Lecture Notes: Combinatorics (Course by: Federico Ardila)

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1 Multisets

A multiset is a set with possibly repeated elements.
$\{1, 2, 2, 2, 4, 5, 5, 7\} =: \{1^1, 2^3, 4^1, 5^2, 7^1\}$ is a multiset, for instance.
For multisets we count repetitions so the cardinality here would be 8 rather than 5.
$(\binom{S}{k})$ is the multisets of size k on S, $(\binom{n}{k}) = |(\binom{S}{k})|$). $(\binom{n}{k})$ could be called "n multi choose k".

Proposition

$(\binom{n}{k}) = \binom{n+k-1}{k}$
Proof:
If k multiset S on $[n]$ has a_i parts equal to i then if I add up $a_1 + a_2 + \cdots\ a_n = k$, making it a weak n-composition of k (since 0s are allowed). $\square$

2 Multisets and GFs

The multivariant generating function for multisets on $[n]$ is given by

$$\sum_{v:[n]\to\mathbb{N}} x_1^{v(1)} x_2^{v(2)} \cdots x_n^{v(n)}$$
$$= (1 + x_1 + x_1^2 + \cdots)(1 + x_2 + x_2^2 + \cdots) \cdots (1 + x_n + x_n^2 + \cdots)$$
$$= \frac{1}{1 - x_1} \cdots \frac{1}{1 - x_n}$$

Let $x_1 = x_2 = \cdots = x_n = x$

$$\sum_{v:[n]\to\mathbb{N}} x^{v(1)+v(2)+\cdots+v(n)} = (\frac{1}{1-x})^n$$
$$= \sum_{\text{Multisets on}[n]} x^{|Multiset|} = (1-x)^n$$
$$= \sum_{k=0} (\left(\binom{n}{k}\right)) x^k = \sum_{k=0} \binom{-n}{k} (-1)^k x^k$$

The binomial theorem states $(1+x)^\alpha = \sum_{k=0} \binom{\alpha}{k} x^k$, $\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$ because if α is an integer it works and if it is real it still makes sense. Doing a taylor expansion will show why the binomial theorem holds for $\alpha \in \mathbb{R}$ so we don't have to worry about $\binom{-n}{k} = (-1)^k(\left(\binom{n}{k}\right))$. This is an example for what we call a "combinatorial resipositry theorem". What this shows us that even though it sounds nonsensical to ask how many k-subsets of a set sized -n exist, it still means something in combinatorics.

3 Multinomial coefficients

The multinomial coefficient $\binom{n}{a_1, a_2, \cdots, a_k}$ is the number of ways of splitting a set of size n into a set of size $a_1, a_2, \cdots, a_k$.

Example

$\binom{n}{k, n-k} = \binom{n}{k}$

Proposition

The number of permutations of a multiset $\{1^{a_1}, 2^{a_2}, \cdots, k^{a_k}\}$ of size n is $\binom{n}{a_1, \cdots, a_k}$
Proof:
To make a permutation of size n I must choose a_1 positions for 1 to go, a_2 positions for 2 to go and so on. Split $[n]$ into sets — where the 1s go, 2s go, etc. $\square$

Proposition

$\binom{n}{a_1, \cdots, a_k} = \frac{n!}{a_1! a_2! \cdots a_k!}$
Proof:
The multinomial coefficient $\binom{n}{a_1, \cdots, a_k}$ represents the number of ways to partition a set of n elements into k subsets of sizes $a_1, a_2, \ldots, a_k$, where each subset contains identical elements. Alternatively, it also counts the number of distinct permutations of a multiset with elements occurring with frequencies $a_1, a_2, \ldots, a_k$.
We proceed by considering the step-by-step placement of the elements:

1. First, choose a_1 positions for the first type of element (e.g., the ones). Since the order among these elements does not matter, the number of ways to do this is:

$$\binom{n}{a_1} = \frac{n!}{a_1!(n-a_1)!}.$$

2. Next, from the remaining $n - a_1$ positions, choose a_2 positions for the second type of element. Again, the order among these elements does not matter, so we have:

$$\binom{n-a_1}{a_2} = \frac{(n-a_1)!}{a_2!(n-a_1-a_2)!}.$$

3. This process continues for all k types of elements until all positions are filled. The last group of elements is automatically placed, leaving us with:

$$\binom{n-a_1-\cdots-a_{k-1}}{a_k} = \frac{(n-a_1-\cdots-a_{k-1})!}{a_k!(n-a_1-\cdots-a_k)!}.$$

Multiplying all these binomial coefficients together, we obtain:

$$\binom{n}{a_1}\binom{n-a_1}{a_2}\binom{n-a_1-a_2}{a_3}\cdots\binom{n-a_1-\cdots-a_{k-1}}{a_k}.$$

Expanding each binomial coefficient in factorial form, we get:

$$\frac{n!}{a_1!(n-a_1)!}\cdot\frac{(n-a_1)!}{a_2!(n-a_1-a_2)!}\cdots\frac{(n-a_1-\cdots-a_{k-1})!}{a_k!(n-a_1-\cdots-a_k)!}.$$

Observing that all intermediate factorial terms cancel out, we are left with:

$$\frac{n!}{a_1!a_2!\cdots a_k!}.$$

Thus, we have shown that:

$$\binom{n}{a_1,\cdots,a_k} = \frac{n!}{a_1!a_2!\cdots a_k!}.$$

□

Example

How many distinct ways can we rearrange the letters in the word `MISSISSIPPI`?
Solution: Since some letters repeat, we cannot simply use the factorial formula for permutations of distinct objects. Instead, we must account for repeated letters using the multiset method.
Step 1: Identifying the Frequencies The word `MISSISSIPPI` consists of 11 letters with the following frequencies:

$$M:1,\quad I:4,\quad S:4,\quad P:2.$$

Since some letters are identical, the number of unique rearrangements is given by the multinomial coefficient:

$$\frac{11!}{1!4!4!2!}.$$

Step 2: Computing the Multinomial Coefficient

$$11! = 39916800,$$
$$4! = 24, \quad 2! = 2, \quad 1! = 1,$$
$$4!4!2!1! = (24 \cdot 24 \cdot 2 \cdot 1) = 1152,$$
$$\frac{39916800}{1152} = 34650.$$

Thus, the number of distinct ways to rearrange the letters in `MISSISSIPPI` is

$$\mathbf{34650}.$$

Multinomial Theorem

$$(x_1 + \cdots + x_k)^n = \sum_{a_1, \cdots, a_k} \binom{n}{a_1 \cdots a_k} x_1^{a_1} \cdots x_k^{a_k}$$

Proposition

In a box of size $a_1 \times a_2 \times \cdots \times a_k$, the number of shortes lattice paths from one corner to the opposite corner is $\binom{a_1+a_2+\cdots+a_k}{a_1,a_2\cdots,a_k}$.

Proof:

A shortest lattice path from one corner $(0, 0, \ldots, 0)$ to the opposite corner $(a_1, a_2, \ldots, a_k)$ consists of exactly $a_1 + a_2 + \cdots + a_k$ total steps, where: - a_1 steps move in the first coordinate direction, - a_2 steps move in the second coordinate direction, - …, - a_k steps move in the kth coordinate direction.

Each shortest path is uniquely determined by the sequence of moves taken. Since we must choose a_1 of the $a_1 + a_2 + \cdots + a_k$ total steps to move in the first direction, then a_2 steps to move in the second direction, and so on, the number of such sequences is given by the multinomial coefficient:

$$\binom{a_1 + a_2 + \cdots + a_k}{a_1, a_2, \ldots, a_k} = \frac{(a_1 + a_2 + \cdots + a_k)!}{a_1!a_2! \ldots a_k!}.$$

Thus, the number of shortest lattice paths is exactly $\binom{a_1+a_2+\cdots+a_k}{a_1,a_2,\ldots,a_k}$. $\square$

4 Combinatorial Identities

$\binom{n}{k} = \binom{n}{n-k}$, which is easy to prove algebraically. Combinatorially we ask why is the number of k-subsets of n equal to the number of (n-k)-subsets of n? Well there must exist a bijection between k-subsets and (n-k)-subsets which is to take the k-subset's complement which would be the (n-k)-subset. $\binom{[n]}{k} \leftrightarrow \binom{[n]}{n-k} \Leftrightarrow A \leftrightarrow [n] - A$ in other words.

$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$. Combinatorially we say we want to count (k+1) subsets of [n+1]. If n+1 is not n: $\binom{n}{k+1}$. If n+1 is n: $\binom{n}{k}$. Since either happens we add them.

Pascal's Triangle

In row 1 there is one element; 1. In the second there are two 1s and so on. In each row n the first element is $\binom{n}{0} = 1$ then n choose 2, and so on until you reach the middle element then go backwards until you get n choose 0 again. The previous identity is why Pascal's Triangle works, making it a nice visual for the identity.

$\binom{m+n}{k} = \displaystyle\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}$. Count the subsets of a set of m blue and n red elements, or some other distinction. If my subset has i blue elements it has k-i red elements $\rightarrow \binom{m}{i}\binom{n}{k-i}$. I could be any value so we add over all possible values of i, giving us the sum.