u\| + r + 1$. Da er

$$B_{r'}(0) = \{v \in U : \|v\| < \|u\| + r + 1\}$$

Viss vi betrakter en $v \in \overline{B}_r(u)$ så får vi at

$$
\begin{aligned}
\|v\| &= \|v - u + u\| \leq \|v - u\| + \|u\| \\
&\leq r + \|u\| < r + \|u\| + 1
\end{aligned}
$$

så $v \in B_{r'}(0)$ og siden $B_r(u) \subseteq \overline{B}_r(u)$, så gjelder er både den åpne- og lukkede kulen begrensede. □

Bevis av (b). Vi ønsker å vise at alle åpne kuler er åpne mengder.

La $B_r(u) = \{v \in U : \|v - u\| < r\}$ for et vektorrom U og en vilkårlig $r > 0$. Vi må så vise at det finnes en $s > 0$ slik at $B_s(v) \subseteq B_r(u)$ for enhver $v \in B_r(u)$.

For en vilkårlig $v \in B_r(u)$, la $s = r - \|v - u\| > 0$. Da har vi for enhver $v' \in B_s(v)$ at

$$\begin{aligned} \left\|v' - u\right\| &\leq \left\|v - v'\right\| + \|v - u\| < s + \|v - u\| = r \\ &\Rightarrow v' \in B_r(u) \Rightarrow B_s(v) \subseteq B_r(u) \end{aligned}$$

□