Lecture Notes: Enumerative Combinatorics (Lecture 4)

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1 Counting Permutations by Cycle Type

In this lecture, we explore the enumeration of permutations based on their cycle structure.

1.1 Cycle Type

The **cycle type** of a permutation is a partition of n that describes the lengths of its disjoint cycles. For example, the permutation $(1\,2)(3\,4\,5)$ has cycle type $(2, 3)$.

1.2 Counting Permutations with a Given Cycle Type

To count the number of permutations with a specific cycle type $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where λ_i are the lengths of the cycles, we use the formula:

$$\frac{n!}{z_\lambda}$$

where $z_\lambda = \prod_{i=1}^{k} m_i!\lambda_i^{m_i}$ and m_i is the number of parts of λ equal to i.

2 Records in Permutations

A **record** in a permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ is an element π_i that is larger than all previous elements $\pi_1, \pi_2, \ldots, \pi_{i-1}$.

2.1 Counting Records

The expected number of records in a random permutation of n elements is $\sum_{i=1}^{n} \frac{1}{i}$, which approximates $\ln(n) + \gamma$ for large n, where γ is the Euler-Mascheroni constant.

3 Inversions in Permutations

An **inversion** in a permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ is a pair (i, j) such that $i < j$ and $\pi_i > \pi_j$.

3.1 Counting Inversions

The number of inversions in a permutation is a measure of its ”sortedness.” A permutation with zero inversions is the identity permutation, while the maximum number of inversions is $\binom{n}{2}$, corresponding to the reverse permutation.