Simulations in AC Tokamak Ramp Downs

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Oct 2023

A thesis submitted for the degree of
Bachelor of Mathematical Sciences (Honours)
of the Australian National University

Australian National University

For George, Paul, Ringo and John.

Declaration

The work in this thesis is my own except where otherwise stated.

Thomas Malcolm.

Acknowledgements

First and foremost I'd like to thank Matthew Hole, my supervisor. I came into this year a little daunted at the prospect of tackling a physics-based project with a physics background not extending past "A Brief History of Time". Your patience and willingness to explain concepts to me I'll forever appreciate, and your energetic approach to problem solving made me feel both at home and excited for the work we have done in this thesis.

In a similar vein, thank you to the rest of the Plasma Theory and Modelling group for always being willing to answers questions when I had them. You guys were definitely my saving grace at times, and I only wish I spent more time getting to know you all. I should give a special shout out to Nick, Sandra, Dean and Josh specifically for being so accommodating, helpful, and kind in answering my questions when I would pop around.

I'd like to thank the other Honours students for helping make this a fantastic year. Our little nook of MSI proved very distracting, but in the best possible way. As such I would of course expect you to take part responsibility for the late night writing sessions at the end of the year - but you also get credit for making them enjoyable.

Last but certainly not least, to Kirsten and Graeme. Thank you for your support these last couple years, and the infinite kindness you give to all those around you. However silly it is, your encouragement genuinely helped motivate me at times, and I always got a bit of joy explaining my thesis and whatever problem I was working on at the time to you.

Abstract

Contents

Notation and terminology

Notation

$L^2(\mathbb{R}^n)$	Space of L^2 integrable functions on $\mathbb{R}^n$
B_0	On-axis magnetic field (Toroidal magnetic field strength)
R_0	Major radius (Tokamak)
a	Minor radius (Tokamak)
ψ	Normalised poloidal magnetic flux function
Ψ	Poloidal magnetic flux function
J_ϕ	Toroidal current density
j_ϕ	Normalised toroidal current density (normalised w.r.t. B_0)
I_ρ	Normalised toroidal current density (normalised w.r.t. maximum value)
p	Plasma pressure density
β	**Check with wesson to confirm**

Terminology

GS / GSE	Grad-Shafranov equation
GSH	Grad-Shafranov-Helmholtz equation
MHD	Magnetohydrodynamics

Chapter 1

Introduction

1.1 Plasma Science

This thesis is multidisciplenary by nature, incorporating aspects of mathematics, physics and computer science throughout the various challenges faced in reaching the conclusions we have.

Before burying ourselves in the thick of our results we will cover the requisite knowledge for working in the plasma science simulation space. First we'll discuss what the physical object of our attention ("plasma") actually is, with a brief discussion on the design of fusion reactors, where we will emphasise the structures that specifically relate to our interests. In the background chapter we will expound on this by building the mathematical theory underpinning physical processes within the fusion reactor, providing us a way to reason about the behaviour of plasma (and related processes) inside a fusion reactor, or, more accurately, approximate their behaviour via simulation.

1.1.1 I'm a Mathematician... what is "plasma"?

While an initially daunting topic, fear not fearful mathematician, for many of the inherently physical behaviours in plasma science we require can be expressed in terms of our dearest mathematical expressions – differential equations! But first, what actually is "plasma"? Webster's dictionary defines plasma to be "a green faintly translucent quartz" [3]. While I'm sure there isn't no relation between crystals and our investigations, this is unfortunately, not the recipient of our affection.

When plasma is referred to in everyday conversation it is often noted as being the "4th state of matter". To introduce slightly more rigour, plasma is an extension of the gaseous state of matter, where its energy (read: temperature) is increased sufficiently high that the electrons are no longer bound by the electromagnetic force to the atom's nucleus REFERENCE. The resulting substance is a "pool" of cations (the positively charged nuclei), and electrons (negatively charged), that exhibits interesting properties. It is these properties that we seek to exploit in the search of controlled, sustained fusion reactions.

Plasma is abundant in nature – just not in many places that we as Humans commonly look. Stars are the most immediate example of matter in a plasma state, and are readily viewable (at least for half the day). Lightning strikes are paths through the atmosphere which are ionised, and neon signs work by heating Neon gas within a tube to ionise it REFERENCE.

The question then is, what is "fusion", and how does this relate to plasma? The answer comes back to energy. Analogously to a fission reaction, where energy is released through the division of atoms, one can also fuse two separate atoms together and have large amounts of energy emitted as a bi-product of that fusion. It is (essentially) this extra energy that we wish to harness when harvesting energy from a fusion reactor.

While there are no shortages of elements that can theoretically be used to fuel a fusion reactor, the one most commonly associated with fusion is Deuterium – a stable isotope of Hydrogen that has a neutron in its nucleus (whereas a 'standard' Hydrogen atom contains only a single proton). Analogously, Tritium is an isotope of Hydrogen that has two neutrons in its nucleus (though is much less stable). Consider the fusion of a Deuterium atom with a Tritium atom:

$$ {}^{2}\mathrm{H} + {}^{3}\mathrm{H} \overset{\text{(fusion)}}{\rightarrow} {}^{4}\mathrm{He} + n + 17.59\,\mathrm{MeV} $$

Here two Hydrogen isotopes fuse together to form a single Helium-4 atom, and in the process of doing so release a single neutron and 17.59 MeV of energy. This excess energy is what is so attractive about fusion processes as a sustainable energy source – for such little input we receive a substantial amount of energy, and at that, using one of the most abundant resources available on Earth; water.

The question then becomes, how do we drive this fusion process? If two atoms can fuse as such, why do we not see atoms fusing everywhere around us accompanied by violent explosions destroying all that we've come to know and love? The answer is that we kind of do – just not normally in the places that humans expect to be inhabiting. In fact, we see this happening everyday – for those of us fortunate enough to be able to see the sun that is. Our sun is a large ball of plasma where an estimated 9.3×10^{37} fusion reactions are expected to occur every second [4], and is one of the easiest examples of both plasma as a state of matter, and of a self-sustaining fusion reaction.

How then do two hydrogen atoms fuse – how can we create a sun on Earth? The nucleus of an atom (consisting of positively charged protons, and neutral neutrons) is positively charged, and so two atoms' nuclei will repel each other due to the Coulomb force when pushed together. This is the force we have to overcome to enable a fusion reaction to take place (and what stops the world around us burning!). To overcome this the process is relatively "simple" – we just increase the energy of our atoms so that when they collide they collide with enough energy to overcome this force, allowing the strong force to become dominant, fusing the two atoms. When we energise a mass (take a gas here) of atoms enough, they become ionised however, which is exactly the state of plasma we described earlier. In other words, if we want to reason about the creation, sustainment, and effects of fusion reactions, we need first to understand the dynamics of plasma, the medium in which the fusion reactions take place. From this comes a plethora of questions, ranging from "how do we generate such a plasma?" to "how do we reliably control such a plasma" and "how do we harness such a plasma"? Alas, we digress however, as we do not seek to solve the big problems in fusion science in this here mere mortal thesis. Instead, now equipped with at least a passing knowledge of what constitutes a "plasma" state and what it means for a fusion reaction to take place, we will humbly delve into the inner workings of fusion reactor terminology and design.

1.1.2 Fusion Reactor Design 101

Here we will discuss the important structural aspects of a Tokamak fusion reactor. The term "Tokamak" is a Russian acronym which translates as "toroidal chamber with magnetic coils" [2]. Aptly, a Tokamak is a toroidal object which is used as a vessell for plasma which is driven via external magnetic coils.

In reality, reactors do not often have such nice geometry. Figure 1.1

Diagram

Poloidal vs toroidal flux

Magnet positioning

Heating of plasma

Confinement

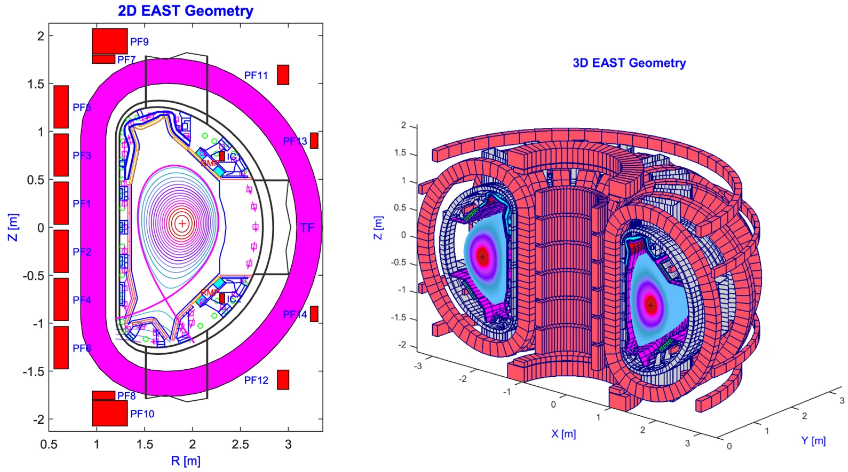

Figure 1.1: EAST fusion reactor geometry. This is a 3D model of its cross section, highlighting the irregular shape

REs

1.2 What problem does this thesis address?

Talk about paper Matthew and Artur released. Presence of residual REs.

1.2.1 What results do we seek?

To provide a theoretical basis for exploring the observed anomalies. To see if theory supports the observation

1.3 Project Progression

Work inspired by Matthew's paper

First expanded MHD equations linearly with a perturbation treatment

Mathematical PDE theory developed by Wang. Identified errors in paper, fixed.

Developed a simulation to reproduce results. Then went other direction, solving for parameters for data.

Simulated current inversion.

ISTTOK data matching.

Feasibility

1.4 Structure of Thesis

TODO (after writing it)

Chapter 2

Background

2.1 Magnetohydrodynamics

The term “magnetohydrodynamics” (MHD) is a portmanteau of two physical concepts which are used to model plasmas inside fusion reactors: the “magneto” term comes from “magnetic field”, and “hydrodynamics” indicates a a fluid dynamics component. Put together, magnetohydrodynamics is the study of electrically conductive materials that behave like fluids. Essentially it provides a way to model the behaviour of a (considerably volumous!) mass of particles, and their electrodynamic forces, as if it were a fluid, as opposed to having to model individual particle interactions. This idea was first introducd by Hannes Alfvén in 1970, for which he earned the Nobel prize [1]!

Much modern research uses some variation of an MHD model (and the derived Grad-Shafranov Equation, which we will soon be introduced to) for a number of reasons, not least being its comparative computational efficiency. One primary benefit of treating our plasma as a fluid being that we avoid modelling the behaviour of each individual particle in said plasma, a simplification which becomes especially important when we consider the order of number of particles we would have to simulate is of order $\sim 10^{20}$ - far too much for the author's ThinkPad to even contemplate!

Here we will build the relevant MHD background for this thesis, deriving the MHD equations from first principles, and explaning the assumptions we make to reduce them to a simplified state known as “ideal MHD”. From there we will look at the PDE which models the behaviour of a plasma inside a Tokamak, the “Grad-Shafranov Equation”. We will then note some pitfalls of using this model to describe AC configuration Tokamaks (as we are investigating), and finish with a discussion on runaway electrons (RE).

2.1.1 MHD Theory

Given MHD is the marriage of fluid dynamics with electrodynamics, it is only natural to begin our study looking at the equations which describe electrodynamic behaviour — Maxwell's equations describe the interaction between magnetic fields $\vec{B}(\vec{r},t)$, electric fields $\vec{E}(\vec{r},t)$, and the current density $j(\vec{r},t)$ which induces them, where $\vec{r} \in \mathbb{R}^3$ is a position vector and $t \in \mathbb{R}$ describes time. Thus, we introduce Maxwell's equations:

Definition 2.1. Maxwell's equations are given [6]:

$$\nabla \times \vec{B} = \mu_0 j + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \tag{2.1}$$

$$\nabla \times E = -\frac{\partial \vec{B}}{\partial t} \tag{2.2}$$

$$\nabla \cdot \vec{B} = 0 \tag{2.3}$$

$$\nabla \cdot E = \frac{\rho_c}{\epsilon_0} \tag{2.4}$$

The functions driving change in this system are $\rho_c(\vec{r},t)$, the electric charge density, and $j(\vec{r},t)$, the electric current density. We also have μ_0, the free-space magnetic permeability (in henry m^{-1}); ϵ_0, the free-space permittivity; and c, the speed of light.

These equations give us a way to reason about the electric and magnetic fields if we're given some descriptor for the current we're passing through some medium. We are now to introduce the fluid dynamics component to our system. A fluid's mass density can be given by summing over the effects of individual "species" of particles (e.g. electrons) in the fluid:

$$\rho_c = \sum_\sigma m_\sigma n_\sigma$$

and its current density similarly:

$$\vec{j}(\vec{r},t) = \sum_\sigma n_\sigma q_\sigma \vec{u}_\sigma$$

where σ describes a particle species, m_σ describes its mass, n_σ its number density (a measurement of concentration for the given particle species in a pre-defined volume – akin to Avogadro's constant), q_σ describes its electric charge, and $\vec{u}_\sigma$ the mean velocity of this species of particle in the fluid.

Fluid Dynamics

Notation 2.2. A common simplification in notation for fluids is made in using the *Lagrangian derivative*, given:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \left(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla \right)$$

It describes the total change in a volume within a fluid as it moves throughout said fluid. It is essentially a change in reference frame for a derivative - where a regular derivative might descirbe, for example, how a particle moves with respect to time in its surroundings, its Lagrangian will take into account the motion of the fluid the particle is immersed in as well.

We begin with conservation of mass, also known as the "continuity equation".

Definition 2.3 (The continuity equation)**.** The below relates how mass density, ρ, changes with respect to the motion of a fluid element.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \vec{u}) \tag{2.5}$$

The derivation of the above comes from a surface integral over a volume with an outward and inward flux, and an application of Gauss' flux law. For a full derivation, see pg. 19 - 21 of [5].

Remark 2.4. The above is a PDE with four variables: ρ, the mass density of the medium, and $\vec{u}$, the velocity of the fluid. This renders the system not closed, and thus too general for an analytic solution - we have more unknowns than we have equations [5]. Later we will introduce other equations to our system to apply more restrictions, and make assumptions about the physicality of the system which will reduce these dependencies, and make it determined ("closed").

Remark 2.5. We can rewrite (2.5) with a Lagrangian frame of reference, as

$$\frac{\mathrm{D}}{\mathrm{D}t}\rho = -\rho \nabla \cdot \vec{u} \tag{2.6}$$

$$\iff \frac{\mathrm{D}}{\mathrm{D}t}\rho + \rho \nabla \cdot \vec{u} = 0 \tag{2.7}$$

Equation (2.5) (and equivalently (2.7)) tells us that the mass of our fluid is conserved for motion of a volume element of our fluid - one assumption we make for our model. Next we'll discuss fluid motion as described by Newton for a fluid element:

Definition 2.6 (Newtonian Fluid Motion)**.** Newton's law for a fluid specifies:

$$\rho \frac{\mathrm{D}}{\mathrm{D}t} \vec{u} = \vec{F} \tag{2.8}$$

where $\rho(\vec{r}, t)$ is the mass density of the fluid, $\vec{u}(\vec{r}, t)$ describes the velocity of the fluid element, and $\vec{F}(\vec{r}, t)$ describes the force per unit volume acting on the fluid element [5].

The forces acting on particles within a fluid can be split into two types

- Gravitational

  Here, $\vec{F}_g = \rho \vec{g}$, where $\vec{g}$ is the gravitational acceleration. This should hark back to high school physics, though note this is a vector here as we care about the direction gravity accelerates the fluid element in, and relativistic effects can be an important consideration for high mass systems. This is more relevant for cases that you are using the MHD equations to describe the dynamics of large systems, such as a star. Unsurprisingly, this is less relevant for our case of plasmas within relatively miniscule Tokamaks.

- Electromagnetic

  This is the interesting part for us. As we assume our fluid is capable of conducting electricity (it's a plasma after all), there are electromagnetic forces operating within the fluid that affect the behaviour of the particles that the fluid consists of. The electromagnetic forces themselves can be split into two types, the **electric force** given by $\vec{F}_q = \rho_c \vec{E}$, and the **Lorentz force**, given $\vec{F}_L = \vec{j} \times \vec{B}$ (where $\vec{B}(\vec{r}, t)$ describes the magnetic field)

Taking these forces into account, we can describe the motion of an element of our fluid moving with velocity $\vec{u}$ via:

$$\rho \frac{\mathrm{D}}{\mathrm{D}t} \vec{u} = \vec{j} \times \vec{B} + \rho_c \vec{E} - \nabla p + \rho \vec{g} \tag{2.9}$$

Remark 2.7. Here the $\rho \vec{g}$ term could be abstracted further away into a stress tensor, as described by equation 4.20 of [5]. These pressures however are largely negligible when dealing with the scale we do in Tokamak plasmas however, and are thus ignored. We will soon drop the gravitational consideration as well anyway, but include it here for now for completeness.

Remark 2.8. The equation we have introduced is a function of six variables in its complex form (with stress tensor included), though even in this form we still have more variables than we do equations (1). Similar to before, this is not constrained sufficiently to consider it a closed system.

We next relate a plasma's pressure to its motion. We will simply present it here, though important notes in its derivation are that we assume the plasma behaves as an ideal gas (which is to say the only interaction between particles within the plasma are via elastic collisions with each other, or the boundaries of the container it is contained within). This is equivalent to saying that energy in the system depends only on the pressure. Thus, the energy equation is given:

Definition 2.9 (Energy Equation)**.** Where $\vec{p}(\vec{r}, t)$ describes the pressure of our fluid:

$$\frac{\mathrm{D}}{\mathrm{D}t}p = -\Gamma p \nabla \cdot \vec{u} + (\Gamma - 1)\left[-\nabla \cdot \vec{q} + \vec{\Pi} : \nabla \vec{u} + \eta \vec{J}^2\right] \tag{2.10}$$

where Γ describes "abiabatic index" (a known constant for plasmas), q is the heat flux through the boundary of the volume; η is the electrical resistivity of the fluid; and $\vec{\Pi}$ is the viscous stress tensor (the component which we replaced with $\rho\vec{g}$ earlier), and will soon ignore again.

The equations we've looked at constitute what are known as the fluid equations:

Definition 2.10 (Fluid Equations)**.**

$$\frac{\mathrm{D}}{\mathrm{D}t}\rho + \nabla \cdot \rho \vec{u} = 0 \tag{2.11}$$

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\vec{u} = \vec{j} \times \vec{B} + \rho_c \vec{E} - \nabla p + \rho \vec{g} \tag{2.12}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}p = -\Gamma p \nabla \cdot \vec{u} + (\Gamma - 1)\left[-\nabla \cdot \vec{q} + \vec{\Pi} : \nabla \vec{u} + \eta \vec{j}^2\right] \tag{2.13}$$

As reiterated a couple times now, these equations form an unclosed system, and are thus undetermined. To resolve this we introduce some constraints that come from electrodynamic forces, and a couple other relations that lead to a closed system.

Electrodynamics

As it currently stands, the input variables for the fluid equations are $\rho(\vec{r}, t)$ and $p(\vec{r}, t)$. We note that the electric field $\vec{E}$ and the magnetic field $\vec{B}$ are generated by the electric charge density, ρ_c, and the current density $\vec{j}$. This is where Maxwell's equations come into play. By combining Maxwell's equations with the fluid equation given above, we achieve the MHD model. The only piece to our puzzle missing is to tie the motion of the fluid (through $\vec{u}$) to the behaviour of the electric and magnetic fields. This is done via *Ohm's* law:

$$\vec{E} + \vec{u} \times \vec{B} = \eta \vec{j} \tag{2.14}$$

Remark 2.11. Note that the above is technically a lie, as it does not take into account relativistic effects, though for simplicity our MHD model ignores these.

Definition 2.12 (MHD Equations)**.** The MHD equations can then be summarised:

$$\frac{\mathrm{D}}{\mathrm{D}t}\rho = 0 \tag{2.15}$$

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\vec{u} = -\nabla p + \vec{j} \times \vec{B} + \nabla \cdot \Pi \tag{2.16}$$

$$\frac{\mathrm{D}}{\mathrm{D}t}p = -\Gamma p \nabla \cdot \vec{u} + (\Gamma - 1)\left[-\nabla \cdot \vec{q} + \vec{\Pi} : \nabla \vec{u} + \eta \vec{j}^2\right] \tag{2.17}$$

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} \tag{2.18}$$

$$\mu_0 \vec{j} = \nabla \times \vec{B} \tag{2.19}$$

$$\vec{E} + \vec{u} \times \vec{B} = \eta \vec{j} \tag{2.20}$$

Remark 2.13. The MHD equations as presented above constitute 14 equations with 27 unknowns. The breakdown is as such:

- ρ : 1 unknown
- $\vec{u}$: 3 unknowns
- p: 1 unknown
- $\vec{\Pi}$: 9 unknowns
- $\vec{j}$: 3 unknowns
- $\vec{B}$: 3 unknowns

- $\vec{E}$: 3 unknowns
- $\vec{q}$: 3 unknowns
- η : 1 unknown

The above is obviously insufficiently constrained for purposes of identifying a solution. We will skip a large amount of the work required to reduce the above to a closed system, though for details see lecture 7 of [5]. For now, we will comment on two reduced MHD models:

Resistive MHD

The resistive MHD model comes about by setting $\vec{q} = 0$ and $\vec{\Pi} = 0$. Here, we have $\eta \neq 0$ notably. The model is given:

Definition 2.14 (Resistive MHD)**.**

$$\frac{\mathrm{D}}{\mathrm{D}t}\rho = 0 \tag{2.21}$$
$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\vec{u} = -\nabla p + \vec{j} \times \vec{B} \tag{2.22}$$
$$\frac{\mathrm{D}}{\mathrm{D}t}p = -\Gamma p \nabla \cdot \vec{u} \tag{2.23}$$
$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} \tag{2.24}$$
$$\mu_0 \vec{j} = \nabla \times \vec{B} \tag{2.25}$$
$$\vec{E} + \vec{u} \times \vec{B} = \eta \vec{j} \tag{2.26}$$

Remark 2.15. The most notable effect of resistive MHD is that allowing for electrons to diffuse allows the resulting magnetic field lines to reconnect, which leads to breaks in the magnetic field line topology. This can lead to the generation of fast particles, i.e., runaway electrons.

Ideal MHD

The ideal MHD equations are one step removed from the resistive MHD model — in fact they are equivalent, only for the ideal MHD case we also ignore resistivity. Thus, set $\eta = 0$, and we obtain the ideal MHD equations:

Definition 2.16 (Ideal MHD)**.**

$$\frac{\mathrm{D}}{\mathrm{D}t}\rho = 0 \tag{2.27}$$
$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\vec{u} = -\nabla p + \vec{j}\times\vec{B} \tag{2.28}$$
$$\frac{\mathrm{D}}{\mathrm{D}t}p = -\Gamma p \nabla\cdot\vec{u} \tag{2.29}$$
$$\frac{\partial \vec{B}}{\partial t} = -\nabla\times\vec{E} \tag{2.30}$$
$$\mu_0\vec{j} = \nabla\times\vec{B} \tag{2.31}$$
$$\vec{E} + \vec{u}\times\vec{B} = 0 \tag{2.32}$$

Remark 2.17. In doing this, we have removed the dependency on $\vec{\Pi}$, η and $\vec{q}$, which accounts for 13 unknowns. This brings the total number of equations to 14 (or 8 if you make a substitution for Ohm's law) with 14 (8) unknowns. Thus, under the ideal MHD model, the system is closed.

Brief discussion on what resistivity is? Tie into GSE

2.2 Grad-Shafranov Equation

What is it

What do its components describe

Solutions to GS equation?

2.2.1 Derivation

Go through derivation of it

2.3 AC Configuration Tokamaks

Tokamak's normally in DC mode. What does it mean for a fusion reactor to operate in AC mode

2.3.1 Physical Differences

Plasma current (talk: operation of toroidal coils to regulate current)

2.3.2 Why Bother?

What benefits are there to using an AC design over DC?

Talk about confinement time, stability, etc

2.3.3 Runaway Electrons

Generation Mechanisms

Here we will discuss how runaway electrons come to exist within a Tokamak's fusion cycle.

RE Detection

Why we care

Chapter 3

Current Reversal Theory

3.1 Perturbation Methods

3.1.1 What is a Perturbation Treatment?

When it comes to solving differential equations, the ideal scenario is that you find an analytic solution, Ψ. This, however, is generally a difficult problem, and especially so in the case of differential equations. A great deal of research goes into trying to find analytic solutions to PDEs, which is no less true for the Grad-Shafranov equation and its variations. Though such efforts are often futile, or require making assumptions about properties of your solution which may not be representative of what you're trying to show, other means of obtaining the $\Psi(\vec{x})$ for $\vec{x} \in \Omega$ have been developed.

A traditional approach to this is to use some numerical code to approximate solution values. There are various codes which exist for this purpose specifically with regards to the GSE equation, such as HELENA REFERENCE. These codes come in many flavours - some implement a particle simulation, which can provide precision with simulation, though are computationally costly. Others exploit various properties of the variant of GSE they target and may be computationally more feasible, but are restricted in scope, application, and/or reliability because of their assumptions.

A common problem that numerical methods seek to solve is that of time evolution. Simulations which evolve a system through time, for obvious reasons, do not discard the time dependency component in their derivation of the GSE. This, however, often makes them computationally expensive as a result, while also increasing the difficulty of finding analytic solutions to the system.

For this thesis, given the availability of an analytic solution to a variation of the GSE that specifically relates to current reversals, we employ a hybrid approach. In our simulation we wish to observe changes in the system through a current ramp down - this necessitates a time dependency component to our solution. However, there are a couple points to note:

- Time evolution simulations are expensive
- Existing literature on time evolution GSE do not take into account the possibility for current reversal (e.g. resistivity)
- The GSH variant we utilise (which has yet to be introduced) is not time dependent, i.e., was not derived taking into account a time component

There are a couple possible avenues we could explore from here. We could implement a numerical code for the time evolution, for example a particle simulation, however this is for all intents and purposes computationally infeasible given the hardware available (a beloved but dilapidated Thinkpad T440p). We could attempt to find an analytic solution to the GSE with time dependence (or an approximation to one), but given the mountain of research existing in this space and the relative inexperience of the author, this would likely be a futile task. The third, and most lucrative option, is to use an existing analytic solution to a variation of the GSE that accounts for current reversal, and explore if small changes to its state are sufficiently "good enough" approximations of .

This is where perturbation theory comes in. Intuitively we would expect a plasma to vary "smoothly" - nature rarely behaves in instantaneous ways. The analytic solution we have describes the instantaneous state (a slice in time) of our reactor's state. We may anticipate then that if we were to change something in our system by a "sufficiently small amount", that our system may react to that change of state in a way that approximates how it would if it were really varying through time. We will have the ability to determine the state of our system for some given parameters, and so the question becomes: can we vary our input in a way such that the behaviour of the analytic solution to the GSH is an approximation of a time evolution of the plasma. We are able to compare our results to experimental data, and then exploit our work to answer further questions.

3.1.2 Regular Perturbation Theory

`https://jmahaffy.sdsu.edu/courses/f19/math537/beamer/perturb.pdf` `https:`

`//www.ucl.ac.uk/~ucahhwi/LTCC/section2-3-perturb-regular.pdf` `https://math.byu.edu/~bakker/Math521/Lectures/M521Lec15.pdf`

3.1.3 Ideal MHD Perturbation

At the beginning of this thesis we introduced it as being interdisciplenary in nature. Keeping in that spirit, at times here we will put on our physicists cap and seemingly arbitrarily remove terms we no longer wish to have. We kindly ask that any mathematicians reading these sections avert their attention so as to maintain sanity.

In Chapter 2 we derived the ideal MHD equations, given:

$$j \times B = \nabla p \tag{3.1}$$

$$\mu_0 j = \nabla \times B \tag{3.2}$$

$$\nabla \dot{B} = 0 \tag{3.3}$$

These, notably, are the time independent variation. We wish to look at the effect of a small ($\epsilon > 0$) time perturbation to this system, and so will return to this derivation, without discarding the time component. We will begin with Maxwell's equations:

Resistivity Note

While when using the ideal MHD equations we make many simplifications, there is one that we should note perticularly given what we are trying to model. Resistivity is a phenomenon

Talk about AC reactors, acceleration of electrons important

Explain why perturbation theory is justified for resitivity

Allegory below

A perhaps more intuitive analogy is that of swimming. Imagine you are stationary under water in a pool, floating, and you look at your arm extended out in front of you. Focus on the feeling of the water moving around your arm - this will emulate the resistivity that electrons would feel. If you move your arm abruptly, perhaps in a cutting motion, then you will feel the pressure of the water push against your arm, and perhaps you have to use noticably more force to overcome this pressure. Now imagine instead that you don't abruptly cut, but instead very slowly, almost imperceptibly, move your arm through the water.

You will notice that the feeling of water *resisting* your arms motion will not be as noticeable.

NOTE: perhaps it's better told from the perspective of an external observer? Otherwise this might imply that electron speeds should differ, not that the observed force is different.

3.2 Grad-Shafranov-Helmholtz Equation

3.2.1 Helmholtz Equations

What is a Helmholtz equation

3.2.2 GSH Equation

How do we apply it to the Grad-Shafranov equation

The work here will largely follow the paper by Wang and Yu REFERENCE, with some key notes and modifications. We are by now familiar with the GSE:

INSERT GSE

Wang provide a slightly different (though equivalent) formulation:

Proposition 3.1. *First, we will introduce some normalisation terms. Let* $j_\phi = \frac{J_\phi \mu_0 a}{B_0}$ *be the toroidal current density normalised with respect to the magnetic field, recalling that* J_ϕ *is the toroidal current density,* μ_0 *is free-space magnetic permeability, and* B_0 *is the on-axis magnetic field strength (here, the strength of the magnetic field evaluted at the centre of the cross section, where* $x = a$*).*

The GSE can then equivalently be stated:

$$\left(x \frac{\partial}{\partial x} \frac{1}{x} \frac{\partial}{\partial x} + \frac{\partial^2}{\partial z^2} \right) \psi = -\frac{1}{2} x^2 \frac{\mathrm{d}\beta}{\mathrm{d}\psi} - \frac{1}{2} \frac{\mathrm{d}g^2}{\mathrm{d}\psi} = -x j_\phi \tag{3.4}$$

where

$$a_1 = -\frac{1}{2} \frac{\mathrm{d}\beta}{\mathrm{d}\psi} \tag{3.5}$$

$$a_2 - \alpha^2 \psi = -\frac{1}{2} \frac{\mathrm{d}g^2}{\mathrm{d}\psi} \tag{3.6}$$

Here, we have some normalised terms: $\psi(x, z) = \frac{\Psi(x,z)}{B_0 a^2}$ *is the normalised poloidal magnetic flux function. The* β *function is given* $\beta(\psi) = \frac{2\mu_0 p(\psi)}{B_0^2}$*, and* $g(\psi) = \frac{F(\psi)}{B_0 a}$*.*

Remark 3.2. We highlight here the parameter tuple (a_1, a_2, α). We will see soon that these parameters can be used to determine our system. For now we simply note their significance so that the reader (that's you!) may pay them special attention through the rest of the following working.

At the boundary of a fusion reactor we expect there to be no / minimal poloidal magnetic flux, i.e. that the field strength dissipates as we approach the edge of the reactor. This justifies the boundary condition we will impose: $\psi|_b = 0$.

3.2.3 Analytic Solution and Derivation

3.3 Current Reversal Systems

3.3.1 Current and Pressure Density Profiles

3.3.2 Example Configurations

Chapter 4

Numerical Model Fitting

4.1 Non-Linear Optimisation

4.1.1 Least Squares

4.1.2 Optimisation Algorithms

Minimising least squares.

The Usual Suspects

Newton's method

Gradient descent

Bounded vs non-bounded optimisation

MMA Algorithm

Method of Moving Asymptotes (MMA)

Provide explanation of LD-MMA algorithm.

4.2 GSH Parameter Fitting

4.2.1 Optimisation Function

4.2.2 Parameter Space

Graphs showing effect of different parameters

Non-reliance on α (with exception of $\frac{1}{\alpha}$ where $\alpha = 0$ thing)

4.2.3 Convergence Difficulties

Initial difficulties with normalisation.

4.3 Simulated Current Reversals

4.3.1 Method of Reversal

4.3.2 Results and Explanations

Chapter 5

Comparison with Experiment

5.1 ISTTOK

5.1.1 A Brief History of ISTTOK

5.1.2 Reactor Specification

5.2

Chapter 6

Blue Skies and Horizons

6.1 Conclusions

6.2 Further Work

6.2.1 Simulated Electric Field via Fake Solenoids

In a meeting while presenting my findings to the Plasma Science group, I posed the question of extracting electric field information from the data we had available. There are many benefits to being able to describe the electric field for the confinement time of a plasma, however the most significant to our purpose is to positively identify the brith of runaway electrons. This information however is not readily available with the system we worked with.

David Pfefferle proposed a method of simulating the presence of an electric field instead. His idea was to introduce infinitesimal solenoids at the centre of magnetic islands, which would each contribute produce their own electric field. These would then interact, with the idea that the product would be an approximation to the expected electric field for a given state.

The physical justification for this is that the solenoid's magnetic field is emulating the magnetic field produced by current densities, which are themselves informed by magnetic islands. A toroidal current will produce an electric field, including a poloidal component, which will influence the behaviour of runaway electrons. Thus, this approach effectively emulates the presence of a poloidal electric field using position and strength information of current densities.

This approach could utilise work done by Nicholas Bohlsen in identifying the presence of magnetic islands from poloidal magnetic flux data in his thesis.

6.2.2 Comparison with ISTTOK vloop data

6.2.3 Retrieval of

Appendix A

Appendix title goes here

Blah blah blah...

Blah blah blah...

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