Simulations in AC Tokamak Ramp Downs

Thomas Malcolm

Sep 2023

A thesis submitted for the degree of
Bachelor of Mathematical Sciences (Honours)
of the Australian National University

For George, Paul, Ringo and John.

Declaration

The work in this thesis is my own except where otherwise stated.

Thomas Malcolm.

Acknowledgements

First and foremost I'd like to thank Matthew Hole, my supervisor. I came into this year a little daunted at the prospect of tackling a physics-based project with a physics background not extending past "A Brief History of Time". Your patience and willingness to explain concepts to me I'll forever appreciate, and your energetic approach to problem solving made me feel both at home and excited for the work we have done in this thesis.

In a similar vein, thank you to the rest of the Plasma Theory and Modelling group for always being willing to answers questions when I had them. You guys were definitely my saving grace at times, and I only wish I spent more time getting to know you all. I should give a special shout out to Nick, Sandra, Dean and Josh specifically for being so accommodating, helpful, and kind in answering my questions whenever I would pop around, even if it was just for me to rant.

I'd like to thank the other Honours students for helping make this a fantastic year - I won't name everyone like Joe, but do extend my gratitutde to everyone nonetheless. Our little nook of MSI proved very distracting, but in the best possible way. As such I would of course expect you to take part responsibility for the late night writing sessions toward the end of the year - but you also get credit for making them enjoyable.

Last but certainly not least, to Kirsten and Graeme. Thank you for your support these last few years, and the infinite kindness you give to all those around you. However silly it is, your encouragement genuinely helped motivate me at times, and I always got a bit of joy explaining my thesis and whatever problem I was working on at the time to you.

Abstract

Contents

Notation and terminology

Notation

$L^2(\mathbb{R}^n)$	Space of L^2 integrable functions on $\mathbb{R}^n$.
B_0	On-axis magnetic field (Toroidal magnetic field strength)
R_0	Major radius (Tokamak)
a	Minor radius (Tokamak)
ψ	Normalised poloidal magnetic flux function
Ψ	Poloidal magnetic flux function
J_ϕ	Toroidal current density
j_ϕ	Normalised toroidal current density (normalised w.r.t. B_0)
I_ρ	Normalised toroidal current density (normalised w.r.t. maximum value)
p	Plasma pressure density
β	**Check with wesson to confirm**

Terminology

GS(E)	Grad-Shafranov equation
GSH	Grad-Shafranov-Helmholtz equation
MHD	Magnetohydrodynamics

Chapter 1

Introduction

1.1 Plasma Science

This thesis is multidisciplenary by nature, incorporating aspects of mathematics, physics and computer science throughout the various challenges faced in reaching the conclusions we have.

Before burying ourselves in the thick of our results we will cover the requisite knowledge for working in the plasma science simulation space. First we'll discuss what the physical object of our attention ("plasma") actually is, with a brief discussion on the design of fusion reactors, where we will emphasise the structures that specifically relate to our interests. In the background chapter we will expound on this by building the mathematical theory underpinning physical processes within the fusion reactor, providing us a way to reason about the behaviour of plasma (and related processes) inside a fusion reactor, or, more accurately, approximate their behaviour via simulation.

1.1.1 I'm a Mathematician... what is "plasma"?

While an initially daunting topic, fear not fearful mathematician, for many of the inherently physical behaviours in plasma science we require can be expressed in terms of our dearest mathematical expressions – differential equations! But first, what actually is "plasma"? Webster's dictionary defines plasma to be "a green faintly translucent quartz" [1]. While I'm sure there isn't no relation between crystals and our investigations, this is unfortunately, not the recipient of our affection.

When plasma is referred to in everyday conversation it is often noted as being the "4th state of matter". To introduce slightly more rigour, plasma is an extension of the gaseous state of matter, where its energy (read: temperature) is increased sufficiently high that the electrons are no longer bound by the electromagnetic force to the atom's nucleus REFERENCE. The resulting substance is a "pool" of cations (the positively charged nuclei), and electrons (negatively charged), that exhibits interesting properties. It is these properties that we seek to exploit in the search of controlled, sustained fusion reactions.

Plasma is abundant in nature – just not in many places that we as Humans commonly look. Stars are the most immediate example of matter in a plasma state, and are readily viewable (at least for half the day). Lightning strikes are paths through the atmosphere which are ionised, and neon signs work by heating Neon gas within a tube to ionise it REFERENCE.

The question then is, what is "fusion", and how does this relate to plasma? The answer comes back to energy. Analogously to a fission reaction, where energy is released through the division of atoms, one can also fuse two separate atoms together and have large amounts of energy emitted as a bi-product of that fusion. It is (essentially) this extra energy that we wish to harness when harvesting energy from a fusion reactor (which we'll discuss in the next part). When two

HYDROGEN FUSION EQUATION

How do two hydrogen atoms fuse? This process can occur most easily when in a plasma state, as repellant forces are minimised.

1.1.2 Fusion Reactor Design 101

Diagram

Poloidal vs toroidal flux

Magnet positioning

Heating of plasma

Confinement

REs

1.2 What problem does this thesis address?

Talk about paper Matthew and Artur released. Presence of residual REs.

1.2.1 What results do we seek?

To provide a theoretical basis for exploring the observed anomalies. To see if theory supports the observation

1.3 Project Progression

Work inspired by Matthew's paper

First expanded MHD equations linearly with a perturbation treatment

Mathematical PDE theory developed by Wang. Identified errors in paper, fixed.

Developed a simulation to reproduce results. Then went other direction, solving for parameters for data.

Simulated current inversion.

ISTTOK data matching.

Feasibility

1.4 Structure of Thesis

TODO (after writing it)

Chapter 2

Background

2.1 Magnetohydrodynamics

Study of magnetism as a fluid. Physically everything is driven by particles and fields, however this is computationally infeasible. So MHD theory was developed, marrying fluid dynamics with Maxwell's equations

2.1.1 MHD Theory

Lagrangian operator

Maxwell's equations

Assumptions and simplifications (leading into ideal MHD)

2.1.2 Ideal MHD

Simplifications made to derive ideal MHD equations

Use in GS equation

Physical implications

2.2 Grad-Shafranov Equation

What is it

What do its components describe

Solutions to GS equation

2.2.1 Derivation

Go through derivation of it

2.3 AC Configuration Tokamaks

Tokamak's normally in DC mode. What does it mean for a fusion reactor to operate in AC mode

2.3.1 Physical Differences

Plasma current (talk: operation of toroidal coils to regulate current)

2.3.2 Why Bother?

What benefits are there to using an AC design over DC?

Talk about confinement time, stability, etc

Chapter 3

Current Reversal Theory

3.1 Perturbation Methods

3.1.1 What is a Perturbation Treatment?

When it comes to solving differential equations, the ideal scenario is that you find an analytic solution, Ψ. This, however, is generally a difficult problem, and especially so in the case of differential equations. A great deal of research goes into trying to find analytic solutions to PDEs, which is no less true for the Grad-Shafranov equation and its variations. Though such efforts are often futile, or require making assumptions about properties of your solution which may not be representative of what you're trying to show, other means of obtaining the $\Psi(\vec{x})$ for $\vec{x} \in \Omega$ have been developed.

A traditional approach to this is to use some numerical code to approximate solution values. There are various codes which exist for this purpose specifically with regards to the GSE equation, such as HELENA REFERENCE. These codes come in many flavours - some implement a particle simulation, which can provide precision with simulation, though are computationally costly. Others exploit various properties of the variant of GSE they target and may be computationally more feasible, but are restricted in scope, application, and/or reliability because of their assumptions.

A common problem that numerical methods seek to solve is that of time evolution. Simulations which evolve a system through time, for obvious reasons, do not discard the time dependency component in their derivation of the GSE. This, however, often makes them computationally expensive as a result, while also increasing the difficulty of finding analytic solutions to the system.

For this thesis, given the availability of an analytic solution to a variation of the GSE that specifically relates to current reversals, we employ a hybrid approach. In our simulation we wish to observe changes in the system through a current ramp down - this necessitates a time dependency component to our solution. However, there are a couple points to note:

- Time evolution simulations are expensive
- Existing literature on time evolution GSE do not take into account the possibility for current reversal (e.g. resistivity)
- The GSH variant we utilise (which has yet to be introduced) is not time dependent, i.e., was not derived taking into account a time component

There are a couple possible avenues we could explore from here. We could implement a numerical code for the time evolution, for example a particle simulation, however this is for all intents and purposes computationally infeasible given the hardware available (a beloved but dilapidated Thinkpad T440p). We could attempt to find an analytic solution to the GSE with time dependence (or an approximation to one), but given the mountain of research existing in this space and the relative inexperience of the author, this would likely be a futile task. The third, and most lucrative option, is to use an existing analytic solution to a variation of the GSE that accounts for current reversal, and explore if small changes to its state are sufficiently "good enough" approximations of .

This is where perturbation theory comes in. Intuitively we would expect a plasma to vary "smoothly" - nature rarely behaves in instantaneous ways. The analytic solution we have describes the instantaneous state (a slice in time) of our reactor's state. We may anticipate then that if we were to change something in our system by a "sufficiently small amount", that our system may react to that change of state in a way that approximates how it would if it were really varying through time. We will have the ability to determine the state of our system for some given parameters, and so the question becomes: can we vary our input in a way such that the behaviour of the analytic solution to the GSH is an approximation of a time evolution of the plasma. We are able to compare our results to experimental data, and then exploit our work to answer further questions.

3.1.2 Regular Perturbation Theory

`https://jmahaffy.sdsu.edu/courses/f19/math537/beamer/perturb.pdf` `https:`

`//www.ucl.ac.uk/~ucahhwi/LTCC/section2-3-perturb-regular.pdf` `https://math.byu.edu/~bakker/Math521/Lectures/M521Lec15.pdf`

3.1.3 Ideal MHD Perturbation

At the beginning of this thesis we introduced it as being interdisciplenary in nature. Keeping in that spirit, at times here we will put on our physicists cap and seemingly arbitrarily remove terms we no longer wish to have. We kindly ask that any mathematicians reading these sections avert their attention so as to maintain sanity.

In Chapter 2 we derived the ideal MHD equations, given:

$$j \times B = \nabla p \tag{3.1}$$

$$\mu_0 j = \nabla \times B \tag{3.2}$$

$$\nabla \dot{B} = 0 \tag{3.3}$$

These, notably, are the time independent variation. We wish to look at the effect of a small ($\epsilon > 0$) time perturbation to this system, and so will return to this derivation, without discarding the time component. We will begin with Maxwell's equations:

Resistivity Note

While when using the ideal MHD equations we make many simplifications, there is one that we should note perticularly given what we are trying to model. Resistivity is a phenomenon

Talk about AC reactors, acceleration of electrons important

Explain why perturbation theory is justified for resitivity

Allegory below

A perhaps more intuitive analogy is that of swimming. Imagine you are stationary under water in a pool, floating, and you look at your arm extended out in front of you. Focus on the feeling of the water moving around your arm - this will emulate the resistivity that electrons would feel. If you move your arm abruptly, perhaps in a cutting motion, then you will feel the pressure of the water push against your arm, and perhaps you have to use noticably more force to overcome this pressure. Now imagine instead that you don't abruptly cut, but instead very slowly, almost imperceptibly, move your arm through the water.

You will notice that the feeling of water *resisting* your arms motion will not be as noticeable.

NOTE: perhaps it's better told from the perspective of an external observer? Otherwise this might imply that electron speeds should differ, not that the observed force is different.

3.2 Grad-Shafranov-Helmholtz Equation

3.2.1 Helmholtz Equations

What is a Helmholtz equation

3.2.2 GSH Equation

How do we apply it to the Grad-Shafranov equation

The work here will largely follow the paper by Wang and Yu REFERENCE, with some key notes and modifications. We are by now familiar with the GSE:

INSERT GSE

Wang provide a slightly different (though equivalent) formulation:

Proposition 3.1. *First, we will introduce some normalisation terms. Let* $j_\phi = \frac{J_\phi \mu_0 a}{B_0}$ *be the toroidal current density normalised with respect to the magnetic field, recalling that* J_ϕ *is the toroidal magnetic field,* μ_0 *is free-space magnetic permeability, and* B_0 *is the on-axis magnetic field strength (here, the strength of the magnetic field evaluted at the centre of the cross section, where* $x = a$*).*

The GSE can then equivalently be stated:

$$\left(x\frac{\partial}{\partial x}\frac{1}{x}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial z^2} \right)\psi = -\frac{1}{2}x^2\frac{\mathrm{d}\beta}{\mathrm{d}\psi} - \frac{1}{2}\frac{\mathrm{d}g^2}{\mathrm{d}\psi} = -xj_\phi \tag{3.4}$$

where

$$a_1 = -\frac{1}{2}\frac{\mathrm{d}\beta}{\mathrm{d}\psi} \tag{3.5}$$

$$a_2 - \alpha^2\psi = -\frac{1}{2}\frac{\mathrm{d}g^2}{\mathrm{d}\psi} \tag{3.6}$$

Here, we have some normalised terms: $\psi(x, z) = \frac{\Psi(x,z)}{B_0 a^2}$ *is the normalised poloidal magnetic flux function. The* β *function is given* $\beta(\psi) = \frac{2\mu_0 p(\psi)}{B_0^2}$*, and* $g(\psi) = \frac{F(\psi)}{B_0 a}$*.*

Remark 3.2. We highlight here the parameter tuple (a_1, a_2, α). We will see soon that these parameters can be used to determine our system. For now we simply note their significance so that the reader (that's you!) may pay them special attention through the rest of the following working.

At the boundary of a fusion reactor we expect there to be no / minimal poloidal magnetic flux, i.e. that the field strength dissipates as we approach the edge of the reactor. This justifies the boundary condition we will impose: $\psi|_b = 0$.

3.2.3 Analytic Solution and Derivation

3.3 Current Reversal Systems

3.3.1 Current and Pressure Density Profiles

3.3.2 Example Configurations

Chapter 4

Numerical Model Fitting

4.1 Non-Linear Optimisation

4.1.1 Least Squares

4.1.2 Optimisation Algorithms

Minimising least squares.

The Usual Suspects

Newton's method

Gradient descent

Bounded vs non-bounded optimisation

MMA Algorithm

Method of Moving Asymptotes (MMA)

Provide explanation of LD-MMA algorithm.

4.2 GSH Parameter Fitting

4.2.1 Optimisation Function

4.2.2 Parameter Space

Graphs showing effect of different parameters

Non-reliance on α (with exception of $\frac{1}{\alpha}$ where $\alpha = 0$ thing)

4.2.3 Convergence Difficulties

Initial difficulties with normalisation.

4.3 Simulated Current Reversals

4.3.1 Method of Reversal

4.3.2 Results and Explanations

Chapter 5

Comparison with Experiment

5.1 ISTTOK

5.1.1 A Brief History of ISTTOK

5.1.2 Reactor Specification

5.2

Chapter 6

Blue Skies and Horizons

6.1 Further Work

6.1.1 Simulated Electric Field via Fake Solenoids

In a meeting while presenting my findings to the Plasma Science group, I posed the question of extracting electric field information from the data we had available. There are many benefits to being able to describe the electric field for the confinement time of a plasma, however the most significant to our purpose is to positively identify the brith of runaway electrons. This information however is not readily available with the system we worked with.

David Pfefferle proposed a method of simulating the presence of an electric field instead. His idea was to introduce infinitesimal solenoids at the centre of magnetic islands, which would each contribute produce their own electric field. These would then interact, with the idea that the product would be an approximation to the expected electric field for a given state.

The physical justification for this is that the solenoid's magnetic field is emulating the magnetic field produced by current densities, which are themselves informed by magnetic islands. A toroidal current will produce an electric field, including a poloidal component, which will influence the behaviour of runaway electrons. Thus, this approach effectively emulates the presence of a poloidal electric field using position and strength information of current densities.

This approach could utilise work done by Nicholas Bohlsen in identifying the presence of magnetic islands from poloidal magnetic flux data in his thesis.

6.2 Problems Encountered

6.2.1

6.2.2

Appendix A

Appendix title goes here

Blah blah blah...

Blah blah blah...

Bibliography

[1] Merriam-Webster. Plasma, 2023. https://www.merriam-webster.com/dictionary/plasma.