Lecture Notes: Abstract Algebra — Homomorphisms (Course by: Alvaro Lozano-Robledo)

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Definition 1. *Let G and G' be groups. Then a homomorphism from G to G' $\phi : G \to G'$ is a mapping that respects groups structure such that for $a, b \in G$, $\phi(a \star_G b) = \phi(a) \star_{G'} \phi(b)$*

Notice that how we can then define isomorphisms as bijective homomorphisms.
Example 1: Let G be any group, $g \in G$, we can define $\phi : \mathbb{Z} \to G$, $n \to g^n$.

$$\phi(n + m) = g^{n+m} = g^n g^m = \phi(n)\phi(m)$$

Notice how if we take G to be any group this is not guaranteed to be a bijection, but it still respects group structure like isomorphisms do.
Example 2: $G = \mathbb{Z}/_6\mathbb{Z}, g = 1 \bmod 6$, $\phi : \mathbb{Z} \to G$, $n \to n \bmod 6$ makes a surjective homomorphism. Notice how $\phi(1) = \phi(7)$, making ϕ non-injective.
Example 3: $G = GL(2, \mathbb{R}), H = \mathbb{R}^*$. $\psi : GL(2, \mathbb{R}) \to \mathbb{R}^*$, $\psi(A) = det(A)$. ψ is well-defined because $A \in G \Rightarrow \psi(A) = det(A) \neq 0$. ψ respects structure, because of relationship between matrix multiplication and determinants:

$$\psi(AB) = det(AB) = det(A)det(B) = \psi(A)\psi(B)$$

Properties of Homomorphisms

Proposition 1. *Let $\psi : G \to H$ be a homomorphism. Then*

1. $\psi(e_G) = e_H$
2. $\forall \psi(g^{-1}) = \psi(g)^{-1}$
3. *$J \subseteq G$ is a subgroup, then $\psi[J] \subseteq H$ is a subgroup*
4. *$K \subseteq H$ is a subgroup, then $\psi^{-1}[K] \subseteq G$*
5. *If $K \triangleleft H$ normal subgroup, then $\psi^{-1}[K] \triangleleft G$*

Proof (1). $\psi : G \to H$ homomorphism.

$$\begin{gathered} h = \psi(e_G) = \psi(e_G \star_G e_G \\ = \psi(e_G) \star_H \psi(e_G) = h \star_H h \\ \therefore h = h \star_H h \\ \Rightarrow e_H = h \end{gathered}$$

Thus $\psi(e_G) = e_H$. $\square$

Proof (2). Let $g \in G$,

$$\begin{gathered} e_H = \psi(e_G) = \psi(g \star_G g^{-1}) \\ = \psi(g) \star_H \psi(g^{-1}) = e_H \\ \Rightarrow \psi(g)^{-1} = \psi(g^{-1}) \end{gathered}$$

$\square$

Proof (3). $J \subseteq G$ a subgroup. Let $K = \psi[J]$. J being a subgroup of G means:

$$\forall j_1, j_2 \in J : j_1 j_2^{-1} \in J$$

Let $k_1, k_2 \in K = \psi[J]$, thus $\exists j_1, j_2 \in J$ such that $k_i = \psi(k_i)$. Then

$$\begin{gathered} k_1 k_2^{-1} = \psi(j_1)\psi(j2)^{-1} = \psi(j_1)\psi(j_2^{-1}) \\ = \psi(j_1 j_2^{-1}) \in \psi[J] = K \end{gathered}$$

$\square$

Proof (4). Define $J = \psi^{-1}[K] = \{g \in G \mid \psi(g) \in K\}$.
Identity: Since $e_H \in K$ (as K is a subgroup), and $\psi(e_G) = e_H$, it follows that $e_G \in \psi^{-1}[K]$.
Closure: Let $g_1, g_2 \in \psi^{-1}[K]$, meaning that $\psi(g_1), \psi(g_2) \in K$. Since K is a subgroup, it is closed under multiplication, so $\psi(g_1)\psi(g_2) \in K$. Since ψ is a homomorphism, we have:

$$\psi(g_1 g_2) = \psi(g_1)\psi(g_2) \in K.$$

Thus, $g_1 g_2 \in \psi^{-1}[K]$, proving closure.
Inverses: Let $g \in \psi^{-1}[K]$, so that $\psi(g) \in K$. Since K is a subgroup, it contains inverses, meaning that $\psi(g)^{-1} \in K$. By the homomorphism property, we know $\psi(g^{-1}) = \psi(g)^{-1}$, and thus $\psi(g^{-1}) \in K$. This implies $g^{-1} \in \psi^{-1}[K]$.
Since $\psi^{-1}[K]$ satisfies the identity, closure, and inverses, it is a subgroup of G. $\square$

Proof (5). Let $x \in \psi^{-1}[K]$, so that $\psi(x) \in K$. Since K is normal in H, we know that for any $h \in H$ and $k \in K$, we have $hkh^{-1} \in K$. Applying this to $h = \psi(g)$ and $k = \psi(x)$, we get:

$$\psi(g)\psi(x)\psi(g)^{-1} \in K.$$

Using the homomorphism property, we rewrite this as:

$$\psi(gxg^{-1}) \in K.$$

This means $gxg^{-1} \in \psi^{-1}[K]$, proving that $\psi^{-1}[K]$ is normal in G. $\square$