Homework 2

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CS331 Algorithms and Complexity

Problem Q1(a). What is the number of topological orderings in this directed graph?
Solution: 2^{n+1}

Problem Q1(b). Let T be a tree such that every node in T has either 2 children or 0 children. If T has $n \geq 1$ leaves, prove that the total number of nodes in T is $2n - 1$.

Proof. **Base case:** (n = 1)
The tree must have precisely 1 node, $2n - 1 = 2 - 1 = 1$.
Suppose the tree has more than 1 node.
By the definition of tree, the graph must be connected.
The tree has precisely 1 leaf $l \implies$ every other node must have 2 children.
Suppose l is connected to p, which must have 2 children.
Each descendent of p must have 2 children, meaning the tree must have an infinite number of nodes.
Contradiction
Induction Hypothesis:
Suppose a tree T has n leaves.
Remove 2 leaves with the same parent to form tree T'.
By the definition of a leaf, each leaf is connected to precisely 1 other node (parent).
By removing 2 leaves, precisely 2 edges are removed.
$\therefore$ the number of nodes in T' is $2n - 3$.

$\square$

Problem Q2. Let s be the vertex of a connected undirected graph G. Let $T^B_{G,s}$ and $T^D_{G,s}$ respectively be the trees obtained by running BFS and DFS on graph G starting at node s. Prove

$$T^B_{G,s} \equiv T^D_{G,s} \implies G \text{ is acyclic}$$

Lemma 1: $T^B_{G,s} \equiv T^D_{G,s} \implies T^B_{G,s}$ has no non-tree edges.

Proof. Suppose vertices $v_1, v_2 \in T^B_{G,s}$ are connected by a non-tree edge e.
In DFS, either v_1 or v_2 must have been explored first, because vertices are explored in a strictly non-parallel fashion.
Let v_p be the vertex which was explored first in DFS, and v_c be the other vertex.
If v_p was explored first, v_c must not have been explored when v_p was added to the DFS tree.
Since v_c was not explored, the edge $e = (v_p, v_c)$ must be added to the DFS tree by the DFS algorithm.
$\therefore e \in T^D_{G,s} \wedge T^B_{G,s} \implies T^B_{G,s} \not\equiv T^D_{G,s}$

$\square$

Proof. By lemma 1, $T^B_{G,s}$ has no non-tree edges.
This implies $T^B_{G,s} = G$
By definition, if G is a tree, G is acyclic.

$\square$

Problem Q3. Given n images and m unambiguous matches, design an algorithm that runs in $O(m+n)$ time and uniquely labels n images as either A or B, such that two images reported to be the same by ImgComp get the same label, and two images reported to be different by ImgComp get different labels. Prove the algorithm's runtime and correctness.

Problem Q4.