

# Mat210 Advanced Discrete Mathematics Notes

Thobias Høivik

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## 1 Pre-Semester Start – Cardinality

The following chapter contains notes based on what I think the course will cover in the first week (week 33). According to the syllabus, cardinality is mentioned early, so this section will review some basics.

### Definition 1.1: Cardinality

Let  $A$  and  $B$  be sets. We say  $A$  and  $B$  have the same *cardinality*, written  $|A| = |B|$ , if there exists a bijection  $f : A \rightarrow B$ . If no such bijection exists, the sets have different cardinalities.

### Example 1.1

Let  $A = \{1, 2\}$ ,  $B = \{3, 4\}$ . While this is a trivial example, we can show that there are as many elements in  $A$  as in  $B$  by constructing a function  $f : A \rightarrow B$  and showing that  $f$  is a bijection.

*Proof that  $|A| = |B|$ .* Let  $f : A \rightarrow B$  be defined by

$$f(n) = n + 2.$$

Let  $x, y \in A$  and suppose  $f(x) = f(y)$ . Then

$$f(x) = f(y)$$

$$x + 2 = y + 2$$

$$x = y.$$

Thus,  $f$  is injective.

Now let  $b \in B$ . Then  $b - 2 \in A$ , since  $B = \{3, 4\}$  and subtracting 2 yields values in  $A = \{1, 2\}$ . So for every  $b \in B$ , there exists  $a = b - 2 \in A$  such that  $f(a) = b$ . Hence,  $f$  is surjective.

Since  $f$  is both injective and surjective, it is a bijection, and therefore  $|A| = |B|$ .  $\square$

### Definition 1.2: Finite and Infinite Sets

A set  $A$  is *finite* if there exists a natural number  $n \in \mathbb{N}$  such that  $|A| = |\{1, 2, \dots, n\}|$ . Otherwise,  $A$  is *infinite*.

### Definition 1.3: Countably Infinite

A set  $A$  is *countably infinite* if there exists a bijection  $f : \mathbb{N} \rightarrow A$ . A set is *countable* if it is finite or countably infinite.

### Definition 1.4: Uncountable Set

A set  $A$  is *uncountable* if it is not countable; that is, there does not exist a bijection from  $\mathbb{N}$  to  $A$ .

### Example 1.2

The set  $\mathbb{R}$  is famously uncountable, as is rigorously demonstrated in any introductory analysis course (e.g., via Cantor's diagonal argument).

**Definition 1.5: Power Set**

Let  $A$  be a set. The *power set* of  $A$ , denoted  $\mathcal{P}(A)$ , is the set of all subsets of  $A$ .

**Theorem 1.1: Cantor's Theorem**

For any set  $A$ , we have  $|\mathcal{P}(A)| > |A|$ . In particular, there is no surjection from  $A$  onto  $\mathcal{P}(A)$ .

*Proof.* It suffices to show that there cannot exist a surjective function  $f : A \rightarrow \mathcal{P}(A)$ . Suppose, for contradiction, that such a surjective function  $f$  exists. Define the set

$$B = \{a \in A \mid a \notin f(a)\}.$$

Then  $B \subseteq A$ , so  $B \in \mathcal{P}(A)$ . Since  $f$  is surjective, there exists  $b \in A$  such that  $f(b) = B$ . We now ask: is  $b \in B$ ?

- If  $b \in B$ , then by the definition of  $B$ ,  $b \notin f(b) = B$ , a contradiction.
- If  $b \notin B$ , then by the definition of  $B$ ,  $b \in f(b) = B$ , again a contradiction.

In either case, we reach a contradiction. Therefore, our assumption that  $f$  is surjective must be false. Hence, there is no surjection from  $A$  onto  $\mathcal{P}(A)$ , and so

$$|\mathcal{P}(A)| > |A|.$$

□

After showing that the power set is strictly larger, we usually demonstrate that

$$|\mathcal{P}(A)| = 2^{|A|} > |A|$$

even for infinite sets. However, for infinite cardinals, exponentiation behaves differently than for finite numbers. For example,  $2^{\aleph_0} = \mathfrak{c} = |\mathbb{R}|$ .

**Problem 1.1**

Prove that  $|\mathbb{N}| = |\mathbb{Z}|$ , assuming  $0 \in \mathbb{N}$ .

*Proof of Problem 1.1.* We will construct a bijection  $f : \mathbb{N} \rightarrow \mathbb{Z}$ .

Define:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

We first show that  $f$  is injective. Suppose  $f(x) = f(y)$ .

**Case 1:** Both  $x$  and  $y$  are even. Then:

$$\frac{x}{2} = \frac{y}{2} \Rightarrow x = y.$$

**Case 2:** Both  $x$  and  $y$  are odd. Then:

$$-\frac{x+1}{2} = -\frac{y+1}{2} \Rightarrow x+1 = y+1 \Rightarrow x = y.$$

**Case 3:** One is even, one is odd. Then  $f(x) \in \mathbb{Z}_{\geq 0}$ ,  $f(y) \in \mathbb{Z}_{< 0}$ , so  $f(x) \neq f(y)$ . Hence,  $f$  is injective.

Now we show that  $f$  is surjective. Let  $z \in \mathbb{Z}$ . We find  $n \in \mathbb{N}$  such that  $f(n) = z$ :

**Case 1:**  $z \geq 0$ . Then let  $n = 2z$ . Since  $z \in \mathbb{Z}_{\geq 0}$ ,  $n \in \mathbb{N}$ , and  $f(n) = z$ .

**Case 2:**  $z < 0$ . Then let  $n = -2z - 1$ . Since  $z \in \mathbb{Z}_{<0}$ ,  $n \in \mathbb{N}$ , and:

$$f(n) = -\frac{n+1}{2} = -\frac{(-2z-1)+1}{2} = -\frac{-2z}{2} = z.$$

In both cases, such an  $n \in \mathbb{N}$  exists, so  $f$  is surjective.

Thus,  $f$  is a bijection and  $|\mathbb{N}| = |\mathbb{Z}|$ . □

## 2 Tasks from 7.4

### Problem 2.1: Task 17

Show that  $\mathbb{Q}$  is dense along the number line by showing that given two rational numbers  $r_1$  and  $r_2$  with  $r_1 < r_2$ , there exists a rational number  $x$  such that  $r_1 < x < r_2$ .

*Proof.* Let  $r_1, r_2 \in \mathbb{Q}$  such that  $r_1 < r_2$ . Consider the average of these numbers

$$\begin{aligned} x &= \frac{r_1 + r_2}{2} \\ &= \frac{\frac{a}{b} + \frac{c}{d}}{2} \\ &= \frac{a + c}{2bd} \end{aligned}$$

Clearly,  $x$  is a rational number since if  $a, b, c, d \in \mathbb{Z}$  then  $a + c \in \mathbb{Z}$  and  $2bd \in \mathbb{Z}$ . Furthermore

$$\begin{aligned} 2r_1 < r_1 + r_2 &\Rightarrow r_1 < \frac{r_1 + r_2}{2} = x \\ r_1 + r_2 < 2r_2 &\Rightarrow \frac{r_1 + r_2}{2} = x < r_2 \end{aligned}$$

Thus we have that  $x$  is a rational number satisfying the desired property. Hence  $\mathbb{Q}$  is dense along the number line.  $\square$

### Problem 2.2: Task 26

Prove that any uncountably infinite set  $A$  has a countably infinite subset.

*Proof.* Let  $A$  be a set such that  $|A| > \aleph_0$ . To construct a countably infinite subset we proceed by induction as follows:

Let  $a_0 \in A$  be the first element. Then for our next element choose some element  $a_1 \in A \setminus \{a_0\}$ . We know  $A \setminus \{a_0\}$  is non-empty since  $A$  is infinite. If we have  $n$  elements in our subset take the subsequent element to be

$$a_{n+1} \in A \setminus \{a_0, a_1, \dots, a_n\}$$

As mentioned earlier,  $A$  take away  $\{a_1, \dots, a_n\}$  leaves a non-empty set and  $a_{n+1}$  is an available element of this set, meaning we can introduce it to our subset. Then, by mathematical induction, we get a sequence which is itself a type of subset  $\{a_i : i \in \mathbb{N}\}$ . Clearly we can construct a bijection

$$f : \mathbb{N} \rightarrow \{a_i : i \in \mathbb{N}\}$$

such that  $f(i) = a_i$ . Note that this procedure of making infinitely many choices, means using a weak form of the Axiom of Choice.  $\square$

### Problem 2.3: Task 27

Let  $A$  and  $B$  be sets such that  $|A| = \aleph_0$ . Prove that if there exists some  $g : A \rightarrow B$  surjection, then  $B$  is countable.

*Proof.* We will proceed by proving that if there exists some surjection from one set  $\Gamma$  to another set  $\Delta$ , then  $|\Gamma| \geq |\Delta|$ . With this it follows that  $B$  is countable, assuming the conditions set in the problem description. Suppose  $\phi : \Gamma \rightarrow \Delta$  is surjective, i.e.

$$\forall \delta \in \Delta, \exists \gamma \in \Gamma \text{ s.t. } \phi(\gamma) = \delta$$

Since we assume  $\phi$  is well-defined,  $\phi(\gamma)$  goes to one and only one  $\delta \in \Delta$ . Since  $\phi$  is surjective, for any  $\delta \in \Delta$  there must be at least one  $\gamma$  mapped to  $\delta$ . As stated, no  $\gamma$  can map to more than one  $\delta$ . Therefore, for each  $\delta$  to have some  $\gamma$  which maps to it there must be at least as many  $\gamma \in \Gamma$  as there are  $\delta \in \Delta$ . In other words,

$$|\Gamma| \geq |\Delta|$$

With this fact, and given that we have sets  $A, B$  where  $|A| = \aleph_0$  and a surjection  $g : A \rightarrow B$  it must be the case that

$$|B| \leq |A| = \aleph_0$$

which is what it means to be countable. □

#### Problem 2.4: Task 32

Prove that the cartesian product of  $\mathbb{Z}$  with itself,  $\mathbb{Z} \times \mathbb{Z}$ , is countably infinite.

*Proof.* To show that  $\mathbb{Z}^2$  is countably infinite we must show that it is infinite ( $|\mathbb{Z}^2| \geq \aleph_0$ ), and it is countable ( $|\mathbb{Z}^2| \leq \aleph_0$ ), in other words,

$$|\mathbb{Z}^2| = \aleph_0$$

First we show  $\mathbb{Z}^2$  is infinite. This should be obvious since  $\mathbb{Z}$  is infinite, but to demonstrate this rigorously consider the function  $\pi_1 : \mathbb{Z}^2 \rightarrow \mathbb{Z}$ , defined as follows:

$$\pi_1(a, b) = a$$

Clearly,  $\pi_1$  is well-defined, since  $(a, b)$  is mapped to a unique  $a \in \mathbb{Z}$ . Also,  $\pi_1$  is surjective, since for any  $a \in \mathbb{Z}$ , there exists an infinite amount of elements in  $\mathbb{Z}^2$  such that  $\pi_1(a, b) = a$ . Thus we have shown that we can project  $\mathbb{Z}^2$  onto an infinite set  $\mathbb{Z}$ . Hence  $\mathbb{Z}^2$  is infinite. In other words:  $|\mathbb{Z}^2| \geq \aleph_0$ .

Now we show that there is a surjection from the naturals to  $\mathbb{Z}^2$ . First define a bijection  $h : \mathbb{Z} \rightarrow \mathbb{N}$  by

$$h(n) = \begin{cases} 2n, & n \geq 0, \\ -2n-1, & n < 0. \end{cases}$$

Let  $\pi : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  be the Cantor pairing function

$$\pi(a, b) = \frac{(a+b)(a+b+1)}{2} + b,$$

which is a bijection. Its inverse  $\pi^{-1} : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$  can be written explicitly: for  $n \in \mathbb{N}$  set

$$w = \left\lfloor \frac{\sqrt{8n+1}-1}{2} \right\rfloor, \quad t = \frac{w(w+1)}{2}, \quad b = n - t, \quad a = w - b,$$

so  $\pi^{-1}(n) = (a, b)$ .

Now define  $s : \mathbb{N} \rightarrow \mathbb{Z}^2$  by

$$s(n) = (h^{-1}(a), h^{-1}(b)) \quad \text{where } (a, b) = \pi^{-1}(n).$$

(Here  $h^{-1} : \mathbb{N} \rightarrow \mathbb{Z}$  exists because  $h$  is a bijection.)

To see  $s$  is surjective, take any  $(x, y) \in \mathbb{Z}^2$ . Let  $a = h(x)$  and  $b = h(y)$ . Put  $m = \pi(a, b) \in \mathbb{N}$ . Then  $\pi^{-1}(m) = (a, b)$ , hence

$$s(m) = (h^{-1}(a), h^{-1}(b)) = (x, y).$$

Thus every element of  $\mathbb{Z}^2$  has a preimage under  $s$ , so  $s$  is surjective.

Consequently  $|\mathbb{Z}^2| \leq |\mathbb{N}| = \aleph_0$ . (Since  $\mathbb{Z}^2$  projects onto  $\mathbb{Z}$ , we also have  $|\mathbb{Z}^2| \geq \aleph_0$ , so in fact  $|\mathbb{Z}^2| = \aleph_0$ .)

□

### Problem 2.5: Task 38

Suppose  $A_1, A_2, \dots$  is an infinite sequence of countable sets. Prove that

$$\bigcup_{i=1}^{\infty} A_i$$

is countable.

*Proof.* We intend to show that the countably infinite union of countable sets is countable.

Let  $A_1, A_2, \dots$  be a sequence of countable sets.

Recall that

$$\bigcup_{i=1}^{\infty} A_i = \{x : x \in A_i, i \in \mathbb{Z}_+\}.$$

Since  $A_i$  is countable and  $\mathbb{Z}_+$  is countable, there exists a surjection  $g_i : \mathbb{Z}_+ \rightarrow A_i$ . Recall also that  $\mathbb{Z}_+ \times \mathbb{Z}_+$  is countable. Therefore if we can construct a surjective  $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \rightarrow \bigcup_{i=1}^{\infty} A_i$ , it follows that  $\bigcup_{i=1}^{\infty} A_i$  is countable.

Define  $f(n, m) = g_n(m)$ , where  $g_n$  denotes the surjection from  $\mathbb{Z}_+$  to  $A_n$ . To check surjectivity, let  $x \in \bigcup_{i=1}^{\infty} A_i$ . Then there exists some  $k \in \mathbb{Z}_+$  such that  $x \in A_k$ . Since  $g_k$  is surjective, there exists  $m \in \mathbb{Z}_+$  such that  $g_k(m) = x$ . Hence  $f(k, m) = x$ . Therefore  $f$  is surjective.

Since  $\mathbb{Z}_+ \times \mathbb{Z}_+$  is countable and  $f$  is surjective, it follows that  $\bigcup_{i=1}^{\infty} A_i$  is countable.

□



### 3 Multiplication principle

#### Definition 3.1: Multiplication Principle

If a task can be performed in a sequence of  $k$  steps, and the first step can be performed in  $n_1$  ways, the second in  $n_2$  ways, and so on, then the entire task can be performed in

$$n_1 \times n_2 \times \cdots \times n_k$$

ways.

#### Theorem 3.1: Multiplication Principle

Suppose an experiment consists of two successive stages. If the first stage can be performed in  $m$  ways and, for each of these, the second stage can be performed in  $n$  ways, then the experiment can be performed in

$$m \times n$$

ways. More generally, if there are  $k$  stages with  $n_i$  possible outcomes for stage  $i$ , then the total number of possible outcomes is

$$\prod_{i=1}^k n_i.$$

#### Example 3.1: Outfits

Suppose you have 3 shirts and 2 pairs of pants. Each shirt can be paired with any pair of pants, so the total number of possible outfits is

$$3 \times 2 = 6.$$

#### Example 3.2: License Plates

A license plate consists of 3 letters followed by 3 digits. There are  $26^3$  choices for the letters and  $10^3$  choices for the digits. Hence, the total number of license plates is

$$26^3 \times 10^3.$$

#### Example 3.3: Coin and Die

Suppose you flip a coin and then roll a die. The coin has 2 possible outcomes and the die has 6. By the multiplication principle, the total number of outcomes is

$$2 \times 6 = 12.$$