Financial Econometrics/Time Series Analysis for Finance

GARCH Models and Value-at-Risk

Authors:

Leon Werner (l62259) leon.werner@students.uni-mannheim.de
Małgorzata Leszczyńska (l62232) mleszczynska@student.agh.edu.pl
Mateusz Mulka (l62257) mateuszmulka@student.agh.edu.pl
Pedro Teigão (l62738) pedroteigao02@gmail.com
Thomas Childs (l58013) thomas.m.childs@tecnico.ulisboa.pt

Group ♣

2023/2024 – Summer Semester

Contents

1 Introduction

This report has been prepared in the context of the group project assigned for the courses of Econometrics and Financial Time Series. Over the course of the following sections, two risky financial assets - the stocks of Visa Inc. and Microsoft Corporation (from here on referred to simply as "Visa" and "Microsoft"), will be analysed. While they are listed on different stock exchanges (the NYSE and the NASDAQ, respectively), they are both large-cap stocks, with a great deal of historical, publicly available data. All financial data used in this report was collected from the respective Yahoo Finance API/webpage[1], as of May 1st, 2024.

The structure which will be followed in this report is the same as the structure of the worksheet in which the project questions were stated - however, the numbering of the questions is omitted, for the sake of readability. In particular, section 2 covers the analysis of the time series, from a general overview of their characteristics over time, all the way to the evaluation of the suitability of different candidate GARCH models. The next section, 3, covers the estimation of the Value at Risk (VaR), based on different assumptions regarding the ARMA-GARCH model developed in the previous section for the returns.

2 Time Series Analysis

2.1 Time Series History

In this section, a general overview is given of the main characteristics and behaviour of the time series made up of the closing prices for each of the considered stocks. In this section, the full history of both stocks is considered.

2.1.1 Visa

The time series composed of Visa's closing stock prices can be seen in Figure 1. Overall, the stock presents little volatility between its IPO and around 2019, where a more volatile period seems to have begun. We will now take a look at how the stock price evolution in this period may relate to external events.

Visa started trading on the stock market in March 2008, selling each share for 44$. Right after, the 2008 financial crisis shook the world, but Visa managed to stay strong, and kept growing, with little downwards movement. The company spread its services all over the world, as a provider of electronic payment solutions for businesses and consumers.

In 2010, The Durbin Amendment aimed to regulate debit card interchange fees, which could have hurt Visa (a small slump is clearly visible in figure 1 around this time), but the company was otherwise largely unaffected. Visa kept expanding, by acquiring other companies, such as Visa Europe, in 2016. Significant investments were were also made into the development of new technology for digital payments, such as payments with a mobile phone, or contactless technology.

When COVID-19 hit in 2020, the stock dropped, experiencing one of its most volatile periods yet. Visa quickly switched gears, and as the consumer demand for traditional payment methods slumped, a consequence of worldwide stay-at-home policies, a focus on digital payments enabled the company to stay afloat. As consumer confidence grew and stimulus checks arrived, consumer

[1] The URLs can be found hyperlinked here, for both Visa and Microsoft

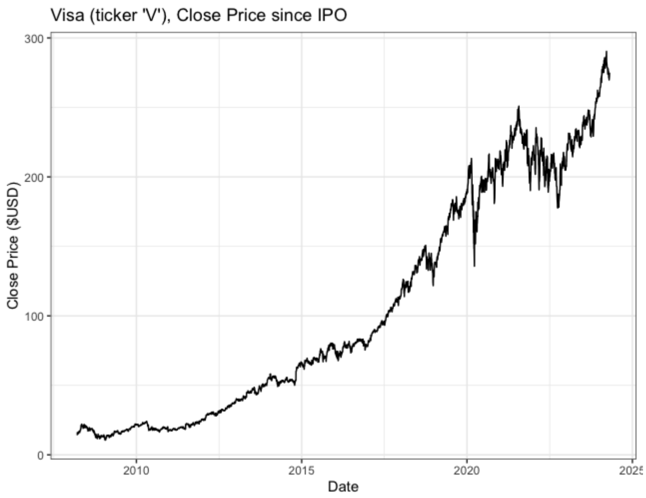


Figure 1: Visa stock price (all history), $USD.

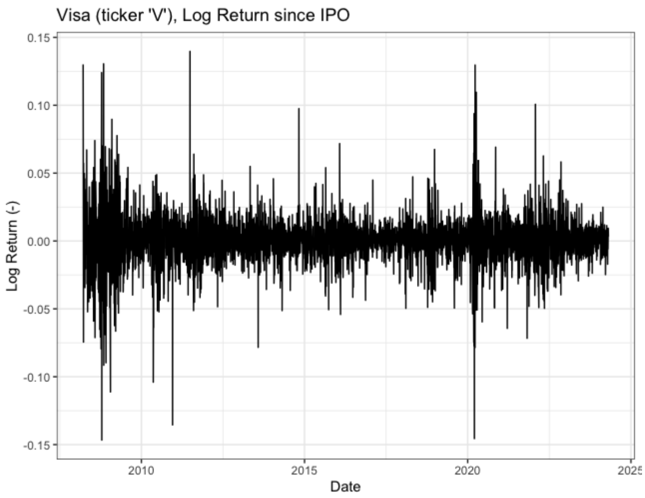


Figure 2: Visa log returns.

spending increased, especially online, which helped propelled Visa's stock price straight back up.

With the economic rebound, mildly damped by the market correction of 2021-2022, Visa surged ahead, backed by increasing consumer spending, and the widespread embrace of digital transactions. This momentum catapulted Visa's stock price to all time heights as of April 2024. The high levels of inflation experienced across different economies, over the last two years, has likely also played a role in fuelling the company's revenue growth, and with it an accelerated increase of its stock price, when compared to previous periods of time.

Figure 2 show the log returns, where it is more clear how the events mentioned above impacted the stock price. Between 2008 and 2013, there was a noticeable higher-volatility period, likely associated with the housing crash of 2008, and the subsequent financial crises which shook the world, and the implementation of new regulations regarding debit card fees. During 2015 until 2020 there was period of lower volatility than the previous one. Starting in 2020, there was a significant spike in volatility, along with another period of heightened volatility, likely linked to the uncertainty introduced by the pandemic, and the subsequent economic recovery.

Also, it's evident from the graph that there are noticeable spikes, which could be attributed to strong/weak quarterly earnings, lawsuits, regulatory constrains, or positive/negative reactions of the market to news such as acquisitions.

2.1.2 Microsoft

Since it its shares were first listed in 1986, at 21$ each, Microsoft has seen a lot of changes. The time series composed of Microsoft's daily close price can be seen in figure 3.

At first, the company performed consistently for a decade, likely because of the success of its early products, such as MS-DOS and Windows 3.0, launched in 1990. The late 1990s saw Microsoft riding the wave of the dot-com bubble, with its stock price soaring to an all-time high of 59.97$ per share in 1999, amidst the tech boom. However, this era also brought challenges, notably the antitrust lawsuit filed by the Department of Justice in 1998, alleging monopolistic practices by Microsoft. Bill Gates also stepped down as CEO around this time in 2000. All these changes, and external factors, contributed to the stock experiencing some draw down

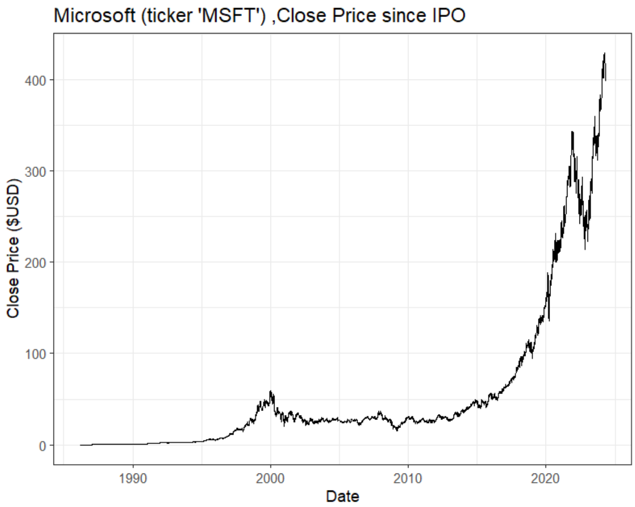


Figure 3: Microsoft stock price (all history), in $USD.

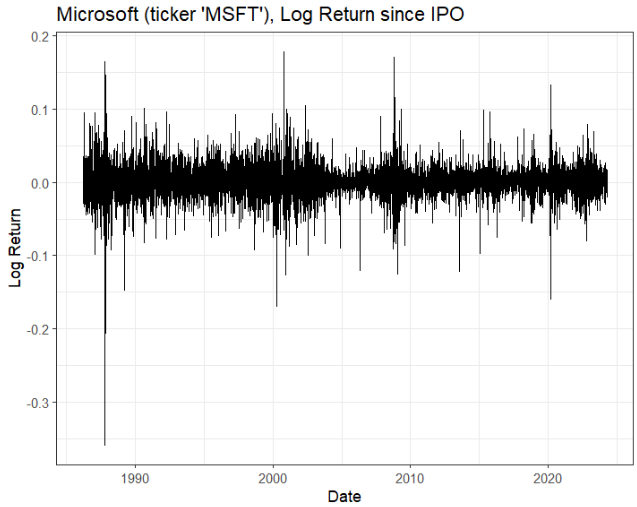


Figure 4: Microsoft log returns.

around this period of time.

The company ventured into new business opportunities such as software, cloud services, and video games. The stock price remained relatively stable throughout the first decade of the 21st century, navigating the financial crises of 2008-2012 relatively well. In 2014, Satya Nadella was selected as the new CEO and his tenure heralded a renewed focus on cloud computing and artificial intelligence. This made the company grow even more, especially with the acquisition of large, renown companies such as LinkedIn (2016) and GitHub (2018). In this period, Microsoft also made significant investments in promising up-and-coming ventures such as OpenAI (2019).

After the initial drop caused by the COVID-19 pandemic disruption, Microsoft's cloud services became even more essential. Coupled with investors' recent interest in AI, this combination of factors propelled its stock price to soar, earning it a place among a prestigious group of high-performing stocks dubbed the "Magnificent 7".

The history of Microsoft is closely tied to fluctuations in its stock price, and can better be analysed by examining the log returns (figure 4). In the 1990s, a significant high-volatility cluster is observed, largely influenced by the inflated pricing during the dot-com bubble, as well as various controversies involving the company throughout the decade. Similarly, at the start of the 2000s, there is also a high-volatility period following the burst of the bubble. The year 2008 saw another spike in volatility due to the financial crisis. Subsequently, volatility remained relatively low, until the onset of the pandemic, which led to a notable increase in volatility once again. The volatility has remained as of April 2024, largely attributed to the AI boom (and Microsoft's connection to OpenAI, the creator of ChatGPT) and the growth of the cloud services segment (Azure) of Microsoft's business.

The noticeable spikes can be attributed to the nature of Microsoft as a tech company, which tends to be more volatile due to the constant need for innovation and expansion. This volatility is exemplified by significant events such as the acquisitions of LinkedIn or GitHub, as well as the company's investments in AI and other ventures.

2.2 Descriptive statistics of the observed financial data

The next part of the analysis is strictly focused on the latest 5 years of data. In this subsection, the descriptive statistics of the financial data are shown, for both assets. The observed behaviour will be related with the expected behaviour known from the stylised facts about financial returns. In sum, we observed:

- Slight negative skewness in the log returns of both assets;
- Heavy-tailed distributions for the log returns of both assets (i.e., high Kurtosis);
- Clear signs of volatility clustering, exhibiting a strongly time-dependent behaviour (furthermore, these appear to be directly correlated with external events, such as the beginning of wars, or the COVID-19 pandemic outbreak);
- Low autocorrelation of the financial returns, after the first lag.

We did not observe:

- Clear signs of a positive risk equity premium;
- Any significant difference in the the standard deviation of each stock's log returns - the observed standard deviation, in both cases around 0.02, is consistent with what we would expect from this class of equities (large-cap US, individual stocks);

Furthermore, at this point we did not look into the presence of calendar effects, of leverage effects, or test for aggregational Gaussianity. (This analysis was conducted later on, in Section 2.11).

2.2.1 Visa

Looking closer to the Visa prices through this period (figure 5), we can clearly see the COVID-19 pandemic in March 2020 - there is a significant drop of the prices. After that moment, as it was mentioned before, Visa started growing again. The growth has a visible volatility - constant ups and downs. Furthermore, the stock price is clearly non-stationary (despite bumps, the general trend is upwards).

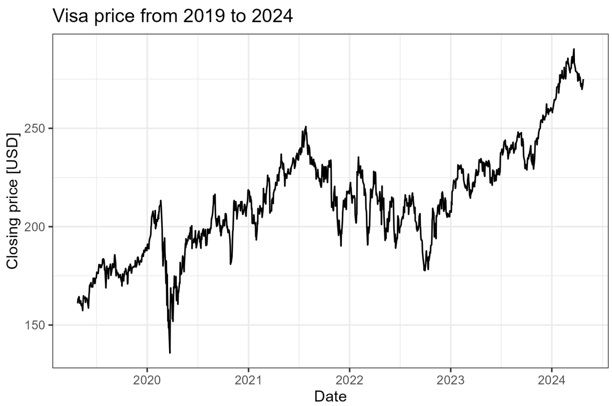


Figure 5: Visa stock price (in the last 5 years), in $USD.

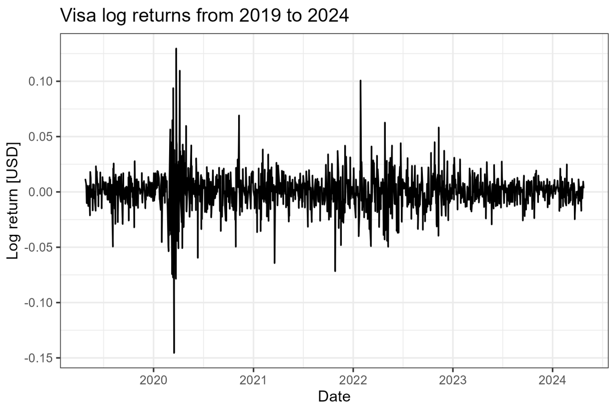
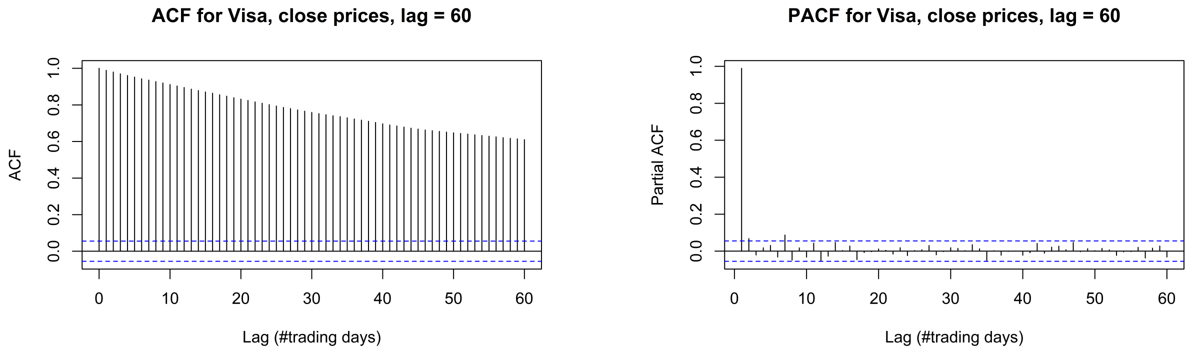


Figure 6: Visa log returns (in the last 5 years).



(a) Visa ACF (prices). (b) Visa PACF (prices).

Figure 7: ACF and PACF, applied directly to Visa's stock price in the analysed period.

mean	std	median	min	max	skew	kurtosis
212.87	27.87	211.77	135.74	290.37	0.34	0.02

Table 1: Visa prices - descriptive statistics.

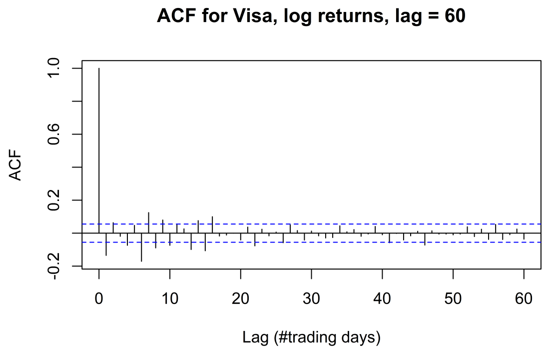
Focusing on the log returns of the close prices, shown in figure 6, and rather unsurprisingly, the greatest volatility is to be noticed around March 2020. There are also other peaks on the graph and the one in the beginning of 2022 can be important - this was when Russia launched the (still ongoing) attack on Ukraine. Nevertheless, data on this graph can give the impression of being stationary, centred at or close to zero. This is consistent with the behaviour we would expect.

Continuing the analysis, it was decided to apply 60 lags to correlograms (both for the ACF and for the PACF), for both prices and log returns. In the price ACF plot, the tail is infinite and tails off following an exponential decay. The PACF plot shows statistical significance only for first lag. The ACF and PACF plots remind in their shape the plots for random walk process and it can be considered as an another indicator for data being non stationary.

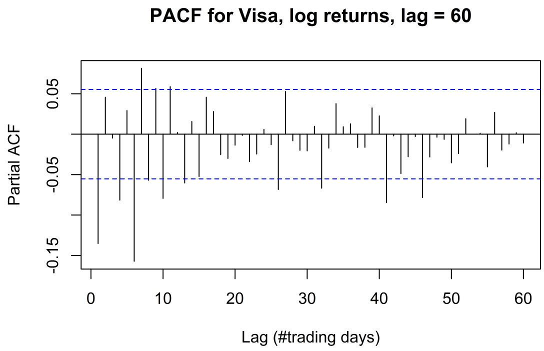
Going to log returns data (figure 8), that was previously labeled as potentially stationary data, in the ACF plot the most statistically significant lag is the first one. Although some of the further lags are also above the default significance level, they are much smaller that the first one, and thus considered to be statistically insignificant. in any case, the ACF is clearly finite in extent. The PACF graph shows significant lags for number 1, 4, 6, 7 and also more, in the next lags. In fact, the PACF appears to be infinite in extension, decaying in the fashion of damped sign waves. If we consider the lag values of the ACF after the first one to be, in fact, statistically insignificant, then the correlograms for the log returns could suggest an $MA(q)$ process.

Table 1 shows the descriptive statistics and frequency distributions. We can see that, for Visa's prices, the mean is around 212 $USD and the standard deviation is more than 27 $USD. The skewness is positive but close to zero, which indicates an almost normal distribution, with a slight right-sided skewness. Kurtosis is almost 0, which indicates a normal distribution.

However, when we take a look at the histogram/frequency distribution (see figure 9), we can see that the closing prices data do not quite behave as a normal distribution, as multiple clear peaks are visible, both below and above the empirical mean.



(a) Visa ACF (log returns).



(b) Visa PACF (log returns).

Figure 8: ACF and PACF, applied to Visa's log returns in the analysed period.

mean	std	median	min	max	skew	kurtosis
0	0.02	0	-0.15	0.13	-0.05	10.09

Table 2: Visa log returns - descriptive statistics.

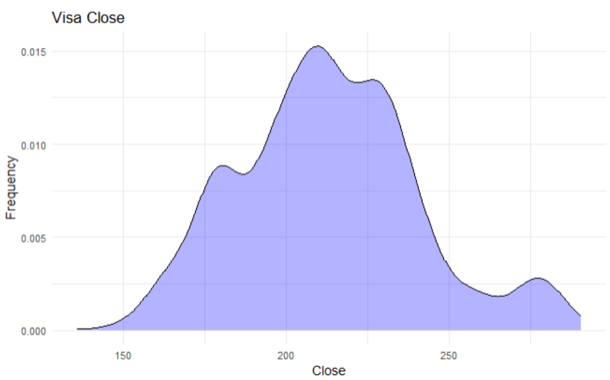


Figure 9: Visa prices histogram - close price frequency.

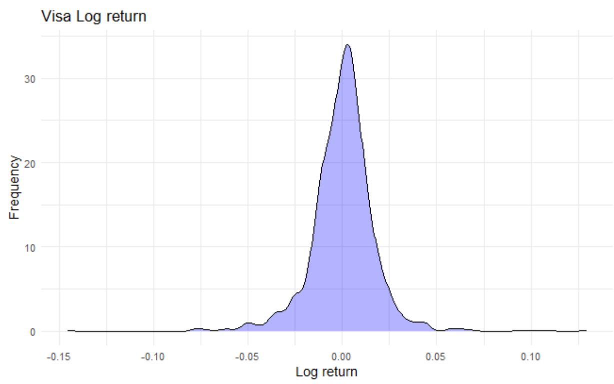


Figure 10: Visa returns histogram - log return frequency.

Analysing the log returns (table 2) we see that the mean is equal to zero, the standard deviation is around 0.02 and the skewness as well. The kurtosis, however, is significantly positive (mugh higher than 3), which indicates a leptokurtic distribution. In other words, the data has more outliers than normal distribution (i.e., it is heavy-tailed). The histogram shown in figure 10 shows the characteristics described before.

2.2.2 Microsoft

Similarly to Visa, in Microsoft prices (figure 11) we can observe the first drop down, in the last 5 years, in the beginning of 2020 (caused by pandemic) and after that the prices have been growing. What is important to mention, is the fact that Microsoft prices achieved higher values than Visa until 2022. Also, the drop in the first half of 2022 is more clear and visible, but the values still remain higher. With remaining volatility, the prices has been growing up until April 2024.

The log returns plot of Microsoft (figure 12) has only small differences comparing to the log returns plot of Visa. We can see the significant volatility in March 2020, then in 2022. The

Figure 11: Microsoft stock price (in the last 5 years), in $USD.

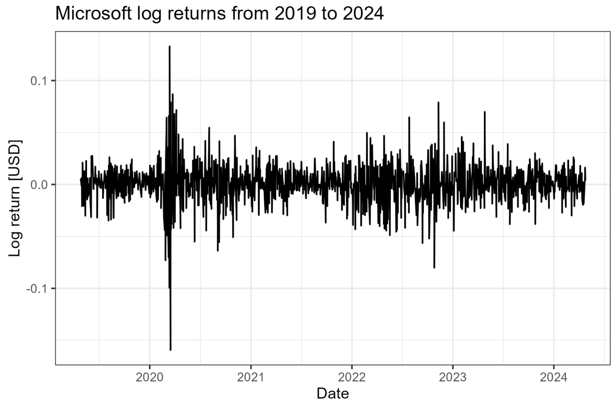
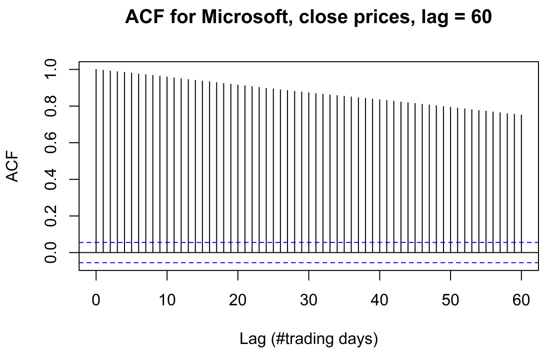


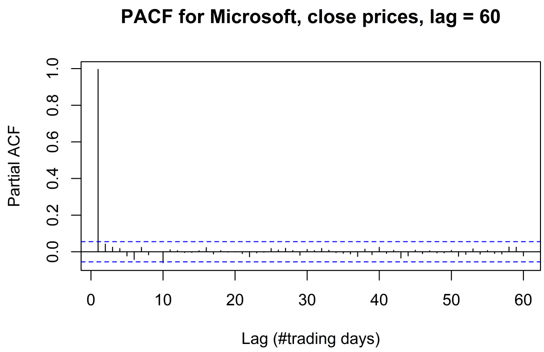
Figure 12: Microsoft log returns (in the last 5 years).

data also gives the impression of being stationary.

The ACF plot for Microsoft prices looks similarly to ACF plot for Visa. The tail is infinite and tails off as a slowly-exponential decay. The PACF plot shows significant first lag, similarly to Visa. The ACF and PACF plots remind in their shape the plots for random walk process and it can be considered as an another indicator for data being non stationary. This behaviour can be easily visualised in figure 13.



(a) Microsoft ACF (prices).

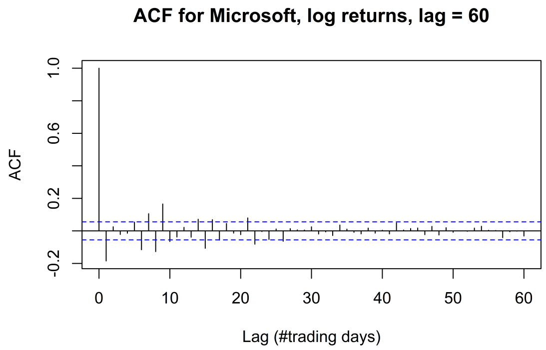


(b) Microsoft PACF (prices).

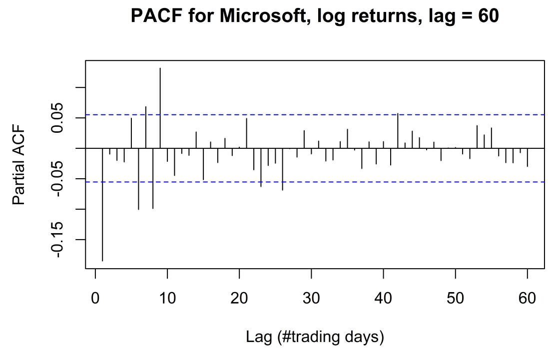
Figure 13: ACF and PACF, applied directly to Microsoft's closing stock prices in the analysed period.

In the ACF for Microsoft's log returns (figure 14), the most significant lag is the lag number 1, decaying so fast that it can be considered finite. Some of the further lags are also above the significance level, however they are much smaller than the first one. The PACF graph shows significant lags number 1, 6, 7 and also more, appearing to continue infinitely. The plot thus once again suggests an $MA(q)$ process, specifically with $q = 1$..

Table 3 and figure 15 show the descriptive statistics and frequency distribution for Microsoft's stock price. Firstly, we can see that for Microsoft prices the mean is around 250 $USD and standard deviation is more than 76 $USD, which can be considered as a significant value. The skewness is positive but close to zero, which indicates an almost normal distribution with a slight right-sided skewness. Kurtosis is negative but also very close to normal distribution. Negativity indicates platykurtic distribution that means the distribution produces fewer and/or



(a) Microsoft ACF (log returns).



(b) Microsoft PACF (log returns).

Figure 14: ACF and PACF, applied to Microsoft's log returns in the analysed period.

mean	std	median	min	max	skew	kurtosis
254.72	76.06	252.99	119.84	429.37	0.15	-0.64

Table 3: Microsoft prices - descriptive statistics.

less extreme outliers than the normal distribution.

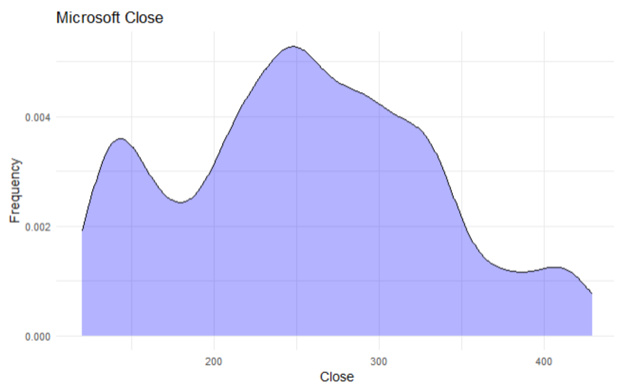


Figure 15: Microsoft price histogram - closing stock price frequency.

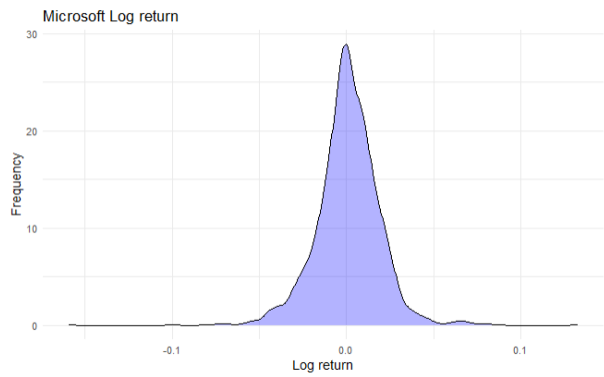


Figure 16: Microsoft Log Return Frequency

Analysing the log returns (table 4 and figure 16), we see that the mean is equal to zero, standard deviation is around 0.2 as expected for this class o equitiy, but skewness and kurtosis differ. The skewness indicates a slight left-sided (negative) distribution and kurtosis indicates leptokurtic distribution, that means the data has more outliers than normal distribution, resulting in heavy tails.

2.3 Stationarity - the Augmented Dickey-Fuller (ADF) Test

In this part of the analysis Augmented Dickey-Fuller test has been performed. The goal of the test is to formally determine whether the data are stationary. Formally, the hypotheses of this test are as follows:

H_0: the time series has one unit root (i.e. it is not stationary)

H_1: the time series has no unit roots (i.e. it is stationary)

mean	std	median	min	max	skew	kurtosis
0	0.02	0	-0.16	0.13	-0.27	7.57

Table 4: Microsoft log returns - descriptive statistics.

The ADF test was performed for log returns of both time series. For both Visa and Microsoft, the obtained p-value was around 0.01. Considering the typical 95% confidence level, corresponding to $\alpha = 0.05$, as the encountered p-value is lower, there is evidence to reject hyptohesis H0, therfore accepting the alternative, H1. In short, the conclusion of applying the ADF test is that, for both Visa and Microsoft, the examined time series, when expressed as log returns, are stationary.

This conclusion aligns with the previous observations about the stationarity of the data, both from Visa and Microsoft. In this case, it was already clearly, even if informally, visible that the log returns presented a stationary behaviour.

2.4 Appropriately modelling the log returns via an ARMA

At this point, it was decided to apply both a simplified Box-Jenkins methodology, and to compare this with the result of the ARIMA() function. The log returns data have been used. It was proven previously that the time series made up of the log returns for both stocks, in this period of time, are stationary. The ARIMA() function is of interest at this point, because it allows to automatically select a model and check the model residuals, in a semi-automated manner; conversely, the Box-Jenkins methodology still leaves some room for interpretation.

As still part of this step, the residuals of the selected model, for each asset, had to be plotted and checked by portmanteau test. Specifically, the Ljung-Box test was used to examine autocorrelations. The Ljung-Box test specifies the following hypotheses:

H_0: $\rho_1 = \rho_2 = ... = \rho_k = 0$, i.e. there is no evidence of residual autocorrelations

H_1: $\rho_1 \neq 0 \vee ... \vee \rho_k \neq 0$, i.e. there is evidence of residual autocorrelation in the tested lags

2.4.1 Visa

As the result of AUTO.ARIMA() function, the best model for the log returns of Visa was indicated to be $ARIMA(3,0,3)$, or, in other words, an $ARMA(3,3)$. The use of this model results in the residuals shown in figure 17.

This value also makes sense if only the Box-Jenkins methodology were to be followed here. The Box-Jenkins methodology is based on the observation of the (sampled) ACF and PACF plots for the log returns we are trying to model. In this specific case, these have already been shown in figure 8. In good rigour, the best approach would be to begin with and $ARMA(p,q)$ with $p = 1$ and $q = 1$, as, in general, the simpler the model, the better. To avoid the cumbersome process of describing every single iteration experimented with, we will just show how the Box-Jenkins approach could be used to try to estimate directly the values of the optimal p and q, and our reasoning could look something like:

1. Based on figure 8, both the ACF and PACF could be considered infinite in extent, suggesting the applicability of a both AR and MA components, as a mixed model;
2. The PACF drops below the statistical significance level (blue dotted line) for the first time after lag $p = 3$, and although it later still exceeds this value for specific lag orders,

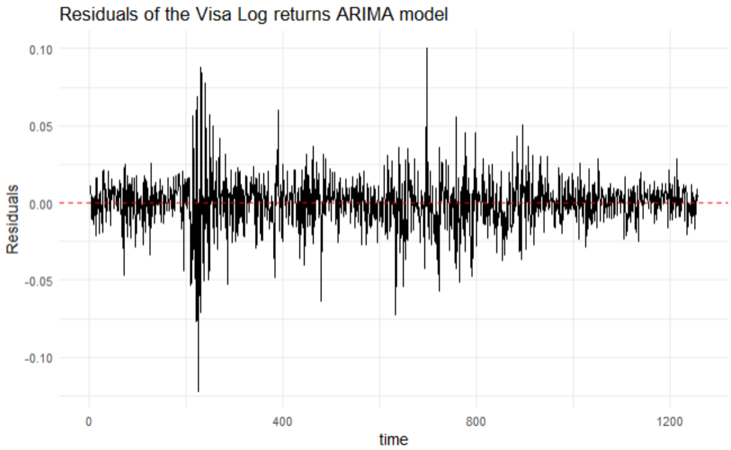
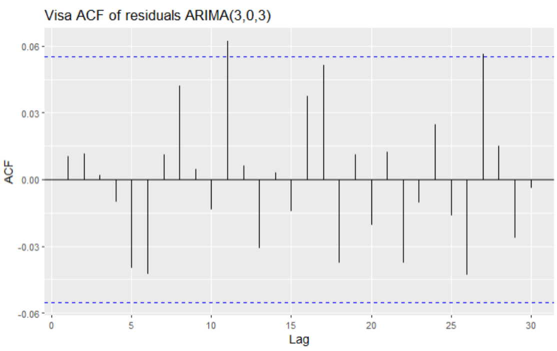
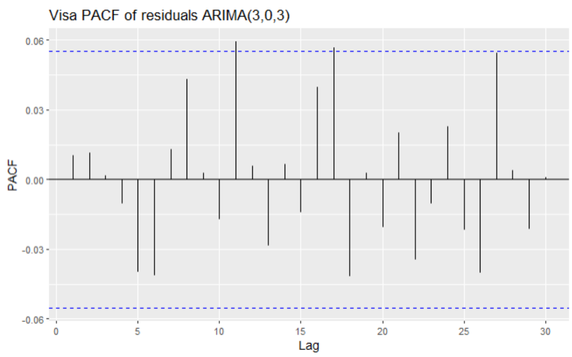


Figure 17: Residuals resulting from modelling Visa's log returns as an $ARMA(3,3)$



(a) Residuals ACF.



(b) Residuals PACF.

Figure 18: Properties of the residuals, obtained when considering an $ARMA(3,3)$, for the log returns of Visa.

it is in principle unnecessary and counterproductive to include higher-order lag terms in the model;

3. A natural first experiment would be the $ARMA(3,1)$ model; with some iteration, and analysing the residuals and fit values, we could find that considering the $ARMA(3,3)$ yields better results.

Applying the computation of the Ljung-Box statistic, resorting to the relevant R functions, the p-value of the Ljung-Box test was 0.6866, significantly above the considered confidence level of $\alpha = 0.05$. Recalling the hypotheses of this test given in section 2.4, we clearly find no reason to reject H0. This means that, fortunately, for this data, the residuals can be seen as uncorrelated. This conclusion is supported by the ACF and PACF for the residuals, which are for the most part below the statistical significance threshold - see figure 18.

2.4.2 Microsoft

As the result of ARIMA() function, the best model for the Microsoft log returns appears to be an $ARIMA(0,0,1)$, i.e., an $MA(1)$. In fact, this was one of the preliminary observations put forth was already said previously, that ACF and PACF plots of the data reminds about $MA(q)$ process, which seems to be supported by the AUTO.ARIMA() method.

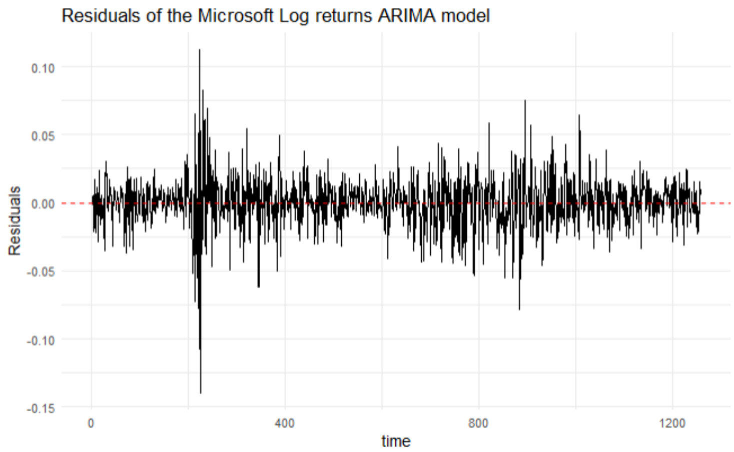


Figure 19: Residuals resulting from modelling Microsoft's log returns as an $MA(1)$

To arrive at this conclusion solely via the Box-Jenkins methodology, our reasoning could look something like the following:

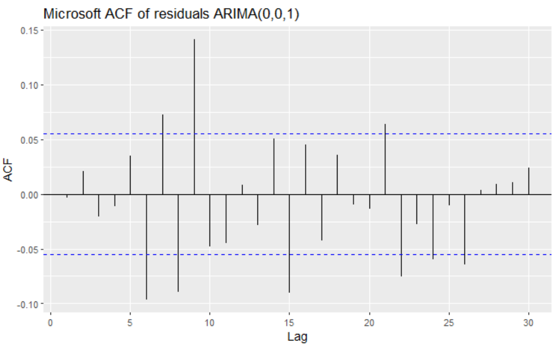
1. Based on figure 14, the ACF looks finite in extend, but the PACF appears infinite in extent, suggesting the applicability of an MA model;
2. The ACF drops below the statistical significance level (blue dotted line) for the first time after lag $p = 2$; however, the difference between the ACF coefficient for lag 1 and for lag 2 is so great (1 vs. -0.2), that a good first approach could be considering initially $p = 1$;
3. A natural first experiment would be the $MA(1)$ model; with some iteration, and analysing the residuals and fit values, the best fit we can find actually turns out to be the $MA(1)$, while having the added benefit of simplicity.

Computing the p-value of the Ljung-Box test, we obtain $4.633e - 9$, which is far below the confidence level of $\alpha = 0.05$. This mean that we find strong evidence to reject H0. Recalling the alternative hypothesis H1, we unfortunately conclude that the residuals obtained with this model are not uncorrelated.

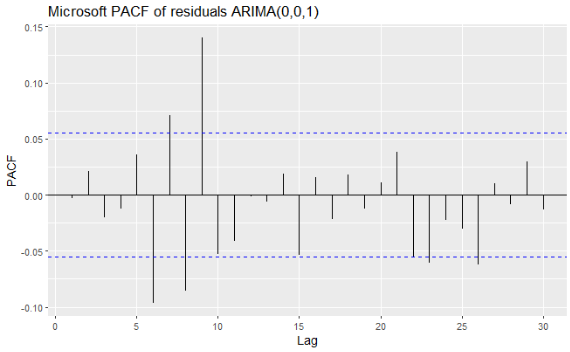
In fact, when analysing the statistical properties of the residuals (figure 20), namely their sampled ACF and PACF plots, there are non-negligible peaks in both the ACF and PACF, which carry statistical relevance.

2.5 ARMA residuals, ARCH effects and optimal ARMA-GARCH model selection

In this part the residuals of the ARIMA models found above, for both Visa (figure 17) and Microsoft (figure 19) are examined. The goal here is to understand if these residuals contain evidence of the presence of ARCH effects. ARCH effects refer to the changing volatility (variance) of residuals in time series models, which varies over time. A good indicator of the presence of ARCH effects is if the squared residuals of the time-series happen to be correlated. The intuitive explanation for this is that, under the hypothesis of volatility clustering, squared



(a) Residuals ACF.



(b) Residuals PACF.

Figure 20: Properties of the residuals, obtained when considering an $MA(1)$, for the log returns of Microsoft.

residuals will necessarily be correlated (a large squared residual points to a large variance; as volatility tends to cluster, more large squared residuals are expected; vice-versa for low squared residuals/low-variance regions).

ARCH effects can be detected by examining the autocorrelation of the squared residuals of a model. Having already computed the residuals above, the autocorrelation of squared residuals was tested for recovering the Ljung-Box test, in this case applied to the squared residuals. In this case, and opting for simpler language, we can express the relevant hypotheses once again as:

H_0: there is no residual autocorrelation in the tested lags

H_1: there is evidence of residual autocorrelation in the tested lags

In order to aid in the selection of the optimal, a for loop was used in the R script. This loop was designed to select the best values for the candidate GARCH model, by selecting the $GARCH(p, q)$ model which minimises information loss. Concretely, the metric used was the Akaike Information Criteria (AIC). This loop tested all possible combinations of $GARCH(p, q)$, with $p, q \in \{0, 1, 2, 3, 4, 5\}$.

2.5.1 Visa

The Box-Ljung test for the squared residuals of the $ARMA(3, 3)$ fitted to Visa's log returns resulted in a p-value lower than $2.2e - 16$, much lower than $\alpha = 0.05$, so we cannot accept H_0. This test confirms that autocorrelation between squared residuals exists and is non-negligible, thus proving the existence of ARCH effects.

Choosing the best GARCH model was based on the iterative process described above, aimed at minimising the AIC value. The best result obtained was for a $GARCH(2, 3)$. Combining the models for the mean and for the variance, we therefore obtain an $ARMA(3, 3)$-$GARCH(2, 3)$.

2.5.2 Microsoft

The Box-Ljung test for the squared residuals of the $ARMA(3, 3)$ fitted to Microsoft's log returns resulted also in a p-value lower than $2.2e - 16$, once again much lower than $\alpha = 0.05$. Rejecting H0, this test confirms that again that the autocorrelation between squared residuals exists and is non-negligible, thus proving the existence of ARCH effects.

Choosing the best GARCH model was based on the iterative process described above, aimed at minimising the AIC value. The best result obtained was for a $GARCH(2,2)$. Combining the models for the mean and for the variance, we therefore obtain an $MA(1)$-$GARCH(2,2)$.

2.6 Adequacy of proposed ARMA-GARCH models

In this section, and actually so far, the evaluation of the ARMA-GARCH models proposed above ($ARMA(3,3)$-$GARCH(2,3)$ for Visa and $MA(1)$-$GARCH(2,2)$ for Microsoft), is conducted based on the assumption that the standardised errors $z_t = \varepsilon_t/\sigma_t$ are i.i.d. random variables, following a normal distribution.

For the two stocks being analysed, the diagnostic checks enumerated below will be performed. The same numbering was kept in the two respective subsections.

1. Checking for autocorrelation left in the standardised residuals of the fitted model, using Ljung-Box test applied to z_t;
2. Checking for remaining ARCH effects left in the sqaured standardised residuals of the fitted model, using Ljung-Box test applied to z_t^2;
3. Computing the sample skewness and kurtosis of z_t and checking if the values are close to the values of the theoretical distribution function;
4. Checking the default Goodness-of-Fit tests provided by

2.6.1 Visa

1. Applying the weighted Ljung-Box model to the standardised residuals obtained with the $ARMA(3,3)$-$GARCH(2,3)$ model, a p-value around 0.19 was obtained, suggesting that there is not strong enough evidence to reject the mean - residuals can be considered uncorrelated.
2. Applying the weighted Ljung-Box model to the standardised squared residuals obtained with the $ARMA(3,3)$-$GARCH(2,3)$ model yields also p-values consistently above the typical confidence levels, suggesting that there is no evidence of remaining ARCH effects.

3. The statistical description of the residuals can be found in table 5. The residuals continue to be slightly negatively skewed, the kurtosis is lower than for the case of only considering the $ARMA(3,3)$ model for the mean, without modelling the volatility, and very close to the "ideal" 3 expected from a normal distribution.
4. Applying the Adjusted Pearson Goodness-of-Fit Test, we find a p-value consistently below the typical confidence level corresponding to $\alpha = 0.05$. Thus, there is evidence to reject the null hypothesis - the obtained standardised residuals do not seem to follow a normal distribution.

mean	std	median	min	max	skew	kurtosis
-0.03	1	0.04	-5.36	5.01	-0.46	2.27

Table 5: Visa residuals - descriptive statistics.

2.6.2 Microsoft

1. Applying the weighted Ljung-Box model to the standardized residuals obtained with the $MA(1)$-$GARCH(2,2)$ model, a p-values consistently above 0.3 are obtained, suggesting strong evidence that the residuals are in fact not correlated.
2. Applying the weighted Ljung-Box model to the standardized squared residuals obtained with the $MA(1)$-$GARCH(2,2)$ model yields p-values between 0.83 and 0.95, suggesting to a high level of confidence that there is no evidence of remaining ARCH effects.
3. The statistical description of the residuals can be found in table 6. The residuals continue to be slightly negatively skewed, and although the kurtosis is lower than was for the case of only considering the $MA(1)$ model for the mean, and also lower that the "ideal" 3 expected from a normal distribution.
4. Applying the Adjusted Pearson Goodness-of-Fit Test, we find a p-value consistently below the typical confidence level corresponding to $\alpha = 0.05$. Thus, there is evidence to reject the null hypothesis - the obtained standardised residuals also do not seem to follow a normal distribution.

mean	std	median	min	max	skew	kurtosis
-0.02	1	-0.04	-3.71	4.05	-0.24	0.83

Table 6: Microsoft residuals - descriptive statistics.

2.7 Explicit ARMA-GARCH equations

In this section, the equations for each of the estimated models are shown, as well as the estimated optimal parameters in the standard $ARMA(p,q)$-$GARCH(m,n)$ form, shown below. All results were rounded to 3 decimal places.

$$r_t = \phi_0 + \phi_1 r_{t-1} + ... + \phi_p r_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}, \varepsilon_t = \sigma_t z_t, z_t \overset{\text{iid}}{\sim} \mathcal{N}(0,\ 1)$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + ... + \alpha_m \varepsilon_{t-m}^2 + \beta_1 \sigma_{t-1}^2 + ... + \beta_n \sigma_{t-n}^2$$

2.7.1 Visa: $ARMA(3,3)$-$GARCH(2,3)$

$$r_t = 0.000 - 0.912 r_{t-1} + 0.783 r_{t-2} + 0.932 r_{t-3} + \varepsilon_t, +0.865\varepsilon_{t-1} - 0.848\varepsilon_{t-2} - -0.951\varepsilon_{t-3},$$
$$\varepsilon_t = \sigma_t z_t, z_t \overset{\text{iid}}{\sim} \mathcal{N}(0,\ 1)$$
$$\sigma_t^2 = 0.000 + 0.067\varepsilon_{t-1}^2 + 0.168 + \varepsilon_{t-2}^2 + 0.109\sigma_{t-1}^2 + 0.000\sigma_{t-2}^2 + 0.629\sigma_{t-3}^2$$

2.7.2 Microsoft: $MA(1)$-$GARCH(2,2)$

$$r_t = 0.001 - 0.097\varepsilon_{t-1}, \varepsilon_t = \sigma_t z_t, z_t \overset{\text{iid}}{\sim} \mathcal{N}(0,\ 1)$$
$$\sigma_t^2 = 0.000 + 0.141\varepsilon_{t-1}^2 + 0.046\varepsilon_{t-2}^2 + 0.000\sigma_{t-1}^2 + 0.781\sigma_{t-2}^2$$

2.8 ARMA residuals, ARCH effects, and optimal ARMA-GARCH model selection based on t-student distribution

In this section, we revisit the residuals of the ARIMA models obtained for both Visa and Microsoft. The aim is to investigate whether these residuals exhibit evidence of ARCH effects, which refer to changing volatility (variance) in time series models. A common indicator of ARCH effects is the correlation between squared residuals.

ARCH effects can be detected by examining the autocorrelation of squared residuals. The relevant hypotheses are:

H_0: There is no residual autocorrelation in the tested lags.

H_1: There is evidence of residual autocorrelation in the tested lags.

Once again, to aid in selecting the optimal model, a for loop was used in the R script to test all possible combinations of $GARCH(p, q)$ models with $p, q \in \{0, 1, 2, 3, 4, 5\}$ and minimize the Akaike Information Criteria (AIC). The differecne was that, this time, the specified distribution was the t-student ("std"), instead of the normal.

2.8.1 Visa

The Box-Ljung test for the squared residuals resulted in a p-value much lower than 0.05, rejecting H_0. This confirms the existence of ARCH effects. The best GARCH model obtained was $GARCH(1, 1)$. Below are given the results of the diagnostics applied to this new fit:

1. Applying the Ljung-Box model to the standardised residuals obtained with the $MA(1)$-$GARCH(1, 1)$ model (considering t-student distribution), a p-value of 0.8646, suggesting strong evidence that the residuals are in fact not autocorrelated.

2. Applying the Ljung-Box model to the standardized squared residuals obtained with the $MA(1)$-$GARCH(1, 1)$ model considering t-student distribution), yields p-value of 0.4586, suggesting to a high level of confidence that there is also no evidence of remaining ARCH effects.

3. The statistical description of the residuals can be found in table 7. The residuals continue to be slightly negatively skewed, and the kurtosis is around 3.

4. Applying the Adjusted Pearson Goodness-of-Fit Test, we find a high p-value, always above 0.34. Considering the typical $\alpha = 0.05$, we accept the null hypothesis H_0. For the Adjusted Pearson Goodness-of-Fit Test, this means that the obtained standardised residuals seem to follow a t-student distribution.

mean	std	median	min	max	skew	kurtosis
-0.06	1	0.02	-5.71	6.03	-0.41	2.94

Table 7: Visa residuals - descriptive statistics (t-student distribution).

Besides obtaining a better goodness-of-fit than when considering a normal distribution, we also obtain a simpler GARCH model, which is, in general, desirable. We show here explicitly the estimated model:

$$r_t = 0.000 - 0.916r_{t-1} + 0.770r_{t-2} + 0.929r_{t-3} + \varepsilon_t, +0.878\varepsilon_{t-1} - 0.823\varepsilon_{t-2} - 0.949\varepsilon_{t-3},$$
$$\varepsilon_t = \sigma_t z_t, z_t \overset{\text{iid}}{\sim} (0,1)$$
$$\sigma_t^2 = 0.000 + 0.101\varepsilon_{t-1}^2 + 0.168 + 0.880\sigma_{t-1}^2$$

2.8.2 Microsoft

Similarly, the Box-Ljung test for the squared residuals resulted in a p-value much lower than 0.05, rejecting H_0. This confirms the existence of ARCH effects. The best GARCH model, i.e. the one that was found to minimize the AIC, was the $GARCH(1,1)$. Below are given the results of the diagnostics applied to this new fit:

1. Applying the weighted Ljung-Box model to the standardised residuals obtained with the $MA(1)$-$GARCH(1,1)$ model (considering t-student distribution), a p-value consistently above 0.47, suggesting strong evidence that the residuals are in fact not correlated.

2. Applying the weighted Ljung-Box model to the standardized squared residuals obtained with the $MA(1)$-$GARCH(1,1)$ model considering t-student distribution), yields p-values between 0.48 and 0.90, suggesting to a high level of confidence that there is no evidence of remaining ARCH effects.

3. The statistical description of the residuals can be found in table 8. The residuals continue to be slightly negatively skewed, and the kurtosis is around 1.

4. Applying the Adjusted Pearson Goodness-of-Fit Test, we find a p-value of around 0.097. Considering $\alpha = 0.05$, there is not enough evidence to reject the null hypothesis - the obtained standardised residuals could follow a t-student distribution.

mean	std	median	min	max	skew	kurtosis
-0.04	0.99	-0.05	-3.93	4.26	-0.23	1.04

Table 8: Microsoft residuals - descriptive statistics (t-student distribution).

Besides obtaining a better goodness-of-fit than when considering a normal distribution, we also obtain a simpler GARCH model, which is in general desirable. The estimated optimal parameters are shown below:

$$r_t = 0.001 - 0.076\varepsilon_{t-1}, \varepsilon_t = \sigma_t z_t, z_t \overset{\text{iid}}{\sim} (0,1)$$
$$\sigma_t^2 = 0.000 + 0.111\varepsilon_{t-1}^2 + 0.869\sigma_{t-1}^2$$

2.9 Static and dynamics forecasts: log returns and volatiliy

In this section, the estimation and forecast samples were selected, in order to obtain both static and dynamic forecasts, for both the log returns and volatility. For each of the studied assets, the two preferred GARCH specifications were used - in this case, the ARMA-GARCH models assuming normal standardised residuals, and the (different) ARMA-GARCH models subsequently obtained when assuming t-student standardised residuals.

In this section, the results are compared by setting the forecast sample size to correspond to the last 200 samples in the 5-year dataset, and the previous data to the estimation window. The static forecasts are obtained by iteratively fitting the ARMA-GARCH model under consideration to the available data, and at each iteration obtaining the one-step-ahead forecast (using a recursive window); and the dynamic forecasts are obtained by forecasting multiple-steps-ahead, all the way to the end of the forecasting period, without taking into account the actual returns registered in the meantime.

2.9.1 Visa

Log Returns

Applying the process previously described, the static and dynamic forecast for the log returns were determined, as well as the results in comparison to the real log returns. In Visa's case, this was done for the $ARMA(3,3)$-$GARCH(2,3)$ (normal) and for the $ARMA(3,3)$-$GARCH(1,1)$ (t-student). Looking at figure 21, it is observed that the model cannot reliably predict the returns of the stock. When comparing the results for the log returns, to the results provided by the volatility forecasts, it is observed that the ARMA-GARCH model predicts volatility more accurately than it does the returns. This outcome is expected, since this class of models are specifically designed to forecast volatility, rather than the returns themselves.

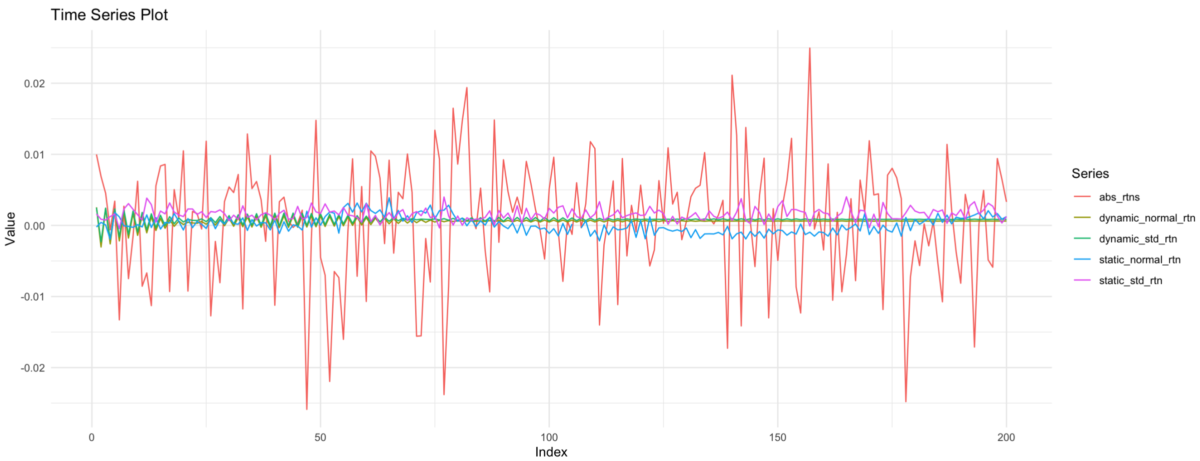


Figure 21: Visa: static and dynamic forecasts for the (log) returns, considering up to 200 steps ahead. In red, the true value of the registered returns; in dark green, the dynamic forecasts considering the normal; in bright green the dynamic forecasts considering the t-student; in light blue the static forecasts considering the normal $GARCH(2,2)$ model; and in purple the static forecasts considering the t-student $GARCH(1,1)$ model.

Volatility

Afterwards, the same process was repeated for the volatility forecasts and plotted. The obtained static and dynamic forecasts were compared to the absolute values of the log return, which is a good proxy for volatility (in fact, in the data used we have no measure of the actual volatility, which would require the availability of intra-daily data, or a more complex estimator). It was observed that for the static forecasts (for both distributions considered), the model accurately predicted the behaviour of the volatility. What can be said as well is that the volatility forecasts provided by the t-student GARCH model seen on figure 22 are smoother than the ones obtained when assuming a normal distribution. In the case of the dynamic forecast for both models, it was found that volatility increases with each forecast step.

This result is not surprising, as it is expected that the uncertainty of a dynamic forecast will increase, the farther we project into the future.

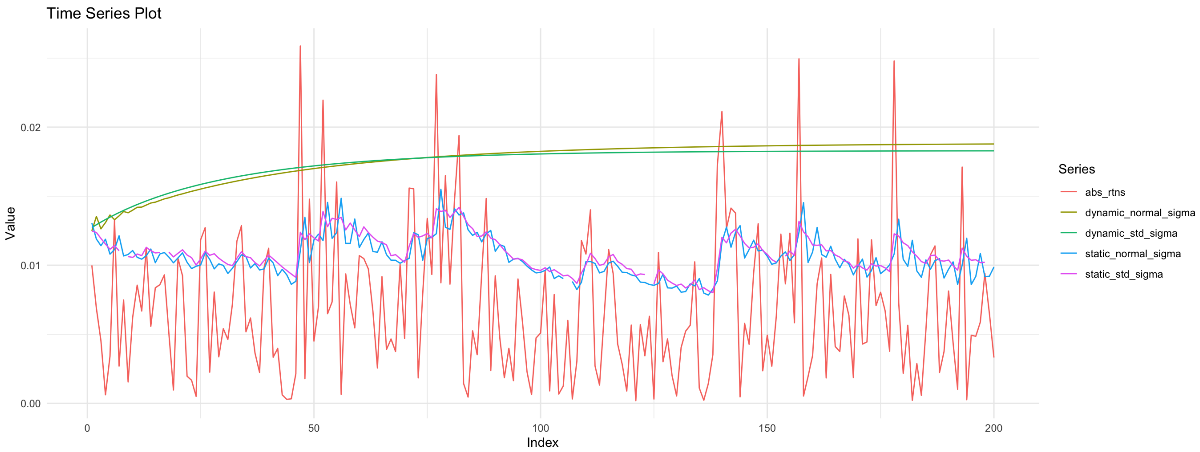


Figure 22: Visa: static and dynamic forecasts for the volatility, considering up to 200 steps ahead. In red, the absolute value of the registered returns (a proxy for the volatility); in dark green, the dynamic forecasts considering the normal; in bright green the dynamic forecasts considering the t-student; in light blue the static forecasts considering the normal $GARCH(2,2)$ model; and in purple the static forecasts considering the t-student $GARCH(1,1)$ model.

2.9.2 Microsoft

Log Returns

Applying the process previously described for the static and dynamic forecasts, the forecast log returns were determined, as well as their comparison to the real log returns. In Microsoft's case, this was done for the $MA(1)$-$GARCH(2,2)$ (normal) and for the $MA(1)$-$GARCH(1,1)$ (t-student). Looking at figure 23, it is observed that the model cannot reliably predict the returns of the stock. When comparing the results using log returns, to the results provided by the volatility forecasts, it is observed that the ARMA-GARCH model predicts volatility more accurately than it does the returns. This outcome is expected, since this class of models are specifically designed to forecast volatility, rather than the returns of a time series.

Volatility

Afterwards, the same process was repeated to the volatility and plotted. The obtained static and dynamic forecasts were compared to the absolute values of the log return (shown in 24), which is a good proxy for volatility (in fact, in the data used we have no measure of the actual volatility, which would require the availability of intra-daily data, or a more complex estimator). It was observed that for the static forecasts (for both distributions considered), the model accurately predicted the behaviour of the volatility. What can be said as well is that the volatility forecasts provided by the t-student GARCH model are smoother than the ones obtained when assuming a normal distribution. In the case of the dynamic forecast for both models, it was found that volatility increases with each forecast step. This result is not surprising, as it is expected that the uncertainty of a dynamic forecast will increase, the farther we project into the future.

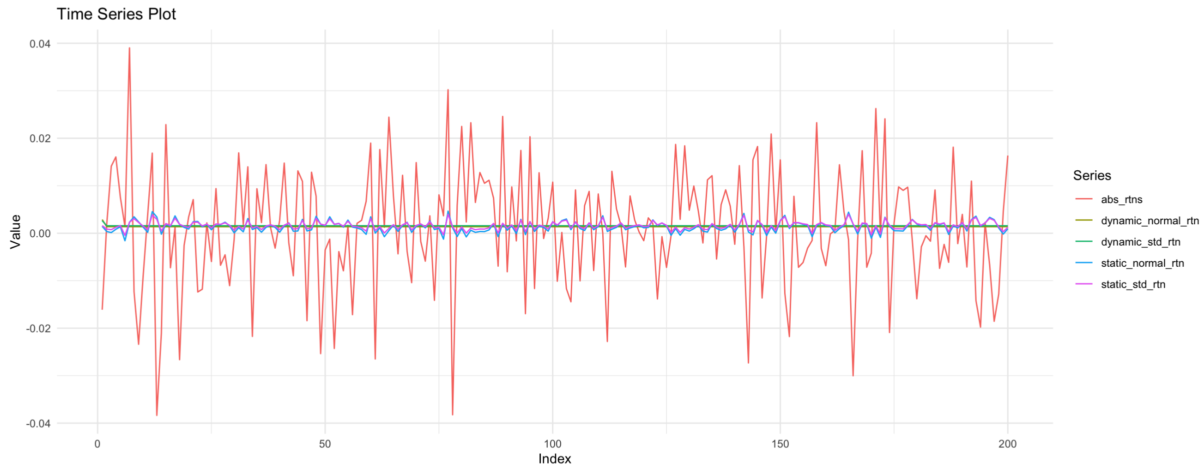


Figure 23: Microsoft: static and dynamic forecasts for the (log) returns, considering up to 200 steps ahead. In red, the true value of the registered returns; in dark green, the dynamic forecasts considering the normal; in bright green the dynamic forecasts considering the t-student; in light blue the static forecasts considering the normal $GARCH(2,2)$ model; and in purple the static forecasts considering the t-student $GARCH(1,1)$ model.

Comparing the volatility forecast for Visa and Microsoft, we observe an interesting result: Unlike Microsoft, Visa's dynamic forecast for the normal distribution exhibits oscillations for the first three time steps. This result is a consequence of the model proposed: for Visa, a $GARCH(2,3)$ model is used to capture the volatility dynamics (and considering higher lag orders), while for Microsoft a $GARCH(2,2)$.

2.10 Explicit computation of 1^{st} and 2^{nd} forecasts

In this section, we chose to focus on the time series of Microsoft's returns. After checking the dynamics of the two models under analysis, the first and second forecasts were determined for Microsoft and are given in Table 9.

Microsoft	1^{st} Forecast		2^{nd} Forecast	
	r_{t+1}	σ_{t+1}	r_{t+2}	σ_{t+2}
$MA(1)-GARCH(2,2)$ (normal distribution for $\hat{r}_t$)	0.0009283	0.0126	0.0012983	0.0140
$MA(1)-GARCH(1,1)$ (t-student distribution for $\hat{r}_t$)	0.001246	0.01347	0.001498	0.01363

Table 9: 1^{st} and 2^{nd} forecasts, for the identified models.

The results were obtained using the UGARCHFORECAST function of the RUGARCH package for a two-step ahead forecast applied to the two models previously determined (for the normal distribution and the Student's t distribution). The derivation for these results is shown below, for the case of the $MA(1)-GARCH(2,2)$ (normal distribution). Recalling the equation for this model, we have:

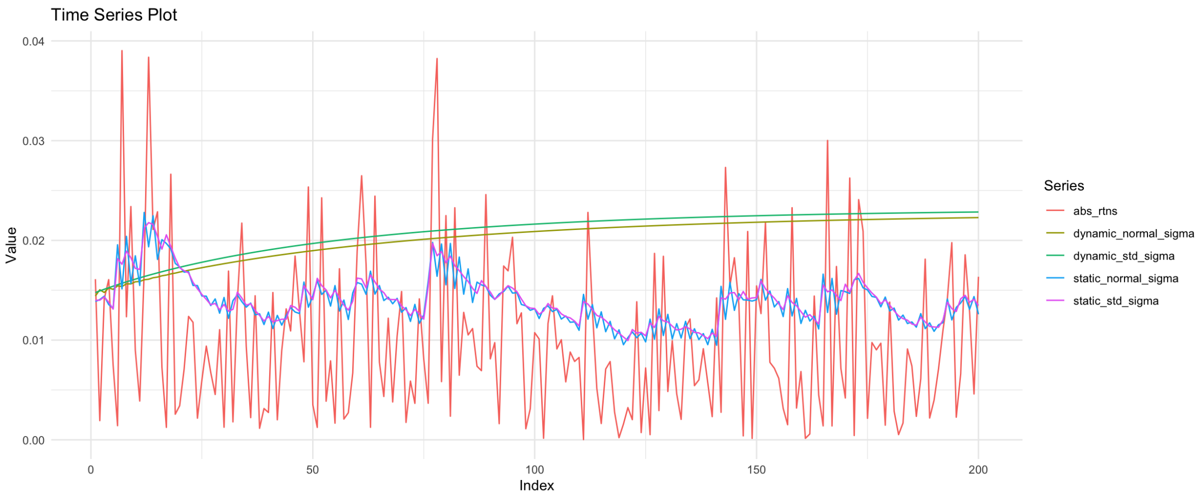


Figure 24: Microsoft: static and dynamic forecasts for the volatility, considering up to 200 steps ahead. In red, the absolute value of the registered returns (a proxy for the volatility); in dark green, the dynamic forecasts considering the normal; in bright green the dynamic forecasts considering the t-student; in light blue the static forecasts considering the normal $GARCH(2,2)$ model; and in purple the static forecasts considering the t-student $GARCH(1,1)$ model.

$$r_t = 0.001 - 0.076\varepsilon_{t-1}, \varepsilon_t = \sigma_t z_t, z_t \overset{\text{iid}}{\sim} (0,1)$$
$$\sigma_t^2 = 0.000 + 0.111\varepsilon_{t-1}^2 + 0.869\sigma_{t-1}^2$$

The optimal 1-step ahead forecast for returns and for volatility are given, respectively, by:

$$\hat{r}_{T+1|T} = E[r_{T+1}|I_T] = E[0.001|I_T] - E[0.097\varepsilon_T|I_T] = 0.001 - 0.097\varepsilon_T$$
$$\hat{\sigma}_{T+1|T}^2 = E[\sigma_{T+1}^2|I_T] = 0.000 + E[0.141\varepsilon_T^2|I_T] + E[0.046\varepsilon_{T-1}^2] + E[0.000\sigma_T^2|I_T] + E[0.781\sigma_{T-1}^2|I_T]$$
$$= 0.000 + 0.141\varepsilon_T^2 + 0.046\varepsilon_{T-1}^2 + 0.000\sigma_T^2 + 0.781\sigma_{T-1}^2$$

The values for ε_T and ε_{T-1} can be obtained directly form the residuals of the model, and the fitted sigma from the output of UGARCHFIT function. Plugging them into the expresssion above (where we use the full optimal coefficients provided by the UGARCHFIT, to avoid loss of numerical precision), we obtain: $\varepsilon_T = 0.003823586$, $\varepsilon_{T-1} = 0.0152213$, $\sigma_T = 0.01408004$, and $\sigma_{T-1} = 0.01379287$.

$$\hat{r}_{T+1|T} = 0.001298 - 0.096757 \times 0.003823586 = 0.00928$$
$$\hat{\sigma}_{T+1|T}^2 = 0.000012 + 0.141080 \times 0.003823586^2 + 0.046458 \times 0.0152213^2 +$$
$$+ 0.000001 \times 0.01408004^2 + 0.780653 \times 0.01379287 = 0.013$$

The optimal 2-step ahead forecast for returns and for volatility is given analogously, but using the optimal 1-step ahead forecast when the information is not available (i.e., when it is necessary to use terms indexed to $T+h, h>0$), and recalling that $\sigma_{T+j|T} = \varepsilon_{T+j}^2$, for $j \geq 0$:

$$
\begin{aligned}
\hat{r}_{T+2|T} &= E[r_{T+2}|I_T] = E[0.001298|I_T] - E[0.097\varepsilon_{T+1}|I_T] = 0.001298 - 0 \\
\hat{\sigma}^2_{T+2|T} &= E[\sigma^2_{T+2}|I_T] = 0.000012 + E[0.141080\varepsilon^2_{T+1}|I_T] + E[0.046458\varepsilon^2_T] + \\
&+ E[0.000001\sigma^2_{T+1}|I_T] + E[0.781\sigma^2_T|I_T] \\
&= 0.000012 + 0.141080\hat{\sigma}^2_{T+1} + 0.046458\varepsilon^2_T + 0.000001\sigma^2_{T+1|T} + 0.781\sigma^2_T \\
&= 0.000012 + 0.141080 \times 0.0126^2 + 0.046458 \times 0.003823586^2 + \\
&+ 0.000001 \times 0.0126^2 + 0.781 \times 0.01408004^2 = 0.014
\end{aligned}
$$

To derive the detailed computations here explictly given for the case of the $MA(1) - GARCH(1,1)$ (t-student distribution), the computations would be exactly the same as above, but simplified (i.e., the GARCH part would have two terms less), and the coefficients should follow the expression below (recovered from 2.8):

$$
\begin{aligned}
&r_t = 0.001 - 0.097\varepsilon_{t-1}, \varepsilon_t = \sigma_t z_t, z_t \overset{\text{iid}}{\sim} \mathcal{N}(0,\,1) \\
&\sigma^2_t = 0.000 + 0.141\varepsilon^2_{t-1} + 0.046\varepsilon^2_{t-2} + 0.000\sigma^2_{t-1} + 0.781\sigma^2_{t-2}
\end{aligned}
$$

2.11 Leverage Effects and Risk Premia

2.11.1 Evidence of nonzero log returns on average

This question is trivial. When taking the expression for the returns, as modelled by the ARMA-GARCH processes, no matter what the distribution assumed for the standardised residuals, there is a nonzero component in the model. For an ARMA process, this is what we would habitually term ϕ_0. For example, for the process used to model the returns of Microsoft, we get $\phi_0 = 0.001$ (in both the normal and t-student case). For Visa, the value is slightly smaller, but also non-zero. Were the returns to be, on average, zero, it would be hard to justify how the stock prices of Visa doubled and Microsoft quadrupled in the 5-year period considered.

2.11.2 Leverage Effects

The idea behind the leverage effect is that negative price movements (negative shocks) have a higher impact on the volatility than positive surprises (positive shocks). In other words, an unusually large negative price movement tends to increase volatility more than the converse positive movement. An sociological explanation is that, when a company's valuation declines (caused by a declining stock price), it's leverage ratio worsens, and therefore leads management to take riskier decisions, which in turn increases volatility.

A simple way to check the presence of this effect is to look at $Corr(r_{t-1}, r_t^2)$.

If $Corr(r_{t-1}, r_t^2) < 0$ (and the more negative this value becomes), then there is evidence of leverage effects.

Computing this value for Microsoft, we obtained $Corr(r_{t-1}, r_t^2) = -0.0274095$. For Visa, we obtained $Corr(r_{t-1}, r_t^2) = -0.06104434$. In both cases, there is strong evidence of leverage effects.

2.11.3 Risk premia

The idea behind the risk premium is that, in general, financial returns should reflect the additional risk undertaken by an investor, when compared with the risk-free return rate. In other words, more volatile assets should have higher expected returns; and from the perspective of a single asset, more volatile periods should present higher returns as well. However, the standard GARCH models do not take this relationship into account, as there are no explanatory variables in the mean equation besides the past returns and shocks. This corresponds to the GARCH-M extension, which can be written, in the general form, as:

$$r_t = \mu_t + \delta g\left(\sigma_t^2\right) + \varepsilon_t, \varepsilon_t = \sigma_t z_t, \quad z_t \overset{\text{iid}}{\sim} (0,1)$$
$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{n} \beta_j \sigma_{t-j}^2$$

In this expression, $g(\sigma_t^2)$ can be selected by the practitioner, but should be a monotonically increasing function of σ_t^2. We will use $g(\sigma_t^2) = \sqrt{\sigma_t^2}$. Assuming the same ARMA-GARCH bases as before, we fitted this new model to Microsoft and to Visa, setting the input ARCHM=TRUE. We obtained surprising values. Microsoft's fit resulted in a value of $\delta = -0.018323$. This value is negative, which surprisingly suggests that there is no evidence of a risk premium for this particular stock. For Visa, a similar conclusion is reached, with a $\delta = -0.034465$, also negative.

3 Value at Risk (VaR) Analysis

3.1 VaR calculations

3.1.1 (i) Constant Model

The only model that can be formed using just unconditional moments is the constant model

$$R_t = \epsilon_t, \epsilon_t \overset{\text{w.n.}}{\sim} N(\mu, \sigma^2),$$

with $\hat{\mu} = \bar{R}_t, \hat{\sigma^2} = Var(R_t)$, for $t \in [1, T]$. Even for a random walk model both the standard deviation of an observation from the preceding one and a last observation would be needed. Both are not given.

To calculate the VaR of each stock, the historical mean and standard deviation (sd) will be estimated and the respective z value of a standard distribution will be scaled by the sd and shifted by the mean. Since the likelihood of the z value can be read off from the normal distribution, the z value corresponding to the likelihood of interest can be chosen.

The scaled and shifted z value corresponds to a maximal log return under the given probability. The typically small log returns of stocks are proportional to percent changes in the stock value. True to this, the maximal monetary value of currency that is at risk can be calculated by multiplying the result of the previous calculation with the value of bought stock.

3.1.1.1 Visa Caclulations.

Calculation for 90% Confidence and $T + 1$:

$\text{Quantile}_{\text{norm}}(0.1) = -1.28, \mu_{\hat{T+1|T}} = 0.0004, \sigma_{\hat{T+1|T}} = 0.0176$

$VaR(\text{log returns})_{10\%,T+1} = 0.0004 + 0.0176 \times -1.28 = -0.0221$

$VaR_{10\%,T+1} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0221 = -220709.87€$

Calculation for 95% Confidence and $T+1$:
Quantile$_{\text{norm}}(0.05) = -1.64, \hat{\mu_{T+1|T}} = 0.0004, \hat{\sigma_{T+1|T}} = 0.0176$
$VaR(\log \text{returns})_{5\%,T+1} = 0.0004 + 0.0176 \times -1.64 = -0.0284$
$VaR_{5\%,T+1} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0284 = -284484.36€$

Calculation for 99% Confidence and $T+1$:
Quantile$_{\text{norm}}(0.01) = -2.33, \hat{\mu_{T+1|T}} = 0.0004, \hat{\sigma_{T+1|T}} = 0.0176$
$VaR(\log \text{returns})_{1\%,T+1} = 0.0004 + 0.0176 \times -2.33 = -0.0404$
$VaR_{1\%,T+1} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0404 = -404114.71€$

Following the modeling assumptions of this subtask, the available stock data and the given sum invested in the Visa stock, the loss incurring over one day will be smaller than -220709.87€, -284484.36€ and -404114.71€ with a confidence level of 90%, 95% and 99%.

The due to simplicity of the model, the VaR further into the future is the same as the 1-day VaR.

3.1.1.2 Microsoft

Caclulations.

Calculation for 90% Confidence and $T+1$:
Quantile$_{\text{norm}}(0.1) = -1.28, \hat{\mu_{T+1|T}} = 0.0009, \hat{\sigma_{T+1|T}} = 0.0192$
$VaR(\log \text{returns})_{10\%,T+1} = 0.0009 + 0.0192 \times -1.28 = -0.0236$
$VaR_{10\%,T+1} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0236 = -236303.82€$

Calculation for 95% Confidence and $T+1$:
Quantile$_{\text{norm}}(0.05) = -1.64, \hat{\mu_{T+1|T}} = 0.0009, \hat{\sigma_{T+1|T}} = 0.0192$
$VaR(\log \text{returns})_{5\%,T+1} = 0.0009 + 0.0192 \times -1.64 = -0.0306$
$VaR_{5\%,T+1} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0306 = -305890.67€$

Calculation for 99% Confidence and $T+1$:
Quantile$_{\text{norm}}(0.01) = -2.33, \hat{\mu_{T+1|T}} = 0.0009, \hat{\sigma_{T+1|T}} = 0.0192$
$VaR(\log \text{returns})_{1\%,T+1} = 0.0009 + 0.0192 \times -2.33 = -0.0436$
$VaR_{1\%,T+1} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0436 = -436424.02€$

Following the modeling assumptions of this subtask, the available stock data and the given sum invested in the Microsoft stock, the loss incurring over one day will be smaller than -236303.82€, -305890.67€ and -436424.02€ with a confidence level of 90%, 95% and 99%.

The due to simplicity of the model, the VaR further into the future is the same as the 1-day VaR.

3.1.2 (ii) AR(0)-GARCH(1,1)

The following GARCH model will be fitted

$$\begin{cases} R_t = \mu_t + \epsilon_t, \epsilon_t = \sigma_t * z_t, z_t \overset{i.i.d.}{\sim} N(0,1) \\ \sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \\ \mu_t = \mu_0 \end{cases}$$

The model fitted in this subtask is more sophisticated than a simple model using only unconditional constants. The model will be specified and fitted to the historical data. The fitted model can then be used to make predictions about the stock returns in the future. The predicted mean and sd of the returns can be used to scale and shift a z value like in the previous subtask. The use of this scaled and shifted z value in the further calculation of the VaR is identical to the previous, current and the later subtasks. Only the one- and five-days ahead VaR will be reported here, the two- to four-days ahead VaR can be seen in A.

3.1.2.1 Visa Caclulations.

Calculation for 90% Confidence and $T+1$:
$\text{Quantile}_{\text{norm}}(0.1) = -1.28, \hat{\mu_{T+1|T}} = 0.0006, \hat{\sigma_{T+1|T}} = 0.0094$
$VaR(\log \text{returns})_{10\%,T+1} = 0.0006 + 0.0094 \times -1.28 = -0.0114$
$VaR_{10\%,T+1} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0114 = -114043.89€$

Calculation for 95% Confidence and $T+1$:
$\text{Quantile}_{\text{norm}}(0.05) = -1.64, \hat{\mu_{T+1|T}} = 0.0006, \hat{\sigma_{T+1|T}} = 0.0094$
$VaR(\log \text{returns})_{5\%,T+1} = 0.0006 + 0.0094 \times -1.64 = -0.0148$
$VaR_{5\%,T+1} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0148 = -148128.33€$

Calculation for 99% Confidence and $T+1$:
$\text{Quantile}_{\text{norm}}(0.01) = -2.33, \hat{\mu_{T+1|T}} = 0.0006, \hat{\sigma_{T+1|T}} = 0.0094$
$VaR(\log \text{returns})_{1\%,T+1} = 0.0006 + 0.0094 \times -2.33 = -0.0212$
$VaR_{1\%,T+1} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0212 = -212065.07€$

Calculation for 90% Confidence and $T+5$:
$\text{Quantile}_{\text{norm}}(0.1) = -1.28, \hat{\mu_{T+5|T}} = 0.0006, \hat{\sigma_{T+5|T}} = 0.0101$
$VaR(\log \text{returns})_{10\%,T+5} = 0.0006 + 0.0101 \times -1.28 = -0.0123$
$VaR_{10\%,T+5} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0123 = -122764.61€$

Calculation for 95% Confidence and $T+5$:
$\text{Quantile}_{\text{norm}}(0.05) = -1.64, \hat{\mu_{T+5|T}} = 0.0006, \hat{\sigma_{T+5|T}} = 0.0101$
$VaR(\log \text{returns})_{5\%,T+5} = 0.0006 + 0.0101 \times -1.64 = -0.0159$
$VaR_{5\%,T+5} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0159 = -159321.26€$

Calculation for 99% Confidence and $T+5$:
$\text{Quantile}_{\text{norm}}(0.01) = -2.33, \hat{\mu_{T+5|T}} = 0.0006, \hat{\sigma_{T+5|T}} = 0.0101$
$VaR(\log \text{returns})_{1\%,T+5} = 0.0006 + 0.0101 \times -2.33 = -0.0228$

$VaR_{1\%,T+5} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.0228 = -227895.44€$

Following the modeling assumptions of this subtask, the available stock data and the given sum invested in the Visa stock, the loss incurring over one day will be smaller than -114043.89€, -148128.33€ and -212065.07€ with a confidence level of 90%, 95% and 99%.

The loss after 5 days will be smaller than $-122764.61€$, $-159321.26€$ and $-227895.44€$ with a confidence level of 90%, 95% and 99%.

3.1.2.2 Microsoft Caclulations.

Calculation for 90% Confidence and $T+1$:
$\text{Quantile}_{\text{norm}}(0.1) = -1.28, \hat{\mu_{T+1|T}} = 0.0013, \hat{\sigma_{T+1|T}} = 0.013$
$VaR(\log \text{returns})_{10\%,T+1} = 0.0013 + 0.013 \times -1.28 = -0.0154$
$VaR_{10\%,T+1} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.0154 = -154418.3€$

Calculation for 95% Confidence and $T+1$:
$\text{Quantile}_{\text{norm}}(0.05) = -1.64, \hat{\mu_{T+1|T}} = 0.0013, \hat{\sigma_{T+1|T}} = 0.013$
$VaR(\log \text{returns})_{5\%,T+1} = 0.0013 + 0.013 \times -1.64 = -0.0202$
$VaR_{5\%,T+1} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.0202 = -201792.7€$

Calculation for 99% Confidence and $T+1$:
$\text{Quantile}_{\text{norm}}(0.01) = -2.33, \hat{\mu_{T+1|T}} = 0.0013, \hat{\sigma_{T+1|T}} = 0.013$
$VaR(\log \text{returns})_{1\%,T+1} = 0.0013 + 0.013 \times -2.33 = -0.0291$
$VaR_{1\%,T+1} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.0291 = -290659.18€$

Calculation for 90% Confidence and $T+5$:
$\text{Quantile}_{\text{norm}}(0.1) = -1.28, \hat{\mu_{T+5|T}} = 0.0013, \hat{\sigma_{T+5|T}} = 0.0136$
$VaR(\log \text{returns})_{10\%,T+5} = 0.0013 + 0.0136 \times -1.28 = -0.0161$
$VaR_{10\%,T+5} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.0161 = -161413.74€$

Calculation for 95% Confidence and $T+5$:
$\text{Quantile}_{\text{norm}}(0.05) = -1.64, \hat{\mu_{T+5|T}} = 0.0013, \hat{\sigma_{T+5|T}} = 0.0136$
$VaR(\log \text{returns})_{5\%,T+5} = 0.0013 + 0.0136 \times -1.64 = -0.0211$
$VaR_{5\%,T+5} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.0211 = -210771.25€$

Calculation for 99% Confidence and $T+5$:
$\text{Quantile}_{\text{norm}}(0.01) = -2.33, \hat{\mu_{T+5|T}} = 0.0013, \hat{\sigma_{T+5|T}} = 0.0136$
$VaR(\log \text{returns})_{1\%,T+5} = 0.0013 + 0.0136 \times -2.33 = -0.0303$
$VaR_{1\%,T+5} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.0303 = -303357.72€$

Following the modeling assumptions of this subtask, the available stock data and the given sum invested in the Microsoft stock, the loss incurring over one day will be smaller than -154418.30€, -201792.70€ and -290659.18€ with a confidence level of 90%, 95% and 99%.

The loss after 5 days will be smaller than -161413.74€, -210771.25€ and -303357.72€ with a confidence level of 90%, 95% and 99%.

Based on this model, the VaR is smaller for investments into Visa than for investments into Microsoft.

The all VaR obtained by this model are smaller than the VaR obtained by the simplistic model in the first subtask. That is due to the small time window, for which the time dependent variance and the VaR is computed. In this case, the time dependent variance is usually smaller than the historic variance.

3.1.3 (iii) AR(1)-GARCH(1,1)

The following GARCH model will be fitted in this subtask

$$\begin{cases} R_t = \mu_t + \epsilon_t, \epsilon_t = \sigma_t * z_t, z_t \overset{i.i.d.}{\sim} N(0,1) \\ \sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \\ \mu_t = \mu_0 + \phi_i R_{t-1} \end{cases}$$

here μ_t is not only modeled as a constant but as a function of the past μ_{t-1}

3.1.3.1 Visa Caclulations.

Calculation for 90% Confidence and $T + 1$:
$\text{Quantile}_{\text{norm}}(0.1) = -1.28, \mu_{T\hat{+}1|T} = 0.0005, \sigma_{T\hat{+}1|T} = 0.0094$
$VaR(\log \text{returns})_{10\%,T+1} = 0.0005 + 0.0094 \times -1.28 = -0.0115$
$VaR_{10\%,T+1} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0115 = -114877.12€$

Calculation for 95% Confidence and $T + 1$:
$\text{Quantile}_{\text{norm}}(0.05) = -1.64, \mu_{T\hat{+}1|T} = 0.0005, \sigma_{T\hat{+}1|T} = 0.0094$
$VaR(\log \text{returns})_{5\%,T+1} = 0.0005 + 0.0094 \times -1.64 = -0.0149$
$VaR_{5\%,T+1} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0149 = -148974.66€$

Calculation for 99% Confidence and $T + 1$:
$\text{Quantile}_{\text{norm}}(0.01) = -2.33, \mu_{T\hat{+}1|T} = 0.0005, \sigma_{T\hat{+}1|T} = 0.0094$
$VaR(\log \text{returns})_{1\%,T+1} = 0.0005 + 0.0094 \times -2.33 = -0.0213$
$VaR_{1\%,T+1} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0213 = -212935.98€$

Calculation for 90% Confidence and $T + 5$:
$\text{Quantile}_{\text{norm}}(0.1) = -1.28, \mu_{T\hat{+}5|T} = 0.0006, \sigma_{T\hat{+}5|T} = 0.0101$
$VaR(\log \text{returns})_{10\%,T+5} = 0.0006 + 0.0101 \times -1.28 = -0.0123$
$VaR_{10\%,T+5} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0123 = -122596.95€$

Calculation for 95% Confidence and $T + 5$:
$\text{Quantile}_{\text{norm}}(0.05) = -1.64, \mu_{T\hat{+}5|T} = 0.0006, \sigma_{T\hat{+}5|T} = 0.0101$
$VaR(\log \text{returns})_{5\%,T+5} = 0.0006 + 0.0101 \times -1.64 = -0.0159$
$VaR_{5\%,T+5} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0159 = -159133.89€$

Calculation for 99% Confidence and $T + 5$:
$\text{Quantile}_{\text{norm}}(0.01) = -2.33, \mu_{T\hat{+}5|T} = 0.0006, \sigma_{T\hat{+}5|T} = 0.0101$

$VaR(\log \text{returns})_{1\%,T+5} = 0.0006 + 0.0101 \times -2.33 = -0.0228$
$VaR_{1\%,T+5} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0228 = -227671.12€$

Following the modeling assumptions of this subtask, the available stock data and the given sum invested in the Visa stock, the loss incurring over one day will be smaller than -114877.12€, -148974.66€ and -212935.98€ with a confidence level of 90%, 95% and 99%.

The loss after 5 days will be smaller than -122596.95€, -159133.89€ and -227671.12€ with a confidence level of 90%, 95% and 99%.

3.1.3.2 Microsoft Caclulations.

Calculation for 90% Confidence and $T+1$:
$\text{Quantile}_{\text{norm}}(0.1) = -1.28, \hat{\mu_{T+1|T}} = 0.0011, \hat{\sigma_{T+1|T}} = 0.0133$
$VaR(\log \text{returns})_{10\%,T+1} = 0.0011 + 0.0133 \times -1.28 = -0.0159$
$VaR_{10\%,T+1} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0159 = -159257.48€$

Calculation for 95% Confidence and $T+1$:
$\text{Quantile}_{\text{norm}}(0.05) = -1.64, \hat{\mu_{T+1|T}} = 0.0011, \hat{\sigma_{T+1|T}} = 0.0133$
$VaR(\log \text{returns})_{5\%,T+1} = 0.0011 + 0.0133 \times -1.64 = -0.0207$
$VaR_{5\%,T+1} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0207 = -207448.05€$

Calculation for 99% Confidence and $T+1$:
$\text{Quantile}_{\text{norm}}(0.01) = -2.33, \hat{\mu_{T+1|T}} = 0.0011, \hat{\sigma_{T+1|T}} = 0.0133$
$VaR(\log \text{returns})_{1\%,T+1} = 0.0011 + 0.0133 \times -2.33 = -0.0298$
$VaR_{1\%,T+1} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0298 = -297845.55€$

Calculation for 90% Confidence and $T+5$:
$\text{Quantile}_{\text{norm}}(0.1) = -1.28, \hat{\mu_{T+5|T}} = 0.0013, \hat{\sigma_{T+5|T}} = 0.0138$
$VaR(\log \text{returns})_{10\%,T+5} = 0.0013 + 0.0138 \times -1.28 = -0.0164$
$VaR_{10\%,T+5} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0164 = -163731.38€$

Calculation for 95% Confidence and $T+5$:
$\text{Quantile}_{\text{norm}}(0.05) = -1.64, \hat{\mu_{T+5|T}} = 0.0013, \hat{\sigma_{T+5|T}} = 0.0138$
$VaR(\log \text{returns})_{5\%,T+5} = 0.0013 + 0.0138 \times -1.64 = -0.0214$
$VaR_{5\%,T+5} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0214 = -213758.49€$

Calculation for 99% Confidence and $T+5$:
$\text{Quantile}_{\text{norm}}(0.01) = -2.33, \hat{\mu_{T+5|T}} = 0.0013, \hat{\sigma_{T+5|T}} = 0.0138$
$VaR(\log \text{returns})_{1\%,T+5} = 0.0013 + 0.0138 \times -2.33 = -0.0308$
$VaR_{1\%,T+5} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0308 = -307601.04€$

Following the modeling assumptions of this subtask, the available stock data and the given sum invested in the Microsoft stock, the loss incurring over one day will be smaller than -154418.30€, -201792.70€ and -290659.18€ with a confidence level of 90%, 95% and 99%.

The loss after 5 days will be smaller than -163731.38€, -213758.49€ and -307601.04€ with a confidence level of 90%, 95% and 99%.

The estimated VaR don't really differ between the previous model which modeled the mean as constant, and the current one, which models the mean of the time series using an AR(1). The reason for this indifference could be, that the conditional mean of stock returns cannot really be predicted from preceding returns. It is already a good aproximation to model this conditional mean as constant. This is consitent with financial theory.

3.1.4 (iv) AR(1)-GARCH(1,1) with t-Student innovations

In this subtask, innovations of R_t will not be modeled after a Normal distribution but a t-Student distibution (std)

3.1.4.1 Visa Caclulations.

Calculation for 90% Confidence and $T+1$:
$\text{Quantile}_{\text{std}(5)}(0.1) = -1.46, \mu_{T\hat{+}1|T} = 0.0009, \sigma_{T\hat{+}1|T} = 0.0096$
$VaR(\log \text{returns})_{10\%,T+1} = 0.0009 + 0.0096 \times -1.46 = -0.0131$
$VaR_{10\%,T+1} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€\times -0.0131 = -130610.68€$

Calculation for 95% Confidence and $T+1$:
$\text{Quantile}_{\text{std}(5)}(0.05) = -1.99, \mu_{T\hat{+}1|T} = 0.0009, \sigma_{T\hat{+}1|T} = 0.0096$
$VaR(\log \text{returns})_{5\%,T+1} = 0.0009 + 0.0096 \times -1.99 = -0.0181$
$VaR_{5\%,T+1} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€\times -0.0181 = -180885.55€$

Calculation for 99% Confidence and $T+1$:
$\text{Quantile}_{\text{std}(5)}(0.01) = -3.27, \mu_{T\hat{+}1|T} = 0.0009, \sigma_{T\hat{+}1|T} = 0.0096$
$VaR(\log \text{returns})_{1\%,T+1} = 0.0009 + 0.0096 \times -3.27 = -0.0304$
$VaR_{1\%,T+1} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€\times -0.0304 = -304332.17€$

Calculation for 90% Confidence and $T+5$:
$\text{Quantile}_{\text{std}(5)}(0.1) = -1.46, \mu_{T\hat{+}5|T} = 0.001, \sigma_{T\hat{+}5|T} = 0.0103$
$VaR(\log \text{returns})_{10\%,T+5} = 0.001 + 0.0103 \times -1.46 = -0.0141$
$VaR_{10\%,T+5} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€\times -0.0141 = -140990.25€$

Calculation for 95% Confidence and $T+5$:
$\text{Quantile}_{\text{std}(5)}(0.05) = -1.99, \mu_{T\hat{+}5|T} = 0.001, \sigma_{T\hat{+}5|T} = 0.0103$
$VaR(\log \text{returns})_{5\%,T+5} = 0.001 + 0.0103 \times -1.99 = -0.0195$
$VaR_{5\%,T+5} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€\times -0.0195 = -195229.37€$

Calculation for 99% Confidence and $T+5$:
$\text{Quantile}_{\text{std}(5)}(0.01) = -3.27, \mu_{T\hat{+}5|T} = 0.001, \sigma_{T\hat{+}5|T} = 0.0103$
$VaR(\log \text{returns})_{1\%,T+5} = 0.001 + 0.0103 \times -3.27 = -0.0328$
$VaR_{1\%,T+5} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€\times -0.0328 = -328409.98€$

Following the modeling assumptions of this subtask, the available stock data and the given sum invested in the Visa stock, the loss incurring over one day will be smaller than -130610.68€, -180885.55€ and -304332.17€ with a confidence level of 90%, 95% and 99%.

The loss after 5 days will be smaller than -140990.25€, -195229.37€ and -328409.98€ with a confidence level of 90%, 95% and 99%.

3.1.4.2 Microsoft Caclulations.

Calculation for 90% Confidence and $T+1$:

$\text{Quantile}_{\text{std}(7)}(0.1) = -1.41, \hat{\mu_{T+1|T}} = 0.0014, \hat{\sigma_{T+1|T}} = 0.0134$

$VaR(\log \text{returns})_{10\%,T+1} = 0.0014 + 0.0134 \times -1.41 = -0.0176$

$VaR_{10\%,T+1} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0176 = -176045.17€$

Calculation for 95% Confidence and $T+1$:

$\text{Quantile}_{\text{std}(7)}(0.05) = -1.88, \hat{\mu_{T+1|T}} = 0.0014, \hat{\sigma_{T+1|T}} = 0.0134$

$VaR(\log \text{returns})_{5\%,T+1} = 0.0014 + 0.0134 \times -1.88 = -0.024$

$VaR_{5\%,T+1} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.024 = -239879.38€$

Calculation for 99% Confidence and $T+1$:

$\text{Quantile}_{\text{std}(7)}(0.01) = -2.97, \hat{\mu_{T+1|T}} = 0.0014, \hat{\sigma_{T+1|T}} = 0.0134$

$VaR(\log \text{returns})_{1\%,T+1} = 0.0014 + 0.0134 \times -2.97 = -0.0386$

$VaR_{1\%,T+1} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0386 = -385621.58€$

Calculation for 90% Confidence and $T+5$:

$\text{Quantile}_{\text{std}(7)}(0.1) = -1.41, \hat{\mu_{T+5|T}} = 0.0015, \hat{\sigma_{T+5|T}} = 0.0141$

$VaR(\log \text{returns})_{10\%,T+5} = 0.0015 + 0.0141 \times -1.41 = -0.0183$

$VaR_{10\%,T+5} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0183 = -183242.66€$

Calculation for 95% Confidence and $T+5$:

$\text{Quantile}_{\text{std}(7)}(0.05) = -1.88, \hat{\mu_{T+5|T}} = 0.0015, \hat{\sigma_{T+5|T}} = 0.0141$

$VaR(\log \text{returns})_{5\%,T+5} = 0.0015 + 0.0141 \times -1.88 = -0.025$

$VaR_{5\%,T+5} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.025 = -249994.08€$

Calculation for 99% Confidence and $T+5$:

$\text{Quantile}_{\text{std}(7)}(0.01) = -2.97, \hat{\mu_{T+5|T}} = 0.0015, \hat{\sigma_{T+5|T}} = 0.0141$

$VaR(\log \text{returns})_{1\%,T+5} = 0.0015 + 0.0141 \times -2.97 = -0.0402$

$VaR_{1\%,T+5} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0402 = -402396.65€$

Following the modeling assumptions of this subtask, the available stock data and the given sum invested in the Microsoft stock, the loss incurring over one day will be smaller than -176045.17€, -239879.38€ and -385621.58€ with a confidence level of 90%, 95% and 99%.

The loss after 5 days will be smaller than -183242.66€, -249994.08€ and -402396.65€ with a confidence level of 90%, 95% and 99%.

The VaR obtained by the current models using t-stundent innovations are much larger than the ones obtained by the similar models using normaly distributed innovations. This is caused by the shift of likelohood towards the tails in t-student distributions. This can reflect an increased likelihood of extreme events like negative price shocks.

3.1.5 (v) Other ARMA-GARCH specification

3.1.5.1 Visa An AR(0)-GARCH(2,3) model with normal innovations is used to obtain VaR values in this subtask. The model was chosen based on the problem statement and the Box-Jenkins-Methodology

Calculation for 90% Confidence and $T+1$:

$\text{Quantile}_{\text{std}(5)}(0.1) = -1.46, \hat{\mu_{T+1|T}} = 0.001, \hat{\sigma_{T+1|T}} = 0.0096$
$VaR(\text{log returns})_{10\%,T+1} = 0.001 + 0.0096 \times -1.46 = -0.0131$
$VaR_{10\%,T+1} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.0131 = -131014.75€$

Calculation for 95% Confidence and $T+1$:

$\text{Quantile}_{\text{std}(5)}(0.05) = -1.98, \hat{\mu_{T+1|T}} = 0.001, \hat{\sigma_{T+1|T}} = 0.0096$
$VaR(\text{log returns})_{5\%,T+1} = 0.001 + 0.0096 \times -1.98 = -0.0182$
$VaR_{5\%,T+1} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.0182 = -181531.61€$

Calculation for 99% Confidence and $T+1$:

$\text{Quantile}_{\text{std}(5)}(0.01) = -3.27, \hat{\mu_{T+1|T}} = 0.001, \hat{\sigma_{T+1|T}} = 0.0096$
$VaR(\text{log returns})_{1\%,T+1} = 0.001 + 0.0096 \times -3.27 = -0.0305$
$VaR_{1\%,T+1} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.0305 = -305375.38€$

Calculation for 90% Confidence and $T+5$:

$\text{Quantile}_{\text{std}(5)}(0.1) = -1.46, \hat{\mu_{T+5|T}} = 0.001, \hat{\sigma_{T+5|T}} = 0.0103$
$VaR(\text{log returns})_{10\%,T+5} = 0.001 + 0.0103 \times -1.46 = -0.014$
$VaR_{10\%,T+5} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.014 = -140149.55€$

Calculation for 95% Confidence and $T+5$:

$\text{Quantile}_{\text{std}(5)}(0.05) = -1.98, \hat{\mu_{T+5|T}} = 0.001, \hat{\sigma_{T+5|T}} = 0.0103$
$VaR(\text{log returns})_{5\%,T+5} = 0.001 + 0.0103 \times -1.98 = -0.0194$
$VaR_{5\%,T+5} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.0194 = -193941.91€$

Calculation for 99% Confidence and $T+5$:

$\text{Quantile}_{\text{std}(5)}(0.01) = -3.27, \hat{\mu_{T+5|T}} = 0.001, \hat{\sigma_{T+5|T}} = 0.0103$
$VaR(\text{log returns})_{1\%,T+5} = 0.001 + 0.0103 \times -3.27 = -0.0326$
$VaR_{1\%,T+5} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.0326 = -325815.67€$

Following the modeling assumptions of this subtask, the available stock data and the given sum invested in the Visa stock, the loss incurring over one day will be smaller than -131014.75€, -181531.61€ and -305375.38€ with a confidence level of 90%, 95% and 99%.

The loss after 5 days will be smaller than -140149.55€, -193941.91€ and -325815.67€ with a confidence level of 90%, 95% and 99%.

3.1.5.2 Microsoft An AR(0)-GARCH(2,2) model with normal innovations is used to obtain VaR values in this subtask. The model was chosen based on the problem statement and the Box-Jenkins-Methodology

Calculation for 90% Confidence and $T+1$:

$\text{Quantile}_{\text{std}(7)}(0.1) = -1.41, \hat{\mu_{T+1|T}} = 0.0015, \hat{\sigma_{T+1|T}} = 0.0132$

$VaR(\log\text{returns})_{10\%,T+1} = 0.0015 + 0.0132 \times -1.41 = -0.0171$

$VaR_{10\%,T+1} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0171 = -171297.95€$

Calculation for 95% Confidence and $T+1$:

$\text{Quantile}_{\text{std}(7)}(0.05) = -1.89, \hat{\mu_{T+1|T}} = 0.0015, \hat{\sigma_{T+1|T}} = 0.0132$

$VaR(\log\text{returns})_{5\%,T+1} = 0.0015 + 0.0132 \times -1.89 = -0.0234$

$VaR_{5\%,T+1} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0234 = -234064.61€$

Calculation for 99% Confidence and $T+1$:

$\text{Quantile}_{\text{std}(7)}(0.01) = -2.97, \hat{\mu_{T+1|T}} = 0.0015, \hat{\sigma_{T+1|T}} = 0.0132$

$VaR(\log\text{returns})_{1\%,T+1} = 0.0015 + 0.0132 \times -2.97 = -0.0378$

$VaR_{1\%,T+1} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0378 = -377535.41€$

Calculation for 90% Confidence and $T+5$:

$\text{Quantile}_{\text{std}(7)}(0.1) = -1.41, \hat{\mu_{T+5|T}} = 0.0015, \hat{\sigma_{T+5|T}} = 0.0139$

$VaR(\log\text{returns})_{10\%,T+5} = 0.0015 + 0.0139 \times -1.41 = -0.0181$

$VaR_{10\%,T+5} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0181 = -181407.55€$

Calculation for 95% Confidence and $T+5$:

$\text{Quantile}_{\text{std}(7)}(0.05) = -1.89, \hat{\mu_{T+5|T}} = 0.0015, \hat{\sigma_{T+5|T}} = 0.0139$

$VaR(\log\text{returns})_{5\%,T+5} = 0.0015 + 0.0139 \times -1.89 = -0.0248$

$VaR_{5\%,T+5} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0248 = -247582.23€$

Calculation for 99% Confidence and $T+5$:

$\text{Quantile}_{\text{std}(7)}(0.01) = -2.97, \hat{\mu_{T+5|T}} = 0.0015, \hat{\sigma_{T+5|T}} = 0.0139$

$VaR(\log\text{returns})_{1\%,T+5} = 0.0015 + 0.0139 \times -2.97 = -0.0399$

$VaR_{1\%,T+5} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0399 = -398843.04€$

Following the modeling assumptions of this subtask, the available stock data and the given sum invested in the Microsoft stock, the loss incurring over one day will be smaller than -171297.95€, -234064.61€ and -377535.41€ with a confidence level of 90%, 95% and 99%.

The loss after 5 days will be smaller than -181407.55€, -247582.23€ and -398843.04€ with a confidence level of 90%, 95% and 99%.

3.2 Difficulties with the econometric approach

The econometric approach of estimating the VaR of investments using ARMA-GARCH models can be a useful tool for portfolio managers to assess and manage risk. However, several challenges and limitations may arise in its practical implementation. Below we list a few:

Firstly, and likely the most evident limitation, is that choosing the appropriate ARMA-GARCH model specification can be hard. Selecting the wrong model specification can lead to significantly different estimates of the VaR. In the same fashion, small changes to the procedure of parameter estimation can result in significant variations in VaR estimates, making it challenging for portfolio managers to rely on these measures for decision-making.

Another important limitation is that the possibility of occurence of especially extreme events is generally not adequately represented in the ARMA-GARCH models, particularly if they are not captured in the data used for fitting. This could leave portfolios vulnerable to unexpected losses during larger-than-expected market downturns. Moreover, specific assumptions, usually made while modelling, could be wrong. In reality, markets can exhibit non-linear dependencies, fat tails in the error distributions, and non-stationary time-dependent return means. All of those can challenge the accuracy of the model's forecasts.

Furthermore, once a model is found and estimated, assessing its accuracy is also a challenge. Since the models are supposed to predict potential, although unlikely, changes, it is hard to back-test the model's performance against any ground truth.

Lastly, estimating ARMA-GARCH models and computing VaR can be computationally intensive. This especially holds for multivariate models in large portfolios, and with high-frequency data. Although we did not experience this last type of issues in the scope of the current project, portfolio managers should ensure that they have both the necessary computational resources and relevant expertise to implement these techniques effectively.

Appendices

A Appendix 1: VaR for T+2, T+3 and T+4

A.1 VaR for T+2, T+3 and T+4 for AR(0)-GARCH(1,1)

A.1.0.1 Visa

Caclulations.

Calculation for 90% Confidence and $T+2$:
$\text{Quantile}_{\text{norm}}(0.1) = -1.28, \hat{\mu_{T+2|T}} = 0.0006, \hat{\sigma_{T+2|T}} = 0.0096$
$VaR(\text{log returns})_{10\%,T+2} = 0.0006 + 0.0096 \times -1.28 = -0.0116$
$VaR_{10\%,T+2} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0116 = -116326.52€$

Calculation for 95% Confidence and $T+2$:
$\text{Quantile}_{\text{norm}}(0.05) = -1.64, \hat{\mu_{T+2|T}} = 0.0006, \hat{\sigma_{T+2|T}} = 0.0096$
$VaR(\text{log returns})_{5\%,T+2} = 0.0006 + 0.0096 \times -1.64 = -0.0151$
$VaR_{5\%,T+2} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0151 = -151058.05€$

Calculation for 99% Confidence and $T+2$:
$\text{Quantile}_{\text{norm}}(0.01) = -2.33, \hat{\mu_{T+2|T}} = 0.0006, \hat{\sigma_{T+2|T}} = 0.0096$
$VaR(\text{log returns})_{1\%,T+2} = 0.0006 + 0.0096 \times -2.33 = -0.0216$
$VaR_{1\%,T+2} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0216 = -216208.64€$

Calculation for 90% Confidence and $T+3$:
$\text{Quantile}_{\text{norm}}(0.1) = -1.28, \hat{\mu_{T+3|T}} = 0.0006, \hat{\sigma_{T+3|T}} = 0.0097$
$VaR(\text{log returns})_{10\%,T+3} = 0.0006 + 0.0097 \times -1.28 = -0.0119$
$VaR_{10\%,T+3} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0119 = -118538.19€$

Calculation for 95% Confidence and $T+3$:
$\text{Quantile}_{\text{norm}}(0.05) = -1.64, \hat{\mu_{T+3|T}} = 0.0006, \hat{\sigma_{T+3|T}} = 0.0097$
$VaR(\text{log returns})_{5\%,T+3} = 0.0006 + 0.0097 \times -1.64 = -0.0154$
$VaR_{5\%,T+3} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0154 = -153896.7€$

Calculation for 99% Confidence and $T+3$:
$\text{Quantile}_{\text{norm}}(0.01) = -2.33, \hat{\mu_{T+3|T}} = 0.0006, \hat{\sigma_{T+3|T}} = 0.0097$
$VaR(\text{log returns})_{1\%,T+3} = 0.0006 + 0.0097 \times -2.33 = -0.022$
$VaR_{1\%,T+3} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.022 = -220223.39€$

Calculation for 90% Confidence and $T+4$:
$\text{Quantile}_{\text{norm}}(0.1) = -1.28, \hat{\mu_{T+4|T}} = 0.0006, \hat{\sigma_{T+4|T}} = 0.0099$
$VaR(\text{log returns})_{10\%,T+4} = 0.0006 + 0.0099 \times -1.28 = -0.0121$
$VaR_{10\%,T+4} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0121 = -120682.99€$

Calculation for 95% Confidence and $T+4$:
$\text{Quantile}_{\text{norm}}(0.05) = -1.64, \hat{\mu_{T+4|T}} = 0.0006, \hat{\sigma_{T+4|T}} = 0.0099$

$VaR(\log \text{returns})_{5\%,T+4} = 0.0006 + 0.0099 \times -1.64 = -0.0157$
$VaR_{5\%,T+4} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.0157 = -156649.52€$

Calculation for 99% Confidence and $T+4$:
Quantile$_{\text{norm}}(0.01) = -2.33, \hat{\mu_{T+4|T}} = 0.0006, \hat{\sigma_{T+4|T}} = 0.0099$
$VaR(\log \text{returns})_{1\%,T+4} = 0.0006 + 0.0099 \times -2.33 = -0.0224$
$VaR_{1\%,T+4} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.0224 = -224116.75€$

A.1.0.2 Microsoft Caclulations.

Calculation for 90% Confidence and $T+2$:
Quantile$_{\text{norm}}(0.1) = -1.28, \hat{\mu_{T+2|T}} = 0.0013, \hat{\sigma_{T+2|T}} = 0.0132$
$VaR(\log \text{returns})_{10\%,T+2} = 0.0013 + 0.0132 \times -1.28 = -0.0156$
$VaR_{10\%,T+2} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.0156 = -156238.14€$

Calculation for 95% Confidence and $T+2$:
Quantile$_{\text{norm}}(0.05) = -1.64, \hat{\mu_{T+2|T}} = 0.0013, \hat{\sigma_{T+2|T}} = 0.0132$
$VaR(\log \text{returns})_{5\%,T+2} = 0.0013 + 0.0132 \times -1.64 = -0.0204$
$VaR_{5\%,T+2} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.0204 = -204128.44€$

Calculation for 99% Confidence and $T+2$:
Quantile$_{\text{norm}}(0.01) = -2.33, \hat{\mu_{T+2|T}} = 0.0013, \hat{\sigma_{T+2|T}} = 0.0132$
$VaR(\log \text{returns})_{1\%,T+2} = 0.0013 + 0.0132 \times -2.33 = -0.0294$
$VaR_{1\%,T+2} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.0294 = -293962.67€$

Calculation for 90% Confidence and $T+3$:
Quantile$_{\text{norm}}(0.1) = -1.28, \hat{\mu_{T+3|T}} = 0.0013, \hat{\sigma_{T+3|T}} = 0.0133$
$VaR(\log \text{returns})_{10\%,T+3} = 0.0013 + 0.0133 \times -1.28 = -0.0158$
$VaR_{10\%,T+3} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.0158 = -158009.37€$

Calculation for 95% Confidence and $T+3$:
Quantile$_{\text{norm}}(0.05) = -1.64, \hat{\mu_{T+3|T}} = 0.0013, \hat{\sigma_{T+3|T}} = 0.0133$
$VaR(\log \text{returns})_{5\%,T+3} = 0.0013 + 0.0133 \times -1.64 = -0.0206$
$VaR_{5\%,T+3} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.0206 = -206401.79€$

Calculation for 99% Confidence and $T+3$:
Quantile$_{\text{norm}}(0.01) = -2.33, \hat{\mu_{T+3|T}} = 0.0013, \hat{\sigma_{T+3|T}} = 0.0133$
$VaR(\log \text{returns})_{1\%,T+3} = 0.0013 + 0.0133 \times -2.33 = -0.0297$
$VaR_{1\%,T+3} =$ Value in Euros $\times VaR(\log returns) = 10000000€\times -0.0297 = -297177.91€$

Calculation for 90% Confidence and $T+4$:
Quantile$_{\text{norm}}(0.1) = -1.28, \hat{\mu_{T+4|T}} = 0.0013, \hat{\sigma_{T+4|T}} = 0.0135$
$VaR(\log \text{returns})_{10\%,T+4} = 0.0013 + 0.0135 \times -1.28 = -0.016$

$VaR_{10\%,T+4} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.016 = -159733.96€$

Calculation for 95% Confidence and $T+4$:
Quantile$_{\text{norm}}(0.05) = -1.64, \hat{\mu_{T+4|T}} = 0.0013, \hat{\sigma_{T+4|T}} = 0.0135$
$VaR(\log \text{returns})_{5\%,T+4} = 0.0013 + 0.0135 \times -1.64 = -0.0209$
$VaR_{5\%,T+4} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0209 = -208615.27€$

Calculation for 99% Confidence and $T+4$:
Quantile$_{\text{norm}}(0.01) = -2.33, \hat{\mu_{T+4|T}} = 0.0013, \hat{\sigma_{T+4|T}} = 0.0135$
$VaR(\log \text{returns})_{1\%,T+4} = 0.0013 + 0.0135 \times -2.33 = -0.03$
$VaR_{1\%,T+4} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.03 = -300308.48€$

A.2 VaR for T+2, T+3 and T+4 for AR(1)-GARCH(1,1)

A.2.0.1 Visa Caclulations.

Calculation for 90% Confidence and $T+2$:
Quantile$_{\text{norm}}(0.1) = -1.28, \hat{\mu_{T+2|T}} = 0.0006, \hat{\sigma_{T+2|T}} = 0.0096$
$VaR(\log \text{returns})_{10\%,T+2} = 0.0006 + 0.0096 \times -1.28 = -0.0116$
$VaR_{10\%,T+2} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0116 = -116213.31€$

Calculation for 95% Confidence and $T+2$:
Quantile$_{\text{norm}}(0.05) = -1.64, \hat{\mu_{T+2|T}} = 0.0006, \hat{\sigma_{T+2|T}} = 0.0096$
$VaR(\log \text{returns})_{5\%,T+2} = 0.0006 + 0.0096 \times -1.64 = -0.0151$
$VaR_{5\%,T+2} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0151 = -150948.86€$

Calculation for 99% Confidence and $T+2$:
Quantile$_{\text{norm}}(0.01) = -2.33, \hat{\mu_{T+2|T}} = 0.0006, \hat{\sigma_{T+2|T}} = 0.0096$
$VaR(\log \text{returns})_{1\%,T+2} = 0.0006 + 0.0096 \times -2.33 = -0.0216$
$VaR_{1\%,T+2} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0216 = -216106.96€$

Calculation for 90% Confidence and $T+3$:
Quantile$_{\text{norm}}(0.1) = -1.28, \hat{\mu_{T+3|T}} = 0.0006, \hat{\sigma_{T+3|T}} = 0.0097$
$VaR(\log \text{returns})_{10\%,T+3} = 0.0006 + 0.0097 \times -1.28 = -0.0118$
$VaR_{10\%,T+3} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0118 = -118425.27€$

Calculation for 95% Confidence and $T+3$:
Quantile$_{\text{norm}}(0.05) = -1.64, \hat{\mu_{T+3|T}} = 0.0006, \hat{\sigma_{T+3|T}} = 0.0097$
$VaR(\log \text{returns})_{5\%,T+3} = 0.0006 + 0.0097 \times -1.64 = -0.0154$
$VaR_{5\%,T+3} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0154 = -153779.32€$

Calculation for 99% Confidence and $T+3$:
Quantile$_{\text{norm}}(0.01) = -2.33, \hat{\mu_{T+3|T}} = 0.0006, \hat{\sigma_{T+3|T}} = 0.0097$
$VaR(\log \text{returns})_{1\%,T+3} = 0.0006 + 0.0097 \times -2.33 = -0.022$

$VaR_{1\%,T+3} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.022 = -220097.65€$

Calculation for 90% Confidence and $T+4$:
$\text{Quantile}_{\text{norm}}(0.1) = -1.28, \hat{\mu_{T+4|T}} = 0.0006, \hat{\sigma_{T+4|T}} = 0.0099$
$VaR(\log \text{returns})_{10\%,T+4} = 0.0006 + 0.0099 \times -1.28 = -0.0121$
$VaR_{10\%,T+4} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0121 = -120541.24€$

Calculation for 95% Confidence and $T+4$:
$\text{Quantile}_{\text{norm}}(0.05) = -1.64, \hat{\mu_{T+4|T}} = 0.0006, \hat{\sigma_{T+4|T}} = 0.0099$
$VaR(\log \text{returns})_{5\%,T+4} = 0.0006 + 0.0099 \times -1.64 = -0.0156$
$VaR_{5\%,T+4} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0156 = -156495.43€$

Calculation for 99% Confidence and $T+4$:
$\text{Quantile}_{\text{norm}}(0.01) = -2.33, \hat{\mu_{T+4|T}} = 0.0006, \hat{\sigma_{T+4|T}} = 0.0099$
$VaR(\log \text{returns})_{1\%,T+4} = 0.0006 + 0.0099 \times -2.33 = -0.0224$
$VaR_{1\%,T+4} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0224 = -223939.51€$

A.2.0.2 Microsoft Caclulations.

Calculation for 90% Confidence and $T+2$:
$\text{Quantile}_{\text{norm}}(0.1) = -1.28, \hat{\mu_{T+2|T}} = 0.0013, \hat{\sigma_{T+2|T}} = 0.0134$
$VaR(\log \text{returns})_{10\%,T+2} = 0.0013 + 0.0134 \times -1.28 = -0.0159$
$VaR_{10\%,T+2} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0159 = -158767.68€$

Calculation for 95% Confidence and $T+2$:
$\text{Quantile}_{\text{norm}}(0.05) = -1.64, \hat{\mu_{T+2|T}} = 0.0013, \hat{\sigma_{T+2|T}} = 0.0134$
$VaR(\log \text{returns})_{5\%,T+2} = 0.0013 + 0.0134 \times -1.64 = -0.0207$
$VaR_{5\%,T+2} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0207 = -207435.64€$

Calculation for 99% Confidence and $T+2$:
$\text{Quantile}_{\text{norm}}(0.01) = -2.33, \hat{\mu_{T+2|T}} = 0.0013, \hat{\sigma_{T+2|T}} = 0.0134$
$VaR(\log \text{returns})_{1\%,T+2} = 0.0013 + 0.0134 \times -2.33 = -0.0299$
$VaR_{1\%,T+2} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0299 = -298728.66€$

Calculation for 90% Confidence and $T+3$:
$\text{Quantile}_{\text{norm}}(0.1) = -1.28, \hat{\mu_{T+3|T}} = 0.0013, \hat{\sigma_{T+3|T}} = 0.0135$
$VaR(\log \text{returns})_{10\%,T+3} = 0.0013 + 0.0135 \times -1.28 = -0.0161$
$VaR_{10\%,T+3} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0161 = -160591.11€$

Calculation for 95% Confidence and $T+3$:
$\text{Quantile}_{\text{norm}}(0.05) = -1.64, \hat{\mu_{T+3|T}} = 0.0013, \hat{\sigma_{T+3|T}} = 0.0135$
$VaR(\log \text{returns})_{5\%,T+3} = 0.0013 + 0.0135 \times -1.64 = -0.021$
$VaR_{5\%,T+3} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.021 = -209723.98€$

Calculation for 99% Confidence and $T+3$:
$\text{Quantile}_{\text{norm}}(0.01) = -2.33, \hat{\mu_{T+3|T}} = 0.0013, \hat{\sigma_{T+3|T}} = 0.0135$
$VaR(\log \text{returns})_{1\%,T+3} = 0.0013 + 0.0135 \times -2.33 = -0.0302$
$VaR_{1\%,T+3} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0302 = -301889.08€$

Calculation for 90% Confidence and $T+4$:
$\text{Quantile}_{\text{norm}}(0.1) = -1.28, \hat{\mu_{T+4|T}} = 0.0013, \hat{\sigma_{T+4|T}} = 0.0136$
$VaR(\log \text{returns})_{10\%,T+4} = 0.0013 + 0.0136 \times -1.28 = -0.0162$
$VaR_{10\%,T+4} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0162 = -162173.23€$

Calculation for 95% Confidence and $T+4$:
$\text{Quantile}_{\text{norm}}(0.05) = -1.64, \hat{\mu_{T+4|T}} = 0.0013, \hat{\sigma_{T+4|T}} = 0.0136$
$VaR(\log \text{returns})_{5\%,T+4} = 0.0013 + 0.0136 \times -1.64 = -0.0212$
$VaR_{5\%,T+4} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0212 = -211759€$

Calculation for 99% Confidence and $T+4$:
$\text{Quantile}_{\text{norm}}(0.01) = -2.33, \hat{\mu_{T+4|T}} = 0.0013, \hat{\sigma_{T+4|T}} = 0.0136$
$VaR(\log \text{returns})_{1\%,T+4} = 0.0013 + 0.0136 \times -2.33 = -0.0305$
$VaR_{1\%,T+4} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0305 = -304773.66€$

A.3 VaR for T+2, T+3 and T+4 for AR(1)-GARCH(1,1) with t-Stundent innovations

A.3.0.1 Visa Caclulations.

Calculation for 90% Confidence and $T+2$:
$\text{Quantile}_{\text{std}(5)}(0.1) = -1.46, \hat{\mu_{T+2|T}} = 0.001, \hat{\sigma_{T+2|T}} = 0.0098$
$VaR(\log \text{returns})_{10\%,T+2} = 0.001 + 0.0098 \times -1.46 = -0.0133$
$VaR_{10\%,T+2} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0133 = -132845.59€$

Calculation for 95% Confidence and $T+2$:
$\text{Quantile}_{\text{std}(5)}(0.05) = -1.99, \hat{\mu_{T+2|T}} = 0.001, \hat{\sigma_{T+2|T}} = 0.0098$
$VaR(\log \text{returns})_{5\%,T+2} = 0.001 + 0.0098 \times -1.99 = -0.0184$
$VaR_{5\%,T+2} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0184 = -184166.93€$

Calculation for 99% Confidence and $T+2$:
$\text{Quantile}_{\text{std}(5)}(0.01) = -3.27, \hat{\mu_{T+2|T}} = 0.001, \hat{\sigma_{T+2|T}} = 0.0098$
$VaR(\log \text{returns})_{1\%,T+2} = 0.001 + 0.0098 \times -3.27 = -0.031$
$VaR_{1\%,T+2} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.031 = -310183.12€$

Calculation for 90% Confidence and $T+3$:
$\text{Quantile}_{\text{std}(5)}(0.1) = -1.46, \hat{\mu_{T+3|T}} = 0.001, \hat{\sigma_{T+3|T}} = 0.01$
$VaR(\log \text{returns})_{10\%,T+3} = 0.001 + 0.01 \times -1.46 = -0.0136$

$VaR_{10\%,T+3} =$ Value in Euros $\times VaR(\log returns) = 10000000$€$\times -0.0136 = -135672$€

Calculation for 95% Confidence and $T+3$:
Quantile$_{\text{std}(5)}(0.05) = -1.99, \hat{\mu_{T+3|T}} = 0.001, \hat{\sigma_{T+3|T}} = 0.01$
$VaR(\log \text{returns})_{5\%,T+3} = 0.001 + 0.01 \times -1.99 = -0.0188$
$VaR_{5\%,T+3} =$ Value in Euros $\times VaR(\log returns) = 10000000$€$\times -0.0188 = -188001.27$€

Calculation for 99% Confidence and $T+3$:
Quantile$_{\text{std}(5)}(0.01) = -3.27, \hat{\mu_{T+3|T}} = 0.001, \hat{\sigma_{T+3|T}} = 0.01$
$VaR(\log \text{returns})_{1\%,T+3} = 0.001 + 0.01 \times -3.27 = -0.0316$
$VaR_{1\%,T+3} =$ Value in Euros $\times VaR(\log returns) = 10000000$€$\times -0.0316 = -316492.34$€

Calculation for 90% Confidence and $T+4$:
Quantile$_{\text{std}(5)}(0.1) = -1.46, \hat{\mu_{T+4|T}} = 0.001, \hat{\sigma_{T+4|T}} = 0.0102$
$VaR(\log \text{returns})_{10\%,T+4} = 0.001 + 0.0102 \times -1.46 = -0.0138$
$VaR_{10\%,T+4} =$ Value in Euros $\times VaR(\log returns) = 10000000$€$\times -0.0138 = -138377.96$€

Calculation for 95% Confidence and $T+4$:
Quantile$_{\text{std}(5)}(0.05) = -1.99, \hat{\mu_{T+4|T}} = 0.001, \hat{\sigma_{T+4|T}} = 0.0102$
$VaR(\log \text{returns})_{5\%,T+4} = 0.001 + 0.0102 \times -1.99 = -0.0192$
$VaR_{5\%,T+4} =$ Value in Euros $\times VaR(\log returns) = 10000000$€$\times -0.0192 = -191679.08$€

Calculation for 99% Confidence and $T+4$:
Quantile$_{\text{std}(5)}(0.01) = -3.27, \hat{\mu_{T+4|T}} = 0.001, \hat{\sigma_{T+4|T}} = 0.0102$
$VaR(\log \text{returns})_{1\%,T+4} = 0.001 + 0.0102 \times -3.27 = -0.0323$
$VaR_{1\%,T+4} =$ Value in Euros $\times VaR(\log returns) = 10000000$€$\times -0.0323 = -322556.47$€

A.3.0.2 Microsoft Caclulations.

Calculation for 90% Confidence and $T+2$:
Quantile$_{\text{std}(7)}(0.1) = -1.41, \hat{\mu_{T+2|T}} = 0.0015, \hat{\sigma_{T+2|T}} = 0.0136$
$VaR(\log \text{returns})_{10\%,T+2} = 0.0015 + 0.0136 \times -1.41 = -0.0177$
$VaR_{10\%,T+2} =$ Value in Euros $\times VaR(\log returns) = 10000000$€$\times -0.0177 = -176748.92$€

Calculation for 95% Confidence and $T+2$:
Quantile$_{\text{std}(7)}(0.05) = -1.88, \hat{\mu_{T+2|T}} = 0.0015, \hat{\sigma_{T+2|T}} = 0.0136$
$VaR(\log \text{returns})_{5\%,T+2} = 0.0015 + 0.0136 \times -1.88 = -0.0241$
$VaR_{5\%,T+2} =$ Value in Euros $\times VaR(\log returns) = 10000000$€$\times -0.0241 = -241347.07$€

Calculation for 99% Confidence and $T+2$:
Quantile$_{\text{std}(7)}(0.01) = -2.97, \hat{\mu_{T+2|T}} = 0.0015, \hat{\sigma_{T+2|T}} = 0.0136$
$VaR(\log \text{returns})_{1\%,T+2} = 0.0015 + 0.0136 \times -2.97 = -0.0389$
$VaR_{1\%,T+2} =$ Value in Euros $\times VaR(\log returns) = 10000000$€$\times -0.0389 = -388833.43$€

Calculation for 90% Confidence and $T+3$:
$\text{Quantile}_{\text{std}(7)}(0.1) = -1.41, \hat{\mu_{T+3|T}} = 0.0015, \hat{\sigma_{T+3|T}} = 0.0138$
$VaR(\log \text{returns})_{10\%,T+3} = 0.0015 + 0.0138 \times -1.41 = -0.0179$
$VaR_{10\%,T+3} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0179 = -179053.2€$

Calculation for 95% Confidence and $T+3$:
$\text{Quantile}_{\text{std}(7)}(0.05) = -1.88, \hat{\mu_{T+3|T}} = 0.0015, \hat{\sigma_{T+3|T}} = 0.0138$
$VaR(\log \text{returns})_{5\%,T+3} = 0.0015 + 0.0138 \times -1.88 = -0.0244$
$VaR_{5\%,T+3} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0244 = -244391.47€$

Calculation for 99% Confidence and $T+3$:
$\text{Quantile}_{\text{std}(7)}(0.01) = -2.97, \hat{\mu_{T+3|T}} = 0.0015, \hat{\sigma_{T+3|T}} = 0.0138$
$VaR(\log \text{returns})_{1\%,T+3} = 0.0015 + 0.0138 \times -2.97 = -0.0394$
$VaR_{1\%,T+3} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0394 = -393567.64€$

Calculation for 90% Confidence and $T+4$:
$\text{Quantile}_{\text{std}(7)}(0.1) = -1.41, \hat{\mu_{T+4|T}} = 0.0015, \hat{\sigma_{T+4|T}} = 0.0139$
$VaR(\log \text{returns})_{10\%,T+4} = 0.0015 + 0.0139 \times -1.41 = -0.0181$
$VaR_{10\%,T+4} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0181 = -181176.27€$

Calculation for 95% Confidence and $T+4$:
$\text{Quantile}_{\text{std}(7)}(0.05) = -1.88, \hat{\mu_{T+4|T}} = 0.0015, \hat{\sigma_{T+4|T}} = 0.0139$
$VaR(\log \text{returns})_{5\%,T+4} = 0.0015 + 0.0139 \times -1.88 = -0.0247$
$VaR_{5\%,T+4} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0247 = -247231.96€$

Calculation for 99% Confidence and $T+4$:
$\text{Quantile}_{\text{std}(7)}(0.01) = -2.97, \hat{\mu_{T+4|T}} = 0.0015, \hat{\sigma_{T+4|T}} = 0.0139$
$VaR(\log \text{returns})_{1\%,T+4} = 0.0015 + 0.0139 \times -2.97 = -0.0398$
$VaR_{1\%,T+4} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.0398 = -398046.1€$

A.4 VaR for T+2, T+3 and T+4 for other ARMA-GARCH specification

A.4.0.1 Visa Caclulations.

Calculation for 90% Confidence and $T+2$:
$\text{Quantile}_{\text{std}(5)}(0.1) = -1.46, \hat{\mu_{T+2|T}} = 0.001, \hat{\sigma_{T+2|T}} = 0.0096$
$VaR(\log \text{returns})_{10\%,T+2} = 0.001 + 0.0096 \times -1.46 = -0.013$
$VaR_{10\%,T+2} = \text{ Value in Euros } \times VaR(\log returns) = 10000000€ \times -0.013 = -130319.29€$

Calculation for 95% Confidence and $T+2$:
$\text{Quantile}_{\text{std}(5)}(0.05) = -1.98, \hat{\mu_{T+2|T}} = 0.001, \hat{\sigma_{T+2|T}} = 0.0096$
$VaR(\log \text{returns})_{5\%,T+2} = 0.001 + 0.0096 \times -1.98 = -0.0181$

$VaR_{5\%,T+2} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0181 = -180586.77€$

Calculation for 99% Confidence and $T+2$:
Quantile$_{\text{std}(5)}(0.01) = -3.27, \mu_{T\hat{+}2|T} = 0.001, \sigma_{T\hat{+}2|T} = 0.0096$
$VaR(\log \text{returns})_{1\%,T+2} = 0.001 + 0.0096 \times -3.27 = -0.0304$
$VaR_{1\%,T+2} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0304 = -303819.19€$

Calculation for 90% Confidence and $T+3$:
Quantile$_{\text{std}(5)}(0.1) = -1.46, \mu_{T\hat{+}3|T} = 0.001, \sigma_{T\hat{+}3|T} = 0.0101$
$VaR(\log \text{returns})_{10\%,T+3} = 0.001 + 0.0101 \times -1.46 = -0.0137$
$VaR_{10\%,T+3} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0137 = -137016.17€$

Calculation for 95% Confidence and $T+3$:
Quantile$_{\text{std}(5)}(0.05) = -1.98, \mu_{T\hat{+}3|T} = 0.001, \sigma_{T\hat{+}3|T} = 0.0101$
$VaR(\log \text{returns})_{5\%,T+3} = 0.001 + 0.0101 \times -1.98 = -0.019$
$VaR_{5\%,T+3} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.019 = -189684.97€$

Calculation for 99% Confidence and $T+3$:
Quantile$_{\text{std}(5)}(0.01) = -3.27, \mu_{T\hat{+}3|T} = 0.001, \sigma_{T\hat{+}3|T} = 0.0101$
$VaR(\log \text{returns})_{1\%,T+3} = 0.001 + 0.0101 \times -3.27 = -0.0319$
$VaR_{1\%,T+3} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0319 = -318804.31€$

Calculation for 90% Confidence and $T+4$:
Quantile$_{\text{std}(5)}(0.1) = -1.46, \mu_{T\hat{+}4|T} = 0.001, \sigma_{T\hat{+}4|T} = 0.0102$
$VaR(\log \text{returns})_{10\%,T+4} = 0.001 + 0.0102 \times -1.46 = -0.0139$
$VaR_{10\%,T+4} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0139 = -138907.79€$

Calculation for 95% Confidence and $T+4$:
Quantile$_{\text{std}(5)}(0.05) = -1.98, \mu_{T\hat{+}4|T} = 0.001, \sigma_{T\hat{+}4|T} = 0.0102$
$VaR(\log \text{returns})_{5\%,T+4} = 0.001 + 0.0102 \times -1.98 = -0.0192$
$VaR_{5\%,T+4} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0192 = -192254.88€$

Calculation for 99% Confidence and $T+4$:
Quantile$_{\text{std}(5)}(0.01) = -3.27, \mu_{T\hat{+}4|T} = 0.001, \sigma_{T\hat{+}4|T} = 0.0102$
$VaR(\log \text{returns})_{1\%,T+4} = 0.001 + 0.0102 \times -3.27 = -0.0323$
$VaR_{1\%,T+4} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0323 = -323037.06€$

A.4.0.2 Microsoft

Caclulations.

Calculation for 90% Confidence and $T+2$:
Quantile$_{\text{std}(7)}(0.1) = -1.41, \mu_{T\hat{+}2|T} = 0.0015, \sigma_{T\hat{+}2|T} = 0.0134$
$VaR(\log \text{returns})_{10\%,T+2} = 0.0015 + 0.0134 \times -1.41 = -0.0175$
$VaR_{10\%,T+2} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0175 = -174679.64€$

Calculation for 95% Confidence and $T+2$:
$\text{Quantile}_{\text{std}(7)}(0.05) = -1.89, \hat{\mu_{T+2|T}} = 0.0015, \hat{\sigma_{T+2|T}} = 0.0134$
$VaR(\log \text{returns})_{5\%,T+2} = 0.0015 + 0.0134 \times -1.89 = -0.0239$
$VaR_{5\%,T+2} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0239 = -238586.28€$

Calculation for 99% Confidence and $T+2$:
$\text{Quantile}_{\text{std}(7)}(0.01) = -2.97, \hat{\mu_{T+2|T}} = 0.0015, \hat{\sigma_{T+2|T}} = 0.0134$
$VaR(\log \text{returns})_{1\%,T+2} = 0.0015 + 0.0134 \times -2.97 = -0.0385$
$VaR_{1\%,T+2} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0385 = -384662.86€$

Calculation for 90% Confidence and $T+3$:
$\text{Quantile}_{\text{std}(7)}(0.1) = -1.41, \hat{\mu_{T+3|T}} = 0.0015, \hat{\sigma_{T+3|T}} = 0.0136$
$VaR(\log \text{returns})_{10\%,T+3} = 0.0015 + 0.0136 \times -1.41 = -0.0177$
$VaR_{10\%,T+3} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0177 = -176945.69€$

Calculation for 95% Confidence and $T+3$:
$\text{Quantile}_{\text{std}(7)}(0.05) = -1.89, \hat{\mu_{T+3|T}} = 0.0015, \hat{\sigma_{T+3|T}} = 0.0136$
$VaR(\log \text{returns})_{5\%,T+3} = 0.0015 + 0.0136 \times -1.89 = -0.0242$
$VaR_{5\%,T+3} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0242 = -241616.24€$

Calculation for 99% Confidence and $T+3$:
$\text{Quantile}_{\text{std}(7)}(0.01) = -2.97, \hat{\mu_{T+3|T}} = 0.0015, \hat{\sigma_{T+3|T}} = 0.0136$
$VaR(\log \text{returns})_{1\%,T+3} = 0.0015 + 0.0136 \times -2.97 = -0.0389$
$VaR_{1\%,T+3} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0389 = -389438.94€$

Calculation for 90% Confidence and $T+4$:
$\text{Quantile}_{\text{std}(7)}(0.1) = -1.41, \hat{\mu_{T+4|T}} = 0.0015, \hat{\sigma_{T+4|T}} = 0.0138$
$VaR(\log \text{returns})_{10\%,T+4} = 0.0015 + 0.0138 \times -1.41 = -0.0179$
$VaR_{10\%,T+4} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0179 = -179214.98€$

Calculation for 95% Confidence and $T+4$:
$\text{Quantile}_{\text{std}(7)}(0.05) = -1.89, \hat{\mu_{T+4|T}} = 0.0015, \hat{\sigma_{T+4|T}} = 0.0138$
$VaR(\log \text{returns})_{5\%,T+4} = 0.0015 + 0.0138 \times -1.89 = -0.0245$
$VaR_{5\%,T+4} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0245 = -244650.52€$

Calculation for 99% Confidence and $T+4$:
$\text{Quantile}_{\text{std}(7)}(0.01) = -2.97, \hat{\mu_{T+4|T}} = 0.0015, \hat{\sigma_{T+4|T}} = 0.0138$
$VaR(\log \text{returns})_{1\%,T+4} = 0.0015 + 0.0138 \times -2.97 = -0.0394$
$VaR_{1\%,T+4} =$ Value in Euros $\times VaR(\log returns) = 10000000€ \times -0.0394 = -394221.83€$