



University of Central Florida

UCF Apocalypse Attack

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2024-04-29

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Contest (1)

```
.bashrc
3 lines

alias c='g++ -Wall -Wconversion -Wfatal-errors -g -std=c++17 \
-fsanitize=undefined,address'
xmodmap -e 'clear lock' -e 'keycode 66=less greater' #caps = <>

hash.sh
6 lines

# Hashes a file, ignoring all whitespace and comments. Use for
# verifying that code was correctly typed.
# Usage:
#   To make executable, run the command: chmod +x hash.sh
#   To execute: ./hash.sh < file.cpp
cpp -dD -P -fpreprocessed | tr -d '[:space:]' | md5sum |cut -c-6
```

Mathematics (2)

2.1 Equations

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The extremum is given by $x = -b/2a$.

$$\begin{aligned} ax + by &= e & x &= \frac{ed - bf}{ad - bc} \\ cx + dy &= f & y &= \frac{af - ec}{ad - bc} \end{aligned}$$

In general, given an equation $Ax = b$, the solution to a variable x_i is given by

$$x_i = \frac{\det A'_i}{\det A}$$

where A'_i is A with the i 'th column replaced by b .

2.2 Recurrences

If $a_n = c_1 a_{n-1} + \dots + c_k a_{n-k}$, and r_1, \dots, r_k are distinct roots of $x^k - c_1 x^{k-1} - \dots - c_k$, there are d_1, \dots, d_k s.t.

$$a_n = d_1 r_1^n + \dots + d_k r_k^n.$$

Non-distinct roots r become polynomial factors, e.g. $a_n = (d_1 n + d_2) r^n$.

2.3 Trigonometry

$$\begin{aligned} \sin(v + w) &= \sin v \cos w + \cos v \sin w \\ \cos(v + w) &= \cos v \cos w - \sin v \sin w \end{aligned}$$

$$\begin{aligned} \tan(v + w) &= \frac{\tan v + \tan w}{1 - \tan v \tan w} \\ \sin v + \sin w &= 2 \sin \frac{v + w}{2} \cos \frac{v - w}{2} \\ \cos v + \cos w &= 2 \cos \frac{v + w}{2} \cos \frac{v - w}{2} \end{aligned}$$

$$(V + W) \tan(v - w)/2 = (V - W) \tan(v + w)/2$$

where V, W are lengths of sides opposite angles v, w .

$$\begin{aligned} a \cos x + b \sin x &= r \cos(x - \phi) \\ a \sin x + b \cos x &= r \sin(x + \phi) \end{aligned}$$

where $r = \sqrt{a^2 + b^2}$, $\phi = \text{atan2}(b, a)$.

2.4 Geometry

2.4.1 Triangles

Side lengths: a, b, c

$$\text{Semiperimeter: } p = \frac{a + b + c}{2}$$

$$\text{Area: } A = \sqrt{p(p-a)(p-b)(p-c)}$$

$$\text{Circumradius: } R = \frac{abc}{4A}$$

$$\text{Inradius: } r = \frac{A}{p}$$

Length of median (divides triangle into two equal-area triangles):
 $m_a = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2}$

Length of bisector (divides angles in two):

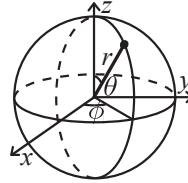
$$s_a = \sqrt{bc \left[1 - \left(\frac{a}{b+c} \right)^2 \right]}$$

$$\text{Law of sines: } \frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} = \frac{1}{2R}$$

$$\text{Law of cosines: } a^2 = b^2 + c^2 - 2bc \cos \alpha$$

$$\text{Law of tangents: } \frac{a+b}{a-b} = \frac{\tan \frac{\alpha+\beta}{2}}{\tan \frac{\alpha-\beta}{2}}$$

2.4.2 Spherical coordinates



$$\begin{aligned} x &= r \sin \theta \cos \phi & r &= \sqrt{x^2 + y^2 + z^2} \\ y &= r \sin \theta \sin \phi & \theta &= \arccos(z / \sqrt{x^2 + y^2 + z^2}) \\ z &= r \cos \theta & \phi &= \text{atan2}(y, x) \end{aligned}$$

2.5 Derivatives/Integrals

$$\begin{aligned} \frac{d}{dx} \arcsin x &= \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx} \arccos x &= -\frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx} \tan x &= 1 + \tan^2 x & \frac{d}{dx} \arctan x &= \frac{1}{1+x^2} \\ \int \tan ax &= -\frac{\ln |\cos ax|}{a} & \int x \sin ax &= \frac{\sin ax - ax \cos ax}{a^2} \\ \int e^{-x^2} &= \frac{\sqrt{\pi}}{2} \text{erf}(x) & \int x e^{ax} dx &= \frac{e^{ax}}{a^2} (ax - 1) \end{aligned}$$

Integration by parts:

$$\int_a^b f(x)g(x)dx = [F(x)g(x)]_a^b - \int_a^b F(x)g'(x)dx$$

2.6 Sums

$$c^a + c^{a+1} + \dots + c^b = \frac{c^{b+1} - c^a}{c - 1}, c \neq 1$$

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(2n+1)(n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

2.7 Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, (-\infty < x < \infty)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, (-1 < x \leq 1)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{2x^3}{32} - \frac{5x^4}{128} + \dots, (-1 \leq x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, (-\infty < x < \infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, (-\infty < x < \infty)$$

2.8 Probability theory

Let X be a discrete random variable with probability $p_X(x)$ of assuming the value x . It will then have an expected value (mean) $\mu = \mathbb{E}(X) = \sum_x x p_X(x)$ and variance $\sigma^2 = V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \sum_x (x - \mathbb{E}(X))^2 p_X(x)$ where σ is the standard deviation. If X is instead continuous it will have a probability density function $f_X(x)$ and the sums above will instead be integrals with $p_X(x)$ replaced by $f_X(x)$.

Expectation is linear:

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$$

For independent X and Y ,

$$V(aX + bY) = a^2 V(X) + b^2 V(Y).$$

2.8.1 Discrete distributions

Binomial distribution

The number of successes in n independent yes/no experiments, each which yields success with probability p is $\text{Bin}(n, p)$, $n = 1, 2, \dots$, $0 \leq p \leq 1$.

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mu = np, \sigma^2 = np(1-p)$$

$\text{Bin}(n, p)$ is approximately $\text{Po}(np)$ for small p .

First success distribution

The number of trials needed to get the first success in independent yes/no experiments, each which yields success with probability p is $\text{Fs}(p)$, $0 \leq p \leq 1$.

$$p(k) = p(1 - p)^{k-1}, \, k = 1, 2, \dots$$

$$\mu = \frac{1}{p}, \, \sigma^2 = \frac{1 - p}{p^2}$$

Poisson distribution

The number of events occurring in a fixed period of time t if these events occur with a known average rate κ and independently of the time since the last event is $\text{Po}(\lambda)$, $\lambda = t\kappa$.

$$p(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \, k = 0, 1, 2, \dots$$

$$\mu = \lambda, \, \sigma^2 = \lambda$$

2.8.2 Continuous distributions
Uniform distribution

If the probability density function is constant between a and b and 0 elsewhere it is $\text{U}(a, b)$, $a < b$.

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \frac{a + b}{2}, \, \sigma^2 = \frac{(b - a)^2}{12}$$

Exponential distribution

The time between events in a Poisson process is $\text{Exp}(\lambda)$, $\lambda > 0$.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\mu = \frac{1}{\lambda}, \, \sigma^2 = \frac{1}{\lambda^2}$$

Normal distribution

Most real random values with mean μ and variance σ^2 are well described by $\mathcal{N}(\mu, \sigma^2)$, $\sigma > 0$.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ then

$$aX_1 + bX_2 + c \sim \mathcal{N}(\mu_1 + \mu_2 + c, a^2\sigma_1^2 + b^2\sigma_2^2)$$

2.9 Markov chains

A *Markov chain* is a discrete random process with the property that the next state depends only on the current state. Let X_1, X_2, \dots be a sequence of random variables generated by the Markov process. Then there is a transition matrix $\mathbf{P} = (p_{ij})$, with $p_{ij} = \text{Pr}(X_n = i | X_{n-1} = j)$, and $\mathbf{p}^{(n)} = \mathbf{P}^n \mathbf{p}^{(0)}$ is the probability distribution for X_n (i.e., $p_i^{(n)} = \text{Pr}(X_n = i)$), where $\mathbf{p}^{(0)}$ is the initial distribution.

π is a stationary distribution if $\pi = \pi \mathbf{P}$. If the Markov chain is *irreducible* (it is possible to get to any state from any state), then $\pi_i = \frac{1}{\mathbb{E}(T_i)}$ where $\mathbb{E}(T_i)$ is the expected time between two visits in state i . π_j / π_i is the expected number of visits in state j between two visits in state i .

For a connected, undirected and non-bipartite graph, where the transition probability is uniform among all neighbors, π_i is proportional to node i 's degree.

A Markov chain is *ergodic* if the asymptotic distribution is independent of the initial distribution. A finite Markov chain is ergodic iff it is irreducible and *aperiodic* (i.e., the gcd of cycle lengths is 1). $\lim_{k \rightarrow \infty} \mathbf{P}^k = \mathbf{1}\pi$.

A Markov chain is an A-chain if the states can be partitioned into two sets \mathbf{A} and \mathbf{G} , such that all states in \mathbf{A} are absorbing ($p_{ii} = 1$), and all states in \mathbf{G} leads to an absorbing state in \mathbf{A} . The probability for absorption in state $i \in \mathbf{A}$, when the initial state is j , is $a_{ij} = p_{ij} + \sum_{k \in \mathbf{G}} a_{ik} p_{kj}$. The expected time until absorption, when the initial state is i , is $t_i = 1 + \sum_{k \in \mathbf{G}} p_{ki} t_k$.

2.9.1 Quadrilaterals

With side lengths a, b, c, d , diagonals e, f , diagonals angle θ , area A and magic flux $F = b^2 + d^2 - a^2 - c^2$:

$$4A = 2ef \cdot \sin \theta = F \tan \theta = \sqrt{4e^2 f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is 180° , $ef = ac + bd$, and $A = \sqrt{(p - a)(p - b)(p - c)(p - d)}$.

Data structures (3)

OrderStatisticTree.h

Description: A set (not multiset!) with support for finding the n'th element, and finding the index of an element. To get a map, change null_type. **Time:** $\mathcal{O}(\log N)$

782797, 16 lines

```
#include <bits/extc++.h>
using namespace __gnu_pbds;

template<class T>
using Tree = tree<T, null_type, less<T>, rb_tree_tag,
    tree_order_statistics_node_update>;
```

```
void example() {
    Tree<int> t, t2; t.insert(8);
    auto it = t.insert(10).first;
    assert(it == t.lower_bound(9));
    assert(t.order_of_key(10) == 1);
    assert(t.order_of_key(11) == 2);
    assert(*t.find_by_order(0) == 8);
    t.join(t2); // assuming T< T2 or T> T2, merge t2 into t
}
```

HashMap.h

Description: Hash map with mostly the same API as unordered_map, but ~3x faster. Uses 1.5x memory. Initial capacity must be a power of 2 (if provided).

d77092, 7 lines

```
#include <bits/extc++.h>
// To use most bits rather than just the lowest ones:
struct chash { // large odd number for C
    const uint64_t C = 11(4e18 * acos(0)) | 71;
    ll operator()(ll x) const { return __builtin_bswap64(x*C); }
};
__gnu_pbds::gp_hash_table<ll,int,chash> h({},{},{},{},{1<<16});
```

PST.h

Description: Persistent segment tree with laziness

Time: $\mathcal{O}(\log N)$ per query, $\mathcal{O}((n+q)\log n)$ memory

3656e8, 39 lines

```
struct PST {
    PST *l = 0, *r = 0;
    int lo, hi;
    ll val = 0, lzadd = 0;
    PST(vl& v, int lo, int hi) : lo(lo), hi(hi) {
        if (lo + 1 < hi) {
            int mid = lo + (hi - lo)/2;
            l = new PST(v, lo, mid); r = new PST(v, mid, hi);
        }
        else val = v[lo];
    }
    ll query(int L, int R) {
        if (R < lo || hi < L) return 0; // idempotent
        if (L <= lo && hi <= R) return val;
        push();
        return l->query(L, R) + r->query(L, R);
    }
    PST * add(int L, int R, ll v) {
        if (R <= lo || hi <= L) return this;
        PST *n;
        if (L <= lo && hi <= R) {
            n = new PST(*this);
            n->val += v;
            n->lzadd += v;
        } else {
            push();
            n = new PST(*this);
            n->l = l->add(L, R, v);
```

```
        n->r = r->add(L, R, v);
    }
    return n;
}
void push() {
    if (lzadd == 0) return;
    l = l->add(lo, hi, lzadd);
    r = r->add(lo, hi, lzadd);
    lzadd = 0;
}
```

UnionFindRollback.h

Description: Disjoint-set data structure with undo. If undo is not needed, skip st, time() and rollback().

Usage: int t = uf.time(); ...; uf.rollback(t);

Time: $\mathcal{O}(\log(N))$

de4ad0, 21 lines

```
struct RollbackUF {
    vi e; vector<pii> st;
    RollbackUF(int n) : e(n, -1) {}
    int size(int x) { return -e[find(x)]; }
    int find(int x) { return e[x] < 0 ? x : find(e[x]); }
    int time() { return sz(st); }
    void rollback(int t) {
        for (int i = time(); i --> t;)
            e[st[i].first] = st[i].second;
        st.resize(t);
    }
    bool join(int a, int b) {
        a = find(a), b = find(b);
        if (a == b) return false;
        if (e[a] > e[b]) swap(a, b);
        st.push_back({a, e[a]});
        st.push_back({b, e[b]});
        e[a] += e[b]; e[b] = a;
        return true;
    }
};
```

LineContainer.h

Description: Container where you can add lines of the form $kx+m$, and query maximum values at points x . Useful for dynamic programming (“convex hull trick”).

Time: $\mathcal{O}(\log N)$

Sec1c7, 30 lines

```
struct Line {
    mutable ll k, m, p;
    bool operator<(const Line& o) const { return k < o.k; }
    bool operator<(ll x) const { return p < x; }
};

struct LineContainer : multiset<Line, less<>> {
    // (for doubles, use inf = 1/.0, div(a,b) = a/b)
    static const ll inf = LLONG_MAX;
    ll div(ll a, ll b) { // floored division
        return a / b - ((a ^ b) < 0 && a % b); }
    bool isect(iterator x, iterator y) {
        if (y == end()) return x->p = inf, 0;
        if (x->k == y->k) x->p = x->m > y->m ? inf : -inf;
        else x->p = div(y->m - x->m, x->k - y->k);
        return x->p >= y->p;
    }
    void add(ll k, ll m) {
        auto z = insert({k, m, 0}), y = z++, x = y;
        while (isect(y, z)) z = erase(z);
        if (x != begin() && isect(--x, y)) isect(x, y = erase(y));
        while ((y = x) != begin() && (--x)->p >= y->p)
            isect(x, erase(y));
```

```
    }
    ll query(ll x) {
        assert(!empty());
        auto l = *lower_bound(x);
        return l.k * x + l.m;
    }
};
```

Treap.h

Description: A short self-balancing tree. It acts as a sequential container with log-time splits/joins, and is easy to augment with additional data.

Time: $\mathcal{O}(\log N)$

635edf, 41 lines

```
struct node {
    int val, prior, sz = 1;
    node *left = nullptr, *right = nullptr;
    node(int val = 0) : val(val), prior(rand()) {}
};

int getSz(node *cur) { return cur ? cur->sz : 0; }
void recalc(node *cur) { cur->sz = getSz(cur->left) + getSz(cur->right) + 1; }

pair<node*, node*> split(node *cur, int v) {
    if (!cur) return {nullptr, nullptr};
    node *left, *right;
    if (getSz(cur->left) >= v) {
        right = cur;
        auto [L, R] = split(cur->left, v);
        left = L, right->left = R;
        recalc(right);
    }
    else {
        left = cur;
        auto [L, R] = split(cur->right, v - getSZ(cur->left) - 1);
        left->right = L, right = R;
        recalc(left);
    }
    return {left, right};
}
```

```
node* merge(node *t1, node *t2) {
    if (!t1 || !t2) return t1 ? t1 : t2;
    node *res;
    if (t1->prior > t2->prior) {
        res = t1;
        res->right = merge(t1->right, t2);
    }
    else {
        res = t2;
        res->left = merge(t1, t2->left);
    }
    recalc(res);
    return res;
}
```

FenwickTree.h

Description: Computes partial sums $a[0] + a[1] + \dots + a[pos - 1]$, and updates single elements $a[i]$, taking the difference between the old and new value.

Time: Both operations are $\mathcal{O}(\log N)$.

e62fac, 22 lines

```
struct FT {
    vector<ll> s;
    FT(int n) : s(n) {}
    void update(int pos, ll dif) { // a[pos] += dif
        for (; pos < sz(s); pos |= pos + 1) s[pos] += dif;
    }
    ll query(int pos) { // sum of values in [0, pos)
```

```
UCF

    ll res = 0;
    for (; pos > 0; pos &= pos - 1) res += s[pos-1];
    return res;
}
int lower_bound(ll sum) { // min pos st sum of [0, pos] >= sum
    // Returns n if no sum is >= sum, or -1 if empty sum is.
    if (sum <= 0) return -1;
    int pos = 0;
    for (int pw = 1 << 25; pw; pw >= 1) {
        if (pos + pw <= sz(s) && s[pos + pw-1] < sum)
            pos += pw, sum -= s[pos-1];
    }
    return pos;
};
```

FenwickTree2d.h

Description: Computes sums a[i,j] for all i<I, j<J, and increases single elements a[i,j]. Requires that the elements to be updated are known in advance (call fakeUpdate() before init()).
Time: $\mathcal{O}(\log^2 N)$. (Use persistent segment trees for $\mathcal{O}(\log N)$).

```
"FenwickTree.h" 157f07, 22 lines

struct FT2 {
    vector<vi> ys; vector<FT> ft;
    FT2(int limx) : ys(limx) {}
    void fakeUpdate(int x, int y) {
        for (; x < sz(ys); x |= x + 1) ys[x].push_back(y);
    }
    void init() {
        for (vi& v : ys) sort(all(v)), ft.emplace_back(sz(v));
    }
    int ind(int x, int y) {
        return (int)(lower_bound(all(ys[x]), y) - ys[x].begin()); }
    void update(int x, int y, ll dif) {
        for (; x < sz(ys); x |= x + 1)
            ft[x].update(ind(x, y), dif);
    }
    ll query(int x, int y) {
        ll sum = 0;
        for (; x; x &= x - 1)
            sum += ft[x-1].query(ind(x-1, y));
        return sum;
    }
};
```

RMQ.h

Description: Range Minimum Queries on an array. Returns min(V[a], V[a + 1], ... V[b - 1]) in constant time.
Usage: RMQ rmq(values);
rmq.query(inclusive, exclusive);
Time: $\mathcal{O}(|V|\log|V| + Q)$

```
template<class T>
struct RMQ {
    vector<vector<T>>> jmp;
    RMQ(const vector<T>& V) : jmp(1, V) {
        for (int pw = 1, k = 1; pw * 2 <= sz(V); pw *= 2, ++k) {
            jmp.emplace_back(sz(V) - pw * 2 + 1);
            rep(j, 0, sz(jmp[k]))
                jmp[k][j] = min(jmp[k - 1][j], jmp[k - 1][j + pw]);
        }
    }
    T query(int a, int b) {
        assert(a < b); // or return inf if a == b
        int dep = 31 - __builtin_clz(b - a);
        return min(jmp[dep][a], jmp[dep][b - (1 << dep)]);
    }
};
```

FenwickTree2d RMQ MoQueries

MoQueries.h

Description: Answer interval or tree path queries by finding an approximate TSP through the queries, and moving from one query to the next by adding/removing points at the ends. If values are on tree edges, change step to add/remove the edge (a,c) and remove the initial add call (but keep in).
Time: $\mathcal{O}(N\sqrt{Q})$

```
a12ef4, 49 lines

void add(int ind, int end) { ... } // add a[ind] (end = 0 or 1)
void del(int ind, int end) { ... } // remove a[ind]
int calc() { ... } // compute current answer

vi mo(vector<pii> Q) {
    int L = 0, R = 0, blk = 350; // ~N/sqrt(Q)
    vi s(sz(Q)), res = s;
#define K(x) pii(x.first/blk, x.second ^ -(x.first/blk & 1))
    iota(all(s), 0);
    sort(all(s), [&](int s, int t){ return K(Q[s]) < K(Q[t]); });
    for (int qi : s) {
        pii q = Q[qi];
        while (L > q.first) add(--L, 0);
        while (R < q.second) add(R++, 1);
        while (L < q.first) del(L++, 0);
        while (R > q.second) del(--R, 1);
        res[qi] = calc();
    }
    return res;
}

vi moTree(vector<array<int, 2>> Q, vector<vi>& ed, int root=0){
    int N = sz(ed), pos[2] = {}, blk = 350; // ~N/sqrt(Q)
    vi s(sz(Q)), res = s, I(N), L(N), R(N), in(N), par(N);
    add(0, 0), in[0] = 1;
    auto dfs = [&](int x, int p, int dep, auto& f) -> void {
        par[x] = p;
        L[x] = N;
        if (dep) I[x] = N++;
        for (int y : ed[x]) if (y != p) f(y, x, !dep, f);
        if (!dep) I[x] = N++;
        R[x] = N;
    };
    dfs(root, -1, 0, dfs);
#define K(x) pii(I[x[0]] / blk, I[x[1]] ^ -(I[x[0]] / blk & 1))
    iota(all(s), 0);
    sort(all(s), [&](int s, int t){ return K(Q[s]) < K(Q[t]); });
    for (int qi : s) rep(end,0,2) {
        int &a = pos[end], b = Q[qi][end], i = 0;
#define step(c) { if (in[c]) { del(a, end); in[a] = 0; } \
                    else { add(c, end); in[c] = 1; } a = c; }
        while (!(L[b] <= L[a] && R[a] <= R[b]))
            I[i++] = b, b = par[b];
        while (a != b) step(par[a]);
        while (i--) step(I[i]);
        if (end) res[qi] = calc();
    }
    return res;
}
```



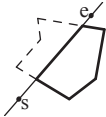
```
    if (cmp(lo, lo + 1) < R)
        a < R&& b >= 0 ? lo = md : hi = md - 1;
    else a < R || b <= 0 ? lo = md : hi = md - 1;
}
return -1; // point strictly inside hull
}
template<class P> pii hullTangents(vector<P>& h, P p) {
    return {getTangent<0>(h, p), getTangent<1>(h, p)};
}
```

PolygonCut.h

Description:
Returns a vector with the vertices of a polygon with every-thing to the left of the line going from s to e cut away.

Usage: vector<P> p = ...;
p = polygonCut(p, P(0,0), P(1,0));

"Point.h", "lineIntersection.h" f2b7d4, 13 lines



```
typedef Point<double> P;
vector<P> polygonCut(const vector<P>& poly, P s, P e) {
    vector<P> res;
    rep(i,0,sz(poly)) {
        P cur = poly[i], prev = i ? poly[i-1] : poly.back();
        bool side = s.cross(e, cur) < 0;
        if (side != (s.cross(e, prev) < 0))
            res.push_back(lineInter(s, e, cur, prev).second);
        if (side)
            res.push_back(cur);
    }
    return res;
}
```

halfplaneIntersection.h

Description: Returns the intersection of halfplanes as a polygon

Time: $\mathcal{O}(n \log n)$

d08058, 43 lines

```
const double eps = 1e-8;
typedef Point<double> P;
struct HalfPlane {
    P s, e, d;
    HalfPlane(P s = P(), P e = P()) : s(s), e(e), d(e - s) {}
    bool contains(P p) { return d.cross(p - s) > -eps; }
    bool operator<(HalfPlane hp) {
        if(abs(d.x) < eps && abs(hp.d.x) < eps)
            return d.y > 0 && hp.d.y < 0;
        bool side = d.x < eps || (abs(d.x) <= eps && d.y > 0);
        bool sideHp = hp.d.x < eps || (abs(hp.d.x) <= eps && hp.d.y > 0);
        if(side != sideHp) return side;
        return d.cross(hp.d) > 0;
    }
    P inter(HalfPlane hp) {
        auto p = hp.s.cross(e, hp.e), q = hp.s.cross(hp.e, s);
        return (s * p + e * q) / d.cross(hp.d);
    }
};

vector<P> hpIntersection(vector<HalfPlane> hps) {
    sort(all(hps));
    int n = sz(hps), l = 1, r = 0;
    vector<HalfPlane> dq(n+1);
    rep(i, 0, n) {
        while(l < r && !hps[i].contains(dq[r].inter(dq[r-1]))) r--;
        while(l < r && !hps[i].contains(dq[l].inter(dq[l+1]))) l++;
        dq[++r] = hps[i];
        if(l < r && abs(dq[r].d.cross(dq[r-1].d)) < eps) {
            if(dq[r].d.dot(dq[r-1].d) < 0) return {};
            r--;
            if(dq[r].contains(hps[i].s)) dq[r] = hps[i];
        }
    }
```

```
    }
    while(l < r - 1 && !dq[l].contains(dq[r].inter(dq[r-1]))) r
        --;
    while(l < r - 1 && !dq[r].contains(dq[l].inter(dq[l+1]))) l
        ++;
    if(l > r - 2) return {};
    vector<P> poly;
    rep(i, l, r)
        poly.push_back(dq[i].inter(dq[i+1]));
    poly.push_back(dq[r].inter(dq[l]));
    return poly;
}
```

centerOfMass.h

Description: Returns the center of mass for a polygon.

Memory: $\mathcal{O}(1)$

Time: $\mathcal{O}(n)$

ccce20, 8 lines

```
template<class P> P polygonCenter(const vector<P>& v) {
    P res(0, 0); double A = 0;
    for (int i = 0, j = sz(v) - 1; i < sz(v); j = i++) {
        res = res + (v[i] + v[j]) * v[j].cross(v[i]);
        A += v[j].cross(v[i]);
    }
    return res / A / 3;
}
```

minkowskiSum.h

Description: returns the minkowski sum of several polygons

13cd02, 30 lines

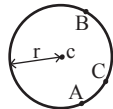
```
template<class P> vector<P> minkSum(vector<vector<P>>& spols) {
    P init(0, 0);
    vector<P> dir;
    for(auto poly: polys) {
        int n = sz(poly);
        if(n == 0) continue;
        init = init + poly[0];
        if(n == 1) continue;
        rep(i, 0, n)
            dir.push_back(poly[(i+1)%n] - poly[i]);
    }
    if(size(dir) == 0) return {init};
    sort(all(dir), [&](P a, P b)->bool {
        bool sideA = a.x > 0 || (a.x == 0 && a.y > 0);
        bool sideB = b.x > 0 || (b.x == 0 && b.y > 0);
        if(sideA != sideB) return sideA;
        return a.cross(b) > 0;
    });
    vector<P> sum;
    P cur = init;
    rep(i, 0, sz(dir)) {
        sum.push_back(cur);
        cur = cur + dir[i];
    }
    return sum;
}
```

4.3 Circles

circumcircle.h

Description:

The circumcircle of a triangle is the circle intersecting all three vertices. ccRadius returns the radius of the circle going through points A, B and C and ccCenter returns the center of the same circle.



```
"Point.h" 1caa3a, 9 lines
typedef Point<double> P;
double ccRadius(const P& A, const P& B, const P& C) {
    return (B-A).dist()*(C-B).dist()*(A-C).dist() /
        abs((B-A).cross(C-A))/2;
}
P ccCenter(const P& A, const P& B, const P& C) {
    P b = C-A, c = B-A;
    return A + (b*c.dist2()-c*b.dist2()).perp()/b.cross(c)/2;
}
```

CircleLine.h

Description: Finds the intersection between a circle and a line. Returns a vector of either 0, 1, or 2 intersection points. P is intended to be Point<double>.

```
"Point.h" e0cfba, 9 lines
template<class P>
vector<P> circleLine(P c, double r, P a, P b) {
    P ab = b - a, p = a + ab * (c-a).dot(ab) / ab.dist2();
    double s = a.cross(b, c), h2 = r*r - s*s / ab.dist2();
    if (h2 < 0) return {};
    if (h2 == 0) return {p};
    P h = ab.unit() * sqrt(h2);
    return {p - h, p + h};
}
```

CircleIntersection.h

Description: Computes the pair of points at which two circles intersect. Returns false in case of no intersection.

```
"Point.h" 84d6d3, 11 lines
typedef Point<double> P;
bool circleInter(P a,P b,double r1,double r2,pair<P, P>* out) {
    if (a == b) { assert(r1 != r2); return false; }
    P vec = b - a;
    double d2 = vec.dist2(), sum = r1+r2, dif = r1-r2,
        p = (d2 + r1*r1 - r2*r2)/(d2*2), h2 = r1*r1 - p*p*d2;
    if (sum*sum < d2 || dif*dif > d2) return false;
    P mid = a + vec*p, per = vec.perp() * sqrt(fmax(0, h2) / d2);
    *out = {mid + per, mid - per};
    return true;
}
```

CirclePolygonIntersection.h

Description: Returns the area of the intersection of a circle with a ccw polygon.

```
Time:  $\mathcal{O}(n)$ 
".../content/geometry/Point.h" alee63, 19 lines
typedef Point<double> P;
#define arg(p, q) atan2(p.cross(q), p.dot(q))
double circlePoly(P c, double r, vector<P> ps) {
    auto tri = [&](P p, P q) {
        auto r2 = r * r / 2;
        P d = q - p;
        auto a = d.dot(p)/d.dist2(), b = (p.dist2()-r*r)/d.dist2();
        auto det = a * a - b;
        if (det <= 0) return arg(p, q) * r2;
        auto s = max(0., -a-sqrt(det)), t = min(1., -a+sqrt(det));
        if (t < 0 || 1 <= s) return arg(p, q) * r2;
        P u = p + d * s, v = p + d * t;
        return arg(p,u) * r2 + u.cross(v)/2 + arg(v,q) * r2;
    };
    auto sum = 0.0;
```



```
    rep(i,0,sz(ps))
        sum += tri(ps[i] - c, ps[(i + 1) % sz(ps)] - c);
    return sum;
}
```

CircleTangents.h

Description: Finds the external tangents of two circles, or internal if r2 is negated. Can return 0, 1, or 2 tangents – 0 if one circle contains the other (or overlaps it, in the internal case, or if the circles are the same); 1 if the circles are tangent to each other (in which case .first = .second and the tangent line is perpendicular to the line between the centers). .first and .second give the tangency points at circle 1 and 2 respectively. To find the tangents of a circle with a point set r2 to 0.

"Point.h"	b0153d, 13 lines
<pre>template<class P> vector<pair<P, P>> tangents(P c1, double r1, P c2, double r2) { P d = c2 - c1; double dr = r1 - r2, d2 = d.dist2(), h2 = d2 - dr * dr; if (d2 == 0 h2 < 0) return {}; vector<pair<P, P>> out; for (double sign : {-1, 1}) { P v = (d * dr + d.perp() * sqrt(h2) * sign) / d2; out.push_back({c1 + v * r1, c2 + v * r2}); } if (h2 == 0) out.pop_back(); return out; }</pre>	

MinimumEnclosingCircle.h

Description: Computes the minimum circle that encloses a set of points.
Time: expected $\mathcal{O}(n)$

"circumcircle.h"	09dd0a, 17 lines
<pre>pair<P, double> mec(vector<P> ps) { shuffle(all(ps), mt19937(time(0))); P o = ps[0]; double r = 0, EPS = 1 + 1e-8; rep(i,0,sz(ps)) if ((o - ps[i]).dist() > r * EPS) { o = ps[i], r = 0; rep(j,0,i) if ((o - ps[j]).dist() > r * EPS) { o = (ps[i] + ps[j]) / 2; r = (o - ps[i]).dist(); rep(k,0,j) if ((o - ps[k]).dist() > r * EPS) { o = ccCenter(ps[i], ps[j], ps[k]); r = (o - ps[i]).dist(); } } } return {o, r}; }</pre>	

4.4 Miscellaneous

ClosestPair.h

Description: Finds the closest pair of points.
Time: $\mathcal{O}(n \log n)$

"Point.h"	ac41a6, 17 lines
<pre>typedef Point<ll> P; pair<P, P> closest(vector<P> v) { assert(sz(v) > 1); set<P> S; sort(all(v), [](P a, P b) { return a.y < b.y; }); pair<ll, pair<P, P>> ret{LLONG_MAX, {P(), P()}}; int j = 0; for (P p : v) { P d{1 + (ll)sqrt(ret.first), 0}; while (v[j].y <= p.y - d.x) S.erase(v[j++]); auto lo = S.lower_bound(p - d), hi = S.upper_bound(p + d); for (; lo != hi; ++lo)</pre>	

"Point.h"	eefdf5, 88 lines
<pre> ret = min(ret, {(*lo - p).dist2(), { *lo, p}}); S.insert(p); } return ret.second; }</pre>	

FastDelaunay.h

Description: Fast Delaunay triangulation. Each circumcircle contains none of the input points. There must be no duplicate points. If all points are on a line, no triangles will be returned. Should work for doubles as well, though there may be precision issues in 'circ'. Returns triangles in order {t[0][0], t[0][1], t[0][2], t[1][0], ... }, all counter-clockwise.

"Point.h"	eefdf5, 88 lines
<pre>typedef Point<ll> P; typedef struct Quad* Q; typedef __int128_t ll1; // (can be ll if coords are < 2e4) P arb(LLONG_MAX, LLONG_MAX); // not equal to any other point</pre>	

<pre>struct Quad { Q rot, o; P p = arb; bool mark; P& F() { return r()->p; } Q& r() { return rot->rot; } Q prev() { return rot->o->rot; } Q next() { return r()->prev(); } } *H; bool circ(P p, P a, P b, P c) { // is p in the circumcircle? ll1 p2 = p.dist2(), A = a.dist2()-p2, B = b.dist2()-p2, C = c.dist2()-p2; return p.cross(a,b)*C + p.cross(b,c)*A + p.cross(c,a)*B > 0; } Q makeEdge(P orig, P dest) { Q r = H ? H : new Quad{new Quad{new Quad{0}}}; H = r->o; r->r()->r() = r; rep(i,0,4) r = r->rot, r->p = arb, r->o = i & 1 ? r : r->r(); r->p = orig; r->F() = dest; return r; } void splice(Q a, Q b) { swap(a->o->rot->o, b->o->rot->o); swap(a->o, b->o); } Q connect(Q a, Q b) { Q q = makeEdge(a->F(), b->p); splice(q, a->next()); splice(q->r(), b); return q; } pair<Q,Q> rec(const vector<P>& s) { if (sz(s) <= 3) { Q a = makeEdge(s[0], s[1]), b = makeEdge(s[1], s.back()); if (sz(s) == 2) return { a, a->r() }; splice(a->r(), b); auto side = s[0].cross(s[1], s[2]); Q c = side ? connect(b, a) : 0; return {side < 0 ? c->r() : a, side < 0 ? c : b->r() }; } #define H(e) e->F(), e->p #define valid(e) (e->F().cross(H(base)) > 0) Q A, B, ra, rb; int half = sz(s) / 2; tie(ra, A) = rec({all(s) - half}); tie(B, rb) = rec({sz(s) - half + all(s)}); while ((B->p.cross(H(A)) < 0 && (A = A->next()) (A->p.cross(H(B)) > 0 && (B = B->r()->o))); Q base = connect(B->r(), A); if (A->p == ra->p) ra = base->r();</pre>	
--	--

<pre> if (B->p == rb->p) rb = base; #define DEL(e, init, dir) Q e = init->dir; if (valid(e)) \ while (circ(e->dir->F(), H(base), e->F())) { \ Q t = e->dir; \ splice(e, e->prev()); \ splice(e->r(), e->r()->prev()); \ e->o = H; H = e; e = t; \ } for (;;) { DEL(LC, base->r(), o); DEL(RC, base, prev()); if (!valid(LC) && !valid(RC)) break; if (!valid(LC) (valid(RC) && circ(H(RC), H(LC)))) base = connect(RC, base->r()); else base = connect(base->r(), LC->r()); } return { ra, rb }; }</pre>	
---	--

<pre>vector<P> triangulate(vector<P> pts) { sort(all(pts)); assert(unique(all(pts)) == pts.end()); if (sz(pts) < 2) return {}; Q e = rec(pts).first; vector<Q> q = {e}; int qi = 0; while (e->o->F().cross(e->F(), e->p) < 0) e = e->o; #define ADD { Q c = e; do { c->mark = 1; pts.push_back(c->p); \ q.push_back(c->r()); c = c->next(); } while (c != e); } ADD; pts.clear(); while (qi < sz(q)) if (!(e = q[qi++])->mark) ADD; return pts; }</pre>	
--	--

PlanarFaceExtraction.h

Description: Given a planar graph and where the points are, extract the set of faces that the graph makes
Time: $\mathcal{O}(E \log E)$

63f230, 39 lines	
<pre>template<class P> vector<vector<P>> extract_faces(vvi adj, vector<P> pts) { int n = sz(pts); #define cmp(i) [&](int pi, int qi) -> bool { \ P p = pts[pi] - pts[i], q = pts[qi] - pts[i]; \ bool sideP = p.y < 0 (p.y == 0 && p.x < 0); \ bool sideQ = q.y < 0 (q.y == 0 && q.x < 0); \ if(sideP != sideQ) return sideP; \ return p.cross(q) > 0; } rep(i, 0, n) sort(all(adj[i]), cmp(i)); vvi ed; rep(i, 0, n) for(int j: adj[i]) ed.emplace_back(i, j); sort(all(ed)); auto get_idx = [&](int i, int j) -> int { return lower_bound(all(ed), pii(i, j))-begin(ed); }; vector<vector<P>> faces; vi used(sz(ed)); rep(i, 0, n) for(int j: adj[i]) { if(used[get_idx(i, j)]) continue; used[get_idx(i, j)] = true; vector<P> face = {pts[i]}; int prv = i, cur = j; while(cur != i) { face.push_back(pts[cur]); auto it = lower_bound(all(adj[cur]), prv, cmp(cur)); if(it == begin(adj[cur]))</pre>	

```
        it = end(adj[cur]);
        prv = cur, cur = *prev(it);
        used[get_idx(prv, cur)] = true;
    }
    faces.push_back(face);
}
#undef cmp
return faces;
}
```

Graphs (5)

5.1 Network flow

MinCostMaxFlow.h

Description: Min-cost max-flow. Negative cost cycles not supported. To obtain the actual flow, look at positive values only.
Time: Approximately $\mathcal{O}(E^2)$, actually $\mathcal{O}(FS)$ where S is the time complexity of the SSSP alg used in find path (in this case SPFA)

```
struct mcmf {
    const ll inf = LLONG_MAX >> 2;
    struct edge {
        int v;
        ll cap, flow, cost;
    };
    int n;
    vector<edge> edges;
    vvi adj; vii par; vi in_q;
    vector<ll> dist, pi;
    mcmf(int n): n(n), adj(n), dist(n), pi(n), par(n), in_q(n) {}
    void add_edge(int u, int v, ll cap, ll cost) {
        int idx = sz(edges);
        edges.push_back({v, cap, 0, cost});
        edges.push_back({u, cap, cap, -cost});
        adj[u].push_back(idx);
        adj[v].push_back(idx ^ 1);
    }
    bool find_path(int s, int t) {
        fill(all(dist), inf);
        fill(all(in_q), 0);
        queue<int> q; q.push(s);
        dist[s] = 0, in_q[s] = 1;
        while(!q.empty()) {
            int cur = q.front(); q.pop();
            in_q[cur] = 0;
            for(int idx: adj[cur]) {
                auto [nxt, cap, fl, wt] = edges[idx];
                ll nxtD = dist[cur] + wt;
                if(fl >= cap || nxtD >= dist[nxt]) continue;
                dist[nxt] = nxtD;
                par[nxt] = {cur, idx};
                if(in_q[nxt]) continue;
                q.push(nxt); in_q[nxt] = 1;
            }
        }
        return dist[t] < inf;
    }
    pair<ll, ll> calc(int s, int t) {
        ll flow = 0, cost = 0;
        while(find_path(s, t)) {
            rep(i, 0, n) pi[i] = min(pi[i] + dist[i], inf);
            ll f = inf;
            for(int i, u, v = t; tie(u, i) = par[v], v != s; v = u)
                f = min(f, edges[i].cap - edges[i].flow);
            flow += f;
            for(int i, u, v = t; tie(u, i) = par[v], v != s; v = u)
                edges[i].flow += f, edges[i^1].flow -= f;
        }
        rep(i, 0, sz(edges)>>1)
            cost += edges[i<<1].cost * edges[i<<1].flow;

        return {flow, cost};
    }
};
```

MinCostMaxFlowDijkstra.h

Description: If SPFA TLEs, swap the find_path function in MCMF with the one below and in_q with seen. If negative edge weights can occur, initialize pi with the shortest path from the source to each node using Bellman-Ford. Negative weight cycles not supported.

```
bool findPath(int s, int t) {
    fill(all(dist), inf);
    fill(all(seen), 0);
    dist[s] = 0;
    __gnu_pbds::priority_queue<pair<ll, int>> pq;
    vector<decltype(pq)::point_iterator> its(n);
    pq.push({0, s});
    while(!pq.empty()) {
        auto [d, cur] = pq.top(); pq.pop(); d *= -1;
        seen[cur] = 1;
        if(dist[cur] < d) continue;
        for(int idx: adj[cur]) {
            auto [nxt, cap, f, wt] = edges[idx];
            ll nxtD = d + wt + pi[cur] - pi[nxt];
            if(f >= cap || nxtD >= dist[nxt] || seen[nxt]) continue;
            dist[nxt] = nxtD;
            par[nxt] = {cur, idx};
            if(its[nxt] == pq.end()) its[nxt] = pq.push({-nxtD, nxt});
            else pq.modify(its[nxt], {-nxtD, nxt});
        }
    }
    rep(i, 0, n) pi[i] = min(pi[i] + dist[i], inf);
    return seen[t];
}
```

Dinic.h

Description: Flow algorithm with complexity $\mathcal{O}(VE \log U)$ where $U = \max|cap|$. $\mathcal{O}(\min(E^{1/2}, V^{2/3})E)$ if $U = 1$; $\mathcal{O}(\sqrt{VE})$ for bipartite matching.

```
struct Dinic {
    struct Edge {
        int to, rev;
        ll c, oc;
        ll flow() { return max(oc - c, 0LL); } // if you need flows
    };
    vi lvl, ptr, q;
    vector<vector<Edge>> adj;
    Dinic(int n) : lvl(n), ptr(n), q(n), adj(n) {}
    void addEdge(int a, int b, ll c, ll rcap = 0) {
        adj[a].push_back({b, sz(adj[b]), c, c});
        adj[b].push_back({a, sz(adj[a]) - 1, rcap, rcap});
    }
    ll dfs(int v, int t, ll f) {
        if (v == t || !f) return f;
        for (int& i = ptr[v]; i < sz(adj[v]); i++) {
            Edge& e = adj[v][i];
            if (lvl[e.to] == lvl[v] + 1)
                if (ll p = dfs(e.to, t, min(f, e.c))) {
                    e.c -= p, adj[e.to][e.rev].c += p;
                    return p;
                }
        }
        return 0;
    }
    ll calc(int s, int t) {
        ll flow = 0; q[0] = s;
        rep(L, 0, 31) do { // 'int L=30' maybe faster for random data
            lvl = ptr = vi(sz(q));
            int qi = 0, qe = lvl[s] = 1;
            while (qi < qe && !lvl[t]) {
                int v = q[qi++];
```

```
for (Edge e : adj[v])
    if (!lvl[e.to] && e.c >> (30 - L))
        q[qe++] = e.to, lvl[e.to] = lvl[v] + 1;
}
while (ll p = dfs(s, t, LLONG_MAX)) flow += p;
} while (lvl[t]);
return flow;
}
bool leftOfMinCut(int a) { return lvl[a] != 0; }
};
```

GlobalMinCut.h

Description: Find a global minimum cut in an undirected graph, as represented by an adjacency matrix.

```
Time:  $\mathcal{O}(V^3)$ 
pair<int, vi> globalMinCut(vector<vi> mat) {
    pair<int, vi> best = {INT_MAX, {}};
    int n = sz(mat);
    vector<vi> co(n);
    rep(i, 0, n) co[i] = {i};
    rep(ph, 1, n) {
        vi w = mat[0];
        size_t s = 0, t = 0;
        rep(it, 0, n-ph) { //  $\mathcal{O}(V^2) \rightarrow \mathcal{O}(E \log V)$  with prio. queue
            w[it] = INT_MIN;
            s = t, t = max_element(all(w)) - w.begin();
            rep(i, 0, n) w[i] += mat[t][i];
        }
        best = min(best, {w[t] - mat[t][t], co[t]});
        co[s].insert(co[s].end(), all(co[t]));
        rep(i, 0, n) mat[s][i] += mat[t][i];
        rep(i, 0, n) mat[i][s] = mat[s][i];
        mat[0][t] = INT_MIN;
    }
    return best;
}
```

GomoryHu.h

Description: Given a list of edges representing an undirected flow graph, returns edges of the Gomory-Hu tree. The max flow between any pair of vertices is given by minimum edge weight along the Gomory-Hu tree path.

```
Time:  $\mathcal{O}(V)$  Flow Computations
"Dinic.h"
typedef array<ll, 3> Edge;
vector<Edge> gomoryHu(int N, vector<Edge> ed) {
    vector<Edge> tree;
    vi par(N);
    rep(i, 1, N) {
        Dinic D(N);
        for (Edge t : ed) D.addEdge(t[0], t[1], t[2], t[2]);
        tree.push_back({i, par[i], D.calc(i, par[i])});
        rep(j, i+1, N)
            if (par[j] == par[i] && D.leftOfMinCut(j)) par[j] = i;
    }
    return tree;
}
```

MatroidIntersection.h

Description: Given two matroids, finds the largest common independent set. For the color and graph matroids, this would be the largest forest where no two edges are the same color. A matroid has 3 functions

- check(int x): returns if current matroid can add x without becoming dependent
- add(int x): adds an element to the matroid (guaranteed to never make it dependent)
- clear(): sets the matroid to the empty matroid

The matroid is given an int representing the element, and is expected to convert it (e.g: the color or the endpoints) Pass the matroid with more expensive add/clear operations to M1.

Time: $R^2 N(M2.add + M1.check + M2.check) + R^3 M1.add + R^2 M1.clear + R N M2.clear$

"../data-structures/UnionFind.h" 9812a7, 60 lines

```
struct ColorMat {
    vi cnt, clr;
    ColorMat(int n, vector<int> clr) : cnt(n), clr(clr) {}
    bool check(int x) { return !cnt[clr[x]]; }
    void add(int x) { cnt[clr[x]]++; }
    void clear() { fill(all(cnt), 0); }
};

struct GraphMat {
    UF uf;
    vector<array<int, 2>> e;
    GraphMat(int n, vector<array<int, 2>> e) : uf(n), e(e) {}
    bool check(int x) { return !uf.sameSet(e[x][0], e[x][1]); }
    void add(int x) { uf.join(e[x][0], e[x][1]); }
    void clear() { uf = UF(sz(uf.e)); }
};

template<class M1, class M2> struct MatroidIsect {
    int n;
    vector<char> iset;
    M1 m1; M2 m2;
    MatroidIsect(M1 m1, M2 m2, int n) : n(n), iset(n + 1), m1(m1), m2(m2) {}

    vi solve() {
        rep(i, 0, n) if (m1.check(i) && m2.check(i))
            iset[i] = true, m1.add(i), m2.add(i);
        while (augment());
        vi ans;
        rep(i, 0, n) if (iset[i]) ans.push_back(i);
        return ans;
    }

    bool augment() {
        vector<int> frm(n, -1);
        queue<int> q({n}); // starts at dummy node
        auto fwdE = [&](int a) {
            vi ans;
            m1.clear();
            rep(v, 0, n) if (iset[v] && v != a) m1.add(v);
            rep(b, 0, n) if (!iset[b] && frm[b] == -1 && m1.check(b))
                ans.push_back(b), frm[b] = a;
            return ans;
        };
        auto backE = [&](int b) {
            m2.clear();
            rep(cas, 0, 2) rep(v, 0, n)
                if ((v == b || iset[v]) && (frm[v] == -1) == cas) {
                    if (!m2.check(v))
                        return cas ? q.push(v), frm[v] = b, v : -1;
                    m2.add(v);
                }
            return n;
        };
        while (!q.empty()) {
            int a = q.front(), c; q.pop();
            for (int b : fwdE(a))
                while ((c = backE(b)) >= 0) if (c == n) {
                    while (b != n) iset[b] ^= 1, b = frm[b];
                }
        }
    }
};

while (!q.empty()) {
    int a = q.front(), c; q.pop();
    for (int b : fwdE(a))
        while ((c = backE(b)) >= 0) if (c == n) {
            while (b != n) iset[b] ^= 1, b = frm[b];
        }
}
```

```
        return true;
    }
}

return false;
};
```

5.2 Matching

hopcroftKarp.h

Description: Fast bipartite matching algorithm. Graph g should be a list of neighbors of the left partition, and $btoa$ should be a vector full of -1's of the same size as the right partition. Returns the size of the matching. $btoa[i]$ will be the match for vertex i on the right side, or -1 if it's not matched.

Usage: vi btoa(m, -1); hopcroftKarp(g, btoa);

Time: $\mathcal{O}(\sqrt{VE})$

f612e4, 42 lines

```
bool dfs(int a, int L, vector<vi>& g, vi& btoa, vi& A, vi& B) {
    if (A[a] != L) return 0;
    A[a] = -1;
    for (int b : g[a]) if (B[b] == L + 1) {
        B[b] = 0;
        if (btoa[b] == -1 || dfs(btoa[b], L + 1, g, btoa, A, B))
            return btoa[b] = a, 1;
    }
    return 0;
}
```

```
int hopcroftKarp(vector<vi>& g, vi& btoa) {
    int res = 0;
    vi A(g.size()), B(btoa.size()), cur, next;
    for (;;) {
        fill(all(A), 0);
        fill(all(B), 0);
        cur.clear();
        for (int a : btoa) if (a != -1) A[a] = -1;
        rep(a, 0, sz(g)) if (A[a] == 0) cur.push_back(a);
        for (int lay = 1;; lay++) {
            bool islast = 0;
            next.clear();
            for (int a : cur) for (int b : g[a]) {
                if (btoa[b] == -1) {
                    B[b] = lay;
                    islast = 1;
                }
                else if (btoa[b] != a && !B[b]) {
                    B[b] = lay;
                    next.push_back(btoa[b]);
                }
            }
            if (islast) break;
            if (next.empty()) return res;
            for (int a : next) A[a] = lay;
            cur.swap(next);
        }
        rep(a, 0, sz(g))
            res += dfs(a, 0, g, btoa, A, B);
    }
}
```

DFSMatching.h

Description: Simple bipartite matching algorithm. Graph g should be a list of neighbors of the left partition, and $btoa$ should be a vector full of -1's of the same size as the right partition. Returns the size of the matching. $btoa[i]$ will be the match for vertex i on the right side, or -1 if it's not matched.

Usage: vi btoa(m, -1); dfsMatching(g, btoa);

Time: $\mathcal{O}(VE)$

522b98, 22 lines

```
bool find(int j, vector<vi>& g, vi& btoa, vi& vis) {
```

```
    if (btoa[j] == -1) return 1;
    vis[j] = 1; int di = btoa[j];
    for (int e : g[di])
        if (!vis[e] && find(e, g, btoa, vis)) {
            btoa[e] = di;
            return 1;
        }
    return 0;
}

int dfsMatching(vector<vi>& g, vi& btoa) {
    vi vis;
    rep(i, 0, sz(g)) {
        vis.assign(sz(btoa), 0);
        for (int j : g[i])
            if (find(j, g, btoa, vis)) {
                btoa[j] = i;
                break;
            }
    }
    return sz(btoa) - (int)count(all(btoa), -1);
}
```

MinimumVertexCover.h

Description: Finds a minimum vertex cover in a bipartite graph. The size is the same as the size of a maximum matching, and the complement is a maximum independent set.

"DFSMatching.h" da4196, 20 lines

```
vi cover(vector<vi>& g, int n, int m) {
    vi match(m, -1);
    int res = dfsMatching(g, match);
    vector<bool> lfound(n, true), seen(m);
    for (int it : match) if (it != -1) lfound[it] = false;
    vi q, cover;
    rep(i, 0, n) if (lfound[i]) q.push_back(i);
    while (!q.empty()) {
        int i = q.back(); q.pop_back();
        lfound[i] = 1;
        for (int e : g[i]) if (!seen[e] && match[e] != -1) {
            seen[e] = true;
            q.push_back(match[e]);
        }
    }
    rep(i, 0, n) if (!lfound[i]) cover.push_back(i);
    rep(i, 0, m) if (seen[i]) cover.push_back(n+i);
    assert(sz(cover) == res);
    return cover;
}
```

WeightedMatching.h

Description: Given a weighted bipartite graph, matches every node on the left with a node on the right such that no nodes are in two matchings and the sum of the edge weights is minimal. Takes cost[N][M], where cost[i][j] = cost for L[i] to be matched with R[j] and returns (min cost, match), where L[i] is matched with R[match[i]]. Negate costs for max cost. Requires $N \leq M$.

Time: $\mathcal{O}(N^2 M)$

1e0fe9, 31 lines

```
pair<int, vi> hungarian(const vector<vi> &a) {
    if (a.empty()) return {0, {}};
    int n = sz(a) + 1, m = sz(a[0]) + 1;
    vi u(n), v(m), p(m), ans(n - 1);
    rep(i, 1, n) {
        p[0] = i;
        int j0 = 0; // add "dummy" worker 0
        vi dist(m, INT_MAX), pre(m, -1);
        vector<bool> done(m + 1);
        do { // dijkstra
            done[j0] = true;
            int i0 = p[j0], j1, delta = INT_MAX;
            rep(j, 1, m) if (!done[j]) {
```

```

    auto cur = a[i0 - 1][j - 1] - u[i0] - v[j];
    if (cur < dist[j]) dist[j] = cur, pre[j] = j0;
    if (dist[j] < delta) delta = dist[j], j1 = j;
}
rep(j,0,m) {
    if (done[j]) u[p[j]] += delta, v[j] -= delta;
    else dist[j] -= delta;
}
j0 = j1;
} while (p[j0]);
while (j0) { // update alternating path
    int j1 = pre[j0];
    p[j0] = p[j1], j0 = j1;
}
}
rep(j,1,m) if (p[j]) ans[p[j] - 1] = j - 1;
return {-v[0], ans}; // min cost
}

```

GeneralMatching.h

Description: Matching for general graphs. Fails with probability N/mod .
Time: $\mathcal{O}(N^3)$

../numerical/MatrixInverse-mod.h cb1912, 40 lines

```

vector<pii> generalMatching(int N, vector<pii>& ed) {
    vector<vector<ll>> mat(N, vector<ll>(N)), A;
    for (pii pa : ed) {
        int a = pa.first, b = pa.second, r = rand() % mod;
        mat[a][b] = r, mat[b][a] = (mod - r) % mod;
    }

    int r = matInv(A = mat), M = 2*N - r, fi, fj;
    assert(r % 2 == 0);

    if (M != N) do {
        mat.resize(M, vector<ll>(M));
        rep(i,0,N) {
            mat[i].resize(M);
            rep(j,N,M) {
                int r = rand() % mod;
                mat[i][j] = r, mat[j][i] = (mod - r) % mod;
            }
        }
    } while (matInv(A = mat) != M);

    vi has(M, 1); vector<pii> ret;
    rep(it,0,M/2) {
        rep(i,0,M) if (has[i])
            rep(j,i+1,M) if (A[i][j] && mat[i][j]) {
                fi = i; fj = j; goto done;
            }
        assert(0); done:
        if (fj < N) ret.emplace_back(fi, fj);
        has[fi] = has[fj] = 0;
        rep(sw,0,2) {
            ll a = modpow(A[fi][fj], mod-2);
            rep(i,0,M) if (has[i] && A[i][fj]) {
                ll b = A[i][fj] * a % mod;
                rep(j,0,M) A[i][j] = (A[i][j] - A[fi][j] * b) % mod;
            }
            swap(fi, fj);
        }
    }
    return ret;
}

```

5.3 DFS algorithms

SCC.h

Description: Finds strongly connected components in a directed graph. If vertices u, v belong to the same component, we can reach u from v and vice versa.

Usage: scc(graph, [&](vi& v) { ... }) visits all components in reverse topological order. comp[i] holds the component index of a node (a component only has edges to components with lower index). ncomps will contain the number of components.

Time: $\mathcal{O}(E+V)$ 76b5c9, 24 lines

```

vi val, comp, z, cont;
int Time, ncomps;
template<class G, class F> int dfs(int j, G& g, F& f) {
    int low = val[j] = ++Time, x; z.push_back(j);
    for (auto e : g[j]) if (comp[e] < 0)
        low = min(low, val[e] ?: dfs(e,g,f));

    if (low == val[j]) {
        do {
            x = z.back(); z.pop_back();
            comp[x] = ncomps;
            cont.push_back(x);
        } while (x != j);
        f(cont); cont.clear();
        ncomps++;
    }
    return val[j] = low;
}
template<class G, class F> void scc(G& g, F f) {
    int n = sz(g);
    val.assign(n, 0); comp.assign(n, -1);
    Time = ncomps = 0;
    rep(i,0,n) if (comp[i] < 0) dfs(i, g, f);
}

```

BiconnectedComponents.h

Description: Finds all biconnected components in an undirected graph, and runs a callback for the edges in each. In a biconnected component there are at least two distinct paths between any two nodes. Note that a node can be in several components. An edge which is not in a component is a bridge, i.e., not part of any cycle.

Usage: int eid = 0; ed.resize(N);
 for each edge (a,b) {
 ed[a].emplace_back(b, eid);
 ed[b].emplace_back(a, eid++); }
 bicomps([&](const vi& edgelist) {...});

Time: $\mathcal{O}(E+V)$ 2965e5, 33 lines

```

vi num, st;
vector<vector<pii>> ed;
int Time;
template<class F>
int dfs(int at, int par, F& f) {
    int me = num[at] = ++Time, e, y, top = me;
    for (auto pa : ed[at]) if (pa.second != par) {
        tie(y, e) = pa;
        if (num[y]) {
            top = min(top, num[y]);
            if (num[y] < me)
                st.push_back(e);
        } else {
            int si = sz(st);
            int up = dfs(y, e, f);
            top = min(top, up);
            if (up == me) {
                st.push_back(e);
                f(vi(st.begin() + si, st.end()));
                st.resize(si);
            }
        }
    }
}

```

```

    }
    else if (up < me) st.push_back(e);
    else { /* e is a bridge */ }
}
}
return top;
}

template<class F>
void bicomps(F f) {
    num.assign(sz(ed), 0);
    rep(i,0,sz(ed)) if (!num[i]) dfs(i, -1, f);
}

```

2sat.h

Description: Calculates a valid assignment to boolean variables a, b, c, \dots to a 2-SAT problem, so that an expression of the type $(a \vee b) \wedge (a \vee c) \wedge (d \vee b) \wedge \dots$ becomes true, or reports that it is unsatisfiable. Negated variables are represented by bit-inversions ($\sim x$).

Usage: TwoSat ts(number of boolean variables);
 ts.either(0, ~3); // Var 0 is true or var 3 is false
 ts.setValue(2); // Var 2 is true
 ts.atMostOne({0, ~1, 2}); // ≤ 1 of vars 0, ~1 and 2 are true
 ts.solve(); // Returns true iff it is solvable
 ts.values[0..N-1] holds the assigned values to the vars
Time: $\mathcal{O}(N+E)$, where N is the number of boolean variables, and E is the number of clauses. 5f9706, 56 lines

```

struct TwoSat {
    int N;
    vector<vi> gr;
    vi values; // 0 = false, 1 = true

    TwoSat(int n = 0) : N(n), gr(2*n) {}

    int addVar() { // (optional)
        gr.emplace_back();
        gr.emplace_back();
        return N++;
    }

    void either(int f, int j) {
        f = max(2*f, -1-2*f);
        j = max(2*j, -1-2*j);
        gr[f].push_back(j^1);
        gr[j].push_back(f^1);
    }

    void setValue(int x) { either(x, x); }

    void atMostOne(const vi& li) { // (optional)
        if (sz(li) <= 1) return;
        int cur = ~li[0];
        rep(i,2,sz(li)) {
            int next = addVar();
            either(cur, ~li[i]);
            either(cur, next);
            either(~li[i], next);
            cur = ~next;
        }
        either(cur, ~li[1]);
    }

    vi val, comp, z; int time = 0;
    int dfs(int i) {
        int low = val[i] = ++time, x; z.push_back(i);
        for(int e : gr[i]) if (!comp[e])
            low = min(low, val[e] ?: dfs(e));
        if (low == val[i]) do {
            x = z.back(); z.pop_back();
        }
    }
}

```

```
    comp[x] = low;
    if (values[x>>1] == -1)
        values[x>>1] = x&1;
} while (x != i);
return val[i] = low;
}

bool solve() {
    values.assign(N, -1);
    val.assign(2*N, 0); comp = val;
    rep(i,0,2*N) if (!comp[i]) dfs(i);
    rep(i,0,N) if (comp[2*i] == comp[2*i+1]) return 0;
    return 1;
}
};
```

EulerWalk.h

Description: Eulerian undirected/directed path/cycle algorithm. Input should be a vector of (dest, global edge index), where for undirected graphs, forward/backward edges have the same index. Returns a list of nodes in the Eulerian path/cycle with src at both start and end, or empty list if no cycle/path exists. To get edge indices back, add .second to s and ret.

Time: $\mathcal{O}(V + E)$

```
780b64, 15 lines
vi eulerWalk(vector<vector<pii>>& gr, int nedges, int src=0) {
    int n = sz(gr);
    vi D(n), its(n), eu(nedges), ret, s = {src};
    D[src]++; // to allow Euler paths, not just cycles
    while (!s.empty()) {
        int x = s.back(), y, e, &it = its[x], end = sz(gr[x]);
        if (it == end){ ret.push_back(x); s.pop_back(); continue; }
        tie(y, e) = gr[x][it++];
        if (!eu[e]) {
            D[x]--, D[y]++;
            eu[e] = 1; s.push_back(y);
        }
    }
    for (int x : D) if (x < 0 || sz(ret) != nedges+1) return {};
    return {ret.rbegin(), ret.rend()};
}
```

DominatorTree.h

Description: Builds a dominator tree on a directed graph. Output tree is a parent array with src as the root.

Time: $\mathcal{O}(V + E)$

```
1d35d2, 46 lines
vi getDomTree(vvi &adj, int src) {
    int n = sz(adj), t = 0;
    vvi revAdj(n), child(n), sdomChild(n);
    vi label(n, -1), revLabel(n), sdom(n), idom(n), par(n), best(n);

    auto dfs = [&](int cur, auto &dfs) -> void {
        label[cur] = t, revLabel[t] = cur;
        sdom[t] = par[t] = best[t] = t; t++;
        for(int nxt: adj[cur]) {
            if(label[nxt] == -1) {
                dfs(nxt, dfs);
                child[label[cur]].push_back(label[nxt]);
            }
            revAdj[label[nxt]].push_back(label[cur]);
        }
    };
    dfs(src, dfs);

    auto get = [&](int x, auto &get) -> int {
        if(par[x] != x) {
            int t = get(par[x], get);
            par[x] = par[par[x]];
            if(sdom[t] < sdom[best[x]]) best[x] = t;
        }
    };
```

```
    }
    return best[x];
};

for(int i = t-1; i >= 0; i--) {
    for(int j: revAdj[i]) sdom[i] = min(sdom[i], sdom[get(j, get)]);
    if(i > 0) sdomChild[sdom[i]].push_back(i);
    for(int j: sdomChild[i]) {
        int k = get(j, get);
        if(sdom[j] == sdom[k]) idom[j] = sdom[j];
        else idom[j] = k;
    }
    for(int j: child[i]) par[j] = i;
}

vi dom(n);
rep(i, 1, t) {
    if(idom[i] != sdom[i]) idom[i] = idom[idom[i]];
    dom[revLabel[i]] = revLabel[idom[i]];
}

return dom;
}
```

5.4 Coloring

EdgeColoring.h

Description: Given a simple, undirected graph with max degree D , computes a $(D + 1)$ -coloring of the edges such that no neighboring edges share a color. (D -coloring is NP-hard, but can be done for bipartite graphs by repeated matchings of max-degree nodes.)

Time: $\mathcal{O}(NM)$

```
e210e2, 31 lines
vi edgeColoring(int N, vector<pii> eds) {
    vi cc(N + 1), ret(sz(eds)), fan(N), free(N), loc;
    for (pii e : eds) ++cc[e.first], ++cc[e.second];
    int u, v, ncols = *max_element(all(cc)) + 1;
    vector<vi> adj(N, vi(ncols, -1));
    for (pii e : eds) {
        tie(u, v) = e;
        fan[0] = v;
        loc.assign(ncols, 0);
        int at = u, end = u, d, c = free[u], ind = 0, i = 0;
        while (d = free[v], !loc[d] && (v = adj[u][d]) != -1)
            loc[d] = ++ind, cc[ind] = d, fan[ind] = v;
        cc[loc[d]] = c;
        for (int cd = d; at != -1; cd ^= c ^ d, at = adj[at][cd])
            swap(adj[at][cd], adj[end = at][cd ^ c ^ d]);
        while (adj[fan[i]][d] != -1) {
            int left = fan[i], right = fan[++i], e = cc[i];
            adj[u][e] = left;
            adj[left][e] = u;
            adj[right][e] = -1;
            free[right] = e;
        }
        adj[u][d] = fan[i];
        adj[fan[i]][d] = u;
        for (int y : {fan[0], u, end})
            for (int& z = free[y] = 0; adj[y][z] != -1; z++);
    }
    rep(i,0,sz(eds))
        for (tie(u, v) = eds[i]; adj[u][ret[i]] != v;) ++ret[i];
    return ret;
}
```

5.5 Heuristics

MaximalCliques.h

Description: Runs a callback for all maximal cliques in a graph (given as a symmetric bitset matrix; self-edges not allowed). Callback is given a bitset representing the maximal clique.

Time: $\mathcal{O}\left(3^{n/3}\right)$, much faster for sparse graphs

```
b0d5b1, 12 lines
typedef bitset<128> B;
template<class F>
void cliques(vector<B>& eds, F f, B P = ~B(), B X={}, B R={}) {
    if (!P.any()) { if (!X.any()) f(R); return; }
    auto q = (P | X)._Find_first();
    auto cand = P & ~eds[q];
    rep(i,0,sz(eds)) if (cand[i]) {
        R[i] = 1;
        cliques(eds, f, P & eds[i], X & eds[i], R);
        R[i] = P[i] = 0; X[i] = 1;
    }
}
```

MaximumClique.h

Description: Quickly finds a maximum clique of a graph (given as symmetric bitset matrix; self-edges not allowed). Can be used to find a maximum independent set by finding a clique of the complement graph.

Time: Runs in about 1s for n=155 and worst case random graphs (p=.90). Runs faster for sparse graphs.

```
f7c0bc, 49 lines
typedef vector<bitset<200>> vb;
struct Maxclique {
    double limit=0.025, pk=0;
    struct Vertex { int i, d=0; };
    typedef vector<Vertex> vv;
    vb e;
    vv V;
    vector<vi> C;
    vi qmax, q, S, old;
    void init(vv& r) {
        for (auto& v : r) v.d = 0;
        for (auto& v : r) for (auto j : r) v.d += e[v.i][j.i];
        sort(all(r), [](auto a, auto b) { return a.d > b.d; });
        int mxD = r[0].d;
        rep(i,0,sz(r)) r[i].d = min(i, mxD) + 1;
    }
    void expand(vv& R, int lev = 1) {
        S[lev] += S[lev - 1] - old[lev];
        old[lev] = S[lev - 1];
        while (sz(R)) {
            if (sz(q) + R.back().d <= sz(qmax)) return;
            q.push_back(R.back().i);
            vv T;
            for(auto v:R) if (e[R.back().i][v.i]) T.push_back({v.i});
            if (sz(T)) {
                if (S[lev]++ / ++pk < limit) init(T);
                int j = 0, mxk = 1, mnk = max(sz(qmax) - sz(q) + 1, 1);
                C[1].clear(), C[2].clear();
                for (auto v : T) {
                    int k = 1;
                    auto f = [&](int i) { return e[v.i][i]; };
                    while (any_of(all(C[k]), f)) k++;
                    if (k > mxk) mxk = k, C[mxk + 1].clear();
                    if (k < mnk) T[j++].i = v.i;
                    C[k].push_back(v.i);
                }
                if (j > 0) T[j - 1].d = 0;
                rep(k,mnk,mxk + 1) for (int i : C[k])
                    T[j].i = i, T[j++].d = k;
                expand(T, lev + 1);
            } else if (sz(q) > sz(qmax)) qmax = q;
            q.pop_back(), R.pop_back();
        }
    }
};
```



```
    }
}
vi maxClique() { init(V), expand(V); return qmax; }
Maxclique(vb connn) : e(connn), C(sz(e)+1), S(sz(C)), old(S) {
    rep(i,0,sz(e)) V.push_back({i});
}
};
```

MaximumIndependentSet.h

Description: To obtain a maximum independent set of a graph, find a max clique of the complement. If the graph is bipartite, see MinimumVertex-Cover.

5.6 Trees

BinaryLifting.h

Description: Calculate power of two jumps in a tree, to support fast upward jumps and LCAs. Assumes the root node points to itself.
Time: construction $\mathcal{O}(N \log N)$, queries $\mathcal{O}(\log N)$

```
vector<vi> treeJump(vi& P){
    int on = 1, d = 1;
    while(on < sz(P)) on *= 2, d++;
    vector<vi> jmp(d, P);
    rep(i,1,d) rep(j,0,sz(P))
        jmp[i][j] = jmp[i-1][jmp[i-1][j]];
    return jmp;
}

int jmp(vector<vi>& tbl, int nod, int steps){
    rep(i,0,sz(tbl))
        if(steps&(1<<i)) nod = tbl[i][nod];
    return nod;
}

int lca(vector<vi>& tbl, vi& depth, int a, int b) {
    if (depth[a] < depth[b]) swap(a, b);
    a = jmp(tbl, a, depth[a] - depth[b]);
    if (a == b) return a;
    for (int i = sz(tbl); i--;) {
        int c = tbl[i][a], d = tbl[i][b];
        if (c != d) a = c, b = d;
    }
    return tbl[0][a];
}
```

LCA.h

Description: Data structure for computing lowest common ancestors in a tree (with 0 as root). C should be an adjacency list of the tree, either directed or undirected.
Time: $\mathcal{O}(N \log N + Q)$

```
"/data-structures/RMQ.h"
0f62fb, 21 lines

struct LCA {
    int T = 0;
    vi time, path, ret;
    RMQ<int> rmq;

    LCA(vector<vi>& C) : time(sz(C)), rmq((dfs(C,0,-1), ret)) {}
    void dfs(vector<vi>& C, int v, int par) {
        time[v] = T++;
        for (int y : C[v]) if (y != par) {
            path.push_back(v), ret.push_back(time[v]);
            dfs(C, y, v);
        }
    }

    int lca(int a, int b) {
        if (a == b) return a;
```

```
        tie(a, b) = minmax(time[a], time[b]);
        return path[rmq.query(a, b)];
    }
    //dist(a,b){return depth[a] + depth[b] - 2*depth[lca(a,b)];}
};
```

CentroidDecomp.h

Description: Calls callback function on undirected forest for each centroid
Usage: centroid(adj, [&](const vector<vector<int>>& adj, int cent) { ... });
Time: $\mathcal{O}(n \log n)$

```
c4d5af, 32 lines

template <class F> struct centroid {
    vector<vector<int>>> adj;
    F f;
    vector<int> sub_sz;
    centroid(const vector<vector<int>>& a_adj, F a_f)
        : adj(a_adj), f(a_f), sub_sz(ssize(adj), -1) {
        for (int i = 0; i < ssize(adj); i++)
            if (sub_sz[i] == -1) dfs(i);
    }
    void calc_sz(int u, int p) {
        sub_sz[u] = 1;
        for (int v : adj[u])
            if (v != p)
                calc_sz(v, u), sub_sz[u] += sub_sz[v];
    }
    void dfs(int u) {
        calc_sz(u, -1);
        for (int p = -1, sz_root = sub_sz[u];) {
            auto big_ch = find_if(begin(adj[u]), end(adj[u]), [&](int v) {
                return v != p && 2 * sub_sz[v] > sz_root;
            });
            if (big_ch == end(adj[u])) break;
            p = u, u = *big_ch;
        }
        f(adj, u);
        for (int v : adj[u]) {
            iter_swap(find(begin(adj[v]), end(adj[v]), u), rbegin(adj[v]));
            adj[v].pop_back();
            dfs(v);
        }
    }
};
```

CompressTree.h

Description: Given a rooted tree and a subset S of nodes, compute the minimal subtree that contains all the nodes by adding all (at most $|S| - 1$) pairwise LCA's and compressing edges. Returns a list of (par, orig_index) representing a tree rooted at 0. The root points to itself.
Time: $\mathcal{O}(|S| \log |S|)$

```
"LCA.h"
9775a0, 21 lines

typedef vector<pair<int, int>> vpi;
vpi compressTree(LCA& lca, const vi& subset) {
    static vi rev; rev.resize(sz(lca.time));
    vi li = subset, &T = lca.time;
    auto cmp = [&](int a, int b) { return T[a] < T[b]; };
    sort(all(li), cmp);
    int m = sz(li)-1;
    rep(i,0,m) {
        int a = li[i], b = li[i+1];
        li.push_back(lca.lca(a, b));
    }
    sort(all(li), cmp);
    li.erase(unique(all(li)), li.end());
    rep(i,0,sz(li)) rev[li[i]] = i;
    vpi ret = {pii(0, li[0])};
```

```
rep(i,0,sz(li)-1) {
    int a = li[i], b = li[i+1];
    ret.emplace_back(rev[lca.lca(a, b)], b);
}
return ret;
}
```

HLD.h

Description: Decomposes a tree into vertex disjoint heavy paths and light edges such that the path from any leaf to the root contains at most $\log(n)$ light edges. Code does additive modifications and max queries, but can support commutative segtree modifications/queries on paths and subtrees. Takes as input the full adjacency list. VALS_EDGES being true means that values are stored in the edges, as opposed to the nodes. All values initialized to the segtree default. Root must be 0.
Time: $\mathcal{O}((\log N)^2)$

```
"/data-structures/LazySegmentTree.h"
6f34db, 46 lines

template <bool VALS_EDGES> struct HLD {
    int N, tim = 0;
    vector<vi> adj;
    vi par, siz, depth, rt, pos;
    Node *tree;
    HLD(vector<vi> adj_)
        : N(sz(adj_)), adj(adj_), par(N, -1), siz(N, 1), depth(N),
          rt(N),pos(N),tree(new Node(0, N)){ dfsSz(0); dfsHld(0); }
    void dfsSz(int v) {
        if (par[v] != -1) adj[v].erase(find(all(adj[v]), par[v]));
        for (int& u : adj[v]) {
            par[u] = v, depth[u] = depth[v] + 1;
            dfsSz(u);
            siz[v] += siz[u];
            if (siz[u] > siz[adj[v][0]]) swap(u, adj[v][0]);
        }
    }
    void dfsHld(int v) {
        pos[v] = tim++;
        for (int u : adj[v]) {
            rt[u] = (u == adj[v][0] ? rt[v] : u);
            dfsHld(u);
        }
    }
    template <class B> void process(int u, int v, B op) {
        for (; rt[u] != rt[v]; v = par[rt[v]]) {
            if (depth[rt[u]] > depth[rt[v]]) swap(u, v);
            op(pos[rt[v]], pos[v] + 1);
        }
        if (depth[u] > depth[v]) swap(u, v);
        op(pos[u] + VALS_EDGES, pos[v] + 1);
    }
    void modifyPath(int u, int v, int val) {
        process(u, v, [&](int l, int r) { tree->add(l, r, val); });
    }
    int queryPath(int u, int v) { // Modify depending on problem
        int res = -1e9;
        process(u, v, [&](int l, int r) {
            res = max(res, tree->query(l, r));
        });
        return res;
    }
    int querySubtree(int v) { // modifySubtree is similar
        return tree->query(pos[v] + VALS_EDGES, pos[v] + siz[v]);
    }
};
```

LinkCutTree.h

Description: Represents a forest of unrooted trees. Nodes are 1-indexed. You can add and remove edges (as long as the result is still a forest). You can also do path sum, subtree sum, and LCA queries, which depend on the current root.

Time: All operations take amortized $\mathcal{O}(\log N)$.

9aa6da, 105 lines

```
struct SplayTree {
    struct Node {
        int ch[2] = {0, 0}, p = 0;
        ll self = 0, path = 0;           // Path aggregates
        ll sub = 0, vir = 0;             // Subtree aggregates
        bool flip = 0;                  // Lazy tags
    };
    vector<Node> T;

    SplayTree(int n) : T(n + 1) {}

    void push(int x) {
        if (!x || !T[x].flip) return;
        int l = T[x].ch[0], r = T[x].ch[1];

        T[l].flip ^= 1, T[r].flip ^= 1;
        swap(T[x].ch[0], T[x].ch[1]);
        T[x].flip = 0;
    }

    void pull(int x) {
        int l = T[x].ch[0], r = T[x].ch[1]; push(l); push(r);

        T[x].path = T[l].path + T[x].self + T[r].path;
        T[x].sub = T[x].vir + T[l].sub + T[r].sub + T[x].self;
    }

    void set(int x, int d, int y) {
        T[x].ch[d] = y; T[y].p = x; pull(x);
    }

    void splay(int x) {
        auto dir = [&](int x) {
            int p = T[x].p; if (!p) return -1;
            return T[p].ch[0] == x ? 0 : T[p].ch[1] == x ? 1 : -1;
        };
        auto rotate = [&](int x) {
            int y = T[x].p, z = T[y].p, dx = dir(x), dy = dir(y);
            set(y, dx, T[x].ch[!dx]);
            set(x, !dx, y);
            if (~dy) set(z, dy, x);
            T[x].p = z;
        };
        for (push(x); ~dir(x); ) {
            int y = T[x].p, z = T[y].p;
            push(z); push(y); push(x);
            int dx = dir(x), dy = dir(y);
            if (~dy) rotate(dx != dy ? x : y);
            rotate(x);
        }
    };

    struct LinkCut : SplayTree {
        LinkCut(int n) : SplayTree(n) {}

        int access(int x) {
            int u = x, v = 0;
            for (; u; v = u, u = T[u].p) {
                splay(u);
                int& ov = T[u].ch[1];
                T[u].vir += T[ov].sub;
                T[u].vir -= T[v].sub;
            }
        }
    };
};
```

```
        ov = v; pull(u);
    }
    return splay(x), v;
}

void reroot(int x) {
    access(x); T[x].flip ^= 1; push(x);
}

void Link(int u, int v) {
    reroot(u); access(v);
    T[v].vir += T[u].sub;
    T[u].p = v; pull(v);
}

void Cut(int u, int v) {
    reroot(u); access(v);
    T[v].ch[0] = T[u].p = 0; pull(v);
}

// Rooted tree LCA. Returns 0 if u and v arent connected.
int LCA(int u, int v) {
    if (u == v) return u;
    access(u); int ret = access(v);
    return T[u].p ? ret : 0;
}

// Query subtree of u where v is outside the subtree.
ll Subtree(int u, int v) {
    reroot(v); access(u); return T[u].vir + T[u].self;
}

// Query path [u..v]
ll Path(int u, int v) {
    reroot(u); access(v); return T[v].path;
}

// Update vertex u with value v
void Update(int u, ll v) {
    access(u); T[u].self = v; pull(u);
}
};
```

DirectedMST.h

Description: Finds a minimum spanning tree/arborescence of a directed graph, given a root node. If no MST exists, returns -1.

Time: $\mathcal{O}(E \log V)$

"/data-structures/UnionFindRollback.h" 39e620, 60 lines

```
struct Edge { int a, b; ll w; };
struct Node {
    Edge key;
    Node *l, *r;
    ll delta;
    void prop() {
        key.w += delta;
        if (l) l->delta += delta;
        if (r) r->delta += delta;
        delta = 0;
    }
    Edge top() { prop(); return key; }
};

Node *merge(Node *a, Node *b) {
    if (!a || !b) return a ? b;
    a->prop(), b->prop();
    if (a->key.w > b->key.w) swap(a, b);
    swap(a->l, (a->r = merge(b, a->r)));
    return a;
}

void pop(Node*& a) { a->prop(); a = merge(a->l, a->r); }
```

```
pair<ll, vi> dmst(int n, int r, vector<Edge>& g) {
    RollbackUF uf(n);
    vector<Node*> heap(n);
    for (Edge e : g) heap[e.b] = merge(heap[e.b], new Node(e));
    ll res = 0;
    vi seen(n, -1), path(n), par(n);
    seen[r] = r;
    vector<Edge> Q(n), in(n, {-1,-1}), comp;
    deque<tuple<int, int, vector<Edge>>> cys;
    rep(s,0,n) {
        int u = s, qi = 0, w;
        while (seen[u] < 0) {
            if (!heap[u]) return {-1,{};};
            Edge e = heap[u]->top();
            heap[u]->delta -= e.w, pop(heap[u]);
            Q[qi] = e, path[qi++] = u, seen[u] = s;
            res += e.w, u = uf.find(e.a);
            if (seen[u] == s) {
                Node* cyc = 0;
                int end = qi, time = uf.time();
                do cyc = merge(cyc, heap[w = path[--qi]]);
                while (uf.join(u, w));
                u = uf.find(u), heap[u] = cyc, seen[u] = -1;
                cys.push_front({u, time, {&Q[qi], &Q[end]}});
            }
        }
        rep(i,0,qi) in[uf.find(Q[i].b)] = Q[i];
    }

    for (auto& [u,t,comp] : cys) { // restore sol (optional)
        uf.rollback(t);
        Edge inEdge = in[u];
        for (auto& e : comp) in[uf.find(e.b)] = e;
        in[uf.find(inEdge.b)] = inEdge;
    }
    rep(i,0,n) par[i] = in[i].a;
    return {res, par};
}
```


Numerical Methods (6)

6.1 Polynomials and recurrences

Polynomial.h

c9b7b0, 17 lines

```
struct Poly {
    vector<double> a;
    double operator()(double x) const {
        double val = 0;
        for (int i = sz(a); i--;) (val *= x) += a[i];
        return val;
    }
    void diff() {
        rep(i,1,sz(a)) a[i-1] = i*a[i];
        a.pop_back();
    }
    void divroot(double x0) {
        double b = a.back(), c; a.back() = 0;
        for(int i=sz(a)-1; i--;) c = a[i], a[i] = a[i+1]*x0+b, b=c;
        a.pop_back();
    }
};
```

PolyRoots.h

Description: Finds the real roots to a polynomial.
Usage: polyRoots({{2,-3,1}},-1e9,1e9) // solve x^2-3x+2 = 0
Time: $\mathcal{O}(n^2 \log(1/\epsilon))$

"Polynomial.h"b00bfe, 23 lines

```
vector<double> polyRoots(Poly p, double xmin, double xmax) {
    if (sz(p.a) == 2) { return {-p.a[0]/p.a[1]}; }
    vector<double> ret;
    Poly der = p;
    der.diff();
    auto dr = polyRoots(der, xmin, xmax);
    dr.push_back(xmin-1);
    dr.push_back(xmax+1);
    sort(all(dr));
    rep(i,0,sz(dr)-1) {
        double l = dr[i], h = dr[i+1];
        bool sign = p(l) > 0;
        if (sign ^ (p(h) > 0)) {
            rep(it,0,60) { // while (h - l > 1e-8)
                double m = (l + h) / 2, f = p(m);
                if ((f <= 0) ^ sign) l = m;
                else h = m;
            }
            ret.push_back((l + h) / 2);
        }
    }
    return ret;
}
```

PolyInterpolate.h

Description: Given n points $(x[i], y[i])$, computes an $n-1$ -degree polynomial p that passes through them: $p(x) = a[0] * x^0 + \dots + a[n-1] * x^{n-1}$. For numerical precision, pick $x[k] = c * \cos(k/(n-1) * \pi), k = 0 \dots n-1$.
Time: $\mathcal{O}(n^2)$

08bf48, 13 lines

```
typedef vector<double> vd;
vd interpolate(vd x, vd y, int n) {
    vd res(n), temp(n);
    rep(k,0,n-1) rep(i,k+1,n)
        y[i] = (y[i] - y[k]) / (x[i] - x[k]);
    double last = 0; temp[0] = 1;
    rep(k,0,n) rep(i,0,n) {
        res[i] += y[k] * temp[i];
        swap(last, temp[i]);
        temp[i] -= last * x[k];
    }
```

```
    }
    return res;
}
```

BerlekampMassey.h

Description: Recovers any n -order linear recurrence relation from the first $2n$ terms of the recurrence. Useful for guessing linear recurrences after brute-forcing the first terms. Should work on any field, but numerical stability for floats is not guaranteed. Output will have size $\leq n$.
Usage: berlekampMassey({0, 1, 1, 3, 5, 11}) // {1, 2}
Time: $\mathcal{O}(N^2)$

"../number-theory/ModPow.h"96548b, 20 lines

```
vector<ll> berlekampMassey(vector<ll> s) {
    int n = sz(s), L = 0, m = 0;
    vector<ll> C(n), B(n), T;
    C[0] = B[0] = 1;

    ll b = 1;
    rep(i,0,n) { ++m;
        ll d = s[i] % mod;
        rep(j,1,L+1) d = (d + C[j] * s[i - j]) % mod;
        if (!d) continue;
        T = C; ll coef = d * modpow(b, mod-2) % mod;
        rep(j,m,n) C[j] = (C[j] - coef * B[j - m]) % mod;
        if (2 * L > i) continue;
        L = i + 1 - L; B = T; b = d; m = 0;
    }

    C.resize(L + 1); C.erase(C.begin());
    for (ll& x : C) x = (mod - x) % mod;
    return C;
}
```

LinearRecurrence.h

Description: Generates the k 'th term of an n -order linear recurrence $S[i] = \sum_j S[i - j - 1]tr[j]$, given $S[0 \dots n - 1]$ and $tr[0 \dots n - 1]$. Faster than matrix multiplication. Useful together with Berlekamp-Massey.
Usage: linearRec({0, 1}, {1, 1}, k) // k 'th Fibonacci number
Time: $\mathcal{O}(n^2 \log k)$

f4e444, 26 lines

```
typedef vector<ll> Poly;
ll linearRec(Poly S, Poly tr, ll k) {
    int n = sz(tr);

    auto combine = [&](Poly a, Poly b) {
        Poly res(n * 2 + 1);
        rep(i,0,n+1) rep(j,0,n+1)
            res[i + j] = (res[i + j] + a[i] * b[j]) % mod;
        for (int i = 2 * n; i > n; --i) rep(j,0,n)
            res[i - 1 - j] = (res[i - 1 - j] + res[i] * tr[j]) % mod;
        res.resize(n + 1);
        return res;
    };

    Poly pol(n + 1), e(pol);
    pol[0] = e[1] = 1;

    for (++k; k; k /= 2) {
        if (k % 2) pol = combine(pol, e);
        e = combine(e, e);
    }

    ll res = 0;
    rep(i,0,n) res = (res + pol[i + 1] * S[i]) % mod;
    return res;
}
```

6.2 Optimization

GoldenSectionSearch.h

Description: Finds the argument minimizing the function f in the interval $[a,b]$ assuming f is unimodal on the interval, i.e. has only one local minimum. The maximum error in the result is eps . Works equally well for maximization with a small change in the code. See TernarySearch.h in the Various chapter for a discrete version.
Usage: double func(double x) { return 4+x+.3*x*x; }
double xmin = gss(-1000,1000,func);
Time: $\mathcal{O}(\log((b - a)/\epsilon))$

31d45b, 14 lines

```
double gss(double a, double b, double (*f)(double)) {
    double r = (sqrt(5)-1)/2, eps = 1e-7;
    double x1 = b - r*(b-a), x2 = a + r*(b-a);
    double f1 = f(x1), f2 = f(x2);
    while (b-a > eps)
        if (f1 < f2) { //change to > to find maximum
            b = x2; x2 = x1; f2 = f1;
            x1 = b - r*(b-a); f1 = f(x1);
        } else {
            a = x1; x1 = x2; f1 = f2;
            x2 = a + r*(b-a); f2 = f(x2);
        }
    return a;
}
```

HillClimbing.h

Description: Poor man's optimization for unimodal functions

8eecaf, 14 lines

```
typedef array<double, 2> P;

template<class F> pair<double, P> hillClimb(P start, F f) {
    pair<double, P> cur(f(start), start);
    for (double jmp = 1e9; jmp > 1e-20; jmp /= 2) {
        rep(j,0,100) rep(dx,-1,2) rep(dy,-1,2) {
            P p = cur.second;
            p[0] += dx*jmp;
            p[1] += dy*jmp;
            cur = min(cur, make_pair(f(p), p));
        }
    }
    return cur;
}
```

IntegrateAdaptiveTyler.h

Description: Gets area under a curve

e7beba, 17 lines

```
#define approx(a, b) (b-a) / 6 * (f(a) + 4 * f((a+b) / 2) + f(b))

template<class F>
ld adapt (F &f, ld a, ld b, ld A, int iters) {
    ld m = (a+b) / 2;
    ld A1 = approx(a, m), A2 = approx(m, b);
    if (!iters && (abs(A1 + A2 - A) < eps || b-a < eps))
        return A;
    ld left = adapt(f, a, m, A1, max(iters-1, 0));
    ld right = adapt(f, m, b, A2, max(iters-1, 0));
    return left + right;
}

template<class F>
ld integrate(F f, ld a, ld b, int iters = 0) {
    return adapt(f, a, b, approx(a, b), iters);
}
```

25c1ac, 12 lines

aa8530, 68 lines

bd5cec, 15 lines

3313dc, 18 lines

44c9ab, 38 lines

fa2d7a, 34 lines

```
typedef bitset<1000> bs;
```

```
int solveLinear(vector<bs>& A, vi& b, bs& x, int m) {
    int n = sz(A), rank = 0, br;
    assert(m <= sz(x));
    vi col(m); iota(all(col), 0);
    rep(i,0,n) {
        for (br=i; br<n; ++br) if (A[br].any()) break;
        if (br == n) {
            rep(j,i,n) if(b[j]) return -1;
            break;
        }
        int bc = (int)A[br]._Find_next(i-1);
        swap(A[i], A[br]);
        swap(b[i], b[br]);
        swap(col[i], col[bc]);
        rep(j,0,n) if (A[j][i] != A[j][bc]) {
            A[j].flip(i); A[j].flip(bc);
        }
        rep(j,i+1,n) if (A[j][i]) {
            b[j] ^= b[i];
            A[j] ^= A[i];
        }
        rank++;
    }

    x = bs();
    for (int i = rank; i--;) {
        if (!b[i]) continue;
        x[col[i]] = 1;
        rep(j,0,i) b[j] ^= A[j][i];
    }
    return rank; // (multiple solutions if rank < m)
}
```

MatrixInverse.h

Description: Invert matrix A. Returns rank; result is stored in A unless singular (rank < n). Can easily be extended to prime moduli; for prime powers, repeatedly set $A^{-1} = A^{-1}(2I - AA^{-1}) \pmod{p^k}$ where A^{-1} starts as the inverse of A mod p, and k is doubled in each step.

Time: $\mathcal{O}(n^3)$

```
int matInv(vector<vector<double>>& A) {
    int n = sz(A); vi col(n);
    vector<vector<double>> tmp(n, vector<double>(n));
    rep(i,0,n) tmp[i][i] = 1, col[i] = i;

    rep(i,0,n) {
        int r = i, c = i;
        rep(j,i,n) rep(k,i,n)
            if (fabs(A[j][k]) > fabs(A[r][c]))
                r = j, c = k;
        if (fabs(A[r][c]) < 1e-12) return i;
        A[i].swap(A[r]); tmp[i].swap(tmp[r]);
        rep(j,0,n)
            swap(A[j][i], A[j][c]), swap(tmp[j][i], tmp[j][c]);
        swap(col[i], col[c]);
        double v = A[i][i];
        rep(j,i+1,n) {
            double f = A[j][i] / v;
            A[j][i] = 0;
            rep(k,i+1,n) A[j][k] -= f*A[i][k];
            rep(k,0,n) tmp[j][k] -= f*tmp[i][k];
        }
        rep(j,i+1,n) A[i][j] /= v;
        rep(j,0,n) tmp[i][j] /= v;
        A[i][i] = 1;
    }
}
```

```
for (int i = n-1; i > 0; --i) rep(j,0,i) {
    double v = A[j][i];
```

```
    rep(k,0,n) tmp[j][k] -= v*tmp[i][k];
}

rep(i,0,n) rep(j,0,n) A[col[i]][col[j]] = tmp[i][j];
return n;
}
```

MatrixInverse-mod.h

Description: Invert matrix A modulo a prime. Returns rank; result is stored in A unless singular (rank < n). For prime powers, repeatedly set $A^{-1} = A^{-1}(2I - AA^{-1}) \pmod{p^k}$ where A^{-1} starts as the inverse of A mod p, and k is doubled in each step.

Time: $\mathcal{O}(n^3)$

```
"../number-theory/ModPow.h"
int matInv(vector<vector<ll>>& A) {
    int n = sz(A); vi col(n);
    vector<vector<ll>> tmp(n, vector<ll>(n));
    rep(i,0,n) tmp[i][i] = 1, col[i] = i;

    rep(i,0,n) {
        int r = i, c = i;
        rep(j,i,n) rep(k,i,n) if (A[j][k]) {
            r = j; c = k; goto found;
        }
        return i;
    found:
        A[i].swap(A[r]); tmp[i].swap(tmp[r]);
        rep(j,0,n) swap(A[j][i], A[j][c]), swap(tmp[j][i], tmp[j][c]);
        swap(col[i], col[c]);
        ll v = modpow(A[i][i], mod - 2);
        rep(j,i+1,n) {
            ll f = A[j][i] * v % mod;
            A[j][i] = 0;
            rep(k,i+1,n) A[j][k] = (A[j][k] - f*A[i][k]) % mod;
            rep(k,0,n) tmp[j][k] = (tmp[j][k] - f*tmp[i][k]) % mod;
        }
        rep(j,i+1,n) A[i][j] = A[i][j] * v % mod;
        rep(j,0,n) tmp[i][j] = tmp[i][j] * v % mod;
        A[i][i] = 1;
    }

    for (int i = n-1; i > 0; --i) rep(j,0,i) {
        ll v = A[j][i];
        rep(k,0,n) tmp[j][k] = (tmp[j][k] - v*tmp[i][k]) % mod;
    }

    rep(i,0,n) rep(j,0,n)
        A[col[i]][col[j]] = tmp[i][j] % mod + (tmp[i][j] < 0 ? mod : 0);
    return n;
}
```

Tridiagonal.h

Description: x = tridiagonal(d, p, q, b) solves the equation system

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} d_0 & p_0 & 0 & 0 & \cdots & 0 \\ q_0 & d_1 & p_1 & 0 & \cdots & 0 \\ 0 & q_1 & d_2 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{n-3} & d_{n-2} & p_{n-2} \\ 0 & 0 & \cdots & 0 & q_{n-2} & d_{n-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{pmatrix}.$$

This is useful for solving problems on the type

$$a_i = b_i a_{i-1} + c_i a_{i+1} + d_i, 1 \leq i \leq n,$$

where a_0, a_{n+1}, b_i, c_i and d_i are known. a can then be obtained from

$$\{a_i\} = \text{tridiagonal}(\{1, -1, -1, \dots, -1, 1\}, \{0, c_1, c_2, \dots, c_n\}, \{b_1, b_2, \dots, b_n, 0\}, \{a_0, d_1, d_2, \dots, d_n, a_{n+1}\}).$$

Fails if the solution is not unique. If $|d_i| > |p_i| + |q_{i-1}|$ for all i , or $|d_i| > |p_{i-1}| + |q_i|$, or the matrix is positive definite, the algorithm is numerically stable and neither tr nor the check for diag[i] == 0 is needed.

Time: $\mathcal{O}(N)$

```
typedef double T;
vector<T> tridiagonal(vector<T> diag, const vector<T>& super,
    const vector<T>& sub, vector<T> b) {
    int n = sz(b); vi tr(n);
    rep(i,0,n-1) {
        if (abs(diag[i]) < 1e-9 * abs(super[i])) { // diag[i] == 0
            b[i+1] -= b[i] * diag[i+1] / super[i];
            if (i+2 < n) b[i+2] -= b[i] * sub[i+1] / super[i];
            diag[i+1] = sub[i]; tr[++i] = 1;
        } else {
            diag[i+1] -= super[i]*sub[i]/diag[i];
            b[i+1] -= b[i]*sub[i]/diag[i];
        }
    }
    for (int i = n; i--;) {
        if (tr[i]) {
            swap(b[i], b[i-1]);
            diag[i-1] = diag[i];
            b[i] /= super[i-1];
        } else {
            b[i] /= diag[i];
            if (i) b[i-1] -= b[i]*super[i-1];
        }
    }
    return b;
}
```

JacobianMatrix.h

Description: Makes Jacobian Matrix using finite differences

```
template<class F, class T>
vector<vector<T>> makeJacobian(F &f, vector<T> &x) {
    int n = sz(x);
    vector<vector<T>> J(n, vector<T>(n));
    vector<T> fX0 = f(x);
    rep(i, 0, n) {
        x[i] += eps;
        vector<T> fX1 = f(x);
        rep(j, 0, n){
            J[j][i] = (fX1[j] - fX0[j]) / eps;
        }
        x[i] -= eps;
    }
    return J;
}
```

NewtonsMethod.h

Description: Solves a system on non-linear equations

```
jacobianMatrix.h
template<class F, class T>
void solveNonlinear(F f, vector<T> &x){
    int n = sz(x);
    rep(iter, 0, 100) {
        vector<vector<T>> J = makeJacobian(f, x);
        matInv(J);
        vector<T> dx = J * f(x);
        x = x - dx;
    }
}
```

6.4 Fourier transforms

FastFourierTransform.h

Description: $\text{fft}(a)$ computes $\hat{f}(k) = \sum_x a[x] \exp(2\pi i \cdot kx/N)$ for all k . N must be a power of 2. Useful for convolution: $\text{conv}(a, b) = c$, where $c[x] = \sum a[i]b[x-i]$. For convolution of complex numbers or more than two vectors: FFT, multiply pointwise, divide by n , reverse(start+1, end), FFT back. Rounding is safe if $(\sum a_i^2 + \sum b_i^2) \log_2 N < 9 \cdot 10^{14}$ (in practice 10^{16} ; higher for random inputs). Otherwise, use NTT/FFTMod.
Time: $\mathcal{O}(N \log N)$ with $N = |A| + |B|$ ($\sim 1s$ for $N = 2^{22}$)

```
00ced6, 35 lines
typedef complex<double> C;
typedef vector<double> vd;
void fft(vector<C>& a) {
    int n = sz(a), L = 31 - __builtin_clz(n);
    static vector<complex<long double>> R(2, 1);
    static vector<C> rt(2, 1); // (^ 10% faster if double)
    for (static int k = 2; k < n; k *= 2) {
        R.resize(n); rt.resize(n);
        auto x = polar(1.0L, acos(-1.0L) / k);
        rep(i,k,2*k) rt[i] = R[i] = i&1 ? R[i/2] * x : R[i/2];
    }
    vi rev(n);
    rep(i,0,n) rev[i] = (rev[i / 2] | (i & 1) << L) / 2;
    rep(i,0,n) if (i < rev[i]) swap(a[i], a[rev[i]]);
    for (int k = 1; k < n; k *= 2)
        for (int i = 0; i < n; i += 2 * k) rep(j,0,k) {
            C z = rt[j+k] * a[i+j+k]; // (25% faster if hand-rolled)
            a[i + j + k] = a[i + j] - z;
            a[i + j] += z;
        }
}
vd conv(const vd& a, const vd& b) {
    if (a.empty() || b.empty()) return {};
    vd res(sz(a) + sz(b) - 1);
    int L = 32 - __builtin_clz(sz(res)), n = 1 << L;
    vector<C> in(n), out(n);
    copy(all(a), begin(in));
    rep(i,0,sz(b)) in[i].imag(b[i]);
    fft(in);
    for (C& x : in) x *= x;
    rep(i,0,n) out[i] = in[-i & (n - 1)] - conj(in[i]);
    fft(out);
    rep(i,0,sz(res)) res[i] = imag(out[i]) / (4 * n);
    return res;
}
```

FastFourierTransformMod.h

Description: Higher precision FFT, can be used for convolutions modulo arbitrary integers as long as $N \log_2 N \cdot \text{mod} < 8.6 \cdot 10^{14}$ (in practice 10^{16} or higher). Inputs must be in $[0, \text{mod})$.
Time: $\mathcal{O}(N \log N)$, where $N = |A| + |B|$ (twice as slow as NTT or FFT)

```
b82773, 22 lines
"FastFourierTransform.h"
typedef vector<ll> vl;
template<int M> vl convMod(const vl &a, const vl &b) {
    if (a.empty() || b.empty()) return {};
    vl res(sz(a) + sz(b) - 1);
    int B=32-__builtin_clz(sz(res)), n=1<<B, cut=int(sqrt(M));
    vector<C> L(n), R(n), outs(n), outl(n);
    rep(i,0,sz(a)) L[i] = C((int)a[i] / cut, (int)a[i] % cut);
    rep(i,0,sz(b)) R[i] = C((int)b[i] / cut, (int)b[i] % cut);
    fft(L), fft(R);
    rep(i,0,n) {
        int j = -i & (n - 1);
        outl[j] = (L[i] + conj(L[j])) * R[i] / (2.0 * n);
        outs[j] = (L[i] - conj(L[j])) * R[i] / (2.0 * n) / 1i;
    }
    fft(outl), fft(outs);
    rep(i,0,sz(res)) {
```

```
    ll av = ll(real(outl[i])+.5), cv = ll(imag(outs[i])+.5);
    ll bv = ll(imag(outl[i])+.5) + ll(real(outs[i])+.5);
    res[i] = ((av % M * cut + bv) % M * cut + cv) % M;
}
return res;
}
```

NumberTheoreticTransform.h

Description: $\text{ntt}(a)$ computes $\hat{f}(k) = \sum_x a[x]g^{xk}$ for all k , where $g = \text{root}^{(\text{mod}-1)/N}$. N must be a power of 2. Useful for convolution modulo specific nice primes of the form $2^a b + 1$, where the convolution result has size at most 2^a . For arbitrary modulo, see FFTMod. $\text{conv}(a, b) = c$, where $c[x] = \sum a[i]b[x-i]$. For manual convolution: NTT the inputs, multiply pointwise, divide by n , reverse(start+1, end), NTT back. Inputs must be in $[0, \text{mod})$.

```
Time:  $\mathcal{O}(N \log N)$ 
"../number-theory/ModPow.h"
ced03d, 33 lines
const ll mod = (119 << 23) + 1, root = 62; // = 998244353
// For p < 2^30 there is also e.g. 5 << 25, 7 << 26, 479 << 21
// and 483 << 21 (same root). The last two are > 10^9.
typedef vector<ll> vl;
void ntt(vl &a) {
    int n = sz(a), L = 31 - __builtin_clz(n);
    static vl rt(2, 1);
    for (static int k = 2, s = 2; k < n; k *= 2, s++) {
        rt.resize(n);
        ll z[] = {1, modpow(root, mod >> s)};
        rep(i,k,2*k) rt[i] = rt[i / 2] * z[i & 1] % mod;
    }
    vi rev(n);
    rep(i,0,n) rev[i] = (rev[i / 2] | (i & 1) << L) / 2;
    rep(i,0,n) if (i < rev[i]) swap(a[i], a[rev[i]]);
    for (int k = 1; k < n; k *= 2)
        for (int i = 0; i < n; i += 2 * k) rep(j,0,k) {
            ll z = rt[j + k] * a[i + j + k] % mod, &ai = a[i + j];
            a[i + j + k] = ai - z + (z > ai ? mod : 0);
            ai += (ai + z >= mod ? z - mod : z);
        }
}
vl conv(const vl &a, const vl &b) {
    if (a.empty() || b.empty()) return {};
    int s = sz(a) + sz(b) - 1, B = 32 - __builtin_clz(s), n = 1 << B;
    int inv = modpow(n, mod - 2);
    vl L(a), R(b), out(n);
    L.resize(n), R.resize(n);
    ntt(L), ntt(R);
    rep(i,0,n) out[-i & (n - 1)] = (ll)L[i] * R[i] % mod * inv % mod;
    ntt(out);
    return {out.begin(), out.begin() + s};
}
```

FastSubsetTransform.h

Description: Transform to a basis with fast convolutions of the form $c[z] = \sum_{z=x \oplus y} a[x] \cdot b[y]$, where \oplus is one of AND, OR, XOR. The size of a must be a power of two.
Time: $\mathcal{O}(N \log N)$

```
464cf3, 16 lines
void FST(vi& a, bool inv) {
    for (int n = sz(a), step = 1; step < n; step *= 2) {
        for (int i = 0; i < n; i += 2 * step) rep(j,i,i+step) {
            int &u = a[j], &v = a[j + step]; tie(u, v) =
                inv ? pii(v - u, u) : pii(v, u + v); // AND
                inv ? pii(v, u - v) : pii(u + v, u); // OR
                pii(u + v, u - v); // XOR
        }
    }
}
```

```
    if (inv) for (int& x : a) x /= sz(a); // XOR only
}
vi conv(vi a, vi b) {
    FST(a, 0); FST(b, 0);
    rep(i,0,sz(a)) a[i] *= b[i];
    FST(a, 1); return a;
}
```

Number theory (7)

7.1 Modular arithmetic

ModInverse.h
Description: Pre-computation of modular inverses. Assumes $\text{LIM} \leq \text{mod}$ and that mod is a prime.

```
const ll mod = 1000000007, LIM = 200000;
ll* inv = new ll[LIM] - 1; inv[1] = 1;
rep(i,2,LIM) inv[i] = mod - (mod / i) * inv[mod % i] % mod;
```

ModLog.h
Description: Returns the smallest $x > 0$ s.t. $a^x = b \pmod m$, or -1 if no such x exists. $\text{modLog}(a,1,m)$ can be used to calculate the order of a .
Time: $\mathcal{O}(\sqrt{m})$

```
ll modLog(ll a, ll b, ll m) {
    ll n = (ll) sqrt(m) + 1, e = 1, f = 1, j = 1;
    unordered_map<ll, ll> A;
    while (j <= n && (e = f * e % m) != b % m)
        A[e * b % m] = j++;
    if (e == b % m) return j;
    if (__gcd(m, e) == __gcd(m, b))
        rep(i,2,n+2) if (A.count(e = e * f % m))
            return n * i - A[e];
    return -1;
}
```

ModSum.h
Description: Sums of mod'ed arithmetic progressions. $\text{modsum}(\text{to}, c, k, m) = \sum_{i=0}^{\text{to}-1} (ki + c) \% m$. divsum is similar but for floored division.
Time: $\log(m)$, with a large constant.

```
typedef unsigned long long ull;
ull sumsq(ull to) { return to / 2 * ((to-1) | 1); }
```

```
ull divsum(ull to, ull c, ull k, ull m) {
    ull res = k / m * sumsq(to) + c / m * to;
    k %= m; c %= m;
    if (!k) return res;
    ull to2 = (to * k + c) / m;
    return res + (to - 1) * to2 - divsum(to2, m-1 - c, m, k);
}
```

```
ll modsum(ull to, ll c, ll k, ll m) {
    c = ((c % m) + m) % m;
    k = ((k % m) + m) % m;
    return to * c + k * sumsq(to) - m * divsum(to, c, k, m);
}
```

ModMulLL.h
Description: Calculate $a \cdot b \pmod c$ (or $a^b \pmod c$) for $0 \leq a, b \leq c \leq 7.2 \cdot 10^{18}$.
Time: $\mathcal{O}(1)$ for modmul , $\mathcal{O}(\log b)$ for modpow

```
typedef unsigned long long ull;
ull modmul(ull a, ull b, ull M) {
    ll ret = a * b - M * ull(1.L / M * a * b);
    return ret + M * (ret < 0) - M * (ret >= (ll)M);
}
ull modpow(ull b, ull e, ull mod) {
    ull ans = 1;
    for (; e; b = modmul(b, b, mod), e /= 2)
        if (e & 1) ans = modmul(ans, b, mod);
    return ans;
}
```

ModSqrt.h
Description: Tonelli-Shanks algorithm for modular square roots. Finds x s.t. $x^2 = a \pmod p$ ($-x$ gives the other solution).
Time: $\mathcal{O}(\log^2 p)$ worst case, $\mathcal{O}(\log p)$ for most p

```
"ModPow.h"
19a793, 24 lines

ll sqrt(ll a, ll p) {
    a %= p; if (a < 0) a += p;
    if (a == 0) return 0;
    assert(modpow(a, (p-1)/2, p) == 1); // else no solution
    if (p % 4 == 3) return modpow(a, (p+1)/4, p);
    // a^(n+3)/8 or 2^(n+3)/8 * 2^(n-1)/4 works if p % 8 == 5
    ll s = p - 1, n = 2;
    int r = 0, m;
    while (s % 2 == 0)
        ++r, s /= 2;
    while (modpow(n, (p - 1) / 2, p) != p - 1) ++n;
    ll x = modpow(a, (s + 1) / 2, p);
    ll b = modpow(a, s, p), g = modpow(n, s, p);
    for (;;) r = m) {
        ll t = b;
        for (m = 0; m < r && t != 1; ++m)
            t = t * t % p;
        if (m == 0) return x;
        ll gs = modpow(g, 1LL << (r - m - 1), p);
        g = gs * gs % p;
        x = x * gs % p;
        b = b * g % p;
    }
}
```

7.2 Primality

FastEratosthenes.h
Description: Prime sieve for generating all primes smaller than LIM.
Time: $\text{LIM} = 1e9 \approx 1.5s$

```
const int LIM = 1e6;
bitset<LIM> isPrime;
vi eratosthenes() {
    const int S = (int)round(sqrt(LIM)), R = LIM / 2;
    vi pr = {2}, sieve(S+1); pr.reserve((int)(LIM/log(LIM)*1.1));
    vector<pii> cp;
    for (int i = 3; i <= S; i += 2) if (!sieve[i]) {
        cp.push_back({i, i * i / 2});
        for (int j = i * i; j <= S; j += 2 * i) sieve[j] = 1;
    }
    for (int L = 1; L <= R; L += S) {
        array<bool, S> block{};
        for (auto &[p, idx] : cp)
            for (int i=idx; i < S+L; idx = (i+=p)) block[i-L] = 1;
        rep(i,0,min(S, R - L))
            if (!block[i]) pr.push_back((L + i) * 2 + 1);
    }
    for (int i : pr) isPrime[i] = 1;
    return pr;
}
```

LinearSieve.h
Description: Finds smallest prime factor of each integer
Time: $\mathcal{O}(N)$

```
const int LIM = 1000000;
vi lp(LIM+1), primes;

rep(i, 2, LIM + 1) {
    if (lp[i] == 0) primes.push_back(lp[i] = i);
    for (int j = 0; j < sz(primes) && i * primes[j] <= LIM &&
        primes[j] <= lp[i]; ++j)
        lp[i * primes[j]] = primes[j];
}
```

MillerRabin.h
Description: Deterministic Miller-Rabin primality test. Guaranteed to work for numbers up to $7 \cdot 10^{18}$; for larger numbers, use Python and extend A randomly.
Time: 7 times the complexity of $a^b \pmod c$.

```
"ModMulLL.h"
60dcd1, 12 lines

bool isPrime(ull n) {
    if (n < 2 || n % 6 % 4 != 1) return (n | 1) == 3;
    ull A[] = {2, 325, 9375, 28178, 450775, 9780504, 1795265022},
        s = __builtin_ctzll(n-1), d = n >> s;
    for (ull a : A) { // ^ count trailing zeroes
        ull p = modpow(a%n, d, n), i = s;
        while (p != 1 && p != n - 1 && a % n && i--)
            p = modmul(p, p, n);
        if (p != n-1 && i != s) return 0;
    }
    return 1;
}
```

Factor.h
Description: Pollard-rho randomized factorization algorithm. Returns prime factors of a number, in arbitrary order (e.g. $2299 \rightarrow \{11, 19, 11\}$).
Time: $\mathcal{O}(n^{1/4})$, less for numbers with small factors.

```
"ModMulLL.h", "MillerRabin.h"
a33cf6, 18 lines

ull pollard(ull n) {
    auto f = [n](ull x) { return modmul(x, x, n) + 1; };
    ull x = 0, y = 0, t = 30, prd = 2, i = 1, q;
    while (t++ % 40 || __gcd(prd, n) == 1) {
        if (x == y) x = ++i, y = f(x);
        if ((q = modmul(prd, max(x,y) - min(x,y), n))) prd = q;
        x = f(x), y = f(f(y));
    }
    return __gcd(prd, n);
}

vector<ull> factor(ull n) {
    if (n == 1) return {};
    if (isPrime(n)) return {n};
    ull x = pollard(n);
    auto l = factor(x), r = factor(n / x);
    l.insert(l.end(), all(r));
    return l;
}
```

GetFactors.h
Description: Gets all factors of a number N given the prime factorization of the number.
Time: $\mathcal{O}(\sqrt[3]{N})$

```
void getFactors(auto &pF, auto &primes, auto &factors, int i = 0, int n = 1) {
    if(i == sz(pF)) {
        factors.push_back(n);
        return;
    }

    for(int j = 0, pow = 1; i <= pf[j]; j++, pow *= primes[j])
        getFactors(pF, primes, factors, i+1, n * pow);
}
```

7.3 Divisibility

euclid.h
Description: Finds two integers x and y , such that $ax + by = \gcd(a, b)$. If you just need gcd, use the built in $_gcd$ instead. If a and b are coprime, then x is the inverse of $a \pmod b$.

```
ll euclid(ll a, ll b, ll &x, ll &y) {
    if (!b) return x = 1, y = 0, a;
```



```
    ll d = euclid(b, a % b, y, x);
    return y -= a/b * x, d;
}
```

CRT.h
Description: Chinese Remainder Theorem. crt(a, m, b, n) computes x such that $x \equiv a \pmod m$, $x \equiv b \pmod n$. If $|a| < m$ and $|b| < n$, x will obey $0 \leq x < \text{lcm}(m, n)$. Assumes $mn < 2^{62}$.
Time: $\log(n)$

```
"euclid.h"                                04d93a, 7 lines

ll crt(ll a, ll m, ll b, ll n) {
    if (n > m) swap(a, b), swap(m, n);
    ll x, y, g = euclid(m, n, x, y);
    assert((a - b) % g == 0); // else no solution
    x = (b - a) % n * x % n / g * m + a;
    return x < 0 ? x + m*n/g : x;
}
```

7.3.1 Bézout’s identity

For $a \neq, b \neq 0$, then $d = \gcd(a, b)$ is the smallest positive integer for which there are integer solutions to

$ax + by = d$

If (x, y) is one solution, then all solutions are given by

$$\left(x + \frac{kb}{\gcd(a,b)}, y - \frac{ka}{\gcd(a,b)}\right), \quad k \in \mathbb{Z}$$

phiFunction.h
Description: Euler’s ϕ function is defined as $\phi(n) := \#$ of positive integers n that are coprime with n . $\phi(1) = 1$, p prime $\Rightarrow \phi(p^k) = (p - 1)p^{k-1}$, m, n coprime $\Rightarrow \phi(mn) = \phi(m)\phi(n)$. If $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ then $\phi(n) = (p_1 - 1)p_1^{k_1-1} \dots (p_r - 1)p_r^{k_r-1}$. $\phi(n) = n \cdot \prod_{p|n} (1 - 1/p)$. $\sum_{d|n} \phi(d) = n$, $\sum_{1 \leq k \leq n, \gcd(k,n)=1} k = n\phi(n)/2, n > 1$ **Euler’s thm:** a, n coprime $\Rightarrow a^{\phi(n)} \equiv 1 \pmod n$. **Fermat’s little thm:** p prime $\Rightarrow a^{p-1} \equiv 1 \pmod p \forall a$.

```
cf7d6d, 8 lines

const int LIM = 5000000;
int phi[LIM];

void calculatePhi() {
    rep(i, 0, LIM) phi[i] = i&1 ? i : i/2;
    for (int i = 3; i < LIM; i += 2) if(phi[i] == i)
        for (int j = i; j < LIM; j += i) phi[j] -= phi[j] / i;
}
```

7.4 Fractions

ContinuedFractions.h
Description: Given N and a real number $x \geq 0$, finds the closest rational approximation p/q with $p, q \leq N$. It will obey $|p/q - x| \leq 1/qN$. For consecutive convergents, $p_{k+1}q_k - q_{k+1}p_k = (-1)^k$. (p_k/q_k alternates between $> x$ and $< x$.) If x is rational, y eventually becomes ∞ ; if x is the root of a degree 2 polynomial the a ’s eventually become cyclic.
Time: $\mathcal{O}(\log N)$

```
dd6c5e, 21 lines

typedef double d; // for N ~ 1e7; long double for N ~ 1e9
pair<ll, ll> approximate(d x, ll N) {
    ll LP = 0, LQ = 1, P = 1, Q = 0, inf = LLONG_MAX; d y = x;
    for (;;) {
        ll lim = min(P ? (N-LP) / P : inf, Q ? (N-LQ) / Q : inf),
            a = (ll)floor(y), b = min(a, lim),
            NP = b*P + LP, NQ = b*Q + LQ;
        if (a > b) {
```

```
// If b > a/2, we have a semi-convergent that gives us a
// better approximation; if b = a/2, we *may* have one.
// Return {P, Q} here for a more canonical approximation.
return (abs(x - (d)NP / (d)NQ) < abs(x - (d)P / (d)Q)) ?
    make_pair(NP, NQ) : make_pair(P, Q);
}
if (abs(y = 1/(y - (d)a)) > 3*N) {
    return {NP, NQ};
}
LP = P; P = NP;
LQ = Q; Q = NQ;
}
```

FracBinarySearch.h
Description: Given f and N , finds the smallest fraction $p/q \in [0, 1]$ such that $f(p/q)$ is true, and $p, q \leq N$. You may want to throw an exception from f if it finds an exact solution, in which case N can be removed.
Usage: fracBS([](Frac f) { return f.p>=3*f.q; }, 10); // {1, 3}
Time: $\mathcal{O}(\log(N))$

```
27ab3e, 25 lines

struct Frac { ll p, q; };

template<class F>
Frac fracBS(F f, ll N) {
    bool dir = 1, A = 1, B = 1;
    Frac lo{0, 1}, hi{1, 1}; // Set hi to 1/0 to search (0, N]
    if (f(lo)) return lo;
    assert(f(hi));
    while (A || B) {
        ll adv = 0, step = 1; // move hi if dir, else lo
        for (int si = 0; step; (step *= 2) >>= si) {
            adv += step;
            Frac mid{lo.p * adv + hi.p, lo.q * adv + hi.q};
            if (abs(mid.p) > N || mid.q > N || dir == !f(mid)) {
                adv -= step; si = 2;
            }
            hi.p += lo.p * adv;
            hi.q += lo.q * adv;
            dir = !dir;
            swap(lo, hi);
            A = B; B = !!adv;
        }
        return dir ? hi : lo;
    }
}
```

7.5 Pythagorean Triples

The Pythagorean triples are uniquely generated by

$a = k \cdot (m^2 - n^2), \quad b = k \cdot (2mn), \quad c = k \cdot (m^2 + n^2),$

with $m > n > 0, k > 0, m \perp n$, and either m or n even.

7.6 Primes

$p = 962592769$ is such that $2^{21} \mid p - 1$, which may be useful. For hashing use 970592641 (31-bit number), 31443539979727 (45-bit), 3006703054056749 (52-bit). There are 78498 primes less than 1 000 000.

Primitive roots exist modulo any prime power p^a , except for $p = 2, a > 2$, and there are $\phi(\phi(p^a))$ many. For $p = 2, a > 2$, the group $\mathbb{Z}_{2^a}^\times$ is instead isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{a-2}}$.

7.7 Estimates

$\sum_{d|n} d = \mathcal{O}(n \log \log n)$.

The number of divisors of n is at most around 100 for $n < 5e4$, 500 for $n < 1e7$, 2000 for $n < 1e10$, 200 000 for $n < 1e19$.

7.8 Mobius Function

$$\mu(n) = \begin{cases} 0 & n \text{ is not square free} \\ 1 & n \text{ has even number of prime factors} \\ -1 & n \text{ has odd number of prime factors} \end{cases}$$

Mobius Inversion:

$$g(n) = \sum_{d|n} f(d) \Leftrightarrow f(n) = \sum_{d|n} \mu(d)g(n/d)$$

Other useful formulas/forms:

$\sum_{d|n} \mu(d) = [n = 1]$ (very useful)

$g(n) = \sum_{n|d} f(d) \Leftrightarrow f(n) = \sum_{n|d} \mu(d/n)g(d)$

$g(n) = \sum_{1 \leq m \leq n} f(\lfloor \frac{n}{m} \rfloor) \Leftrightarrow f(n) = \sum_{1 \leq m \leq n} \mu(m)g(\lfloor \frac{n}{m} \rfloor)$

Combinatorial (8)

8.1 Permutations

8.1.1 Factorial

n	1	2	3	4	5	6	7	8	9	10
$n!$	1	2	6	24	120	720	5040	40320	362880	3628800
n	11	12	13				14	15	16	17
$n!$	4.0e7	4.8e8	6.2e9	8.7e10	1.3e12	2.1e13	3.6e14			
n	20	25	30	40	50	100	150	171		
$n!$	2e18	2e25	3e32	8e47	3e64	9e157	6e262	>DBL_MAX		

IntPerm.h
Description: Permutation -> integer conversion. (Not order preserving.)
Integer -> permutation can use a lookup table.
Time: $\mathcal{O}(n)$

```
044568, 6 lines
int permToInt(vi& v) {
    int use = 0, i = 0, r = 0;
    for(int x:v) r = r * ++i + __builtin_popcount(use & -(1<<x)),
        use |= 1 << x; // (note: minus, not ~!)
    return r;
}
```

8.1.2 Cycles

Let $g_S(n)$ be the number of n -permutations whose cycle lengths all belong to the set S . Then

$$\sum_{n=0}^\infty g_S(n) \frac{x^n}{n!} = \exp\left(\sum_{n\in S} \frac{x^n}{n}\right)$$

8.1.3 Derangements

Permutations of a set such that none of the elements appear in their original position.

$$D(n) = (n-1)(D(n-1)+D(n-2)) = nD(n-1) + (-1)^n = \left\lfloor \frac{n!}{e} \right\rfloor$$

8.1.4 Burnside’s lemma

Given a group G of symmetries and a set X , the number of elements of X up to symmetry equals

$$\frac{1}{|G|} \sum_{g\in G} |X^g|,$$

where X^g are the elements fixed by g ($g.x = x$).

If $f(n)$ counts “configurations” (of some sort) of length n , we can ignore rotational symmetry using $G = \mathbb{Z}_n$ to get

$$g(n) = \frac{1}{n} \sum_{k=0}^{n-1} f(\gcd(n,k)) = \frac{1}{n} \sum_{k|n} f(k) \phi(n/k).$$

8.2 Partitions and subsets

8.2.1 Partition function

Number of ways of writing n as a sum of positive integers, disregarding the order of the summands.

$$p(0) = 1, \quad p(n) = \sum_{k\in\mathbb{Z}\setminus\{0\}} (-1)^{k+1} p(n-k(3k-1)/2)$$

$$p(n) \sim 0.145/n \cdot \exp(2.56\sqrt{n})$$

n	0	1	2	3	4	5	6	7	8	9	20	50	100
$p(n)$	1	1	2	3	5	7	11	15	22	30	627	~2e5	~2e8

8.2.2 Lucas’ Theorem

Let n, m be non-negative integers and p a prime. Write $n = n_k p^k + \dots + n_1 p + n_0$ and $m = m_k p^k + \dots + m_1 p + m_0$. Then $\binom{n}{m} \equiv \prod_{i=0}^k \binom{n_i}{m_i} \pmod{p}$.

8.2.3 Binomials

multinomial.h

```
a0a312, 6 lines
Description: Computes  $\binom{k_1+\dots+k_n}{k_1,k_2,\dots,k_n} = \frac{(\sum k_i)!}{k_1!k_2!\dots k_n!}$ .
11 multinomial(vi& v) {
    ll c = 1, m = v.empty() ? 1 : v[0];
    rep(i,1,sz(v)) rep(j,0,v[i])
        c = c * ++m / (j+1);
    return c;
}
```

8.3 General purpose numbers

8.3.1 Bernoulli numbers

EGF of Bernoulli numbers is $B(t) = \frac{t}{e^t-1}$ (FFT-able).

$$B[0,\dots] = [1, -\tfrac{1}{2}, \tfrac{1}{6}, 0, -\tfrac{1}{30}, 0, \tfrac{1}{42}, \dots]$$

Sums of powers:

$$\sum_{i=1}^n i^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k \cdot (n+1)^{m+1-k}$$

Euler-Maclaurin formula for infinite sums:

$$\begin{aligned} \sum_{i=m}^\infty f(i) &= \int_m^\infty f(x)dx - \sum_{k=1}^\infty \frac{B_k}{k!} f^{(k-1)}(m) \\ &\approx \int_m^\infty f(x)dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \frac{f'''(m)}{720} + O(f^{(5)}(m)) \end{aligned}$$

8.3.2 Stirling numbers of the first kind

Number of permutations on n items with k cycles.

$$\begin{aligned} c(n,k) &= c(n-1,k-1) + (n-1)c(n-1,k), \quad c(0,0) = 1 \\ \sum_{k=0}^n c(n,k) x^k &= x(x+1)\dots(x+n-1) \end{aligned}$$

$$\begin{aligned} c(8,k) &= 8, 0, 5040, 13068, 13132, 6769, 1960, 322, 28, 1 \\ c(n,2) &= 0, 0, 1, 3, 11, 50, 274, 1764, 13068, 109584, \dots \end{aligned}$$

8.3.3 Eulerian numbers

Number of permutations $\pi \in S_n$ in which exactly k elements are greater than the previous element. k j:s s.t. $\pi(j) > \pi(j+1)$, $k+1$ j:s s.t. $\pi(j) \geq j$, k j:s s.t. $\pi(j) > j$.

$$E(n,k) = (n-k)E(n-1,k-1) + (k+1)E(n-1,k)$$

$$E(n,0) = E(n,n-1) = 1$$

$$E(n,k) = \sum_{j=0}^k (-1)^j \binom{n+1}{j} (k+1-j)^n$$

8.3.4 Stirling numbers of the second kind

Partitions of n distinct elements into exactly k groups.

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

$$S(n,1) = S(n,n) = 1$$

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n$$

8.3.5 Bell numbers

Total number of partitions of n distinct elements. $B(n) = 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \dots$ For p prime,

$$B(p^m + n) \equiv mB(n) + B(n+1) \pmod{p}$$

8.3.6 Labeled unrooted trees

on n vertices: n^{n-2}
on k existing trees of size n_i : $n_1 n_2 \dots n_k n^{k-2}$
with degrees d_i : $(n-2)! / ((d_1-1)! \dots (d_n-1)!)$

8.3.7 Catalan numbers

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!}$$

$$C_0 = 1, \quad C_{n+1} = \frac{2(2n+1)}{n+2} C_n, \quad C_{n+1} = \sum C_i C_{n-i}$$

$$C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \dots$$

- sub-diagonal monotone paths in an $n \times n$ grid.
- strings with n pairs of parenthesis, correctly nested.
- binary trees with with $n+1$ leaves (0 or 2 children).
- ordered trees with $n+1$ vertices.
- ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines.
- permutations of $[n]$ with no 3-term increasing subseq.

Strings (9)

KMP.h

Description: pi[x] computes the length of the longest prefix of s that ends at x, other than s[0...x] itself (abacaba -> 0010123). Can be used to find all occurrences of a string.

Time: $\mathcal{O}(n)$

```

vi pi(const string& s) {
    vi p(sz(s));
    rep(i,1,sz(s)) {
        int g = p[i-1];
        while (g && s[i] != s[g]) g = p[g-1];
        p[i] = g + (s[i] == s[g]);
    }
    return p;
}

```

```

vi match(const string& s, const string& pat) {
    vi p = pi(pat + '\0' + s), res;
    rep(i,sz(p)-sz(s),sz(p))
        if (p[i] == sz(pat)) res.push_back(i - 2 * sz(pat));
    return res;
}

```

Zfunc.h

Description: z[x] computes the length of the longest common prefix of s[i] and s, except z[0] = 0. (abacaba -> 0010301)

Time: $\mathcal{O}(n)$

```

vi Z(string S) {
    vi z(sz(S));
    int l = -1, r = -1;
    rep(i,1,sz(S)) {
        z[i] = i >= r ? 0 : min(r - i, z[i - l]);
        while (i + z[i] < sz(S) && S[i + z[i]] == S[z[i]])
            z[i]++;
        if (i + z[i] > r)
            l = i, r = i + z[i];
    }
    return z;
}

```

Manacher.h

Description: For each position in a string, computes p[0][i] = half length of longest even palindrome around pos i, p[1][i] = longest odd (half rounded down).

Time: $\mathcal{O}(N)$

```

array<vi, 2> manacher(const string& s) {
    int n = sz(s);
    array<vi,2> p = {vi(n+1), vi(n)};
    rep(z,0,2) for (int i=0,l=0,r=0; i < n; i++) {
        int t = r-i+!z;
        if (i<r) p[z][i] = min(t, p[z][l+t]);
        int L = i-p[z][i], R = i+p[z][i]-!z;
        while (L>=1 && R+1<n && s[L-1] == s[R+1])
            p[z][i]++, L--, R++;
        if (R>r) l=L, r=R;
    }
    return p;
}

```

Eertree.h

Description: Generates an eertree on str. cur is accurate at the end of the main loop before the final assignment to t.

Time: $\mathcal{O}(|S|)$

```

struct Eertree {

```

```

    vector<int> slink = {0, 0}, len = {-1, 0};
    vector<vector<int>> down;
    int cur = 0, t = 0;
    Eertree(string &str) : down(2, vector<int>(26, -1)) {
        for (int i = 0; i < str.size(); i++) {
            char c = str[i]; int ci = c - 'a';
            while (t <= 0 || str[t-1] != c)
                t = i - len[cur = slink[cur]];
            if (down[cur][ci] == -1) {
                down[cur][ci] = slink.size();
                down.emplace_back(26, -1);
                len.push_back(len[cur] + 2);
                if (len.back() > 1) {
                    do t = i - len[cur = slink[cur]];
                    while(t <= 0 || str[t-1] != c);
                    slink.push_back(down[cur][ci]);
                } else slink.push_back(1);
                cur = slink.size() - 1;
            } else cur = down[cur][ci];
            t = i - len[cur] + 1;
        }
    }
};

```

MinRotation.h

Description: Finds the lexicographically smallest rotation of a string.

Usage: rotate(v.begin(), v.begin()+minRotation(v), v.end());

Time: $\mathcal{O}(N)$

```

int minRotation(string s) {
    int a=0, N=sz(s); s += s;
    rep(b,0,N) rep(k,0,N) {
        if (a+k == b || s[a+k] < s[b+k]) {b += max(0, k-1); break;}
        if (s[a+k] > s[b+k]) {a = b; break;}
    }
    return a;
}

```

SuffixArray.h

Description: Builds suffix array for a string. sa[i] is the starting index of the suffix which is *i*'th in the sorted suffix array. The returned vector is of size *n* + 1, and sa[0] = *n*. The lcp array contains longest common prefixes for neighbouring strings in the suffix array: lcp[i] = lcp(sa[i], sa[i-1]), lcp[0] = 0. The input string must not contain any zero bytes.

Time: $\mathcal{O}(n \log n)$

```

struct SuffixArray {
    vi sa, lcp;
    SuffixArray(string& s, int lim=256) { // or basic_string<int>
        int n = sz(s) + 1, k = 0, a, b;
        vi x(all(s)+1), y(n), ws(max(n, lim)), rank(n);
        sa = lcp = y, iota(all(sa), 0);
        for (int j = 0, p = 0; p < n; j = max(1, j * 2), lim = p) {
            p = j, iota(all(y), n - j);
            rep(i,0,n) if (sa[i] >= j) y[p++] = sa[i] - j;
            fill(all(ws), 0);
            rep(i,0,n) ws[x[i]]++;
            rep(i,1,lim) ws[i] += ws[i - 1];
            for (int i = n; i--;) sa[--ws[x[y[i]]]] = y[i];
            swap(x, y), p = 1, x[sa[0]] = 0;
            rep(i,1,n) a = sa[i - 1], b = sa[i], x[b] =
                (y[a] == y[b] && y[a + j] == y[b + j]) ? p - 1 : p++;
        }
        rep(i,1,n) rank[sa[i]] = i;
        for (int i = 0, j; i < n - 1; lcp[rank[i+]] = k)
            for (k && k--, j = sa[rank[i] - 1];
                s[i + k] == s[j + k]; k++);
    }
};

```

SuffixAutomaton.h

Description: Creates a partial DFA (DAG) that accepts all suffixes, with suffix links. One-to-one map between a path from the root and a substring. len is the longest-length substring ending here. pos is the first index in the string matching here. term is whether this node is a terminal (aka a suffix)

Time: construction takes $\mathcal{O}(N \log K)$, where K = Alphabet Size.

```

struct st { int len, pos, term; st *link; map<char, st*> next;
};

st *suffixAutomaton(string &str) {
    st *last = new st(), *root = last;
    for(auto c : str) {
        st *p = last, *cur = last = new st{last->len + 1, last->len
        };
        while(p && !p->next.count(c))
            p->next[c] = cur, p = p->link;
        if (!p) cur->link = root;
        else {
            st *q = p->next[c];
            if (p->len + 1 == q->len) cur->link = q;
            else {
                st *clone = new st{p->len+1, q->pos, 0, q->link, q->
                    next};
                for (; p && p->next[c] == q; p = p->link)
                    p->next[c] = clone;
                q->link = cur->link = clone;
            }
        }
    }
    while(last) last->term = 1, last = last->link;
    return root;
}

```

SuffixTree.h

Description: Ukkonen's algorithm for online suffix tree construction. Each node contains indices [l, r) into the string, and a list of child nodes. Suffixes are given by traversals of this tree, joining [l, r) substrings. The root is 0 (has l = -1, r = 0), non-existent children are -1. To get a complete tree, append a dummy symbol – otherwise it may contain an incomplete path (still useful for substring matching, though).

Time: $\mathcal{O}(26N)$

```

struct SuffixTree {
    enum { N = 200010, ALPHA = 26 }; // N ~ 2*maxlen+10
    int toi(char c) { return c - 'a'; }
    string a; // v = cur node, q = cur position
    int t[N][ALPHA], l[N], r[N], p[N], s[N], v=0, q=0, m=2;

    void ukkadd(int i, int c) { suff:
        if (r[v]<=q) {
            if (t[v][c]==-1) { t[v][c]=m; l[m]=i;
                p[m++]=v; v=s[v]; q=r[v]; goto suff; }
            v=t[v][c]; q=l[v];
        }
        if (q==-1 || c==toi(a[q])) q++; else {
            l[m+1]=i; p[m+1]=m; l[m]=l[v]; r[m]=q;
            p[m]=p[v]; t[m][c]=m+1; t[m][toi(a[q])]=v;
            l[v]=q; p[v]=m; t[p[m]][toi(a[l[m]])]=m;
            v=s[p[m]]; q=l[m];
            while (q<r[m]) { v=t[v][toi(a[q])]; q+=r[v]-l[v]; }
            if (q==r[m]) s[m]=v; else s[m]=m+2;
            q=r[v]-(q-r[m]); m+=2; goto suff;
        }
    }
}

```

```

SuffixTree(string a) : a(a) {
    fill(r,r+N,sz(a));
    memset(s, 0, sizeof s);
    memset(t, -1, sizeof t);
    fill(t[1],t[1]+ALPHA,0);
}

```



```
s[0] = 1; l[0] = l[1] = -1; r[0] = r[1] = p[0] = p[1] = 0;
rep(i,0,sz(a)) ukkadd(i, toi(a[i]));
}

// example: find longest common substring (uses ALPHA = 28)
pii best;
int lcs(int node, int i1, int i2, int olen) {
    if (l[node] <= i1 && i1 < r[node]) return 1;
    if (l[node] <= i2 && i2 < r[node]) return 2;
    int mask = 0, len = node ? olen + (r[node] - l[node]) : 0;
    rep(c,0,ALPHA) if (t[node][c] != -1)
        mask |= lcs(t[node][c], i1, i2, len);
    if (mask == 3)
        best = max(best, {len, r[node] - len});
    return mask;
}
static pii LCS(string s, string t) {
    SuffixTree st(s + (char)('z' + 1) + t + (char)('z' + 2));
    st.lcs(0, sz(s), sz(s) + 1 + sz(t), 0);
    return st.best;
}
};
```

Hashing.h

Description: Self-explanatory methods for string hashing. 4b8fa1, 19 lines

```
// Arithmetic mod 2^64-1. 2x slower than mod 2^64 and more
// code, but works on evil test data (e.g. Thue-Morse, where
// ABBA... and BAAB... of length 2^10 hash the same mod 2^64).
// "typedef ull H;" instead if you think test data is random,
// or work mod 10^9+7 if the Birthday paradox is not a problem.
typedef uint64_t ull;
struct H {
    ull x; H(ull x=0) : x(x) {}
    H operator+(H o) { return x + o.x + (x + o.x < x); }
    H operator-(H o) { return *this + ~o.x; }
    H operator*(H o) { auto m = (__uint128_t)x * o.x;
        return H((ull)m) + (ull)(m >> 64); }
    ull get() const { return x + !~x; }
    bool operator==(H o) const { return get() == o.get(); }
    bool operator<(H o) const { return get() < o.get(); }
};
static const H C = (11)1e11+3; // (order ~ 3e9; random also ok)

H hashString(string& s){H h{}; for(char c:s) h=h*C+c;return h;}
```

iiiiij HEAD =====

HashInterval.h

Description: Various self-explanatory methods for string hashing. "Hashing.h" 122649, 12 lines

```
struct HashInterval {
    vector<H> ha, pw;
    HashInterval(string& str) : ha(sz(str)+1), pw(ha) {
        pw[0] = 1;
        rep(i,0,sz(str))
            ha[i+1] = ha[i] * C + str[i],
            pw[i+1] = pw[i] * C;
    }
    H hashInterval(int a, int b) { // hash [a, b)
        return ha[b] - ha[a] * pw[b - a];
    }
};
```

AhoCorasick.h

Hashing HashInterval AhoCorasick AhoCorasick-Tyler

Description: Aho-Corasick automaton, used for multiple pattern matching. Initialize with AhoCorasick ac(patterns); the automaton start node will be at index 0. find(word) returns for each position the index of the longest word that ends there, or -1 if none. findAll(–, word) finds all words (up to $N\sqrt{N}$ many if no duplicate patterns) that start at each position (shortest first). Duplicate patterns are allowed; empty patterns are not. To find the longest words that start at each position, reverse all input. For large alphabets, split each symbol into chunks, with sentinel bits for symbol boundaries. Time: construction takes $\mathcal{O}(26N)$, where N = sum of length of patterns. find(x) is $\mathcal{O}(N)$, where N = length of x. findAll is $\mathcal{O}(NM)$. f35677, 66 lines

```
struct AhoCorasick {
    enum {alpha = 26, first = 'A'}; // change this!
    struct Node {
        // (nmatches is optional)
        int back, next[alpha], start = -1, end = -1, nmatches = 0;
        Node(int v) { memset(next, v, sizeof(next)); }
    };
    vector<Node> N;
    vi backp;
    void insert(string& s, int j) {
        assert(!s.empty());
        int n = 0;
        for (char c : s) {
            int& m = N[n].next[c - first];
            if (m == -1) { n = m = sz(N); N.emplace_back(-1); }
            else n = m;
        }
        if (N[n].end == -1) N[n].start = j;
        backp.push_back(N[n].end);
        N[n].end = j;
        N[n].nmatches++;
    }
    AhoCorasick(vector<string>& pat) : N(1, -1) {
        rep(i,0,sz(pat)) insert(pat[i], i);
        N[0].back = sz(N);
        N.emplace_back(0);

        queue<int> q;
        for (q.push(0); !q.empty(); q.pop()) {
            int n = q.front(), prev = N[n].back;
            rep(i,0,alpha) {
                int &ed = N[n].next[i], y = N[prev].next[i];
                if (ed == -1) ed = y;
                else {
                    N[ed].back = y;
                    (N[ed].end == -1 ? N[ed].end : backp[N[ed].start])
                        = N[y].end;
                    N[ed].nmatches += N[y].nmatches;
                    q.push(ed);
                }
            }
        }
    }
    vi find(string word) {
        int n = 0;
        vi res; // ll count = 0;
        for (char c : word) {
            n = N[n].next[c - first];
            res.push_back(N[n].end);
            // count += N[n].nmatches;
        }
        return res;
    }
    vector<vi> findAll(vector<string>& pat, string word) {
        vi r = find(word);
        vector<vi> res(sz(word));
        rep(i,0,sz(word)) {
            int ind = r[i];
```

```
while (ind != -1) {
    res[i - sz(pat[ind]) + 1].push_back(ind);
    ind = backp[ind];
}
return res;
};
```

iiiiiii main

AhoCorasick-Tyler.h

Description: Aho-Corasick automaton, used for multiple pattern matching. Initialize with AhoCorasick ac(patterns); the automaton start node will be at index 0. find(word) returns for each position the index of the longest word that ends there, or -1 if none. findAll(–, word) finds all words (up to $N\sqrt{N}$ many if no duplicate patterns) that start at each position (shortest first). Duplicate patterns are allowed; empty patterns are not. To find the longest words that start at each position, reverse all input. For large alphabets, split each symbol into chunks, with sentinel bits for symbol boundaries. Time: construction takes $\mathcal{O}(26N)$, where N = sum of length of patterns. find(x) is $\mathcal{O}(N)$, where N = length of x. findAll is $\mathcal{O}(NM)$. 647ca9, 47 lines

```
const int ABSIZE = 26;

struct node {
    int nxt[ABSIZE];
    vi ids = {};
    int prv = -1, link = -1;
    char c;
    int linkMemo[ABSIZE];

    node(int prv = -1, char c = '$') : prv(prv), c(c) {
        fill(all(nxt), -1);
        fill(all(linkMemo), -1);
    }
};

vector<node> trie(1);

void addWord(string &s, int id) {
    int cur = 0;
    for(char c: s) {
        int idx = c - 'a';
        if(trie[cur].nxt[idx] == -1) {
            trie[cur].nxt[idx] = sz(trie);
            trie.emplace_back(cur, c);
        }
        cur = trie[cur].nxt[idx];
    }
    trie[cur].ids.push_back(id);
}

int getLink(int cur);

int calc(int cur, char c) {
    int idx = c - 'a';
    auto &ret = trie[cur].linkMemo[idx];
    if(ret != -1) return ret;
    if(trie[cur].nxt[idx] != -1)
        return ret = trie[cur].nxt[idx];
    return ret = cur == 0 ? 0 : calc(getLink(cur), c);
}

int getLink(int cur) {
    auto &ret = trie[cur].link;
    if(ret != -1) return ret;
    if(cur == 0 || trie[cur].prv == 0) return ret = 0;
    return ret = calc(getLink(trie[cur].prv), trie[cur].c);
}
```

}

Various (10)

10.1 Intervals

IntervalContainer.h

Description: Add and remove intervals from a set of disjoint intervals. Will merge the added interval with any overlapping intervals in the set when adding. Intervals are [inclusive, exclusive).

Time: $\mathcal{O}(\log N)$

```
set<pii>::iterator addInterval(set<pii>& is, int L, int R) {
    if (L == R) return is.end();
    auto it = is.lower_bound({L, R}), before = it;
    while (it != is.end() && it->first <= R) {
        R = max(R, it->second);
        before = it = is.erase(it);
    }
    if (it != is.begin() && (--it)->second >= L) {
        L = min(L, it->first);
        R = max(R, it->second);
        is.erase(it);
    }
    return is.insert(before, {L,R});
}
```

```
void removeInterval(set<pii>& is, int L, int R) {
    if (L == R) return;
    auto it = addInterval(is, L, R);
    auto r2 = it->second;
    if (it->first == L) is.erase(it);
    else (int&)it->second = L;
    if (R != r2) is.emplace(R, r2);
}
```

IntervalCover.h

Description: Compute indices of smallest set of intervals covering another interval. Intervals should be [inclusive, exclusive). To support [inclusive, inclusive], change (A) to add || R.empty(). Returns empty set on failure (or if G is empty).

Time: $\mathcal{O}(N \log N)$

```
template<class T>
vi cover(pair<T, T> G, vector<pair<T, T>> I) {
    vi S(sz(I)), R;
    iota(all(S), 0);
    sort(all(S), [&](int a, int b) { return I[a] < I[b]; });
    T cur = G.first;
    int at = 0;
    while (cur < G.second) { // (A)
        pair<T, int> mx = make_pair(cur, -1);
        while (at < sz(I) && I[S[at]].first <= cur) {
            mx = max(mx, make_pair(I[S[at]].second, S[at]));
            at++;
        }
        if (mx.second == -1) return {};
        cur = mx.first;
        R.push_back(mx.second);
    }
    return R;
}
```

ConstantIntervals.h

Description: Split a monotone function on [from, to) into a minimal set of half-open intervals on which it has the same value. Runs a callback g for each such interval.

Usage: constantIntervals(0, sz(v), [&](int x){return v[x];}, [&](int lo, int hi, T val){...});

Time: $\mathcal{O}(k \log \frac{n}{k})$

```
template<class F, class G, class T>
void rec(int from, int to, F& f, G& g, int& i, T& p, T q) {
    if (p == q) return;
    if (from == to) {
        g(i, to, p);
        i = to; p = q;
    } else {
        int mid = (from + to) >> 1;
        rec(from, mid, f, g, i, p, f(mid));
        rec(mid+1, to, f, g, i, p, q);
    }
}

template<class F, class G>
void constantIntervals(int from, int to, F f, G g) {
    if (to <= from) return;
    int i = from; auto p = f(i), q = f(to-1);
    rec(from, to-1, f, g, i, p, q);
    g(i, to, q);
}
```

10.2 Misc. algorithms

LIS.h

Description: Compute indices for the longest increasing subsequence.

Time: $\mathcal{O}(N \log N)$

```
template<class I> vi lis(const vector<I>& S) {
    if (S.empty()) return {};
    vi prev(sz(S));
    typedef pair<I, int> p;
    vector<p> res;
    rep(i,0,sz(S)) {
        // change 0 -> i for longest non-decreasing subsequence
        auto it = lower_bound(all(res), p{S[i], 0});
        if (it == res.end()) res.emplace_back(), it = res.end()-1;
        *it = {S[i], i};
        prev[i] = it == res.begin() ? 0 : (it-1)->second;
    }
    int L = sz(res), cur = res.back().second;
    vi ans(L);
    while (L--) ans[L] = cur, cur = prev[cur];
    return ans;
}
```

FastKnapsack.h

Description: Given N non-negative integer weights w and a non-negative target t, computes the maximum S <= t such that S is the sum of some subset of the weights.

Time: $\mathcal{O}(N \max(w_i))$

```
int knapsack(vi w, int t) {
    int a = 0, b = 0, x;
    while (b < sz(w) && a + w[b] <= t) a += w[b++];
    if (b == sz(w)) return a;
    int m = *max_element(all(w));
    vi u, v(2*m, -1);
    v[a+m-t] = b;
    rep(i,b,sz(w)) {
        u = v;
        rep(x,0,m) v[x+w[i]] = max(v[x+w[i]], u[x]);
        for (x = 2*m; --x > m;) rep(j, max(0,u[x]), v[x])
            v[x-w[j]] = max(v[x-w[j]], j);
    }
    for (a = t; v[a+m-t] < 0; a--);
    return a;
}
```

10.3 Dynamic programming

KnuthDP.h

Description: When doing DP on intervals: $a[i][j] = \min_{i < k < j} (a[i][k] + a[k][j]) + f(i, j)$, where the (minimal) optimal k increases with both i and j , one can solve intervals in increasing order of length, and search $k = p[i][j]$ for $a[i][j]$ only between $p[i][j - 1]$ and $p[i + 1][j]$. This is known as Knuth DP. Sufficient criteria for this are if $f(b, c) \leq f(a, d)$ and $f(a, c) + f(b, d) \leq f(a, d) + f(b, c)$ for all $a \leq b \leq c \leq d$. Consider also: LineContainer (ch. Data structures), monotone queues, ternary search.

Time: $\mathcal{O}(N^2)$

DivideAndConquerDP.h

Description: Given $a[i] = \min_{lo(i) \leq k < hi(i)} (f(i, k))$ where the (minimal) optimal k increases with i , computes $a[i]$ for $i = L..R - 1$.

Time: $\mathcal{O}((N + (hi - lo)) \log N)$

```
struct DP { // Modify at will:
    int lo(int ind) { return 0; }
    int hi(int ind) { return ind; }
    ll f(int ind, int k) { return dp[ind][k]; }
    void store(int ind, int k, ll v) { res[ind] = pii(k, v); }

    void rec(int L, int R, int LO, int HI) {
        if (L >= R) return;
        int mid = (L + R) >> 1;
        pair<ll, int> best(LLONG_MAX, LO);
        rep(k, max(LO, lo(mid)), min(HI, hi(mid)))
            best = min(best, make_pair(f(mid, k), k));
        store(mid, best.second, best.first);
        rec(L, mid, LO, best.second+1);
        rec(mid+1, R, best.second, HI);
    }
    void solve(int L, int R) { rec(L, R, INT_MIN, INT_MAX); }
};
```

10.4 Debugging tricks

- signal(SIGSEGV, [](int) { _Exit(0); }); converts segfaults into Wrong Answers. Similarly one can catch SIGABRT (assertion failures) and SIGFPE (zero divisions). _GLIBCXX_DEBUG failures generate SIGABRT (or SIGSEGV on gcc 5.4.0 apparently).
- feenableexcept(29); kills the program on NaNs (1), 0-divs (4), infinities (8) and denormals (16).

10.5 Optimization tricks

__builtin_ia32_ldmxcsr(40896); disables denormals (which make floats 20x slower near their minimum value).

10.5.1 Bit hacks

- x & -x is the least bit in x.
- for (int x = m; x;) { --x &= m; ... } loops over all subset masks of m (except m itself).
- c = x&-x, r = x+c; (((r^x) >> 2)/c) | r is the next number after x with the same number of bits set.
- rep(b,0,K) rep(i,0,(1 << K)) if (i & 1 << b) D[i] += D[i^(1 << b)]; computes all sums of subsets.

10.5.2 Pragas

- `#pragma GCC optimize ("Ofast")` will make GCC auto-vectorize loops and optimizes floating points better.
- `#pragma GCC target ("avx2")` can double performance of vectorized code, but causes crashes on old machines.
- `#pragma GCC optimize ("trapv")` kills the program on integer overflows (but is really slow).

FastMod.h

Description: Compute $a\%b$ about 5 times faster than usual, where b is constant but not known at compile time. Returns a value congruent to a (mod b) in the range $[0, 2b)$.

```
751a02, 8 lines
typedef unsigned long long ull;
struct FastMod {
    ull b, m;
    FastMod(ull b) : b(b), m(-1ULL / b) {}
    ull reduce(ull a) { // a % b + (0 or b)
        return a - (ull)((__uint128_t(m) * a) >> 64) * b;
    }
};
```

FastInput.h

Description: Read an integer from stdin. Usage requires your program to pipe in input from file.
Usage: ./a.out < input.txt
Time: About 5x as fast as cin/scanf.

```
7b3c70, 17 lines
inline char gc() { // like getchar()
    static char buf[1 << 16];
    static size_t bc, be;
    if (bc >= be) {
        buf[0] = 0, bc = 0;
        be = fread(buf, 1, sizeof(buf), stdin);
    }
    return buf[bc++]; // returns 0 on EOF
}

int readInt() {
    int a, c;
    while ((a = gc()) < 40);
    if (a == '-') return -readInt();
    while ((c = gc()) >= 48) a = a * 10 + c - 48;
    return a - 48;
}
```

BumpAllocator.h

Description: When you need to dynamically allocate many objects and don't care about freeing them. "new X" otherwise has an overhead of something like 0.05us + 16 bytes per allocation.

```
745db2, 8 lines
// Either globally or in a single class:
static char buf[450 << 20];
void* operator new(size_t s) {
    static size_t i = sizeof buf;
    assert(s < i);
    return (void*)&buf[i -= s];
}
void operator delete(void*) {}
```

SmallPtr.h

Description: A 32-bit pointer that points into BumpAllocator memory.

```
2dd6c9, 10 lines
template<class T> struct ptr {
    unsigned ind;
```

```
ptr(T* p = 0) : ind(p ? unsigned((char*)p - buf) : 0) {
    assert(ind < sizeof buf);
}
T& operator*() const { return *(T*)(buf + ind); }
T* operator->() const { return &*this; }
T& operator[](int a) const { return (&*this)[a]; }
explicit operator bool() const { return ind; }
};
```

BumpAllocatorSTL.h

Description: BumpAllocator for STL containers.
Usage: vector<vector<int, small<int>>> ed(N);

```
bb66d4, 14 lines
char buf[450 << 20] alignas(16);
size_t buf_ind = sizeof buf;

template<class T> struct small {
    typedef T value_type;
    small() {}
    template<class U> small(const U&) {}
    T* allocate(size_t n) {
        buf_ind -= n * sizeof(T);
        buf_ind &= 0 - alignof(T);
        return (T*)(buf + buf_ind);
    }
    void deallocate(T*, size_t) {}
};
```

SIMD.h

Description: Cheat sheet of SSE/AVX intrinsics, for doing arithmetic on several numbers at once. Can provide a constant factor improvement of about 4, orthogonal to loop unrolling. Operations follow the pattern `"_mm(256)?name.(si(128|256)|epi(8|16|32|64)|pd|ps)".` Not all are described here; grep for `_mm_` in `/usr/lib/gcc/*/4.9/include/` for more. If AVX is unsupported, try 128-bit operations, "emmintrin.h" and `#define __SSE__` and `__MMX__` before including it. For aligned memory use `_mm_malloc(size, 32)` or `int buf[N] alignas(32)`, but prefer `loadu/storeu`.

```
551b82, 43 lines
#pragma GCC target ("avx2") // or sse4.1
#include "immintrin.h"

typedef __m256i mi;
#define L(x) _mm256_loadu_si256((mi*)&(x))

// High-level/specific methods:
// load(u)?_si256, store(u)?_si256, setzero_si256, _mm_malloc
// blendv_(epi8|ps|pd)(z?y:x), movemask_epi8 (hibits of bytes)
// i32gather_epi32(addr, x, 4): map addr[] over 32-b parts of x
// sad_epu8: sum of absolute differences of u8, outputs 4xi64
// maddubs_epi16: dot product of unsigned i7's, outputs 16xi15
// madd_epi16: dot product of signed i16's, outputs 8xi32
// extractf128_si256(, i) (256->128), cvtsi128_si32 (128->lo32)
// permute2f128_si256(x,x,1) swaps 128-bit lanes
// shuffle_epi32(x, 3*64+2*16+1*4+0) == x for each lane
// shuffle_epi8(x, y) takes a vector instead of an imm

// Methods that work with most data types (append e.g. _epi32):
// set1, blend (i8?x:y), add, adds (sat.), mullo, sub, and/or,
// andnot, abs, min, max, sign(i,x), cmp(gt|eq), unpack(lo|hi)

int sumi32(mi m) { union {int v[8]; mi m;} u; u.m = m;
    int ret = 0; rep(i,0,8) ret += u.v[i]; return ret; }
mi zero() { return _mm256_setzero_si256(); }
mi one() { return _mm256_set1_epi32(-1); }
bool all_zero(mi m) { return _mm256_testz_si256(m, m); }
bool all_one(mi m) { return _mm256_testc_si256(m, one()); }

ll example_filteredDotProduct(int n, short* a, short* b) {
```

```
int i = 0; ll r = 0;
mi zero = _mm256_setzero_si256(), acc = zero;
while (i + 16 <= n) {
    mi va = L(a[i]), vb = L(b[i]); i += 16;
    va = _mm256_and_si256(_mm256_cmpgt_epil6(vb, va), va);
    mi vp = _mm256_madd_epil6(va, vb);
    acc = _mm256_add_epi64(_mm256_unpacklo_epi32(vp, zero),
        _mm256_add_epi64(acc, _mm256_unpackhi_epi32(vp, zero)));
}
union {ll v[4]; mi m;} u; u.m = acc; rep(i,0,4) r += u.v[i];
for (;i<n;++i) if (a[i] < b[i]) r += a[i]*b[i]; // <- equiv
return r;
}
```