Mathematical Logic

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Contents

1 Introduction

These notes are for (introductory) mathematical logic. As of writing this introduction, I have not yet been accepted to take the mathematical logic course (University of Oslo). These notes will be based on the book used in the aforementioned course, as well as resources I can find online (e.g. publicly avaliable courses like UCLA's Math 220A). Therefore the structure of these notes might be a bit scattered and disorganized.

1.1 High level overview / rambling

First-order logic is the standard way to formalize mathematics. For instance Peano arithmetic formalizes number theory, Zermelo-Fraenkel set theory formalizes set theory. In symbolic logic, with formal languages, we don't consider interpretations of symbols. Using structures we can have a notion of validity or satisfaction to logical expressions. With these, we have notions of syntax and semantics.

1.2 Gödel's Completeness Theorem

This is a fundamental result in mathematical logic which states (informally) that a formula with no free variables can be formally deduced from a given set of axioms if and only if it is valid in every structure satisfying these axioms.

2 Languages and Structures

2.1 Languages

What does $\forall x(x > 0 \Rightarrow \exists y(y \cdot y = x))$ mean? While we might recognize that this is a statement that holds in an ordered field where every positive element is a square , it is just a sequence of symbols.

Definition 2.1: First order language

A (first-order) language is a set of symbols $\mathfrak{L}$ composed of two disjoint subsets:

1. The first part (common to all languages) consists of "(" and ")" together with the following logical symbols: the set of variables $\mathcal{V} = \{v_n : n \in \mathbb{N}\}$, the equality symbol "=", connectives "$\neg$", "$\wedge$", the existential quantifier "$\exists$"
2. The second part, called the signature of $\mathfrak{L}$ denoted $\sigma^{\mathfrak{L}}$, consists of the non-logical symbols of $\mathfrak{L}$. It consists of:
   - a set of constant symbols $C^{\mathfrak{L}}$
   - a sequence of sets $\mathscr{F}_n^{\mathfrak{L}}, n \in \mathbb{N}_{\geq 1}$, where elements of this set are called the n-ary function symbols
   - a sequence of sets $\mathscr{R}_n^{\mathfrak{L}}, n \in \mathbb{N}_{\geq 1}$, where elements of this set are called n-ary relation symbols (or, n-ary predicates)

The language $\mathfrak{L}$ is given by the disjoint union of these sets.

Note that the existential quantifier and logical or may be swapped for their negations: $\forall, \vee$ and nothing will change. The choice is a matter of taste. Also note that while we give practical names to the sets in $\mathfrak{L}$, we should attempt not to imbue any sort of interpretation on languages.

Now, a few *remarks*:

1. A language is always infinite. Furthermore, the logical part is countable while the non-logical can have arbitrary cardinality.
2. Abusing the notation: $\mathfrak{L}$ and $\sigma^{\mathfrak{L}}$ may be identified.

Example 2.1

$\mathfrak{L}_{\emptyset}$ the empty language (still contains logical part).
$\mathfrak{L}_{ring} = \{\underline{0}, \underline{1}, +, -, \cdot\}$ the ring language.
$\mathfrak{L}_{ord} = \{<\}$ the order language.
$\mathfrak{L}_{o.ring} := \mathfrak{L}_{ring} \cup \mathfrak{L}_{ord}$ the ordered ring language.
$\mathfrak{L}_{set} = \{\in\}$ the language of set theory.
$\mathfrak{L}_{grp} = \{\cdot, ^{-1}\}, 1$ the language of groups.
$\mathfrak{L}_{graph} = \{E\}$ the language of graphs.
$\mathfrak{L}_{ar} = \{\underline{0}, S, +, \cdot, <\}$ the language of arithmetic.

2.2 Structures

Definition 2.2: Structures

Let $\mathfrak{L}$ be a first-order language. An $\mathfrak{L}$-structure, $\mathfrak{A}$, constists of a non-empty set A (called the base set or universe of $\mathfrak{A}$ together with an element $c^{\mathfrak{A}} \in A$ for each $c \in c^{\mathfrak{L}}$, a function $f^{\mathfrak{A}} : A^n \longrightarrow A$ for each $f \in F_n^{\mathfrak{L}}$, and a subset $R^{\mathfrak{A}} \subseteq A^n$ for each $R \in R_n^{\mathfrak{L}}$. We write $\mathfrak{A} = \left(A, \left(Z^{\mathfrak{A}}\right)_{Z \in \sigma^{\mathfrak{L}}}\right)$. $Z^{\mathfrak{A}}$ is called the interpretation of the symbol $Z \in \sigma^{\mathfrak{L}}$ in the structure $\mathfrak{A}$

Note that there are different ways of formally defining this and the specifics of this can safely be taken for granted later on.

Example 2.2

1. $\mathcal{N} = (\mathbb{N}, 0, \underset{x \mapsto x+1}{S}, +, \cdot, <)$ is an $\mathfrak{L}_{ar}$-structure
2. $C = (\mathbb{C}, 0, 1, +, -, \cdot)$ is an $\mathfrak{L}_{ring}$-structure
3. $R = (\mathbb{R}, 0, 1, +, -, \cdot, <)$ the ordered field of real numbers, is an $\mathfrak{L}_{o.ring}$-structures

Definition 2.3: isomorphism of structures

We say that two $\mathfrak{L}$-structures $\mathfrak{A}, \mathfrak{B}$ are isomorphic, $\mathfrak{A} \cong \mathfrak{B}$, if there exists an isomorphism $F : A \underset{\text{bijection}}{\longrightarrow} B$ between the base sets which commutes with the interpretations of the symbols $\sigma^{\mathfrak{L}}$, that is:

1. $F(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$ for every constant symbol $c \in C^{\mathfrak{L}}$
2. $F(f^{\mathfrak{A}}(a_1, \ldots, a_n)) = f^{\mathfrak{B}}(F(a_1), \ldots, F(a_n))$ for every function symbol $f \in \mathscr{F}_n^{\mathfrak{L}}$, and every tuple $(a_1, \ldots, a_n) \in A^n$
3. $(a_1, \ldots, a_n) \in R^{\mathfrak{A}} \Leftrightarrow (F(a_1), \ldots, F(a_n)) \in R^{\mathfrak{B}}$ for every predicate $R \in \mathscr{R}_n^{\mathfrak{L}}$ and every tuple $(a_1, \ldots, a_n) \in A^n$

3 Terms and Formulas

3.1 Terms

Definition 3.1

A word w over a set (alphabet) E is a finite string $w = a_0 a_1 \dots a_{k-1}$ with $a_i \in E$ for every i. We call k the length of w, and we denote E^* the set of all words over E.

Definition 3.2

Let $\mathfrak{L}$ be a language. The set $\mathcal{T}^{\mathfrak{L}}$ of $\mathfrak{L}$-terms is the smallest subset D of $\mathfrak{L}^*$ containing the variables and the constants of $\mathfrak{L}$, such that if $f \in F_n^{\mathfrak{L}}$ and $t_1, \dots, t_n \in D$, then $f t_1 \dots t_n \in D$.

$$\Rightarrow \mathcal{T}^{\mathfrak{L}} = \bigcup_{n \in \mathbb{N}} \mathcal{T}_n^{\mathfrak{L}}$$

where $\mathcal{T}_0^{\mathfrak{L}} = C^{\mathfrak{L}} \cup \mathcal{V}^{\mathfrak{L}}$ and, inductively

$$\mathcal{T}_{n+1}^{\mathfrak{L}} = \mathcal{T}_n^{\mathfrak{L}} \cup \{f t_1 \dots t_k : k \in \mathbb{N}_{\geq 1}, f \in \mathcal{F}_k^{\mathfrak{L}} \text{ and } t_1, \dots, t_k \in \mathcal{T}_n^{\mathfrak{L}}\}$$

.

Proposition:

Any term $t \in \mathcal{T}^{\mathfrak{L}}$ satisfies one and only one of the following:

1. t is a variable
2. t is a constant symbol
3. there exists a unique integer $n \geq 1$, a unique n-ary function symbol $f \in \mathcal{F}_n^{\mathfrak{L}}$ and a unique sequence $(t_1, \dots, t_n)$ of terms such that $t = f t_1 \dots t_n$

Proof. We proceed by structural induction on the term $t \in \mathcal{T}^{\mathfrak{L}}$, as defined by the inductive construction of terms in a first-order language.

Base cases.

If t is a *variable* (i.e., $t \in \mathcal{V}$), then it satisfies condition (1).

If t is a *constant symbol* (i.e., a 0-ary function symbol), then it satisfies condition (2).

These two cases are mutually exclusive since the sets of variables and constant symbols are disjoint.

Inductive step.

Assume the statement holds for terms $t_1, \dots, t_n \in \mathcal{T}^{\mathfrak{L}}$.

Let $f \in \mathcal{F}_n^{\mathfrak{L}}$ be an n-ary function symbol for some $n \geq 1$, and define a new term

$$t = f t_1 \dots t_n.$$

Then t is neither a variable nor a constant symbol. Furthermore, the structure of the term guarantees that there exists a *unique* integer n, a unique function symbol f, and a unique sequence $(t_1, \dots, t_n)$ of terms such that

$$t = f t_1 \dots t_n,$$

due to the syntactic rules of term formation in first-order logic. Thus, condition (3) holds uniquely.

Exclusivity.

We now verify that the three cases are mutually exclusive:

- A term cannot be both a variable and a constant symbol (by definition of the syntax).
- A term cannot be a variable or constant symbol and simultaneously of the form $f t_1 \dots t_n$ for any $n \geq 1$.
- Finally, due to the unique structure of terms, a term of the form $f t_1 \dots t_n$ cannot be written in any other way.

Therefore, every term falls under exactly one of the three cases, completing the proof.

□

We introduce some *notation* here, for practical purposes: we shall often write $f(t_1, \dots, t_n)$ instead of $f t_1 \dots t_n$. When f is binary we might write $t_1 f t_2$ instead of $f t_1 t_2$. For example: $(x+y) \cdot z$ means $\cdot + xyz$.

Definition 3.3

The height of a term t, denoted $ht(t)$, is defined as the smallest natural number k such that $t \in \mathcal{T}_k^{\mathfrak{L}}$.

From this definition and the unique reading property for terms, it follows that $ht(f(t_1, \dots, t_n)) = 1 + \max\{ht(t_i) : 1 \leq i \leq n\}$.

3.2 Formulas

Definition 3.4: Atomic formula

An atomic $\mathfrak{L}$-formula is one of the following

- a word of the form $t_1 = t_2$, where t_1, t_2 are terms of the language
- a word of the form $R t_1 \dots t_n$, where $R \in \mathcal{R}_n^{\mathfrak{L}}$ and all $t_1, \dots, t_n$ are terms of the language

Then the set $Fml^{\mathfrak{L}}$ of $\mathfrak{L}$-formulas is the smallest subset D of $\mathfrak{L}^*$ that contains all atomic $\mathfrak{L}$-formulas such that if $x \in \mathcal{V}^{\mathfrak{L}}$ and $\phi, \psi \in D$, then $\neg\phi$, $(\phi \wedge \psi)$ and $\exists x \phi$ are all in D.

$$\Rightarrow Fml^{\mathfrak{L}} = \bigcup_{n \in \mathbb{N}} Fml_n^{\mathfrak{L}}$$

where $Fml_0^{\mathfrak{L}}$ is the set of atomic $\mathfrak{L}$-formulas, and inductively

$$Fml_{n+1}^{\mathfrak{L}} := Fml_n^{\mathfrak{L}} \cup \{\neg\phi : \phi \in Fml_n^{\mathfrak{L}}\} \cup \{(\phi \wedge \psi) : \phi, \psi \in Fml_n^{\mathfrak{L}}\} \cup \{\exists x \phi : \phi \in Fml_n^{\mathfrak{L}}, x \in \mathcal{V}^{\mathfrak{L}}\}$$

Note that the inclusion of $Fml_n^{\mathfrak{L}}$ in the union in the inductive definition is technically redundant, but we defined it like this for practicallity in other definitions.

The same can be said for the definition of $\mathcal{T}_n^{\mathfrak{L}}$.

3.3 Exercises from 1.6.1

Problem 3.1: Task 4

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are two $\mathfrak{L}$-structures. We say that they are *isomorphic*, $\mathfrak{A} \cong \mathfrak{B}$, if there exists a bijection $i : A \to A$ such that for each constant symbol c of $C^{\mathfrak{L}}$, $i(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$, for each n-ary function symbol f in $\mathscr{F}_n^{\mathfrak{L}}$, and for each $(a_1, \dots, a_n) \in A^n$, $i(f^{\mathfrak{A}}(a_1, \dots, a_n)) = f^{\mathfrak{B}}(i(a_1), \dots, i(a_n))$, and for each n-ary relation/predicate symbol R in $\mathscr{R}_n^{\mathfrak{L}}$, and for each $(a_1, \dots, a_n) \in A^n$, we have $(a_1, \dots, a_n) \in R^{\mathfrak{A}} \Leftrightarrow (i(a_1), \dots, i(a_n)) \in R^{\mathfrak{B}}$.

A Show that $\cong$ is an equivalence relation.

B Find two different structures for a particular language and prove that they are not isomorphic.

Proof of first. Let $\mathfrak{L}$ be an arbitrary language, and $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be $\mathfrak{L}$-structures.

First we show that $\mathfrak{A} \cong \mathfrak{A}$, i.e. $\cong$ is reflexive. Let $i : A \to A$ be the identity map $i(a) = a$. It is trivial to show that i is bijective, being its own inverse. Then we have $i(c^{\mathfrak{A}}) = c^{\mathfrak{A}}$ satisfying the first property of structure isomorphisms. Furthermore, if we take f to be an n-ary function symbol of the language, and $(a_1, \dots, a_n) \in A^n$, we get that $i(f^{\mathfrak{A}}(a_1, \dots, a_n)) = f^{\mathfrak{A}}(a_1, \dots, a_n)$. Now recognize that $i(a_j) = a_j$ for all j, and we find that $i(f^{\mathfrak{A}}(a_1, \dots, a_n)) = f^{\mathfrak{A}}(i(a_1), \dots, i(a_n))$, showing that the second property holds. Lastly, with R being some n-ary relation symbol in the language and $(a_1, \dots, a_n) \in A^n$, we have, through the same identity argument,

$$(a_1, \dots, a_n) \in R^{\mathfrak{A}} \Leftrightarrow (i(a_1), \dots, i(a_n)) \in R^{\mathfrak{A}}.$$

Thus $\mathfrak{A} \cong \mathfrak{A}$.

Second we show that $\cong$ is symmetric. Suppose $i : A \to B$ is an isomorphism, so $\mathfrak{A} \cong \mathfrak{B}$ via i. Since i is bijective, it has an inverse $j := i^{-1} : B \to A$ which is also a bijection.

We now show that j is a structure isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$.

Constants: For any constant symbol c in the language, we know $i(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$. Applying j to both sides gives:

$$j(c^{\mathfrak{B}}) = j(i(c^{\mathfrak{A}})) = c^{\mathfrak{A}}.$$

So the condition on constants is satisfied.

Functions: Let f be an n-ary function symbol and $(b_1, \dots, b_n) \in B^n$. Since j maps from B to A, we define $a_k := j(b_k)$ for each k. Then $i(a_k) = b_k$ for all k, and:

$$\begin{aligned} j(f^{\mathfrak{B}}(b_1, \dots, b_n)) &= j(f^{\mathfrak{B}}(i(a_1), \dots, i(a_n))) \\ &= j(i(f^{\mathfrak{A}}(a_1, \dots, a_n))) = f^{\mathfrak{A}}(a_1, \dots, a_n), \end{aligned}$$

which shows that:

$$f^{\mathfrak{A}}(j(b_1), \dots, j(b_n)) = j(f^{\mathfrak{B}}(b_1, \dots, b_n)),$$

satisfying the function compatibility condition.

Relations: Let R be an n-ary relation symbol and $(b_1, \dots, b_n) \in B^n$. Let $a_k := j(b_k)$ as before. Then, since $i(a_k) = b_k$, we get:

$$(b_1, \dots, b_n) \in R^{\mathfrak{B}} \Leftrightarrow (i(a_1), \dots, i(a_n)) \in R^{\mathfrak{B}} \Leftrightarrow (a_1, \dots, a_n) \in R^{\mathfrak{A}}$$

by the isomorphism definition for i. Applying j to the tuple and reversing the direction of the implication gives the desired equivalence. Hence $\mathfrak{B} \cong \mathfrak{A}$.

Lastly, we show that $\cong$ is transitive. Suppose $\mathfrak{A} \underset{i}{\cong} \mathfrak{B}$ and $\mathfrak{B} \underset{j}{\cong} \mathfrak{C}$. Define $k := j \circ i : A \to C$, which is bijective since it is the composition of two bijections.

We show that k is a structure isomorphism from $\mathfrak{A}$ to $\mathfrak{C}$.

Constants: For any constant symbol c, we have:

$$k(c^{\mathfrak{A}}) = j(i(c^{\mathfrak{A}})) = j(c^{\mathfrak{B}}) = c^{\mathfrak{C}}.$$

Functions: Let f be an n-ary function symbol and $(a_1, \ldots, a_n) \in A^n$. Then:

$$\begin{aligned} k(f^{\mathfrak{A}}(a_1, \ldots, a_n)) &= j(i(f^{\mathfrak{A}}(a_1, \ldots, a_n))) \\ &= j(f^{\mathfrak{B}}(i(a_1), \ldots, i(a_n))) \\ &= f^{\mathfrak{C}}(j(i(a_1)), \ldots, j(i(a_n))) \\ &= f^{\mathfrak{C}}(k(a_1), \ldots, k(a_n)). \end{aligned}$$

Relations: Let R be an n-ary relation symbol and $(a_1, \ldots, a_n) \in A^n$. Then:

$$(a_1, \ldots, a_n) \in R^{\mathfrak{A}} \Leftrightarrow (i(a_1), \ldots, i(a_n)) \in R^{\mathfrak{B}} \Leftrightarrow (j(i(a_1)), \ldots, j(i(a_n))) \in R^{\mathfrak{C}} \Leftrightarrow (k(a_1), \ldots, k(a_n)) \in R^{\mathfrak{C}}.$$

The reverse direction follows similarly by applying the inverses of j and i. Hence, k preserves all relations.

Thus, $\mathfrak{A} \cong \mathfrak{C}$, and we have shown that $\cong$ is transitive.

Therefore, $\cong$ is an equivalence relation (reflexive, symmetric, transitive). $\square$

Solution to the second (Informal). Let $\mathfrak{L}_{NT} = \{0, S, +, \cdot, E, <\}$ be the language of number theory.

Let $\mathcal{N}, \mathfrak{A}$ be $\mathfrak{L}_{NT}$-structures. More specifically, let $\mathcal{N}$ be the usual structure which models number theory. In $\mathfrak{A}$, let $a +^{\mathfrak{A}} b = a$ for all $a, b \in A$. To show that two structures like these cannot be isomorphic we proceed by contradiction, supposing there does exist an isomorphism $i : \mathbb{N} \to A$.

i is some function that, among other properties, satisfies $i(n+m) = i(n) +^{\mathfrak{A}} i(m)$, much like a group homomorphism, fun fun. Now since $i(n) +^{\mathfrak{A}} i(m) = i(n)$ we have

$$i(n+m) = i(n)$$

Choose $n = 1 = m$ and we get

$$i(2) = i(1)$$

which contradicts injectivity of i.

4 Semantics of first-order formulas

4.1 Unique readability, free variables, notation

Proposition. (Unique readability of formulas) Any $\mathfrak{L}$-formula ϕ satisfies one and only one of the following:

1. ϕ is an atomic formula
2. ϕ is equal to $\neg\psi$ for some unique $\mathfrak{L}$-formula ψ
3. ϕ is equal to $\psi \wedge \mathfrak{X}$ for some unique $\mathfrak{L}$-formulas $\psi, \mathfrak{X}$
4. ϕ is equal to $\exists x\psi$ for some unique variable x and some unique $\mathfrak{L}$-formula ψ

Proof. We proceed by structural induction on the formation of formulas in $\mathfrak{L}$.

Let $Fml_n^{\mathfrak{L}}$ denote the set of formulas of depth at most n:

- Base case: $Fml_0^{\mathfrak{L}}$ consists of atomic formulas.
- Inductive step: $Fml_{n+1}^{\mathfrak{L}}$ consists of formulas obtained by applying negation, conjunction, or existential quantification to formulas in $Fml_n^{\mathfrak{L}}$.

Base case $(n = 0)$**:** Any formula $\phi \in Fml_0^{\mathfrak{L}}$ is, by definition, atomic. Hence, case (1) holds, and the rest do not apply. The decomposition is trivially unique.

Inductive hypothesis: Assume that for all $\psi \in Fml_n^{\mathfrak{L}}$, ψ satisfies exactly one of the four cases of the proposition, with the associated subformulas and variables being unique.

Inductive step: Let $\phi \in Fml_{n+1}^{\mathfrak{L}} \setminus Fml_n^{\mathfrak{L}}$. Then ϕ is constructed as exactly one of the following:

- $\phi = \neg\psi$, where $\psi \in Fml_n^{\mathfrak{L}}$,
- $\phi = \psi \wedge \chi$, where $\psi, \chi \in Fml_n^{\mathfrak{L}}$,
- $\phi = \exists x\,\psi$, where x is a variable and $\psi \in Fml_n^{\mathfrak{L}}$.

Each of these constructions matches exactly one of cases (2), (3), or (4). We now prove uniqueness for each case.

Case 2: Suppose $\phi = \neg\psi_1 = \neg\psi_2$. Then by the syntax of negation, it follows that $\psi_1 = \psi_2$. Hence, ψ is uniquely determined.

Case 3: Suppose $\phi = \psi_1 \wedge \chi_1 = \psi_2 \wedge \chi_2$. Then the only way such a formula can be parsed as a conjunction is with:

$$\psi_1 = \psi_2 \quad \text{and} \quad \chi_1 = \chi_2.$$

Thus, both components are uniquely determined.

Case 4: Suppose $\phi = \exists x_1\,\psi_1 = \exists x_2\,\psi_2$. By the syntax of quantification, it must be that $x_1 = x_2$ and $\psi_1 = \psi_2$. Hence, both the variable and subformula are uniquely determined.

Mutual exclusivity: Finally, we verify that no formula can belong to more than one of the cases (1)–(4).

- Atomic formulas cannot start with $\neg$, $\wedge$, or $\exists$.
- Formulas of the form $\neg\psi$ cannot be written as conjunctions or quantifiers.
- Conjunctions and quantifiers have distinct syntactic structures and cannot be confused.

Hence, each formula ϕ satisfies exactly one of the four cases, and the decomposition is unique in each.

$\square$

Definition 4.1

Let $v_k \in \mathcal{V}^{\mathfrak{L}}$ be a variable. We define: a free occurence of v_k in ϕ:

- If ϕ is atomic (i.e. $t_1 = t_2$ for some terms), all occurences of v_k in ϕ are free.
- If ϕ is of the form $\neg\psi$, then the free occurences of v_k in ϕ are those which are free in ψ.
- If ϕ is of the form $(\psi \wedge \chi)$, then the free occurences of v_k in ϕ are those that are free in ψ and those that are free in χ
- If ϕ is of the form $\exists v_l \psi$ and $l \neq k$, then the free occurences of v_k in ϕ are those which are free in ψ
- If ϕ is of the form $\exists v_k \psi$, then no occurence of v_k in ϕ is free

Occurences of v_k in ϕ that are not free are called bound.
The free variables of ϕ are those having at least one reoccurence in ϕ. Denote $Free(\phi)$ the set of free variables of ϕ.
A sentence is a formula with no free variables.

Example 4.1

If ϕ is the formula $(\exists v_0 v_0 < v_1 \wedge v_0 = v_1)$ (note that there is no use of "(" or ")" in the formula itself).
Then:

- The first two occurences of v_0 are bound
- The third v_0 is free
- All occurences of v_1 are free

Notation. We will use the following abbreviations:

- $(\phi \vee \psi)$ for $\neg(\neg\phi \wedge \neg\psi)$
- $(\phi \Rightarrow \psi)$ for $\neg(\phi \wedge \neg\psi)$
- $(\phi \Leftrightarrow \psi)$ for $((\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi))$
- $\forall x \phi$ for $\neg\exists x \neg\phi$

Note that these do not constitute new symbols in the meaning of formulas in the formation rules, this is simply for convenience. We, formally, stick to the minimum amount of symbols for the full power of first-order logic.

We will also write $\exists x_1,\dots,x_n$ instead of $\exists x_1,\dots,\exists x_n$ and do so for the universal quantifier, also. Furthermore we write $R(t_1,\dots,t_n)$ instead of $Rt_1,\dots,t_n$ as well as write t_1Rt_2 instead of Rt_1t_2 when R is a binary relation. We will write $(\phi_0\wedge\dots\wedge\phi_n)$ or sometimes $\bigwedge_{i=0}^n\phi_i$ instead of $(\dots(\phi_0\wedge\phi_1)\wedge\phi_2)\wedge\dots\phi_n)$, as well as similar abbreviations for disjunction. We will add parentheses, or omit them for readability; with the convention that symbols from $\{\exists,\neg,\forall\}$ bind strongest, followed by $\wedge$, then $\vee$, then the symbols from $\{\Rightarrow,\Leftrightarrow\}$.

Example 4.2

$\forall x\phi\wedge\psi\Rightarrow\chi$ is read as $((\forall x\phi\wedge\psi)\Rightarrow\chi)$, that is $\neg(\forall x\phi\wedge\psi)\wedge\neg\chi)$, and so finally we get

$$\neg((\neg\exists x\neg\phi\wedge\psi)\wedge\neg\chi)$$

4.2 Semantics

Definition 4.2

Let $\mathfrak{A}$ be an $\mathfrak{L}$-structure.

1. An assignment (with value in $\mathfrak{A}$) is a function $\alpha:\mathcal{V}^{\mathfrak{L}}\to A$ from the set of variables to the base set or universe of $\mathfrak{A}$.
2. If α is an assignment and t is an $\mathfrak{L}$-term, we define $t^{\mathfrak{A}}[\alpha]$ by induction on the height of t $ht(t)$:
   - $v_i^{\mathfrak{A}}[\alpha]:=\alpha(v_i)$ (for $v_i\in\mathcal{V}^{\mathfrak{A}}$) and $c^{\mathfrak{A}}[\alpha]:=c^{\mathfrak{A}}$ (for constant symbol $c\in C^{\mathfrak{L}}$)
   - $f(t_1,\dots,t_n)^{\mathfrak{A}}[\alpha]:=f^{\mathfrak{A}}(t_1^{\mathfrak{A}}[\alpha],\dots,t_n^{\mathfrak{A}}[\alpha])$

Lemma 4.1

If two assignments α,β concide on all variables occuring in t, then $t[\alpha]=t[\beta]$.

Notation. If t is a term, we might denote it by $t(x_1,\dots,x_n)$ if the variables x_i are all distinct and all variables having at least one occurence in t all belong to the set $\{x_1,\dots,x_n\}$. If $t(x_1,\dots,x_n)$ and a tuple of elements $a_1,\dots,a_n\in A$, we define $t^{\mathfrak{A}}[a_1,\dots,a_n]$ by $t^{\mathfrak{A}}[\alpha]$ with α defined to be $\alpha(x_i)=a_i$ for all i.

Definition 4.3: Satisfaction of a formula/"Tarski's definition of truth"

Let $\mathfrak{A}$ be an $\mathfrak{L}$-structure. By induction on the height $ht(\phi)$ of a formula we define $\mathfrak{A}\vDash\phi[\alpha]$ (read as "ϕ is satisfied in $\mathfrak{A}$ in α"):

- $\mathfrak{A}\vDash t_1=t_2$ if and only if $t_1^{\mathfrak{A}}[\alpha]=t_2^{\mathfrak{A}}[\alpha]$
- $\mathfrak{A}\vDash Rt_1\dots t_n[\alpha]$ if and only if $(t_1[\alpha],\dots,t_n[\alpha]\in R^{\mathfrak{A}}$
- $\mathfrak{A}\vDash\neg\psi[\alpha]$ if and only if $\nvDash\psi[\alpha]$
- $\mathfrak{A}\vDash(\psi\wedge\chi)[\alpha]$ if and only if $\mathfrak{A}\vDash\psi[\alpha]$ and $\mathfrak{A}\vDash\chi[\alpha]$
- $\mathfrak{A}\vDash\exists x\psi[\alpha]$ if and only if there exists $a\in A$ with $\mathfrak{A}\vDash\psi[\alpha_{a/x}]$, where $\alpha_{a/x}$ denotes the assignment defined by $\alpha_{a/x}(x)=a$ and $\alpha_{a/x}(y)=\alpha(y)$ for all y different from x

Proposition. If two assignments α, β coincide on $Free(\phi)$, then one has:

$$\mathfrak{A} \vDash \phi[\alpha] \text{ if and only if } \mathfrak{A} \vDash \phi[\beta]$$

Proof. By induction on $ht(\phi)$. The case of atomic formulas follows from the earlier lemma about terms.

Inductive step:

Consider the case ϕ is equal to $\exists x \psi$. If we have that $\mathfrak{A} \vDash \phi[\alpha]$, there exists $a \in A$ such that $\mathfrak{A} \vDash \psi[\alpha_{a/x}]$. Any variable $y \neq x$ that is free in ψ is also free in ϕ. Hence we have $\mathfrak{A} \vDash \psi[\beta_{a/x}]$. By the inductive hypothesis, we also have $\mathfrak{A} \vDash \phi[\beta]$. □

Notation. A formula ϕ will be denoted by $\phi(x_1, \ldots, x_n)$ if the variables x_i are all distinct and free in ϕ and they belong to the set $\{x_1, \ldots, x_n\}$. If a formula $\phi(x_1, \ldots, x_n)$ and elements $a_1, \ldots, a_n \in A$ are given, we define $\mathfrak{A} \vDash \phi[a_1, \ldots, a_n]$ by $\mathfrak{A} \vDash \phi[\alpha]$ where α is an assignment such that $\alpha(x_i) = a_i$ (well defined by the previous proposition). Thus the formula $\phi(x_1, \ldots, x_n)$ defines an n-ary relation on the underlying base set A given by $\{(a_1, \ldots, a_n) \in A^n : \mathfrak{A} \vDash \phi[a_1, \ldots, a_n]\}$. In particular, when ϕ is a sentence, then the relation $\mathfrak{A} \vDash \phi$ is interpreted as "ϕ is satisfied or true in $\mathfrak{A}$" or "$\mathfrak{A}$ is a model of ϕ".

5 Substitution

5.1 Substitution of terms into formulas

Earlier we have discussed satisfaction of formulas in structures with assignment functions, $\mathfrak{A} \models \phi[\alpha]$. We may write $\phi(x_1,\dots,x_n)$ if x_i are distinct, free variables in $x_1,\dots,x_n$. Given $a_1,\dots,a_n \in A$, $\mathfrak{A} \models \phi[a_1,\dots,a_n] :\Leftrightarrow \mathfrak{A} \models \phi[\alpha]$ for α some assignment where $\alpha(x_i) = a_i$. Then the formula $\phi(x_1,\dots,x_n)$ defines an n-ary relation on A, given by $\phi[\mathfrak{A}] := \{(a_1,\dots,a_n) \in A^n : \mathfrak{A} \models \phi[a_1,\dots,a_n]\}$.

Definition 5.1

Let $\mathfrak{A}$ be an $\mathfrak{L}$-structure, and $D \subseteq A^n$.

1. The set D is $\emptyset$-definable (or, 0-definable) if $D = \phi[\mathfrak{A}]$ for some formula $\phi(x_1,\dots,x_n)$.
2. Let $B \subseteq A$. Then D is B-definable (or, definable over B) if there exists a formula $\phi(x_1,\dots,x_n,y_1,\dots,y_m)$ and $\overline{b} \in B^m$ such that

$$D = \phi[\mathfrak{A},\overline{b}] := \{\overline{a} \in A^n : \mathfrak{A} \models \phi(a_1,\dots,a_n,b_1,\dots,b_m)\}$$

Intuitively, a subset $D \subseteq A^n$ of an $\mathfrak{L}$-structure $\mathfrak{A}$ is *definable* if it can be described by a formula in the language $\mathfrak{L}$, possibly using parameters from the domain.

- If D is $\emptyset$*-definable* (or 0-definable), then there exists a formula $\phi(x_1,\dots,x_n)$ with no parameters from A such that
$$D = \{\overline{a} \in A^n : \mathfrak{A} \models \phi(\overline{a})\}.$$
In other words, D can be described without naming any specific elements of A beyond those already available through the symbols of $\mathfrak{L}$.
- If D is B*-definable* for some $B \subseteq A$, then there exists a formula $\phi(x_1,\dots,x_n,y_1,\dots,y_m)$ and a tuple $\overline{b} \in B^m$ such that
$$D = \{\overline{a} \in A^n : \mathfrak{A} \models \phi(\overline{a},\overline{b})\}.$$
Here, the formula may refer to specific elements of B through the parameters $\overline{b}$.

Example 5.1

Let $\mathfrak{A} = (\mathbb{N}, 0, S, +, \times)$, where S is the successor function.

1. The set of even natural numbers
$$E = \{n \in \mathbb{N} : \exists y\ (n = y + y)\}$$
is $\emptyset$-definable, since it can be defined by the formula $\phi(x) := \exists y\ (x = y + y)$, which uses no parameters from $\mathbb{N}$.
2. The singleton set $\{3\}$ is not $\emptyset$-definable in this language (unless a constant symbol for 3 is present in $\mathfrak{L}$), but it is definable over $B = \{3\}$ using the formula $\psi(x,y) := (x = y)$ with y interpreted as 3.
3. The set $\{x \in \mathbb{N} : x > 3\}$ is definable over $B = \{3\}$ by the formula $\theta(x,y) := x > y$, again with y interpreted as 3.

When it comes to substitution we may, naively, want to replace every free occurence of some variable x by a term s. For example $\phi(x_0)$ of the form $\exists v_1 \neg(v_1 = v_0)$. Now, what if $s = v_1$. This is bad because we would get

$$\exists v_1 \neg(v_1 = v_1)$$

which is satisfied in no structure.

Definition 5.2

Let $x_0, \ldots, x_r$ be distinct variables and $s_0, \ldots, s_r$ be terms. Then we define the simultaneous substitution of the x_i by the terms s_i in the following manner:

1. Let t be a term. Then $t_{s_0/x_0,\ldots,s_r/x_r} = t_{\overline{s}/\overline{x}}$ is the word obtained by:

$$x_{\overline{s}/\overline{x}} = \begin{cases} x & \text{if } x \neq x_0, \ldots, x_r \\ s_i & \text{if } x = x_i \end{cases}$$

   for $x \in \mathcal{V}^{\mathfrak{L}}$.

$$c_{\overline{s}/\overline{x}} = c$$

   for $c \in C^{\mathfrak{L}}$.

$$[f t^1 \ldots t^n]_{\overline{s}/\overline{x}} = f t^1_{\overline{s}/\overline{x}} \ldots t^n_{\overline{s}/\overline{x}}$$

   for $f \in \mathscr{F}^{\mathfrak{L}}$ and t^i a term.

   By induction on the height of terms, this is well-defined.

2. By induction on the height of a formula:

   $[t = t']_{\overline{s}/\overline{x}} = t_{\overline{s}/\overline{x}} = t'_{\overline{s}/\overline{x}}$.

   $[R t^1 \ldots t^n]_{\overline{s}/\overline{x}} = R t^1_{\overline{s}/\overline{x}} \ldots t^n_{\overline{s}/\overline{x}}$

   $[\neg\psi]_{\overline{s}/\overline{x}} = \neg[\psi]_{\overline{s}/\overline{x}}$

   $(\psi \wedge \chi)_{\overline{s}/\overline{x}} = \psi_{\overline{s}/\overline{x}} \wedge \chi_{\overline{s}/\overline{x}}$

   For quantifiers: let $x_{i_1}, \ldots, x_{i_k}$ $(i_1 < \cdots < i_k)$ be those variables among $x_0, \ldots, x_r$ that are free in $\exists x \psi$ (in particular $x \neq x_{i_1} \neq \ldots \neq x_{i_k}$). Then we define:

$$(\exists x \psi)_{\overline{s}/\overline{x}} = \exists x (\psi_{\overline{s}'/\overline{x}'})$$

   where $\overline{x}' = (x_{i_1}, \ldots, x_{i_k})$ and $\overline{s}' = (s_{i_1}, \ldots, s_{i_k})$.

   If x has some occurence in one of $s_{i_1}, \ldots, s_{i_k}$, we define it to be $\exists u \left[\psi_{\overline{s}/\overline{x}, u/x}\right]$ where u is the first variable appearing in $v_1, v_2, \ldots$ which does not occur in $\exists x \psi, s_{i_1}, \ldots, s_{i_k}$.

Definition 5.3

Let $x_0, \ldots, x_r$ be distinct variables, α an assignment with values in $\mathfrak{A}$ and $a_0, \ldots, a_r$ are elements of A. We define the assignment $\alpha_{a_0/x_0,\ldots,a_r/x_r} = \alpha_{\overline{a}/\overline{x}}$ by $\alpha_{\overline{a}/\overline{x}}(x_i) = a_i$ and $\alpha_{\overline{a}/\overline{x}}(y) = \alpha(y)$ for $y \neq x_i$.

Lemma 5.1

$Let x_0, \ldots, x_r$ be distinct variables, $s_0, \ldots, x_r$ be terms, and α be an assignment with values in $\mathfrak{A}$.

1. For every term t, $t^{\mathfrak{A}}_{\overline{s}/\overline{x}}[\alpha] = t^{\mathfrak{A}}[\alpha_{s_0^{\mathfrak{A}}[\alpha]/x_0,\ldots,s_r^{\mathfrak{A}}[\alpha]/x_r}]$
2. For every formula ϕ, $\mathfrak{A} \vDash \phi_{\overline{s}/\overline{x}}[\alpha]$ if and only if $\mathfrak{A} \vDash \phi[\alpha_{s_0^{\mathfrak{A}}[\alpha]/x_r,\ldots,s_r^{\mathfrak{A}}[\alpha]/x_r}]$

Proof of (1). We proceed by induction on the structure of the term t.

Base case. Suppose t is a variable y.

- If $y = x_i$ for some $i \le r$, then by definition of substitution
$$t_{\overline{s}/\overline{x}} = s_i.$$
Hence
$$t^{\mathfrak{A}}_{\overline{s}/\overline{x}}[\alpha] = s_i^{\mathfrak{A}}[\alpha].$$
On the other hand,
$$t^{\mathfrak{A}}[\alpha_{s_0^{\mathfrak{A}}[\alpha]/x_0,\ldots,s_r^{\mathfrak{A}}[\alpha]/x_r}] = (\alpha_{s_0^{\mathfrak{A}}[\alpha]/x_0,\ldots})(x_i) = s_i^{\mathfrak{A}}[\alpha].$$
Thus the equality holds.
- If $y \notin \{x_0, \ldots, x_r\}$, then $t_{\overline{s}/\overline{x}} = y$, so
$$t^{\mathfrak{A}}_{\overline{s}/\overline{x}}[\alpha] = \alpha(y).$$
Since the modified assignment $\alpha_{s_0^{\mathfrak{A}}[\alpha]/x_0,\ldots,s_r^{\mathfrak{A}}[\alpha]/x_r}$ agrees with α on y, we have
$$t^{\mathfrak{A}}[\alpha_{s_0^{\mathfrak{A}}[\alpha]/x_0,\ldots}] = \alpha(y).$$

Inductive step. Suppose $t = f(t_1, \ldots, t_n)$ with function symbol f. By definition of substitution,
$$t_{\overline{s}/\overline{x}} = f\big((t_1)_{\overline{s}/\overline{x}}, \ldots, (t_n)_{\overline{s}/\overline{x}}\big).$$
Hence
$$t^{\mathfrak{A}}_{\overline{s}/\overline{x}}[\alpha] = f^{\mathfrak{A}}\big((t_1)^{\mathfrak{A}}_{\overline{s}/\overline{x}}[\alpha], \ldots, (t_n)^{\mathfrak{A}}_{\overline{s}/\overline{x}}[\alpha]\big).$$
By the induction hypothesis, for each j,
$$(t_j)^{\mathfrak{A}}_{\overline{s}/\overline{x}}[\alpha] = t_j^{\mathfrak{A}}[\alpha_{s_0^{\mathfrak{A}}[\alpha]/x_0,\ldots,s_r^{\mathfrak{A}}[\alpha]/x_r}].$$
Thus
$$t^{\mathfrak{A}}_{\overline{s}/\overline{x}}[\alpha] = f^{\mathfrak{A}}\big(t_1^{\mathfrak{A}}[\alpha_{\ldots}], \ldots, t_n^{\mathfrak{A}}[\alpha_{\ldots}]\big) = t^{\mathfrak{A}}[\alpha_{s_0^{\mathfrak{A}}[\alpha]/x_0,\ldots,s_r^{\mathfrak{A}}[\alpha]/x_r}].$$
This completes the induction and the proof. □

Proof of (2). We proceed by induction on the height $ht(\phi)$ of the formula ϕ.

Base cases.

- If ϕ is an atomic formula of the form $t_1 = t_2$, then
$$\mathfrak{A} \vDash (t_1 = t_2)_{\overline{s}/\overline{x}}[\alpha] \iff t^{\mathfrak{A}}_{1_{\overline{s}/\overline{x}}}[\alpha] = t^{\mathfrak{A}}_{2_{\overline{s}/\overline{x}}}[\alpha].$$
By part (1), this is equivalent to
$$t_1^{\mathfrak{A}}[\alpha_{s_0^{\mathfrak{A}}[\alpha]/x_0,\ldots,s_r^{\mathfrak{A}}[\alpha]/x_r}] = t_2^{\mathfrak{A}}[\alpha_{s_0^{\mathfrak{A}}[\alpha]/x_0,\ldots,s_r^{\mathfrak{A}}[\alpha]/x_r}],$$
which is precisely
$$\mathfrak{A} \vDash (t_1 = t_2)[\alpha_{s_0^{\mathfrak{A}}[\alpha]/x_0,\ldots,s_r^{\mathfrak{A}}[\alpha]/x_r}].$$

- If ϕ is atomic of the form $R(t_1,\ldots,t_n)$, then

$$\mathfrak{A}\vDash (R(t_1,\ldots,t_n))_{\overline{s}/\overline{x}}[\alpha] \iff (t^{\mathfrak{A}}_{1_{\overline{s}/\overline{x}}}[\alpha],\ldots,t^{\mathfrak{A}}_{n_{\overline{s}/\overline{x}}}[\alpha])\in R^{\mathfrak{A}}.$$

  By part (1), this is equivalent to

$$(t_1^{\mathfrak{A}}[\alpha_{\ldots}],\ldots,t_n^{\mathfrak{A}}[\alpha_{\ldots}])\in R^{\mathfrak{A}},$$

  i.e.

$$\mathfrak{A}\vDash R(t_1,\ldots,t_n)[\alpha_{s_0^{\mathfrak{A}}[\alpha]/x_0,\ldots,s_r^{\mathfrak{A}}[\alpha]/x_r}].$$

Inductive step. Assume the claim holds for all formulas of smaller height than ϕ.

- If $\phi=\neg\psi$, then

$$\mathfrak{A}\vDash(\neg\psi)_{\overline{s}/\overline{x}}[\alpha] \iff \mathfrak{A}\nvDash\psi_{\overline{s}/\overline{x}}[\alpha].$$

  By the induction hypothesis,

$$\mathfrak{A}\nvDash\psi[\alpha_{\ldots}] \iff \mathfrak{A}\vDash(\neg\psi)[\alpha_{\ldots}].$$

- If $\phi=(\psi\wedge\theta)$, then

$$\mathfrak{A}\vDash(\psi\wedge\theta)_{\overline{s}/\overline{x}}[\alpha] \iff \mathfrak{A}\vDash\psi_{\overline{s}/\overline{x}}[\alpha] \text{ and } \mathfrak{A}\vDash\theta_{\overline{s}/\overline{x}}[\alpha].$$

  By induction hypothesis, this is equivalent to

$$\mathfrak{A}\vDash\psi[\alpha_{\ldots}] \text{ and } \mathfrak{A}\vDash\theta[\alpha_{\ldots}],$$

  which is

$$\mathfrak{A}\vDash(\psi\wedge\theta)[\alpha_{\ldots}].$$

  The same reasoning works for $\vee,\to,\leftrightarrow$.

- If $\phi=\forall y\,\psi$, then by definition of substitution

$$(\forall y\,\psi)_{\overline{s}/\overline{x}}=\forall y\,(\psi_{\overline{s}/\overline{x}}),$$

  assuming $y\notin\{x_0,\ldots,x_r\}$ (or else with variable renaming). Then

$$\mathfrak{A}\vDash(\forall y\,\psi)_{\overline{s}/\overline{x}}[\alpha] \iff \text{for all } a\in A,\ \mathfrak{A}\vDash\psi_{\overline{s}/\overline{x}}[\alpha_{a/y}].$$

  By induction hypothesis, this is equivalent to

$$\text{for all } a\in A,\ \mathfrak{A}\vDash\psi[\alpha_{a/y,\,s_0^{\mathfrak{A}}[\alpha]/x_0,\ldots,s_r^{\mathfrak{A}}[\alpha]/x_r}],$$

  which is precisely

$$\mathfrak{A}\vDash\forall y\,\psi[\alpha_{s_0^{\mathfrak{A}}[\alpha]/x_0,\ldots,s_r^{\mathfrak{A}}[\alpha]/x_r}].$$

  The case $\phi=\exists y\,\psi$ is analogous.

Thus by structural induction the equivalence holds for every formula ϕ. □

5.2 Exercises from 1.8.1

Problem 5.1: Task 4

Show that x is always substitutable for x in ϕ.

Proof. Let ϕ be some $\mathfrak{L}$-formula and let $x \in \mathcal{V}^{\mathfrak{L}}$. Let $Fml_n^{\mathfrak{L}}, n \in \mathbb{N}$ denote the $\mathfrak{L}$-formulas with height at most n. We proceed by structural induction on the height of the formula $ht(\phi)$.

For the base case suppose $ht(\phi) = 0$, i.e. $\phi \in Fml_0^{\mathfrak{L}}$ is an atomic formula. Then all occurences of variables in ϕ are free since there are no quantifiers, and so are substitutable.

Suppose, for our inductive hypothesis, that x is substitutable for x, for any formula in $Fml_n^{\mathfrak{L}}$.

Then, ignoring the trivial case where ϕ is atomic, we have 3 cases to check.

1. $\phi :\equiv \neg\psi$
2. $\phi :\equiv (\psi \vee \gamma)$
3. $\phi :\equiv (\forall y(\psi))$

Case 1.

Suppose ϕ is of the form $\neg\psi$, where $\psi \in Fml_n^{\mathfrak{L}}$, and by our inductive hypothesis x is substituable for x in $\psi \in Fml_n^{\mathfrak{L}}$. Then x is substitutable for x in ϕ.

Case 2.

Suppose ϕ is of the form $(\psi \vee \gamma)$, where $\psi, \gamma \in Fml_n^{\mathfrak{L}}$, and by our inductive hypothesis x is substitutable for x in any formula of height n. Then x is substitutable for x in both ψ and γ. Thus x is substitutable for x in ϕ.

Case 3.

Suppose ϕ is of the form $(\forall y(\psi))$, where $\psi \in Fml_n^{\mathfrak{L}}$, and by our inductive hypothesis x is substitutable for x in ψ. Since x is substitutable for x in ψ and y does not occur in x we have that x is substitutable for x in ϕ.

This completes the proof. □

5.3 Exercises from 1.9.1

Problem 5.2: Task 3

Suppose that ϕ is an $\mathfrak{L}$-formula and x is a variable. Prove that ϕ is valid if and only if $(\forall x)(\phi)$ is valid.

Proof. Let $\mathfrak{L}$ be a language, let ϕ be an $\mathfrak{L}$-formula, and let $x \in \mathcal{V}^{\mathfrak{L}}$ be a variable of the language. We show that ϕ is valid if and only if $\forall x \phi$ is valid.

($\Rightarrow$) Suppose ϕ is valid. This means that for every $\mathfrak{L}$-structure $\mathfrak{A}$ and every assignment α, we have

$$\mathfrak{A} \models \phi[\alpha].$$

Let $\mathfrak{A}$ be any $\mathfrak{L}$-structure and α any assignment. To check $\mathfrak{A} \models \forall x \phi[\alpha]$, we must verify that

$$\mathfrak{A} \models \phi[\alpha[a/x]] \quad \text{for every } a \in A.$$

But since ϕ is valid, $\mathfrak{A} \models \phi[\alpha[a/x]]$ holds for every $a \in A$. Thus $\mathfrak{A} \models \forall x \phi[\alpha]$ for all $\mathfrak{A}$ and α, so $\forall x \phi$ is valid.

($\Leftarrow$) Suppose $\forall x \phi$ is valid. This means that for every $\mathfrak{L}$-structure $\mathfrak{A}$ and every assignment α,

$$\mathfrak{A} \models \forall x \phi[\alpha].$$

By the semantics of $\forall$, this means that for every $a \in A$,

$$\mathfrak{A} \models \phi[\alpha[a/x]].$$

In particular, if we take $a = \alpha(x)$, then $\alpha[a/x]$ agrees with α on all variables, so

$$\mathfrak{A} \models \phi[\alpha].$$

Since this holds for all $\mathfrak{A}$ and α, ϕ is valid. □

Problem 5.3: Task 4

a) Assume that $\models (\phi \Rightarrow \psi)$. Show that $\phi \models \psi$.
b) Suppose that ϕ is $x < y$ and ψ is $z < w$. Show that $\phi \models \psi$, but $\not\models (\phi \Rightarrow \psi)$.

Proof of a. Let $\mathfrak{L}$ be a language ϕ, ψ $\mathfrak{L}$-formulas and for any arbitrary $\mathfrak{L}$-structure $\mathfrak{A}$ and assignment α, we are have $\mathfrak{A} \models (\phi \Rightarrow \psi)$. Our goal is to show that for any $\mathfrak{A}$ with assignment α, if $\mathfrak{A} \models \phi$ then $\mathfrak{A} \models \psi$.

As mentioned we have $\mathfrak{A} \models (\phi \Rightarrow \psi)[\alpha]$ for arbitrary structure and assignment. Recall that $\phi \Rightarrow \psi$ is short-hand for $\neg\phi \vee \psi$. Suppose $\mathfrak{A} \models (\neg\phi \vee \psi)[\alpha]$ and $\mathfrak{A} \models \phi[\alpha]$, where structure and assignment are arbitrary, as mentioned. Then by the semantics of disjunction we have $\mathfrak{A} \models \neg\phi[\alpha]$ or $\mathfrak{A} \models \psi[\alpha]$, and $\mathfrak{A} \models \phi[\alpha]$. In particular $\mathfrak{A} \models \neg\phi[\alpha]$ means that $\mathfrak{A} \not\models \phi[\alpha]$, but this is not the case since we have $\mathfrak{A} \models \phi[\alpha]$. Therefore $\mathfrak{A} \models \psi[\alpha]$ must hold. Hence $\phi \models \psi$. □

Proof of b. Let ϕ be the formula $x < y$ and ψ be the formula $z < w$.

Step 1: Show $\phi \models \psi$. Assume $\mathfrak{A}$ is a structure with domain A and interpretation $<^{\mathfrak{A}} \subseteq A \times A$ of the relation symbol $<$. Suppose $\mathfrak{A} \models \phi$, i.e. for every assignment α we have

$$(\alpha(x), \alpha(y)) \in <^{\mathfrak{A}}.$$

Let $a, b \in A$ be arbitrary. Define an assignment α with $\alpha(x) = a$ and $\alpha(y) = b$ (the values of other variables are irrelevant). By assumption, $(a, b) \in <^{\mathfrak{A}}$. Since a, b were arbitrary, this shows that

$$<^{\mathfrak{A}} = A \times A.$$

Now let β be any assignment. Then $(\beta(z), \beta(w)) \in A \times A = <^{\mathfrak{A}}$, so $\mathfrak{A} \models \psi$. Thus every structure that satisfies ϕ also satisfies ψ, so $\phi \models \psi$.

Step 2: Show $\not\models (\phi \to \psi)$. Consider the structure $\mathfrak{A}$ with domain $A = \{0, 1\}$ and $<^{\mathfrak{A}}$ interpreted as the usual strict order $\{(0, 1)\}$. Let α be an assignment with

$$\alpha(x) = 0, \quad \alpha(y) = 1, \quad \alpha(z) = 1, \quad \alpha(w) = 0.$$

Then $\mathfrak{A} \models \phi[\alpha]$ since $0 < 1$, but $\mathfrak{A} \not\models \psi[\alpha]$ since $1 \not< 0$. Hence $\mathfrak{A} \not\models (\phi \to \psi)[\alpha]$, so $\phi \to \psi$ is not valid.

Therefore, $\phi \models \psi$ but $\not\models (\phi \to \psi)$. □

6 Elementary Classes and Equivalence

Definition 6.1

For a set of well-formed-formulas Γ, let $Mod\Gamma$ (sometimes written $M(\Gamma)$) be the class of all structures which model Γ.
A class of structures $\mathscr{K}$ is an elementary class if there exists a well-formed-formula ϕ such that $\mathscr{K} = Mod\phi$.
A class of structures $\mathscr{K}$ is a weakly elementary class if there exists a set of well-formed-formulas Γ such that $\mathscr{K} = Mod\Gamma$.

Definition 6.2

Two structures $\mathfrak{A}, \mathfrak{B}$ are elementary equivalent (written $\mathfrak{A} \equiv \mathfrak{B}$) if, for all sentences ϕ

$$\mathfrak{A} \vDash \phi \Leftrightarrow \mathfrak{B} \vDash \phi$$

Example 6.1

- $(\mathbb{N}, 0, +) \not\equiv (\mathbb{Z}, 0, +)$
- $(\mathbb{Q}, 0, +) \not\equiv (\mathbb{Z}, 0, +)$
- $(\mathbb{Q}, 0, +) \equiv (\mathbb{R}, 0, +)$

The first example can be shown by using the formula for existence of inverses, which $\mathbb{Z}$ has, while $\mathbb{N}$ does not.
For the second we can use the fact that for all x, there exists $\frac{x}{2}$.
The last one seems counterintuitive, but without multiplication and a notion of subsets, the two structures are equivalent.

Definition 6.3

Let $Th(\mathfrak{A})$ be the set of all sentences true in $\mathfrak{A}$.
Note:

$$\mathfrak{A} \equiv \mathfrak{B} \Leftrightarrow Th(\mathfrak{A}) = Th(\mathfrak{B})$$

7 Deductions

This concept is not explored in the supplementary lecture videos I use and I do not feel like copy and pasting theorems and definitions from the whole second chapter of the book, so this section will be mostly problems.

7.1 Exercises from 2.2.1

Problem 7.1: Task 1

Let the collection of non-logical axioms be

$$\Sigma = \{(A(x) \wedge A(x)) \to B(x,y), A(x), B(x,y) \to A(x)\}$$

and let the rule of inference be modus ponens. Determine whether the following are deductions or not.

1. $A(x)$

   $A(x) \wedge A(x)$

   $(A(x) \wedge A(x)) \to B(x,y)$

   $B(x,y)$

2. $B(x,y) \to A(x)$

   $A(x)$

   $B(x,y)$

3. $(A(x) \wedge A(x)) \to B(x,y)$

   $B(x,y) \to A(x)$

   $(A(x) \wedge A(x)) \to A(x)$

Solution.

The first is not valid since we cannot conclude $A(x) \wedge A(x)$ from $A(x)$.

The second is not valid since $B(x,y)$ does not follow from any of the preceding.

The third is valid since we can conclude $B(x,y)$ from the first line. Then we can conclude $A(x)$ from the second line. Then we can conclude $(A(x) \wedge A(x)) \to A(x)$. In other words we can get $A(x)$ from $(A(x) \wedge A(x))$.

Problem 7.2: Task 4

Let $\mathscr{L}$ be a language consisting of a single unary predicate symbol R, and B be the infinite set of axioms

$$
\begin{aligned}
B = \{&R(x_1),\\
&R(x_1) \to R(x_2),\\
&R(x_2) \to R(x_3),\\
&\quad\vdots\\
&R(x_i) \to R(x_{i+1}),\\
&\quad\vdots\\
\}&
\end{aligned}
$$

Using modus ponens, prove by induction that $B \vdash R(x_j)$, for each $j \geq 1$.

Proof. For our base case we use the axiom $R(x_i) \in B$, which gives us $B \vdash R(x_i)$. For our inductive hypothesis, assume $B \vdash R(x_i)$ for some $i \geq 1$. Then since $R(x_i) \to R(x_{i+1})$ we have, via modus ponens, $B \vdash R(x_{i+1})$. Then since $B \vdash R(x_1)$ and $(R(x_i), R(x_i) \to R(x_{i+1}))$ we get $B \vdash R(x_2), B \vdash R(x_3), \ldots, B \vdash R_{x_j}$ for each $j \geq 1$. □

7.2 Tasks from 2.5.1

Problem 7.3: Task 4

Show that, if x is not free in ψ, $(\phi \to \psi) \models \left[(\exists x \phi) \to \psi\right]$.

Proof. Let ϕ, ψ be $\mathfrak{L}$-formulas such that x is not free in ψ.

Assume, for any $\mathfrak{L}$-structure, $\mathfrak{A} \models (\phi \to \psi)[s]$.

Suppose, ignoring the trivial case, $\mathfrak{A} \models (\exists x \phi)[s]$. Then there exists $a \in A$ such that

$$\mathfrak{A} \models \phi[s[x|a]]$$

From $\mathfrak{A} \models \phi$, we know that for every assignment t, if $\mathfrak{A} \models \phi$, then $\mathfrak{A} \models \psi$. In particular, this holds for $t = s[x|a]$. Hence

$$\mathfrak{A} \models \psi[s[x|a]]$$

since ψ is invariant under changes to the value of x.

This completes the proof. □

7.3 Tasks from 2.7.1

Problem 7.4: Task 4

Suppose that η is a sentence. Prove that $\Sigma \vdash \eta$ if and only if $\Sigma \cup (\neg\eta) \vdash [(\forall x)x = x] \wedge \neg[(\forall x)x = x]$. Notice that this exercise tells us that our deductive system allows us to do proofs by contradiction.

Proof. Let η be a sentence.

Suppose $\Sigma \vdash \eta$. Then consider $\Sigma \cup (\neg\eta)$. Clearly $\Sigma \cup (\neg\eta) \vdash \eta$, since the smaller set of non-logical axioms $\Sigma \vdash \eta$. Thus $\Sigma \cup (\neg\eta) \vdash \eta \wedge (\neg\eta)$. Since η has no free variables, $\eta \wedge (\neg\eta)$ is a contradiction. Then, for any formula ϕ,

$$\eta \wedge (\neg\eta) \models \phi$$

more specifically,

$$\eta \wedge (\neg\eta) \models [(\forall x)x = x] \wedge \neg[(\forall x)x = x]$$

By the completeness theorem, we get

$$\eta \wedge (\neg\eta) \vdash [(\forall x)x = x] \wedge \neg[(\forall x)x = x]$$

Since, with our assumptions, $\eta \wedge (\neg\eta)$ is provable from $\Sigma \cup (\neg\eta)$, $[(\forall x)x = x) \wedge \neg[(\forall x)x = x]$ is provable from the same set of non-logical axioms.

Now, the other way, assume $\Sigma \cup (\neg\eta) \vdash [(\forall x)x = x) \wedge \neg[(\forall x)x = x]$. Then, by the soundness theorem,

$$\Sigma \cup (\neg\eta) \models [(\forall x)x = x) \wedge \neg[(\forall x)x = x]$$

which means that for arbitrary structure $\mathfrak{A}$, $\mathfrak{A} \not\models \Sigma \cup (\neg\eta)$ or $\mathfrak{A} \models [(\forall x)x = x) \wedge \neg[(\forall x)x = x]$.

Since no structure can satisfy a contradiction it must be the case that

$$\mathfrak{A} \not\models \Sigma \cup (\neg\eta)$$

So $\Sigma \cup (\neg\eta)$ has no modles which means that it is inconsistent. Then, any contradiction can be proven from it. For example

$$\Sigma \cup (\neg\eta) \vdash (\eta \wedge \neg\eta)$$

which can be rewritten via the deduction theorem to

$$\Sigma \vdash (\neg\eta) \to (\eta \wedge \neg\eta)$$

which can again be rewritten, using the laws of propositional logic, to

$$\Sigma \vdash \eta \vee (\eta \wedge \neg\eta)$$

which simplifies to

$$\Sigma \vdash \eta$$

Since, from any contradiction the implication to some formula is valid, we can pick any contradiction to be provable from an inconsistent set of non-logical axioms. It does not necessarily have to be $[(\forall x)x = x] \wedge \neg[(\forall x)x = x]$. This equips us to do proofs by contradiction since if we prove a contradiction our assumptions must be inconsistent, which alows us to prove the contradiction which simplifies to the thing we want to prove. □

8 3rd chapter, completeness and compactness

9 Tasks from 3.2.1

Problem 9.1: Task 1

Suppose that Σ is inconsistent and ϕ is an $\mathfrak{L}$-formula. Prove that $\Sigma \vdash \phi$.

Proof. Let Σ be an inconsistent set of non-logical axioms and ϕ an $\mathfrak{L}$-formula.

Since Σ is inconsistent we have

$$\Sigma \vdash \perp$$

From our logical axioms Λ, we have

$$\vdash \perp \to \phi$$

for any $\mathfrak{L}$-formula ϕ.

Then, since

$$\Sigma \vdash \perp$$

and

$$\perp \to \phi$$

we have

$$\Sigma \vdash \phi$$

through modus ponens. □

Problem 9.2: Task 2

Assume that $\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \cdots$ are such that each Σ_i is a consistent set of sentences in a language $\mathfrak{L}$. Show that $\bigcup \Sigma_i$ is consistent.

Proof. We prove the claim by contradiction.

Suppose $\Gamma := \bigcup \Sigma_i$ is inconsistent. Then there exists a finite subset $\Delta \subseteq \Gamma$. But Δ is finite, so each sentence in Δ lies in some Σ_i, and because Σ_i forms an increasing chain there is an index N such that $\Delta \subseteq \Sigma_N$. Thus Σ_N is inconsistent since it contains an inconsistent subset, contradicting the assumption that each Σ_i is consistent. Therefore Γ must be consistent. □