# Modeling Growth Using Logistic Equation

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# 1 Introduction

Differential equations describe the development of systems in continuous time, they play important role in solving problems in engineering, physics and biology. There are many types of differential equations, ordinary differential equation, partial differential equation etc. In this project we study the nonlinear differential equation logistic equation, how it is used in population growth, bacteria growth and cell growth since they all grow continuously. We are concerned about nonlinear differential equations since most of the things happening in life are nonlinear. Robert Malthus was the first man who created theoretical treatment of population dynamics in 1798, after that P.F Verhulst converted the Malthus theory into a mathematical model called logistic equation which became a nonlinear differential equation in 1838. His work was neglected until 1920, then the logistic equation was discovered by Raymond Pearl and Lowell Reed and made it well known. Logistic model was used to estimate population in humans, animals and fish production. The logistic growth model explains limitations of resources having an impact on the growth of population. These limitations are indicated as saturation level or population carrying capacity that we find when the population become larger.

## 1.1 Population Models

The simple model of population growth is the one which include exponential growth

$$\frac{dN}{dt} = rN \tag{1}$$

Multiplying the exponential model by the the factor $1-\frac{N}{K}$ we obtain Verhulst logistic differential equation.

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \tag{2}$$

where $K$ = caring capacity (, $r$ = Malthusian parameter (rate of population growth), $N_o$ = size of a population. For simplification we let

$$x = \frac{N}{K}$$

$$\frac{dx}{dt} = \frac{1}{K}\frac{dN}{dt}$$

$$\frac{dx}{dt} = \frac{1}{K}rx(K - N)$$

$$\frac{dx}{dt} = rx(1 - x) \tag{3}$$

The above equation is a first order autonomous nonlinear differential equation, it is first order since the only first derivative of $x$ appears in the equation. The right hand side of the equation depends on $x$ only therefore it is autonomous. Multiplying out the brackets we obtain

$$\frac{dx}{dt} = rx - rx^2 \tag{4}$$

The term $x^2$ makes the Logistic equation nonlinear, since x is very small so x squared term in higher order terms can be neglected. Then we obtain the Malthusian model

$$\frac{dx}{dt} = rx \tag{5}$$

we solve the Malthusian model by separating the variables and integrate with respect to t.

$$\int \frac{1}{x}\frac{dx}{dt}dt = \int rdt$$

$$\int \frac{1}{x}dx = \int rdt$$

$$\ln x = rt + c$$

$$x = c_1 e^{rt}$$

where $c_1 = e^c$, using initial condition $t = 0$ we find $x(0) = x_o$. The solution of equation (5) is given by

$$x(t) = x_o e^{rt} \tag{6}$$

When $r < 0$ we obtain exponential decay, the decay start from maximum and decrease toward zero but it never reaches zero. When $r = 0$ the population remain constant. For $r > 0$ we get exponential growth, population grows lager and larger without any limit. Exponential growth occurs in an environment where individual are few and resources are enough. The population is doubling.

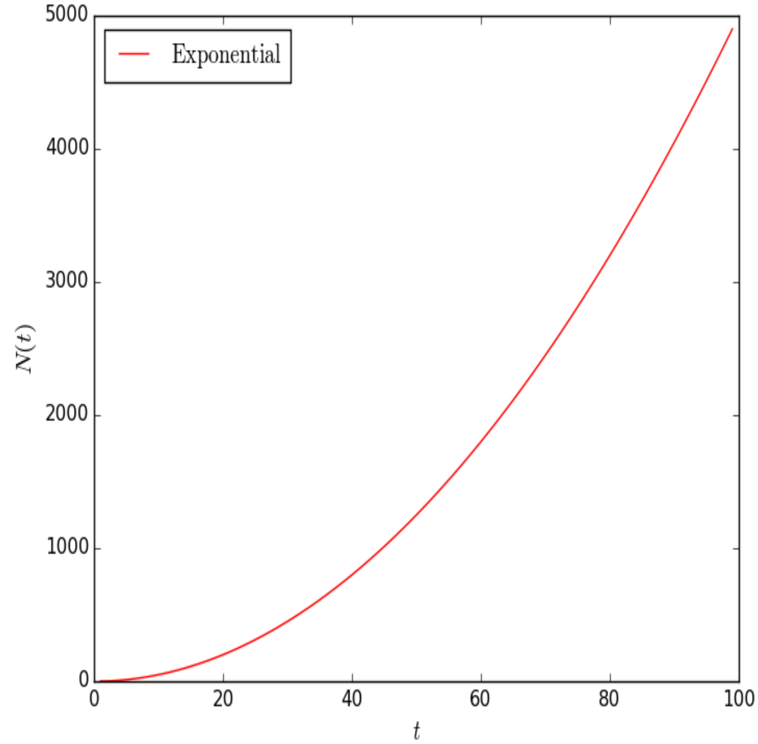


Figure 1: Exponential growth curve for $r > 0$ generated with python

## 1.2 Analytical Solution of a Logistic Equation

In general case equation (3) can be separable iff $x(1-x) \neq 0$, then we have from partial fractions

$$\int \left( \frac{1}{x} + \frac{1}{(1-x)} \right) dx = \int r dt$$

$$\ln|x| - \ln|1-x| = rt + c$$

$$\ln\left| \frac{x}{1-x} \right| = rt + c$$

$$\frac{x}{1-x} = e^{rt+c}$$

$$\frac{x}{1-x} = ke^{rt}$$

$$x = (1-x)ke^{rt}$$

$$x = ke^{rt} - xke^{rt}$$

$$x(1 + ke^{rt}) = ke^{rt}$$

After integrating both sides and solve for x we obtain

$$x(t) = \frac{ke^{rt}}{1 + ke^{rt}}$$

where $k$ is constant which come from integration. The use of initial condition $t = 0$ and solve for $k$ gives

$$k = \frac{x(0)}{1 - x(0)}$$

$$x(t) = \frac{x(0)e^{rt}}{1 - x(0) + x(0)e^{rt}}$$

$$x(t) = \frac{x_o e^{rt}}{(1 - x_o) + x_o e^{rt}}$$

Let $x(t) = \frac{N(t)}{K}$, where $x_o = N_o$
Solving for $N(t)$ and substituting $x_o$ from the above expression we obtain, the analytically solution of a logistic equation.

$$N(t) = \frac{KN_o e^{rt}}{(K - N_o) + N_o e^{rt}}$$

After rearranging the above expression we obtain the solution

$$N(t) = \frac{KN_o}{(K - N_o)e^{-rt} + N_o}$$

### 1.2.1 Properties of a logistic growth equation

$\lim_{t\to\infty} N(t) = K$, the population will eventually reach the carrying capacity. The relative growth rate $\frac{1}{N}\frac{dN}{dt}$ is inversely proportional to size of a population. Rearranging and differentiating the logistic growth equation gives (2)

$$\begin{aligned}
\frac{dN}{dt} &= \frac{r}{K}N(K-N) \\
\frac{d^2N}{dt^2} &= \frac{r}{K}\left[\frac{dN}{dt}(K-N) - N\left(\frac{dN}{dt}\right)\right] \\
&= \frac{r}{K}\frac{dN}{dt}(K-2N) \\
&= r\left(1-\frac{2N}{K}\right)\frac{dN}{dt}
\end{aligned}$$

The inflection point happens at half of a carrying capacity, this is the level of maximum growth. Inflection point occurs when $N = \frac{K}{2}$

Figure 2 represent a logistic curve, first the growth rate is exponential then it becomes closer to the carrying capacity then the growth slow down until it reaches the carrying capacity. when number of individuals increases in an environment the resources become limited.

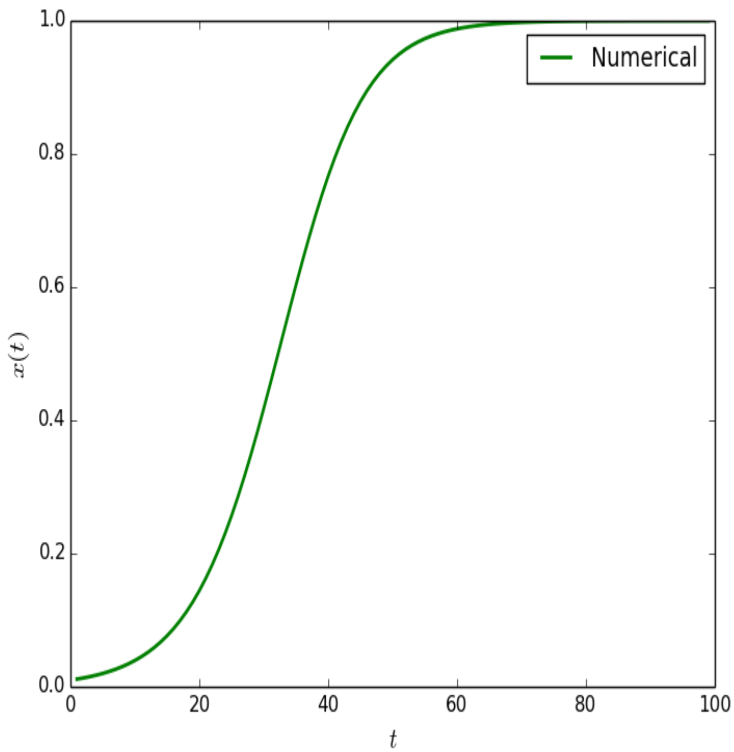


Figure 2: Logistic growth curve for $r = 0.6$ ,$K = 50$, $N_o = 10$

## 1.3 Comparison of the logistic and exponential growth models

In this section we compare the logistic growth and exponential growth models, we want to see their relationship. This relationship is observed by using the data of the result of the experiment conducted by F.G Gause, with protozoa paramecium which is a single celled micro organism. Logistic equation was used to model the data. The table provide his daily count of the population of protozoa. The initial relative growth rate he estimated is 0.7944 and the carrying capacity of 64. From solution of an exponential growth equation (1) we have new solution where the population is denoted by P. Using the given information we have relative growth rate $k = 0.7944$ and the initial population $P_o = 2$, the exponential model is

$$P(t) = P_o e^{kt} = 2e^{0.7944t}$$

The same value k and carrying capacity K =64 is used in a logistic model We represent the solution of the logistic growth equation (7) as

$$P(t) = \frac{K}{1 + Ae^{-kt}} = \frac{64}{1 + Ae^{-0.7944t}} \tag{7}$$

Where $A = \frac{K-P_o}{P_0} = \frac{64-2}{2} = 31$. Now equation 8 becomes

$$P(t) = \frac{K}{1 + 31e^{-0.7944t}}$$

The predicated values were calculated using these above equation (rounded to the nearest integer), they are compared in the table below.

| t(days) | P(observed) | P(logistic model) | P(exponential model) |
|---|---|---|---|
| 0 | 2 | 2 | 2 |
| 1 | 3 | 4 | 4 |
| 2 | 22 | 9 | 10 |
| 3 | 16 | 17 | 22 |
| 4 | 39 | 28 | 48 |
| 5 | 52 | 40 | 106 |
| 6 | 54 | 51 | 235 |
| 7 | 47 | 57 | 520 |
| 8 | 50 | 61 | 1151 |
| 9 | 76 | 62 | 2547 |
| 10 | 69 | 63 | 5637 |
| 11 | 51 | 64 | 12476 |
| 12 | 57 | 64 | 27610 |
| 13 | 70 | 64 | 61105 |
| 14 | 53 | 64 | 135232 |
| 15 | 59 | 64 | 299284 |
| 16 | 57 | 64 | 662349 |

We can see from the figure below that these curves are similar in the beginning and we can see from the table for the first three days the exponential model provide results similar to the logistic model. Population grow slowly for two days in both models. For logistic model population grow rapidly for 6 days, after 6 days the growth slow down because the population becomes lager then there is competition due to limited resources, after 8 days the growth level off and stop at a carrying capacity. The curve of the logistic model is exponential in the beginning before the carrying capacity is reached once the carrying capacity is reached it becomes logistic. Exponential model population grows rapidly without any limit, it is impossible inaccurate after 5 days since it grows to infinity but for the logistic model fit the result reasonable.

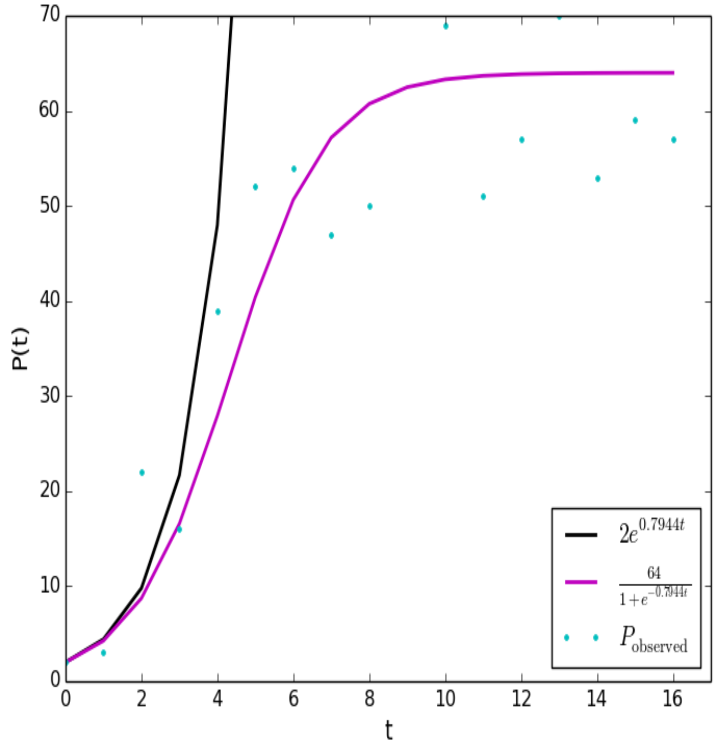


Figure 3: The logistic and exponential models for paramecium data

# 2 Population growth

In this chapter we show why the study of differential equation is important and we evaluate the relationship between the size of population and time. Let the number of individuals in a given area at time t be p(t). The number of individuals at time t+T is p(t+T) so that $p(t+T)-P(t)$ must be the number of individuals that have been added to a population during the interval T. Population growth over time depend on the availability of enough resources, water, food and shelter. The change in the interval T can be written as NT and then

$$p(t+T) - p(t) = NT$$

Rearranging the terms we obtain

$$\frac{p(t+T)-p(t)}{T} = N$$

Taking the limit as T is approaching zero

$$\lim_{T \longrightarrow 0} \frac{p(t+T)-p(t)}{T} = \frac{dp(t)}{dt}$$

$$\frac{dp(t)}{dt} = N \tag{8}$$

N has the following properties, that the change in the size of a population is caused by number of individuals being born. As the time goes on the fertility of parents may change so that more or less offspring are born, this can be caused by birth control or age of a population in humans. The number of individuals born in the interval T may vary as time proceed. Population size change as the time changes. The more individuals are present at time t the more birth is likely to occur. Then N will depend on p(t) also. These help to rewriting our equation as

$$\frac{dp(t)}{dt} = N(t.p(t)) \tag{9}$$

The quantity on which N depends on is a specific growth or per capita growth rate which is $\frac{1}{p}\frac{dp}{dt}$, another way of describing equation is to say that specific growth is $N(t,p)/p$. Some one can expect that the number of birth would be proportional to $p(t)T$ when T is small. If the fertility of parent does not change, the actual number of births in the time interval T can be expressed as $N_o p(t)T$ where $N_o$ is a constant. Then equation becomes

$$\frac{dp(t)}{dt} = N_o p(t) \tag{10}$$

dividing both sides by $p(t)$ we obtain the equation

$$\frac{1}{p(t)}\frac{dp}{dt} = N_o$$

Integrating w.r.t

$$\int \left(\frac{1}{p(t)}\frac{dp}{dt}\right) dt = N_o \int dt$$

$$\int \frac{dp(t)}{p(t)} = N_o t + c$$
$$\ln \mid p(t) \mid = N_o t + c$$
$$p(t) = e^{N_o t} e^c$$

at $t = 0$, $p(0) = p_o$, $c$ arbitrary constant

$$p_o = e^c$$

$$p(t) = p_o e^{N_o t} \tag{11}$$

Equation (9) has been derived on the assumption that only birth occur. In a situation where only death occur the same equation can be reached but now $N_o$ will be negative since the population decreases in the time interval T. It follows from equation (10) that the population decays exponentially with time from its size at $t = 0$. We can assume that the number of deaths in the short time interval T is $D_o p(t)T$. Similarly individuals may enter the given area from outside say I(t)T immigrants in the interval T, the more individuals come in an area the more the population grows. Likewise some may depart from area give rise to E(t)T emigrants, if emigration is greater than immigration the population decreases. Now we can define this equation

$$p(t+T) - P(t) = N_o p(t)T - D_o p(t)T + I(t)T - E(t)T$$

Which leads to

$$\frac{dp(t)}{dt} = (N_o - D_o)p(t) + I(t) - E(t) \tag{12}$$

when $T \to 0$ More generally I and E could be to depend on p so that equation (11), which is often called Verhulst's Differential Equation.

## 2.1 Bacteria growth

In this section we discuss the importance of the study of bacteria growth. Parts of our body such as our skin and large intestine is made of bacteria cells which help us to get nutrition from our food. Bacteria can be used to produce important products, that why it is important to understand the bacterial growth. When studying bacteria growth both exponential growth and logistic growth are considered.

### 2.1.1 Exponential growth for bacteria

The simple model for exponential growth could be $\frac{dX}{dt} = \mu.X$ where $\mu$ represent the birth coefficient and X represent bacteria density.If $\mu$ is a constant greater than zero we have a solution $X(t) = X_0 e^{\mu.t}$. In order to restrict the production of organisms we introduce a variable S which describe the concentration of the nutrients into the dynamical equations.

### 2.1.2 logistic growth for bacteria

We assume that $\frac{dX}{dt} = \mu.X$, with $\mu = \mu(S) = k.S$ and $\frac{dS}{dt} = -\alpha kSX$ this means that each unit of bacteria density produce $kS$ units of offspring per unit time. If $\alpha$ is greater than zero it means that each unit of offspring produced will need $\alpha$ unit of nutrients. This model is identical to our intuition: the term $SX$ determine how often bacteria and food meet, giving the bacteria a chance to consume nutrient particles from the inflow $S_o$ and to reproduce. We get the system of ordinary differential equations:

$$\frac{dX}{dt} = kXS \tag{13}$$

$$\frac{dS}{dt} = -\alpha kSX \tag{14}$$

We try to solve the system of ordinary equations by dividing equation (12) with equation (13) and obtain the differential equation

$$\frac{dX}{dS} = \frac{KXS}{-\alpha KXS}$$

Which simplifies to a differential equation

$$\frac{dX}{dS} = -\frac{1}{\alpha} \tag{15}$$

Solving the differential equation by separation of variables and integrate we obtain

$$\int dS = -\int \alpha dX$$

After integrating we get the solution

$$S = -\alpha X + S_o \tag{16}$$

Substituting the solution into equation (12) we obtain

$$\frac{dX}{dt} = K(S_o - \alpha X)X$$

After rearranging the equation we have

$$\frac{dX}{dt} = KS_o\left(1 - \frac{X}{S_o/\alpha}\right)X$$

Where $r = KS_o$ and $B = S_o/\alpha$
Now the system of ordinary differential equation has been reduced to a logistic equation. From the model of bacterial growth we have derived the logistic equation.

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{B}\right)$$

The factor $(1 - \frac{X}{B})$ is represented by $\mu$, it is called intrinsic growth-speed and B is a carrying capacity.

### 2.1.3 Analyzing the logistic equation

The factor $\frac{-XX}{B}$ represent a crowding effect,it prevent reproduction rate. The factor $(1 - \frac{X}{B})$ is important since it tells us about the sign of $\frac{dX}{dt}$ when $r > 0$, assuming that $X \geqslant 0$.
$X = 0 \Rightarrow \frac{dX}{dt} = 0$, we find the trivial solution which means there is no population.
$0 < X > B \Rightarrow (1 - \frac{X}{B}) > 0$,the population grows.
$X = B \Rightarrow \frac{dX}{dt} = 0$, constant population.
$X > B \Rightarrow \frac{dX}{dt} < 0$, decreasing population.
Solution of (16) is found by separation of variables if $0 < X_o > B$

$$X(t) = \frac{X_o B}{X_o + (B - X_o)e^{-rt}}$$

## 2.2 Modeling Bacteria Growth Using Discrete Approach

Suppose that we want to find the number of bacteria available in an environment. Let N represent the number of bacteria in a population and $t$ be number of days so that $N_t$ is the number of bacteria in days. The population from day zero where $t = 0$ is $N_o$. Let assume that a bacteria population increase by 1 percent in each day, we have to make assumptions about the change. The change from day 0 to day 1 is : $N_1 - N_0$, change from day 1 to day2 :$N_2 - N_1$ and in general change from day t to day $t + 1$ : $N_{t+1} - N_t$. From these assumptions we obtain the difference equation of this form.

$$N_{t+1} - N_t \tag{17}$$

### 2.2.1 Geometric Growth or Malthusian Model

The following assumptions are made: The number of birth between time $t$ and $t + 1$ is proportional to the size of a population $N_t$ at time $t$, so the number of birth between time $t + 1$ and $t$ is $\alpha N_t$ denoting bacteria into a culture. The number of death between $t$ and $t+1$ is also proportional to the size of a population $N_t$ at time $t$, so the number of death between time $t+1$ and $t$ is $\beta N_t$ which denotes bacteria out of a culture. Where $\alpha$ and $\beta$ are positive constants of proportionality. By combining these two assumptions we say that the population at $t + 1$ equal to the population at $t$ plus the births at $t$ to $t + 1$ minus the deaths at $t$ to $t + 1$.

$$N_{t+1} = N_t + \alpha N_t - \beta N_t$$

Factoring $N_t$ on the right we obtain

$$= N_t(1 + \alpha - \beta)$$

$$N_{t+1} = \lambda N_t \tag{18}$$

Letting $\lambda$ be

$$\lambda = 1 + \alpha - \beta$$

where $\lambda$ is the per-capita growth rate of the population, this is a percentage increase or decrease of the population in one unit time. Equation (18) is the Discrete Malthusian Model of population growth and is an example of a difference equation. The Malthusian Model assume that the growth rate

$\lambda$ is constant, independent of t and $N_t$. The difference equation (18) can be solved by first substituting $t = 0$ into equation (18) and obtain the equation

$$N_1 = N_{0+1} = \lambda N_o$$

at time = 1 we have

$$N_2 = N_{1+1} = \lambda N_1 = \lambda(\lambda N_o)$$

$$N_2 = N_{1+1} = \lambda^2 N_o$$

and at time $t = 2$ we obtain

$$N_3 = N_{1+2} = \lambda N_2 = \lambda(\lambda^2 N_o) = \lambda^3 N_o$$

we can see that there is a pattern developing from these equations

$$N_n = N_o \lambda^n$$

Prof by mathematical induction in general

We conclude by principle of mathematical Induction that the solution to a difference equation (16) is

$$N_t = N_o \lambda^t \tag{19}$$

The behavior of the solution (16) to a Malthusian Model when using different values of $\lambda$. If $\lambda > 1$ the population will increases (geometrical) as t increases.If $\lambda < 1$ the population will decrease as t increases. If $\lambda = 1$ the population density will remain constant as t varies.

# 3 Modification of a logistic equation

In this section we are analyzing the growth curves which were developed by extending a logistic growth equation. These growth curves were developed for different purposes, modeling growth of plants, modeling fish weight, modeling population dynamics e.c.t. Their limitations and properties are compared. We develop a generalized form of the logistic growth curve which include these models as special cases.

## 3.1 Generalized logistic equation

$$\frac{dN}{dt} = rN^{\alpha}\left[1 - \left(\frac{N}{K}\right)^{\beta}\right]^{\gamma} \tag{20}$$

Where $\alpha$, $\beta$, $\gamma$ are positive real numbers. In this project we are concerned about positive values of these parameters and that of r, since negative exponents does not give us possible model. Changing the parameters of a generalized logistic equation we obtain different growth models. By differentiating the above equation and setting second derivative to zero we obtain population value at the inflection point.

$$N_{inf} = \left(1 + \frac{\beta\gamma}{\alpha}\right)^{-\frac{1}{\beta}} K$$

### 3.1.1 Von Bertalanffy's growth equation

Von Bertalanffy growth equation was introduced by Von Bertalanffy to model fish population. The logistic equation was modified to model fish weight. Von Bertalanffy represent a different growth by modifying the Verhulst curve in a way that he developed Bernoulli differential equation. Von Bertalanffy growth equation is obtain by substituting the following parameters in a generalized equation, $\alpha = 2/3$, $\beta = 1/3$, $\gamma = 1$. The differential equation is represented as.

$$\frac{dN}{dt} = rN^{2/3}\left[1 - \left(\frac{N}{K}\right)^{1/3}\right]$$

Which has solution

$$N(t) = K\left[1 + \left[1 - \left(\frac{N_o}{K}\right)^{1/3}\right] e^{1/3rtK^{1/3}}\right]^{3}$$

Inflection point occurs at $N_{inf} = \frac{8}{27}K$

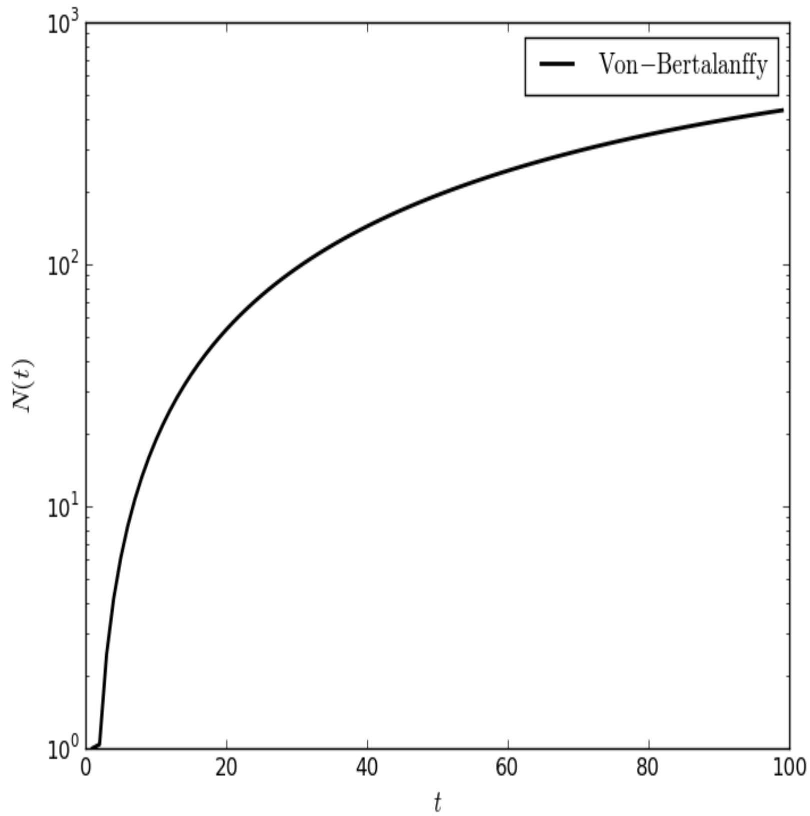


Figure 4: Shows Von Betalanffy weight curve in time t. The parameters are $r = 1$ , $K = 200$, $N_o = 100$, generated using python

### 3.1.2 Richards growth equation

Richards extended the growth equation developed by Von Bertalanffy to model the growth of plants. Richards introduced the equation below which is a special case of the Bernoulli differential equation.

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)^{\beta} \tag{21}$$

Which has solution

$$N(t) = K\left[1 - e^{-\beta rt}\left[1 - \left(\frac{N_o}{K}\right)^{-\beta}\right]\right]^{-\frac{1}{\beta}}$$

Inflection point occurs at

$$N_{inf} = \left(\frac{1}{1+\beta}\right)^{\frac{1}{\beta}} K$$

Richards equation is obtained by substituting the following parameters into a generalized logistic equation. When $\beta = 1$ the Richards equation is equivalent to a Verhulst logistic equation. For $\beta = 0$ the Richards equation becomes the exponential growth. The Richards curve in figure 4 is similar to the exponential growth curve which have a J-shaped curve.

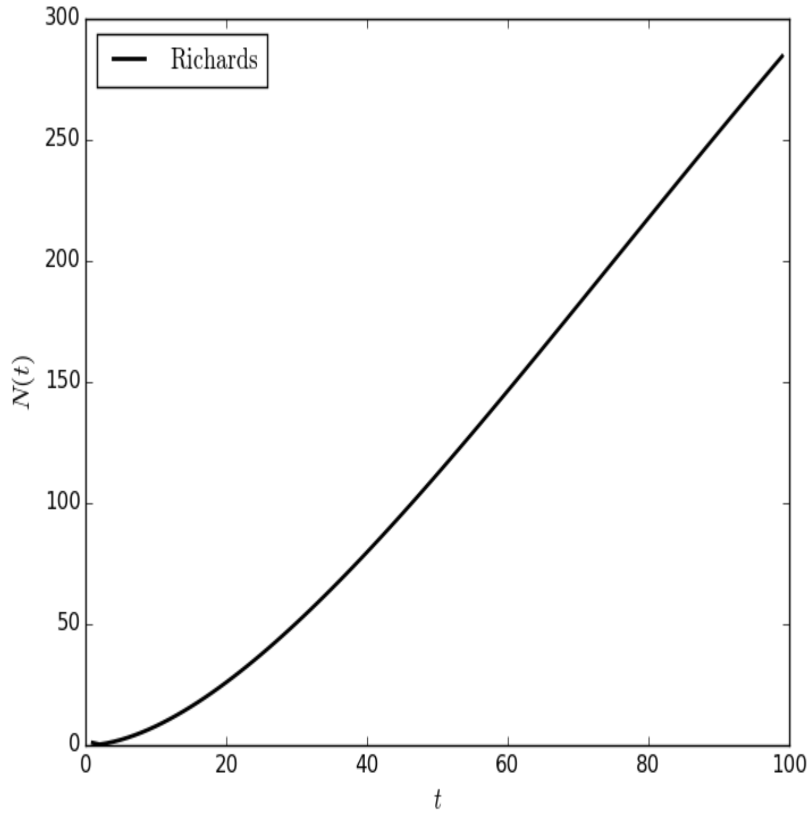


Figure 5: Richards curve for the following parameter: $r = 1$ , $K = 200$, $\beta = 1$, $N_o = 100$, $\alpha = 0.05$, generated using python

#### 3.1.3 Blumberg's Equation

Blumbergs equation is also known as hyper logistic equation, it was introduced by modifying a logistic equation to model population dynamics. He discovered that the most important restriction of a logistic curve was inflexibility of a point of inflection. Blumberg considered the constant intrinsic growth rate r and tried to treat it as a variable independent of time to solve these limitations, which leads to rate where the growth rate is reaching zero.

He introduced a hyper logistic function which is

$$\frac{dN}{dt} = rN^{\alpha}\left(1 - \frac{N}{K}\right)^{\gamma}$$

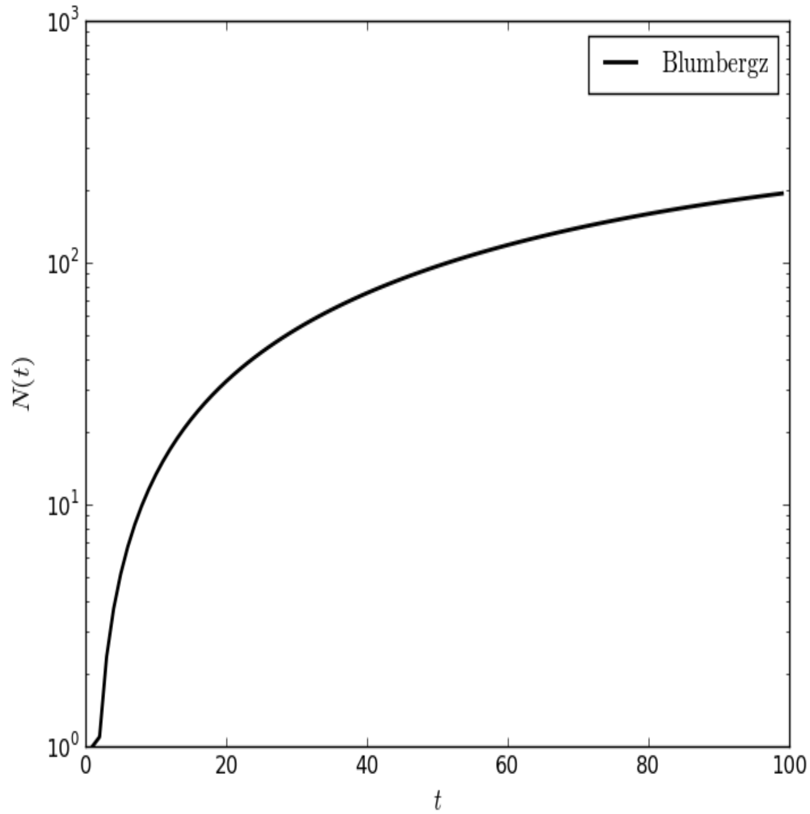


Figure 6: Blumbergs curve for the following parameter: $r = 1$ , $K = 200$, $\beta = 0.5$, $N_o = 100$, $\alpha = 0.5$, generated using python

### 3.1.4 Gompertz Growth Function

Gompertz growth curve is derived from a Gompertz logistic equation of this form.

$$\begin{aligned}\frac{dN}{dt} &= \frac{r}{\beta^{\gamma}} N \left[1 - \left(\frac{N}{K}\right)^{\beta}\right]^{\gamma} \\ &= \frac{rN}{K^{\beta\gamma}} \left(\frac{K^{\beta} - N^{\beta}}{\beta}\right)^{\gamma} \\ &= r'N \left(\frac{K^{\beta} - N^{\beta}}{\beta}\right)\end{aligned}$$

where $r' = \frac{r}{K^{\beta\gamma}}$
taking the limit of $\left(\frac{K^{\beta} - N^{\beta}}{\beta}\right)$ as $\beta \to 0$
Applying L' Hopital's Rule

$$\lim_{\beta \to 0} \left(\frac{K^{\beta} - N^{\beta}}{\beta}\right) = \frac{0}{0}$$

$$\lim_{\beta \to 0} \left(\frac{K^{\beta} - N^{\beta}}{\beta}\right) = \lim_{\beta \to 0} \frac{(K^{\beta} - N^{\beta})'}{(\beta)'}$$

Applying the sum/difference rule

$$\frac{d}{d\beta}(K^{\beta} - N^{\beta}) = K^{\beta} \ln K - N^{\beta} \ln N$$

Applying the common derivative

$$\frac{d}{d\beta}(\beta) = 1$$

$$\begin{aligned}\lim_{\beta \to 0}(K^{\beta} \ln (K) - N^{\beta} \ln (N)) &= \ln (K) - \ln (N) \\ &= \ln \left(\frac{K}{N}\right)\end{aligned}$$

Substituting the above expression to a Gompertz logistic equation we obtain the Gompertz growth function

$$\frac{dN}{dt} = rN \left[\ln \left(\frac{N}{K}\right)\right]^{\gamma}$$