# Simulations in AC Tokamak Ramp Downs

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*For George, Paul, Ringo and John.*

# Declaration

The work in this thesis is my own except where otherwise stated.

Thomas Malcolm.

# Acknowledgements

First and foremost I'd like to thank Matthew Hole, my supervisor. I came into this year a little daunted at the prospect of tackling a physics-based project with a physics background not extending past "A Brief History of Time". Your patience and willingness to explain concepts to me I'll forever appreciate, and your energetic approach to problem solving made me feel both at home and excited for the work we have done in this thesis. Thank you for always taking the time to talk to me, and for being a fantastic supervisor.

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# Abstract

# Contents

# Notation and terminology

### Notation

| | |
|---|---|
| $L^2(\mathbb{R}^n)$ | Space of $L^2$ integrable functions on $\mathbb{R}^n$ |
| $B_0$ | On-axis magnetic field (Toroidal magnetic field strength) |
| $R_0$ | Major radius (Tokamak) |
| $a$ | Minor radius (Tokamak) |
| $\psi$ | Poloidal magnetic flux function |
| $J_\phi$ | Toroidal current density |
| $j_\phi$ | Normalised toroidal current density (normalised w.r.t. $B_0$) |
| $I_\rho$ | Normalised toroidal current density (normalised w.r.t. maximum value) |
| $p$ | Plasma pressure density |

### Terminology

| | |
|---|---|
| GS / GSE | Grad-Shafranov equation |
| GSH | Grad-Shafranov-Helmholtz equation |
| MHD | Magnetohydrodynamics |

# Chapter 1

# Introduction

## 1.1 Plasma Science

This thesis is multidisciplenary by nature, incorporating aspects of mathematics, physics and computer science throughout the various challenges faced in reaching the conclusions we have.

Before burying ourselves in the thick of our results we will cover the requisite knowledge for working in the plasma science simulation space. First we'll discuss what the physical object of our attention ("plasma") actually is, with a brief discussion on the design of fusion reactors, where we will emphasise the structures that specifically relate to our interests. In the background chapter we will expound on this by building the mathematical theory underpinning physical processes within the fusion reactor, providing us a way to reason about the behaviour of plasma (and related processes) inside a fusion reactor, or, more accurately, approximate their behaviour via simulation.

### 1.1.1 I'm a Mathematician... what is "plasma"?

While an initially daunting topic, fear not fearful mathematician, for many of the inherently physical behaviours in plasma science we require can be expressed in terms of our dearest mathematical expressions – differential equations! But first, what actually is "plasma"? Webster's dictionary defines plasma to be "a green faintly translucent quartz" [8]. While I'm sure there isn't no relation between crystals and our investigations, this is unfortunately, not the recipient of our affection.

When plasma is referred to in everyday conversation it is often noted as being the "4th state of matter". To introduce slightly more rigour, plasma is an extension of the gaseous state of matter, where its energy (read: temperature) is increased sufficiently high that the electrons are no longer bound by the electromagnetic force to the atom's nucleus REFERENCE. The resulting substance is a "pool" of cations (the positively charged nuclei), and electrons (negatively charged), that exhibits interesting properties. It is these properties that we seek to exploit in the search of controlled, sustained fusion reactions.

Plasma is abundant in nature – just not in many places that we as Humans commonly look. Stars are the most immediate example of matter in a plasma state, and are readily viewable (at least for half the day). Lightning strikes are paths through the atmosphere which are ionised, and neon signs work by heating Neon gas within a tube to ionise it REFERENCE.

The question then is, what is "fusion", and how does this relate to plasma? The answer comes back to energy. Analogously to a fission reaction, where energy is released through the division of atoms, one can also fuse two separate atoms together and have large amounts of energy emitted as a bi-product of that fusion. It is (essentially) this extra energy that we wish to harness when harvesting energy from a fusion reactor.

While there are no shortages of elements that can theoretically be used to fuel a fusion reactor, the one most commonly associated with fusion is Deuterium – a stable isotope of Hydrogen that has a neutron in its nucleus (whereas a 'standard' Hydrogen atom contains only a single proton). Analogously, Tritium is an isotope of Hydrogen that has two neutrons in its nucleus (though is much less stable). Consider the fusion of a Deuterium atom with a Tritium atom:

$$^{2}\mathrm{H} + {}^{3}\mathrm{H} \overset{\text{(fusion)}}{\rightarrow} {}^{4}\mathrm{He} + n + 17.59\,\mathrm{MeV}$$

Here two Hydrogen isotopes fuse together to form a single Helium-4 atom, and in the process of doing so release a single neutron and 17.59 MeV of energy. This excess energy is what is so attractive about fusion processes as a sustainable energy source – for such little input we receive a substantial amount of energy, and at that, using one of the most abundant resources available on Earth; water.

The question then becomes, how do we drive this fusion process? If two atoms can fuse as such, why do we not see atoms fusing everywhere around us accompanied by violent explosions destroying all that we've come to know and love? The answer is that we kind of do – just not normally in the places that humans expect to be inhabiting. In fact, we see this happening everyday – for those of us fortunate enough to be able to see the sun that is. Our sun is a large ball of plasma where an estimated $9.3 \times 10^{37}$ fusion reactions are expected to occur every second [9], and is one of the easiest examples of both plasma as a state of matter, and of a self-sustaining fusion reaction.

How then do two hydrogen atoms fuse – how can we create a sun on Earth? The nucleus of an atom (consisting of positively charged protons, and neutral neutrons) is positively charged, and so two atoms' nuclei will repel each other due to the Coulomb force when pushed together. This is the force we have to overcome to enable a fusion reaction to take place (and what stops the world around us burning!). To overcome this the process is relatively "simple" – we just increase the energy of our atoms so that when they collide they collide with enough energy to overcome this force, allowing the strong force to become dominant, fusing the two atoms. When we energise a mass (take a gas here) of atoms enough, they become ionised however, which is exactly the state of plasma we described earlier. In other words, if we want to reason about the creation, sustainment, and effects of fusion reactions, we need first to understand the dynamics of plasma, the medium in which the fusion reactions take place. From this comes a plethora of questions, ranging from "how do we generate such a plasma?" to "how do we reliably control such a plasma" and "how do we harness such a plasma"? Alas, we digress however, as we do not seek to solve the big problems in fusion science in this here mere mortal thesis. Instead, now equipped with at least a passing knowledge of what constitutes a "plasma" state and what it means for a fusion reaction to take place, we will humbly delve into the inner workings of fusion reactor terminology and design.

### 1.1.2 Fusion Reactor Design 101

Here we will discuss the important structural aspects of a Tokamak fusion reactor. The term "Tokamak" is a Russian acronym which translates as "toroidal chamber with magnetic coils" [6]. Aptly, a Tokamak is a toroidal object which is used as a vessell for plasma which is driven via external magnetic coils.

In reality, reactors do not often have such nice geometry. Figure 1.1

Diagram

Poloidal vs toroidal flux

Magnet positioning

Heating of plasma

Confinement

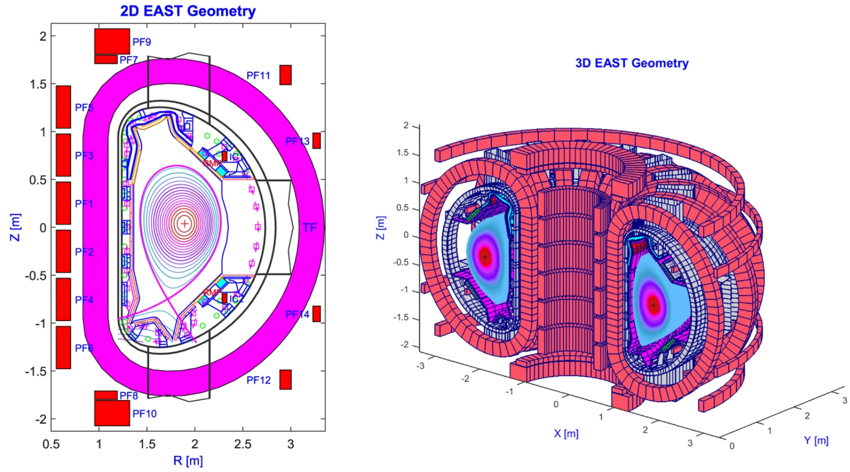


Figure 1.1: EAST fusion reactor geometry. This is a 3D model of its cross section, highlighting the irregular shape [2]

REs

## 1.2 What problem does this thesis address?

Talk about paper Matthew and Artur released. Presence of residual REs.

### 1.2.1 What results do we seek?

To provide a theoretical basis for exploring the observed anomalies. To see if theory supports the observation

## 1.3 Project Progression

I began this project by first doing a lot of reading. Without a formal phsyics education past grade 12, it was important to develop an understanding for the underlying processes that are happening so as to ensure an understanding of the mathematics we deal with. Additionally, much of the mathematics I encountered was new. For example, until this thesis, I had not come across the definition of a PDE before, let alone reasoned about a solution to one and used it to run a simulation. Regardless, the very first step of the process was to read - that began with Wesson's classic textbook "Tokamaks", and a series of lectures given by Matthew Hole at ANU for a fusion science special topics course delivered in 2022, and branched from there.

The inspiration for this thesis (as we stated in the section above, and will frequent throughout the thesis) was a paper published by Artur Malaquias and Matthew Hole, "Experiments during AC current transition in ISTTOK and the hypothesis of ballistic runaway electrons" [7].

Work inspired by Matthew's paper

First expanded MHD equations linearly with a perturbation treatment

Mathematical PDE theory developed by Wang. Identified errors in paper, fixed.

Developed a simulation to reproduce results. Then went other direction, solving for parameters for data.

Simulated current inversion.

ISTTOK data matching.

Feasibility

## 1.4 Structure of Thesis

TODO (after writing it)

# Chapter 2

# Background

## 2.1 Magnetohydrodynamics

The term "magnetohydrodynamics" (MHD) is a portmanteau of two physical concepts which are used to model plasmas inside fusion reactors (among other things): the "**m**agneto" term comes from "magnetic field", and "**h**ydro**d**ynamics" indicates a a fluid dynamics component. Put together, magnetohydrodynamics is the study of electrically conductive materials that behave like fluids. Essentially it provides a way to model the behaviour of a (considerably volumous!) mass of conductive particles, and their electrodynamic forces, as if it were a fluid, as opposed to having to model individual particle interactions. This idea was first introducd by Hannes Alfvén in 1970, for which he earned the Nobel prize [1]!

Much modern research in Tokamak plasma simulation uses some variation of an MHD model (and the derived Grad-Shafranov Equation, which we will soon be introduced to) for a number of reasons, not least being its comparative computational efficiency. One primary benefit of treating our plasma as a fluid being that we avoid modelling the behaviour of each individual particle in said plasma, a simplification which becomes especially important when we consider the order of number of particles we would have to simulate is of order $\sim 10^{20}$ - far too much for the author's ThinkPad T440p to even contemplate!

Here we will build the relevant MHD background for this thesis, deriving the MHD equations from first principles, and explaning the assumptions we make to reduce them to a simplified state known as "ideal MHD". From there we will look at the PDE which models the behaviour of a plasma inside a Tokamak, the "Grad-Shafranov Equation". We will then note some pitfalls of using this model to describe AC configuration Tokamaks (as we are investigating), and finish with a discussion on runaway electrons (RE).

### 2.1.1 MHD Theory

Given MHD is the marriage of fluid dynamics with electrodynamics, it is only natural to begin our study looking at the equations which describe electrodynamic behaviour — Maxwell's equations describe the interaction between magnetic fields $\vec{B}(\vec{r},t)$, electric fields $\vec{E}(\vec{r},t)$, and the current density $j(\vec{r},t)$ which induces them, where $\vec{r} \in \mathbb{R}^3$ is a position vector and $t \in \mathbb{R}$ describes time. Thus, we introduce Maxwell's equations:

**Definition 2.1.** Maxwell's equations are given [13]:

$$\nabla \times \vec{B} = \mu_0 j + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \tag{2.1}$$

$$\nabla \times E = -\frac{\partial \vec{B}}{\partial t} \tag{2.2}$$

$$\nabla \cdot \vec{B} = 0 \tag{2.3}$$

$$\nabla \cdot E = \frac{\rho_c}{\varepsilon_0} \tag{2.4}$$

The functions driving change in this system are $\rho_c(\vec{r},t)$, the electric charge density, and $j(\vec{r},t)$, the electric current density. We also have $\mu_0$, the free-space magnetic permeability (in henry $m^{-1}$); $\varepsilon_0$, the free-space permittivity; and $c$, the speed of light.

These equations give us a way to reason about the electric and magnetic fields if we're given some descriptor for the current we're passing through some medium. We are now to introduce the fluid dynamics component to our system. A fluid's mass density can be given by summing over the effects of individual "species" of particles (e.g. electrons) in the fluid:

$$\rho_c = \sum_\sigma m_\sigma n_\sigma$$

and its current density similarly:

$$\vec{j}(\vec{r},t) = \sum_\sigma n_\sigma q_\sigma \vec{u}_\sigma$$

where $\sigma$ describes a particle species, $m_\sigma$ describes its mass, $n_\sigma$ its number density (a measurement of concentration for the given particle species in a pre-defined volume – akin to Avogadro's constant), $q_\sigma$ describes its electric charge, and $\vec{u}_\sigma$ the mean velocity of this species of particle in the fluid.

### Fluid Dynamics

**Notation 2.2.** A common simplification in notation for fluids is made in using the *Lagrangian derivative*, given:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \left(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla\right)$$

It describes the total change in a volume within a fluid as it moves throughout said fluid. It is essentially a change in reference frame for a derivative - where a regular derivative might descirbe, for example, how a particle moves with respect to time in its surroundings, its Lagrangian will take into account the motion of the fluid the particle is immersed in as well.

We begin with conservation of mass, also known as the "continuity equation".

**Definition 2.3** (The continuity equation)**.** The below relates how mass density, $\rho$, changes with respect to the motion of a fluid element.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \vec{u}) \tag{2.5}$$

The derivation of the above comes from a surface integral over a volume with an outward and inward flux, and an application of Gauss' flux law. For a full derivation, see pg. 19 - 21 of [10].

**Remark 2.4.** The above is a PDE with four variables: $\rho$, the mass density of the medium, and $\vec{u}$, the velocity of the fluid. This renders the system not closed, and thus too general for an analytic solution - we have more unknowns than we have equations [10]. Later we will introduce other equations to our system to apply more restrictions, and make assumptions about the physicality of the system which will reduce these dependencies, and make it determined ("closed").

**Remark 2.5.** We can rewrite (2.5) with a Lagrangian frame of reference, as

$$\frac{\mathrm{D}}{\mathrm{D}t}\rho = -\rho \nabla \cdot \vec{u} \tag{2.6}$$

$$\iff \frac{\mathrm{D}}{\mathrm{D}t}\rho + \rho \nabla \cdot \vec{u} = 0 \tag{2.7}$$

Equation (2.5) (and equivalently (2.7)) tells us that the mass of our fluid is conserved for motion of a volume element of our fluid - one assumption we make for our model. Next we'll discuss fluid motion as described by Newton for a fluid element:

**Definition 2.6** (Newtonian Fluid Motion)**.** Newton's law for a fluid specifies:

$$\rho \frac{\mathrm{D}}{\mathrm{D}t} \vec{u} = \vec{F} \tag{2.8}$$

where $\rho(\vec{r}, t)$ is the mass density of the fluid, $\vec{u}(\vec{r}, t)$ describes the velocity of the fluid element, and $\vec{F}(\vec{r}, t)$ describes the force per unit volume acting on the fluid element [10].

The forces acting on particles within a fluid can be split into two types

- Gravitational

  Here, $\vec{F_g} = \rho\vec{g}$, where $\vec{g}$ is the gravitational acceleration. This should hark back to high school physics, though note this is a vector here as we care about the direction gravity accelerates the fluid element in, and relativistic effects can be an important consideration for high mass systems. This is more relevant for cases that you are using the MHD equations to describe the dynamics of large systems, such as a star. Unsurprisingly, this is less relevant for our case of plasmas within relatively miniscule Tokamaks.

- Electromagnetic

  This is the interesting part for us. As we assume our fluid is capable of conducting electricity (it's a plasma after all), there are electromagnetic forces operating within the fluid that affect the behaviour of the particles that the fluid consists of. The electromagnetic forces themselves can be split into two types, the **electric force** given by $\vec{F_q} = \rho_c\vec{E}$, and the **Lorentz force**, given $\vec{F_L} = \vec{j} \times \vec{B}$ (where $\vec{B}(\vec{r}, t)$ describes the magnetic field)

Taking these forces into account, we can describe the motion of an element of our fluid moving with velocity $\vec{u}$ via:

$$\rho \frac{\mathrm{D}}{\mathrm{D}t} \vec{u} = \vec{j} \times \vec{B} + \rho_c\vec{E} - \nabla p + \rho\vec{g} \tag{2.9}$$

**Remark 2.7.** Here the $\rho\vec{g}$ term could be abstracted further away into a stress tensor, as described by equation 4.20 of [10]. These pressures however are largely negligible when dealing with the scale we do in Tokamak plasmas however, and are thus ignored. We will soon drop the gravitational consideration as well anyway, but include it here for now for completeness.

**Remark 2.8.** The equation we have introduced is a function of six variables in its complex form (with stress tensor included), though even in this form we still have more variables than we do equations (1). Similar to before, this is not constrained sufficiently to consider it a closed system.

We next relate a plasma's pressure to its motion. We will simply present it here, though important notes in its derivation are that we assume the plasma behaves as an ideal gas (which is to say the only interaction between particles within the plasma are via elastic collisions with each other, or the boundaries of the container it is contained within). This is equivalent to saying that energy in the system depends only on the pressure. Thus, the energy equation is given:

**Definition 2.9** (Energy Equation)**.** Where $\vec{p}(\vec{r}, t)$ describes the pressure of our fluid:

$$\frac{\mathrm{D}}{\mathrm{D}t}p = -\gamma p \nabla \cdot \vec{u} + (\gamma - 1)\left[-\nabla \cdot \vec{q} + \vec{\Pi} : \nabla \vec{u} + \eta \vec{J}^2\right] \tag{2.10}$$

where $\gamma$ describes "abiabatic index" (a known constant for plasmas), $q$ is the heat flux through the boundary of the volume; $\eta$ is the electrical resistivity of the fluid; and $\vec{\Pi}$ is the viscous stress tensor (the component which we replaced with $\rho\vec{g}$ earlier), and will soon ignore again.

The equations we've looked at constitute what are known as the fluid equations:

**Definition 2.10** (Fluid Equations)**.**

$$\frac{\mathrm{D}}{\mathrm{D}t}\rho + \nabla \cdot \rho \vec{u} = 0 \tag{2.11}$$

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\vec{u} = \vec{j} \times \vec{B} + \rho_c \vec{E} - \nabla p + \rho \vec{g} \tag{2.12}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}p = -\gamma p \nabla \cdot \vec{u} + (\gamma - 1)\left[-\nabla \cdot \vec{q} + \vec{\Pi} : \nabla \vec{u} + \eta \vec{j}^2\right] \tag{2.13}$$

As reiterated a couple times now, these equations form an unclosed system, and are thus undetermined. To resolve this we introduce some constraints that come from electrodynamic forces, and a couple other relations that lead to a closed system.

### Electrodynamics

As it currently stands, the input variables for the fluid equations are $\rho(\vec{r}, t)$ and $p(\vec{r}, t)$. We note that the electric field $\vec{E}$ and the magnetic field $\vec{B}$ are generated by the electric charge density, $\rho_c$, and the current density $\vec{j}$. This is where Maxwell's equations come into play. By combining Maxwell's equations with the fluid equation given above, we achieve the MHD model. The only piece to our puzzle missing is to tie the motion of the fluid (through $\vec{u}$) to the behaviour of the electric and magnetic fields. This is done via *Ohm's* law:

$$\vec{E} + \vec{u} \times \vec{B} = \eta \vec{j} \tag{2.14}$$

**Remark 2.11.** Note that the above is technically a lie, as it does not take into account relativistic effects, though for simplicity our MHD model ignores these.

**Definition 2.12** (MHD Equations)**.** The MHD equations can then be summarised:

$$\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t}\rho &= -\rho \nabla \cdot \vec{u} && (2.15)\\
\rho \frac{\mathrm{D}}{\mathrm{D}t}\vec{u} &= -\nabla p + \vec{j} \times \vec{B} + \nabla \cdot \Pi && (2.16)\\
\frac{\mathrm{D}}{\mathrm{D}t}p &= -\gamma p \nabla \cdot \vec{u} + (\gamma - 1)\left[-\nabla \cdot \vec{q} + \vec{\Pi} : \nabla \vec{u} + \eta \vec{j}^2\right] && (2.17)\\
\frac{\partial \vec{B}}{\partial t} &= -\nabla \times \vec{E} && (2.18)\\
\mu_0 \vec{j} &= \nabla \times \vec{B} && (2.19)\\
\vec{E} + \vec{u} \times \vec{B} &= \eta \vec{j} && (2.20)
\end{aligned}$$

**Remark 2.13.** The MHD equations as presented above constitute 14 equations with 27 unknowns. The breakdown is as such:

- $\rho$ : 1 unknown
- $\vec{u}$ : 3 unknowns
- $p$: 1 unknown
- $\vec{\Pi}$ : 9 unknowns
- $\vec{j}$ : 3 unknowns
- $\vec{B}$ : 3 unknowns

- $\vec{E}$ : 3 unknowns
- $\vec{q}$ : 3 unknowns
- $\eta$ : 1 unknown

The above is obviously insufficiently constrained for purposes of identifying a solution. We will skip a large amount of the work required to reduce the above to a closed system, though for details see lecture 7 of [10]. For now, we will comment on two reduced MHD models:

### Resistive MHD

The resistive MHD model comes about by setting $\vec{q} = 0$ and $\vec{\Pi} = 0$, in other words saying that we have no external source for current, and . Here, we have $\eta \neq 0$ notably. The model is given:

**Definition 2.14** (Resistive MHD)**.**

$$\frac{\mathrm{D}}{\mathrm{D}t}\rho = -\rho\nabla \cdot \vec{u} \tag{2.21}$$

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\vec{u} = -\nabla p + \vec{j} \times \vec{B} \tag{2.22}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}p = -\gamma p\nabla \cdot \vec{u} \tag{2.23}$$

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} \tag{2.24}$$

$$\mu_0\vec{j} = \nabla \times \vec{B} \tag{2.25}$$

$$\vec{E} + \vec{u} \times \vec{B} = \eta\vec{j} \tag{2.26}$$

**Remark 2.15.** The most notable effect of resistive MHD is that allowing for electrons to diffuse allows the resulting magnetic field lines to reconnect, which leads to breaks in the magnetic field line topology. This can lead to the generation of fast particles, i.e., runaway electrons.

### Ideal MHD

The ideal MHD equations are one step removed from the resistive MHD model — in fact they are equivalent, only for the ideal MHD case we also ignore resistivity. Thus, set $\eta = 0$, and we obtain the ideal MHD equations:

**Definition 2.16** (Ideal MHD).

$$\frac{\mathrm{D}}{\mathrm{D}t}\rho = -\rho\nabla\cdot\vec{u} \tag{2.27}$$
$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\vec{u} = -\nabla p + \vec{j}\times\vec{B} \tag{2.28}$$
$$\frac{\mathrm{D}}{\mathrm{D}t}p = -\gamma p\nabla\cdot\vec{u} \tag{2.29}$$
$$\frac{\partial\vec{B}}{\partial t} = -\nabla\times\vec{E} \tag{2.30}$$
$$\mu_0\vec{j} = \nabla\times\vec{B} \tag{2.31}$$
$$\vec{E} + \vec{u}\times\vec{B} = 0 \tag{2.32}$$

**Remark 2.17.** In doing this, we have removed the dependency on $\vec{\Pi}$, $\eta$ and $\vec{q}$, which accounts for 13 unknowns. This brings the total number of equations to 14 (or 8 if you make a substitution for Ohm's law) with 14 (8) unknowns. Thus, under the ideal MHD model, the system is closed.

**Remark 2.18.** Equation (2.32) is the reduced Ohm's law equation. This presentation of it is a result known as the *frozen flux condition.* Nominally, it tells us that $\frac{\partial\Psi}{\partial t} = 0$ (we will introduce $\Psi$ shortly, but for now take it to be the "magnetic flux function" (which it is) without any context). This is to say that the magnetic flux through any co-moving closed circuit is constant, in other words, field lines are to some extent "attached" to the fluid they are in (read: plasma), and similarly the plasma cannot move across the magnetic field (but the fluid can move along the field lines).

We'll present the ideal MHD equations here again but with their associated names for ease of reference:

| **Continuity Equation** | **Momentum Equation** | **Energy Equation** |
|---|---|---|
| $\frac{\mathrm{D}}{\mathrm{D}t}\rho = -\rho\nabla\cdot\vec{u}$ | $\rho\frac{\mathrm{D}}{\mathrm{D}t}\vec{u} = -\nabla p + \vec{j}\times\vec{B}$ | $\frac{\mathrm{D}}{\mathrm{D}t}p = -\gamma p\nabla\cdot\vec{u}$ |
| **Maxwell-Faraday Equation** | **Ampere's Law** | **Ohm's Law** |
| $\frac{\partial\vec{B}}{\partial t} = -\nabla\times\vec{E}$ | $\mu_0\vec{j} = \nabla\times\vec{B}$ | $\vec{E} + \vec{u}\times\vec{B} = 0$ |

**Resistivity Note:** There is one specific assumption made in ideal MHD that we wish to highlight, and that is the assumption of negligible resistivity within the plasma. In a real plasma we would expect imperfections in the way it conducts, and would expect collisions between particles in the plasma to generate a type of "friction" between each other that inhibits conductivity. When this occurs, the magnetic field lines that are driven by the current will perturb with respect to some diffusion law, leading to changes in magnetic field line topology. These effects occur over some period of time that is dependent on the plasma's composition and geometry of the plasma - thus any plasma model which wishes to accurately represent the dynamics of the electric and magnetic field as a function of time will do well to take these effects into account, especially when it is over a "sufficient" time scale (with special consideration for what "sufficient" actually entails here). It is possible that the timescales on which a simulation is run are too small for resistive effects to meaningfully change field line topology, though this is more a question of acceptable error and initial conditions. In this thesis (as we'll see) we seek to approximate a time evolution, but do so with a series of equilibria, which as such do not take into account resistive effects. We will make note on this again when we come to discuss of our perturbation approach, and similarly when we present our simulated current reversals.

The ideal MHD equations as we stated are what govern many approximations to behaviour of plasmas in general. These in their current form of course make no assumptions about the geometry of the medium in which a plasma exists, nor any properties which may be associated with such a scenario. Next we will take these equations and confine them to the geometry of an axisymmetric Tokamak "reactor", and derive the Grad-Shafranov equation, a model which will let us reason about the poloidal magnetic field structure given some prescribed geometry and conditions.

## 2.2 Grad-Shafranov Equation

The Grad-Shafranov Equation (GSE) concerns a two-dimensional, axisymmetric toroidal plasma under the assumptions of ideal MHD. It is a nonlinear, elliptic partial differential equation (PDE) which presents with a cylindrical coordinate system [11]. We will first begin with some basic definitions surrounding PDEs, partly because the author is not formally educated in them and wishes to demonstrate the amount they have learned over this last year, but largely, realistically, for the obligatory Evans citation. We will then introduce GSE, and explore some of its properties.

**Definition 2.19** (Partial Differential Equation)**.** An expression of the form

$$F(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x) = 0 \quad (x \in U) \tag{2.33}$$

is called a $k^{\text{th}}$-order Partial Differential Equation where

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$$

is provided, and the function

$$u : U \to \mathbb{R}$$

is an unknown [4].

To "solve" a PDE is to find such a function $u$, subject to any boundary conditions $\Gamma$ in $\partial U$ that may exist. A common exercise in the study of PDEs is to simply show existence and uniqueness of a solution to a PDE, with no intention of actually identifying the function itself. In our case, luckily, the GSE has identified solutions for certain assumptions (boundary conditions) imposed on it, one of which we will exploit later in the Grad-Shafranov-Helmholtz equation. Before that however, we'll look at definitions for PDE properties that are associated with the GSE:

**Notation 2.20.** Evans uses the notation $D^k u$ to mean the $k \times k$ matrix of $k^{\text{th}}$ order partial derivatives of $u$. While we use this when presenting the definitions below, we do not follow this convention for the rest of the thesis, reserving $D$ for the Lagrangian derivative instead.

**Definition 2.21** (PDE Linearity Classifications)**.** Let $a_\alpha$ be a function such that $|\alpha| \leq k$, where $k$ is the order of a system of PDEs. Let $f$ be another function. Then:

1. "Linear"

   A PDE as given by (2.33) is said to be "linear" if it is of the form

   $$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = f(x)$$

2. "Semilinear"

   A PDE is said to be "semilinear" if it is of the form

   $$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u + a_0(D^{k-1}u, \ldots, Du, u, x) = 0$$

3. "Quasilinear"

A PDE is said to be "quaslinear" if it is of the form

$$\sum_{|\alpha|=k} a_\alpha \left(D^{k-1}u, \ldots, Du, u, x\right) D^\alpha u + a_0 \left(D^{k-1}u, \ldots, Du, u, x\right) = 0$$

4. "Nonlinear"

   A PDE is said to be "nonlinear" if it depends nonlinearly on the highest order derivatives, i.e. it is of the form

$$\sum_{|\alpha|=k} a_\alpha \left(D^\alpha u, D^{k-1}u, \ldots, Du, x\right) = 0$$

   where the $D^\alpha$ term introduces nonlinearity by $a_\alpha$.

**Definition 2.22** (Elliptic PDE)**.** A second-order PDE with presentation

$$F[u] = F(\cdot, u, Du, D^2u)$$

where $\Gamma = U \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$, $F : \Gamma \to \mathbb{R}$, and $u$ is a solution to $F[u] = 0$ is said to be elliptic at the point $\gamma = (x, z, p, r) \in \Gamma$ if the matrix

$$F_r(\gamma) = [F_{ij}(\gamma)] := \left[F_{r_{ij}}\right] > 0$$

is true, i.e., that the matrix $F_r(\gamma)$ is positive definite for $r \in U$ [12]. Furthermore, the operator $F$ is said to be *elliptic* if this holds true for the set

$$\Gamma_u = \{(x, u(x), Du(x), D^2u(x)) : x \in U\}$$

**Remark 2.23.** Note that the above representation of a second-order PDE, while not how Evans presents it, is a more abstract way of depicting the same thing. Here we can define an operator $F$ to be in terms of our solution $u$ and its first and second order derivatives, instead of as a decoupled statement of otherwise disconnected expressions. This lets us reason about the PDE as a whole more directly, as we have above by using the positive definiteness of the resulting matrix at a point to define ellipticity, instead of having to separately define the form of a second-order PDE and reason about a matrix explicitly in terms of the solution.

Often in our efforts we will deal not with a single PDE, but rather a system of PDEs, and so we formalise this concept as well:

**Definition 2.24** (System of PDEs)**.** An expression of the form

$$\mathbf{F}(D^k u(x), D^{k-1}u(x), \ldots, Du(x), x) = 0 \quad (x \in U) \tag{2.34}$$

is called a $k^{\text{th}}$-order system of Partial Differential Equations where

$$\mathbf{F} : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \cdots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times U \to \mathbb{R}^m$$

is provided, and the function

$$\mathbf{u} : U \to \mathbb{R}^m, \quad \mathbf{u} = (u_1, \ldots, u_m)$$

In fact we have already seen an example of the above in our derivation of the MHD equations ((2.15) - (2.20)), the result of which was a system of PDEs; a system which was determined by virtue of there being as many unknown functions $(u_i)$ as there were PDEs in the system.

We are now armed with the necessary tools to introduce the Grad-Shafranov Equation. Here we present the definition of the GSE as given by Wesson [13].

**Definition 2.25** (Grad-Shafranov Equation)**.** Let $(R, \phi, z)$ be the cylindrical coordinate system we take, where $R$ is the major radial offset, $\phi$ is the toroidal angle, and $z$ is the poloidal vertical offset from the centre of the cross section. $p(\Psi)$ and $f(\Psi)$ are two flux functions of $\Psi$, the poloidal magnetic flux function, and $\mu_0$ is the vacuum magnetic permeability constant. Then the Grad-Shafranov Equation is given:

$$R\frac{\partial}{\partial R}\frac{1}{R}\frac{\partial \Psi}{\partial R} + \frac{\partial^2 \Psi}{\partial z^2} = -\mu_0 R^2 \frac{\partial p(\Psi)}{\partial \Psi} - \mu_0^2 f(\Psi)\frac{\partial f(\Psi)}{\partial \Psi} \tag{2.35}$$

**Remark 2.26.** The equation can be restated more concisely in terms of the elliptic operator

$$\Delta^* \Psi = R\nabla \cdot \left(\frac{1}{R}\nabla\Psi\right)$$

as:

$$\Delta^* \Psi = -\mu_0 R^2 p'(\Psi) - f(\Psi)f'(\Psi)$$

where prime notation denotes partial derivative with respect to $\Psi$.

**Corollary 2.27.** *Associated with the GSE are explicit forms for the magnetic*


Figure 2.1: This should say something meaningful...

*field $\vec{B}$ and current $\vec{j}$:*

$$\vec{B} = \frac{1}{R}\left(\nabla\Psi \times \hat{e}_\phi\right) + \frac{f(\Psi)}{R}\hat{e}_\phi \tag{2.36}$$

$$\mu_0\vec{j} = \frac{1}{R}\frac{\partial f(\Psi)}{\partial\Psi}\left(\nabla\Psi \times \hat{e}_\phi\right) - \left[\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\Psi}{\partial R} + \frac{1}{R}\frac{\partial^2\Psi}{\partial z^2}\right)\right]\hat{e}_\phi \tag{2.37}$$

*where here $\hat{e}_d$ is the unit vector in the direction $d$, as is convention. In their poloidal (subscript $p$) and toroidal (subscript $\phi$) components, these are then explicitly just:*

$$\begin{aligned}
\vec{B}_p &= \frac{1}{R}\left(\nabla\Psi \times \hat{e}_\phi\right)\\
\vec{B}_\phi &= \frac{f(\Psi)}{R}\\
\vec{j}_p &= \frac{1}{\mu_0 R}\left(\nabla f(\Psi) \times \hat{e}_\phi\right)\\
\vec{j}_\phi &= -\frac{1}{\mu_0 R}\Delta^*\Psi
\end{aligned}$$

**Remark 2.28.** The flux function $f(\Psi)$ describes the poloidal flux per radian in $\phi$. Of note, for an axisymmetric torus the total flux function would then simply be $2\pi\Psi$. The other flux function here, $p(\Psi)$, simply describes pressure inside the Tokamak [3].

**Remark 2.29.** We noted in our introduction of the PDE theory that the GSE was an elliptic, nonlinear PDE. We can see the ellipticity immediately by virtue of there being an elliptic operator present in $\Delta^*$, and the nonlinearity we can gather from the fact there are non-linear coefficients of partial derivatives with respect to $\Psi$, for example, the $-f(\Psi)f'(\Psi)$ in the elliptic operator form of the GSE.

The GSE equation is not without its faults, and is, in large, a considerable simplification of the actual physical behaviour of a plasma within a Tokamak – as my supervisor Matthew would say, it is very much so a spherical cow model. To explore why this is so, we'll explain the components of the GSE and the assumptions that are made in reaching it through a derivation with accompanying annotations below.

### 2.2.1 Derivation

Here we present a derivation of the Grad-Shafranov Equation. We begin by assuming the ideal MHD conditions, which immediately gives us that, in our plasma model:

- The plasma particle distribution approximately abides a Maxwellian distribution
- Resistivity is negligible
- There is no external (to the system) current source
- The topology of the magnetic field is fixed relative to the fluid, i.e., the magnetic field and fluid move together (the frozen-in flux theorem)
- Any velocities considered are non-relativistic (notably, this includes electron velocities)

TODO
can use graphics here as well

## 2.3 AC Configuration Tokamaks

Tokamak's normally in DC mode. What does it mean for a fusion reactor to operate in AC mode

### 2.3.1 Physical Differences

Plasma current (talk: operation of toroidal coils to regulate current)

### 2.3.2 Why Bother?

What benefits are there to using an AC design over DC?

Talk about confinement time, stability, etc

### 2.3.3 Runaway Electrons

#### Generation Mechanisms

Here we will discuss how runaway electrons come to exist within a Tokamak's fusion cycle.

#### RE Detection

#### Why we care

# Chapter 3

# Current Reversal Theory

## 3.1 Perturbation Methods

### 3.1.1 What is a Perturbation Treatment?

When it comes to solving differential equations, the ideal scenario is that you find an analytic solution, $\psi$. This, however, is generally a difficult problem, and especially so in the case of differential equations. A great deal of research goes into trying to find analytic solutions to PDEs, which is no less true for the Grad-Shafranov equation and its variations. Though such efforts are often futile, or require making assumptions about properties of your solution which may not be representative of what you're trying to show, other means of obtaining the $\Psi(\vec{x})$ for $\vec{x} \in \Omega$ have been developed.

A traditional approach to this is to use some numerical code to approximate solution values. There are various codes which exist for this purpose specifically with regards to the GSE equation, such as SPEC [5]. These codes come in many flavours - some implement a particle simulation, which can provide precision with simulation, though are computationally costly. Others exploit various properties of the variant of GSE they target and may be computationally more feasible, but are restricted in scope, application, and/or reliability because of their assumptions.

A common problem that numerical methods seek to solve is that of time evolution. Simulations which evolve a system through time, for obvious reasons, do not discard the time dependency component in their derivation of the GSE. A common problem that numerical methods are employed to solve is that of time evolution. Simulations which evolve a system through time however often require time dependence in the system they are trying to solve – something, if you recall from our derivation of the GSE, is not present in our model. This introduces a difficulty for us, as solutions to the GSE explicitly represent plasma equilibria, and so we expect no variation of the system with respect to time.

For this thesis, given the availability of an analytic solution to a variation of the GSE that specifically relates to current reversals, we employ a hybrid approach. In our simulation we wish to observe changes in the system through a current ramp down - this necessitates a time dependency component to our solution. However, there are a couple points to note:

- Time evolution simulations are expensive
- Existing literature on time evolution GSE do not take into account the possibility for current reversal
- The GSH variant we utilise (which has yet to be introduced) is not time dependent, i.e., was not derived taking into account a time component

There are a couple possible avenues we could explore from here. We could implement a numerical code for the time evolution, for example a particle simulation, however this is for all intents and purposes computationally infeasible given the hardware available (a beloved but dilapidated Thinkpad T440p). We could attempt to find an analytic solution to the GSE with time dependence (or an approximation to one), but given the mountain of research existing in this space and the relative inexperience of the author, this would likely be a futile task. The third, and most lucrative option, is to use an existing analytic solution to a variation of the GSE that accounts for current reversal, and explore if small changes to its state are sufficiently "good enough" approximations of our system that it can be used to make statements about experimental data.

This is where perturbation theory comes in. Intuitively we would expect a plasma to vary "smoothly" - nature rarely behaves in instantaneous ways. The analytic solution we will soon have describes the instantaneous state (a slice in time) of our plasma's state. We may anticipate then that if we were to change something in our system by a "sufficiently small amount" (we will later comment on what exactly is meant by sufficiently small), that our system may react to that change of state in a way that approximates how it would if it were really varying through time.

### Resistivity Note

When we derived the ideal MHD equations we compared them to the resistive MHD model, noting that the removal of resistivity would be an important simplification to note. Here we see its importance, as we are now allowing our system to vary with respect to time, which means that there exists some time scale for which resistive effects will begin to impact the behaviour of our plasma.

Our model assumes the ideal MHD assumptions however, which includes dropping resistivity, and so the effect of these resistive instabilities is lost in our model. The question of whether these effects are significant enough to affect the accuracy of our time evolution is one we hope to answer when we come to reviewing our simulations in chapter 5.

A perhaps more intuitive analogy is that of swimming. Imagine you are stationary sitting at the bottom of a pool, and you are looking at your arm extended out in front of you. Focus on the feeling of the water moving around your arm - this will emulate the resistivity that electrons would feel. If you move your arm abruptly, perhaps in a cutting motion, then you will feel the pressure of the water push against your arm, and that might affect how quickly you can move your arm, or if the current of the water is particularly strong, perhaps you notice your arm move not in the direction you intended. Perhaps you also have to use more energy to move your arm than you intended, so it doesn't go as far. Now imagine instead that you don't abruptly cut, but instead, very slowly, almost imperceptibly, move your arm through the water. In this case you may not be aware of any resistance at all - there is no sensation of resistance, your arm moves exactly as fast as you expect it to, and you use exactly the amount of energy to move it as you expected. However, if you do this for long enough, perhaps you will begin to notice the effects more and more. Or maybe by now you'll realise you're out of breath and need to resurface. An instructive graphic is below.

Figure 3.1: Demonstrative graphic for conducting a resistivity experiment at the bottom of a pool.

We will have the ability to determine the state of our system for some given parameters, and so the question becomes: can we vary our input in a way such that the behaviour of the analytic solution to the GSH is an approximation of a time evolution of the plasma. We are able to compare our results to experimental data, and then exploit our work to answer further questions.

### 3.1.2 Regular Perturbation Theory

https://jmahaffy.sdsu.edu/courses/f19/math537/beamer/perturb.pdf https://www.ucl.ac.uk/~ucahhwi/LTCC/section2-3-perturb-regular.pdf https://math.byu.edu/~bakker/Math521/Lectures/M521Lec15.pdf

### 3.1.3 Ideal MHD Perturbation

At the beginning of this thesis we introduced it as being interdisciplenary in nature. Keeping in that spirit, at times here we will put on our physicists cap and seemingly arbitrarily remove terms we no longer wish to have. We kindly ask that any mathematicians reading these sections avert their gaze in such times so as to maintain sanity.

In Chapter 2 we derived the ideal MHD equations, and their subsequent reduction, used in GSE, given:

$$\vec{j} \times \vec{B} = \nabla p \tag{3.1}$$
$$\mu_0 \vec{j} = \nabla \times \vec{B} \tag{3.2}$$
$$\nabla \cdot \vec{B} = 0 \tag{3.3}$$

These by note are time independent. We wish to look at the effect of some small ($\varepsilon > 0$) time perturbation to the original system however, and so will return to their derivation, this time without discarding the time component. Consider the pre-GSE ideal MHD equations again, as given in ((2.27) - (2.32))

$$\frac{\mathrm{D}}{\mathrm{D}t}\rho = -\rho \nabla \cdot \vec{u}$$
$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\vec{u} = -\nabla p + \vec{j} \times \vec{B}$$
$$\frac{\mathrm{D}}{\mathrm{D}t}p = -\gamma p \nabla \cdot \vec{u}$$

with the assumptions

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$$
$$\mu_0 \vec{j} = \nabla \times \vec{B}$$
$$\vec{E} + \vec{u} \times \vec{B} = 0$$

We will apply our perturbation treatment to the first three equations of these (the continuity equation, momentum equation, and energy equation), using the assumptions to aid us. We will being with the continuity equation.

### Continuity Equation Time Perturbation

We restate the continuity equation:

$$\frac{\mathrm{D}}{\mathrm{D}t}\rho = -\rho \nabla \cdot \vec{u}$$

Here we have two functions, $\rho$ and $\vec{u}$, which are both functions of three dimension in space and one of time, i.e.

$$\rho := \rho(\vec{r}, t)$$
$$\vec{u} := \vec{u}(\vec{r}, t)$$

where $\vec{r} \in \mathbb{R}^3, t \in \mathbb{R}$. Here (and throughout these perturbation treatments) we will decouple the time dependency by assuming it to be some epsilon perturbation from an initial state. For example, for mass density ($\rho$), we can let $\rho_0$ describe some initial state that is only dependent on space, and $\rho_1$ a perturbation component similarly dependent only on space. Let $\varepsilon_\rho < 0$ be some small perturbation factor for mass density, and $t$ be the time evolved. Then we prescribe a time-linear perturbation expansion as such:

$$\rho(\vec{r}, t) = \rho_0(\vec{r}) + \varepsilon_\rho t \rho_1(\vec{r}) + \mathcal{O}(\varepsilon_\rho^2)$$

and analogously:

$$\vec{u}(\vec{r}, t) = \vec{u}_0(\vec{r}) + \varepsilon_u t \vec{u}_1(\vec{r}) + \mathcal{O}(\varepsilon_u^2)$$

Here the $t$ term effectively acts as a scaling factor on top of $\varepsilon$. We have also dropped any $\varepsilon$ terms of order greater than 1 in our approximation, that is to say, if we were to be accurate our expansion would actually be:

$$\begin{aligned} \rho(\vec{r}, t) &= \rho_0(\vec{r}) + \varepsilon_\rho t \rho_1(\vec{r}) + \varepsilon_\rho^2 t \rho_2(\vec{r}) + \ldots \\ \vec{u}(\vec{r}, t) &= \vec{u}_0(\vec{r}) + \varepsilon_u t \vec{u}_1(\vec{r}) + \varepsilon_u^2 t \vec{u}_2(\vec{r}) + \ldots \end{aligned}$$

As we said at the start of this chapter however, this is an interdisciplenary thesis, and the above is an example of us putting on our physicist cap and deciding to remove "unnecessary" complexity. Physically we can justify this dropping of terms as we would expect terms of greater order to contribute diminishingly to our system. That is to say, on the time scales we are considering, an approximation of up to $\varepsilon$ should be sufficient for experimental comparisons. As such, we need only to reason about that, and drop any terms on order of $\mathcal{O}(\varepsilon^2)$.

We substitute these perturbation expansions into our equation. For brevity we will drop function parameters in our notation, and first begin with an expansion of the $\rho$ term:

*LHS:*

$$\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t}\rho &= \left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\rho \\
&= \frac{\partial}{\partial t}\rho + \vec{u}\cdot(\nabla\rho) \\
&= \frac{\partial}{\partial t}(\rho_0 + \varepsilon_\rho t\rho_1) + \vec{u}\cdot(\nabla(\rho_0 + \varepsilon_\rho t\rho_1)) \\
&= \frac{\partial}{\partial t}\rho_0 + \varepsilon_\rho\rho_1 + \vec{u}\cdot(\nabla\rho_0) + \varepsilon_\rho t\vec{u}\cdot(\nabla\rho_1)
\end{aligned}$$

noting that $\rho_0$ is not a function of time

$$= \varepsilon_\rho\rho_1 + \vec{u}\cdot(\nabla\rho_0) + \varepsilon_\rho t\vec{u}\cdot(\nabla\rho_1)$$

*RHS:*

$$\begin{aligned}
-\rho\nabla\cdot\vec{u} &= -(\rho_0 + \varepsilon_\rho t\rho_1)(\nabla\cdot\vec{u}) \\
&= -\rho_0(\nabla\cdot\vec{u}) - \varepsilon_\rho t\rho_1(\nabla\cdot\vec{u})
\end{aligned}$$

*Combined:*

$$\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t}\rho &= -\rho\nabla\cdot\vec{u} \\
\varepsilon_\rho\rho_1 + \vec{u}\cdot(\nabla\rho_0) + \varepsilon_\rho t\vec{u}\cdot(\nabla\rho_1) &= -\rho_0(\nabla\cdot\vec{u}) - \varepsilon_\rho t\rho_1(\nabla\cdot\vec{u}) \\
\varepsilon_\rho\rho_1 + \varepsilon_\rho t\left[\vec{u}\cdot(\nabla\rho_1) + \rho_1(\nabla\cdot\vec{u})\right] + \left[\rho_0(\nabla\cdot\vec{u}) + \vec{u}\cdot(\nabla\rho_0)\right] &= 0 \\
\varepsilon_\rho\rho_1 + \nabla\cdot(\rho_0\vec{u}) + \varepsilon_\rho t[\nabla\cdot(\rho_1\vec{u})] &= 0
\end{aligned}$$

Here we have used the divergence property $\nabla\cdot(f\vec{v}) = (\nabla f)\cdot\vec{v} + f(\nabla\cdot\vec{v})$. Note that for the Grad-Shafranov equation we usually (and we shall) assume an incompressible fluid, which mathematically is significant as it gives us the property that $\nabla\cdot(\rho\vec{u}) = 0$. From the above we immediately get that $\varepsilon_\rho\rho_1 = 0$, which just tells us that the time component we introduced has no effect on the system, given the assumptions we make. This is excellent news for us, as it means that small perturbations in our system in a manner that emulates a time evolution should not affect the accuracy of our model, at least insofar as the effects of the continuity equation are concerned, and up to a term of $\mathcal{O}(\varepsilon^2)$. To be confident that we can do this for the Grad-Shafranov equation on a whole, we need to repeat this process for the other ideal MHD equations. Next we'll look at the momentum equation.

### Momentum Equation

We restate the momentum equation here:

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\vec{u} = -\nabla \rho + \vec{j} \times \vec{B}$$

We will follow the same process as we did for the continuity equation. Here we have four equations, however, by introducing one of our GSE assumptions we can reduce this to three immediately. For GSE we assume that our fluid has no velocity, i.e. $\vec{u} = 0$, which gives us that $\frac{\mathrm{D}}{\mathrm{D}t}\vec{u} = 0$. Thus the equation we actually have to expand is:

$$0 = -\nabla \rho + \vec{j} \times \vec{B}$$

The three equations we have are each functions of $\vec{r} \in \mathbb{R}^3$ and $t \in \mathbb{R}$, and can be linearly perturbed in time as we did earlier. As such:

$$\begin{aligned}
\rho(\vec{r}, t) &:= \rho_0(\vec{r}) + \varepsilon_\rho t \rho_1(\vec{r}) + \mathcal{O}(\varepsilon_\rho^2) \\
\vec{B}(\vec{r}, t) &:= \vec{B}_0(\vec{r}) + \varepsilon_B t \vec{B}_1(\vec{r}) + \mathcal{O}(\varepsilon_B^2) \\
\vec{j}(\vec{r}, t) &:= \vec{j}_0(\vec{r}) + \varepsilon_j t \vec{j}_1(\vec{r}) + \mathcal{O}(\varepsilon_j^2)
\end{aligned}$$

Making these substitutions into the momentum equation:

$$\begin{aligned}
\nabla \rho &= \vec{j} \times \vec{B} \\
\nabla(\rho_0 + \varepsilon_\rho t \rho_1) &= \vec{j} \times (\vec{B}_0 + \varepsilon_B t \vec{B}_1) \\
(\nabla \rho_0) + \varepsilon_\rho t (\nabla \rho_1) &= (\vec{j} \times \vec{B}_0) + \varepsilon_B t (\vec{j} \times \vec{B}_1) \\
(\nabla \rho_0) + \varepsilon_\rho t (\nabla \rho_1) &= [(\vec{j}_0 \times \vec{B}_0) + \varepsilon_j t (\vec{j}_1 \times \vec{B}_0)] + \varepsilon_B t [(\vec{j}_0 \times \vec{B}_1) + \varepsilon_j t (\vec{j}_1 \times \vec{B}_1)] \\
(\nabla \rho_0) + \varepsilon_\rho t (\nabla \rho_1) &= \vec{j}_0 \times \vec{B}_0 + \varepsilon_j \varepsilon_B t^2 (\vec{j}_1 \times \vec{B}_1)
\end{aligned}$$

If we collate terms of like order, we get

$$\begin{aligned}
\nabla \rho_0 &= \vec{j}_0 \times \vec{B}_0 \\
\varepsilon_\rho t (\nabla \rho_1) &= \varepsilon_j \varepsilon_B t^2 (\vec{j}_1 \times \vec{B}_1)
\end{aligned}$$

both of which look remarkably like the original momentum equation, which is not to be mistaken for coincidence. Notably, when we take our evolution over "small enough" time scales, the perturbation components here disappear entirely, and we are left with the version of the momentum equation which is used in the regular equilibrium-state derivation of the Grad-Shafranov equation. As such, we would anticipate that small time perturbations associated with the effects of the momentum equation will introduce "small" errors in the resulting GSE solution, though (and as we hope to observe through experiment), hopefully imperceptibly so - as the above expansion seems to suggest.

### The Other Ideal MHD Equations

Isaac Newton is quoted as saying "it causeth my head to ache", which he is claimed to have said in reference to perturbation expansions in trying to determine the orbit of the Moon. Hopefully understandably, we ask that the reader excuse our omitting the working for the more involved of the ideal MHD equations, as to write it all down here would be to commit an expositionary crime. The expansion of the remaining equations however leads to a similar outcome as we found with the continuity equation and the momentum equation - that, up to an $\varepsilon$ factor, the ideal MHD equations do not show significant deviation from their equilibrium counterparts.

### Significance

The purpose of this exercise was to provide some theoretical justification for the simulations we will later run. By showing that we can approximate a linear time evolution of the ideal MHD equations using states of equilibrium, we have shown that the GSE will be resistant to such a treatment itself. What we have not done, is show that the GSE can be evolved with time as such. TODOhave to join this and the below

It is important to remark that the GSE is an equilibrium model - it is constructed under the assumptions of plasma equilibrium, and as we've already remarked on for resistivity, essentially "wishes away" many of the physical effects that would otherwise be introduced by some system that changes with respect to time. To that extent, it could be considered tomfoolery (which, for what it's worth, does befit the author) to assume that the work we've done is a guarantee that we can TODO

TODO

## 3.2 Grad-Shafranov-Helmholtz Equation

### 3.2.1 Helmholtz Equations

What is a Helmholtz equation

### 3.2.2 GSH Equation

How do we apply it to the Grad-Shafranov equation

The work here will largely follow the paper by Wang and Yu REFERENCE, with some key notes and modifications. We are by now familiar with the GSE:

INSERT GSE

Wang provide a slightly different (though equivalent) formulation:

**Proposition 3.1.** *First, we will introduce some normalisation terms. Let* $j_\phi = \frac{J_\phi \mu_0 a}{B_0}$ *be the toroidal current density normalised with respect to the magnetic field, recalling that* $J_\phi$ *is the toroidal current density,* $\mu_0$ *is free-space magnetic permeability, and* $B_0$ *is the on-axis magnetic field strength (here, the strength of the magnetic field evaluted at the centre of the cross section, where* $x = a$*).*

*The GSE can then equivalently be stated:*

$$\left( x\frac{\partial}{\partial x}\frac{1}{x}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial z^2} \right)\psi = -\frac{1}{2}x^2\frac{\mathrm{d}\beta}{\mathrm{d}\psi} - \frac{1}{2}\frac{\mathrm{d}g^2}{\mathrm{d}\psi} = -xj_\phi \tag{3.4}$$

*where*

$$a_1 = -\frac{1}{2}\frac{\mathrm{d}\beta}{\mathrm{d}\psi} \tag{3.5}$$

$$a_2 - \alpha^2\psi = -\frac{1}{2}\frac{\mathrm{d}g^2}{\mathrm{d}\psi} \tag{3.6}$$

*Here, we have some normalised terms:* $\psi(x,z) = \frac{\Psi(x,z)}{B_0 a^2}$ *is the normalised poloidal magnetic flux function. The* $\beta$ *function is given* $\beta(\psi) = \frac{2\mu_0 p(\psi)}{B_0^2}$*, and* $g(\psi) = \frac{F(\psi)}{B_0 a}$.

**Remark 3.2.** We highlight here the parameter tuple $(a_1, a_2, \alpha)$. We will see soon that these parameters can be used to determine our system. For now we simply note their significance so that the reader (that's you!) may pay them special attention through the rest of the following working.

At the boundary of a fusion reactor we expect there to be no / minimal poloidal magnetic flux, i.e. that the field strength dissipates as we approach the edge of the reactor. This justifies the boundary condition we will impose: $\psi|_b = 0$.

### 3.2.3 Analytic Solution and Derivation

## 3.3 Current Reversal Systems

### 3.3.1 Current and Pressure Density Profiles

### 3.3.2 Example Configurations

# Chapter 4

# Numerical Model Fitting

## 4.1 Non-Linear Optimisation

### 4.1.1 Least Squares

### 4.1.2 Optimisation Algorithms

Minimising least squares.

#### The Usual Suspects

Newton's method

Gradient descent

Bounded vs non-bounded optimisation

#### MMA Algorithm

Method of Moving Asymptotes (MMA)

Provide explanation of LD-MMA algorithm.

## 4.2 GSH Parameter Fitting

### 4.2.1 Optimisation Function

### 4.2.2 Parameter Space

Graphs showing effect of different parameters

Non-reliance on $\alpha$ (with exception of $\frac{1}{\alpha}$ where $\alpha = 0$ thing)

### 4.2.3 Convergence Difficulties

Initial difficulties with normalisation.

## 4.3 Simulated Current Reversals

### 4.3.1 Method of Reversal

### 4.3.2 Results and Explanations

# Chapter 5

# Comparison with Experiment

## 5.1 ISTTOK

### 5.1.1 A Brief History of ISTTOK

### 5.1.2 Reactor Specification

## 5.2

# Chapter 6

# Blue Skies and Horizons

## 6.1 Conclusions

## 6.2 Further Work

### 6.2.1 Simulated Electric Field via Fake Solenoids

In a meeting while presenting my findings to the Plasma Science group, I posed the question of extracting electric field information from the data we had available. There are many benefits to being able to describe the electric field for the confinement time of a plasma, however the most significant to our purpose is to positively identify the birth of runaway electrons. This information however is not readily available with the system we worked with.

David Pfefferle proposed a method of simulating the presence of an electric field instead. His idea was to introduce infinitesimal solenoids at the centre of magnetic islands, which would each contribute produce their own electric field. These would then interact, with the idea that the product would be an approximation to the expected electric field for a given state.

The physical justification for this is that the solenoid's magnetic field is emulating the magnetic field produced by current densities, which are themselves informed by magnetic islands. A toroidal current will produce an electric field, including a poloidal component, which will influence the behaviour of runaway electrons. Thus, this approach effectively emulates the presence of a poloidal electric field using position and strength information of current densities.

This approach could utilise work done by Nicholas Bohlsen in using topological data analysis to identify the presence of magnetic islands from poloidal magnetic flux data in his thesis.

### 6.2.2 Comparison with ISTTOK vloop data

### 6.2.3 Retrieval of

# Appendix A

# Appendix title goes here

Blah blah blah...

Blah blah blah...

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