§4 Free quotients

Recap **Def^n** $\sigma \in \mathrm{Stab}(X)$ geometric iff $\forall x \in X$, $\mathcal{O}_x$ is σ-stable.

Lecture 1:

Q4 alb$_X$ not finite $\Rightarrow$ $\exists$ nongeometric stability conditions ?

IDEA Investigate examples which arise as free quotients

ie $\underset{\text{finite}}{G} \overset{\text{free}}{\curvearrowright} X$ alb$_X$ finite

$$\downarrow \pi$$

"free quotient" $Y = X/G$ alb$_Y$ not finite

Last time we proved:

Th^m 4 Suppose G is abelian and $G \curvearrowright \mathcal{D}$.

There are biholomorphisms of (closed) complex submanifolds:

$$\mathrm{Forg}_G^{-1} : \mathrm{Stab}_\Lambda(\mathcal{D})^G \rightleftarrows \mathrm{Stab}_\Lambda(\mathcal{D}_G)^{\hat{G}} : \mathrm{Forg}_{\hat{G}}^{-1}$$

Today: translate this to geometry, and answer Q4 !

§4 Free Quotients

Setup • $\pi : X \longrightarrow X/G =: Y$ free quotient

• G abelian (for simplicity!)

Recall • $D^b(Y) \simeq D^b(X/G)$

• $\pi^* \simeq \mathrm{Forg}_G$, $\pi_* \simeq \mathrm{Inf}_G$

Th^m 5 $(\pi^*)^{-1} : \mathrm{Stab}_\Lambda(X)^G \overset{(\ast)}{\simeq} \mathrm{Stab}_\Lambda(Y)^{\hat{G}} : (\pi_*)^{-1}$

$$\sigma = (P, Z) \longmapsto \sigma' = (P', Z')$$

where $P'(\phi) = \{ E \in D^b(Y) : \pi^*(E) \in P(\phi) \}$

ie. E is σ'-s.s. $\Leftrightarrow$ $\pi^* E$ is σ-s.s.

$$Z'(\phi) = Z \circ \pi^*$$

(⚠ Could also use Polish chunk criterion to π^*, π_* to prove Prop^n 2)

Lecture 4

SIMILARLY $\tau' = (\mathcal{Q}', W') \longleftarrow\!\!| \ \tau = (\mathcal{Q}, W) : (\pi_*)^{-1}$

$\mathcal{Q}'(\phi) = \{ E \in D^b(X) : \pi_*(E) \in \mathcal{Q}(\phi) \}$

E is τ'-s.s. $\iff$ $\pi_* E$ is τ-s.s.

$W'(\phi) = Z \circ \pi_*$

Remark ① Th^m 4 & 5 generalise to nonabelian group actions [D.- Edmund Heng - Anthony Licata]. The hard part is to describe $\mathrm{im}(\pi_*^{-1})$.

② ($\hat{G}$-action) $\pi_* \mathcal{O}_X = \bigoplus_{\chi \in \hat{G}} \mathcal{L}_\chi$, each $\mathcal{L}_\chi$ is a line bundle with $ch_1(\mathcal{L}_\chi) = 0$ (in $NS_{\mathbb{R}}(X)$, "numerically trivial")

Then $\hat{G} \curvearrowright D^b(Y)$ by $- \otimes \mathcal{L}_\chi$.

[ie. $\sigma = (P, Z) \in \mathrm{Stab}(Y)^{\hat{G}} \iff$ • $P(\phi) \otimes \mathcal{L}_\chi = P(\phi)$ $\forall \chi \in \hat{G}$
• $Z(E \otimes \mathcal{L}_\chi) = Z(E)$]

NOW BACK TO Q4 :

Lemma ⊛ preserves geometric stability conditions

proof sketch

($\Rightarrow$) $\sigma \in \mathrm{Stab}(X)^G$ geometric, WTS σ' geometric.

• $y \in Y$, $\pi^* \mathcal{O}_y = \bigoplus_{g \in G} g^* \mathcal{O}_x$

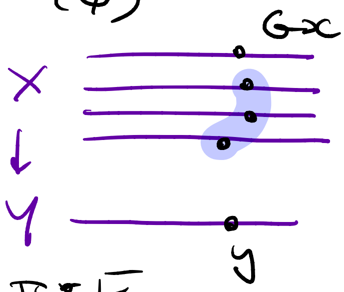
• σ geometric $\Rightarrow$ $\mathcal{O}_x \in P(\phi)$ $\overset{\sigma \text{ G-invariant}}{\Rightarrow}$ $\pi^* \mathcal{O}_y \in P(\phi)$

$\overset{⊛}{\iff}$ $\mathcal{O}_y \in P'(\phi)$

WTS $\mathcal{O}_y$ simple in $P'(\phi)$:

• $\forall E \hookrightarrow \mathcal{O}_y$ in $P'(\phi)$

$\Rightarrow \pi^* E \hookrightarrow \underbrace{\bigoplus_{g \in G} g^* \mathcal{O}_x}_{\oplus \text{ stables}}$ in $P(\phi)$



But any (strict) subobject cannot be of the form $\pi^* E$

$\Rightarrow \mathcal{O}_y$ is simple in $P'(\phi)$

($\Leftarrow$) similar : $\cdot\ \sigma' \in (\mathrm{Stab}(Y))^{\hat{G}}$ geometric

$\Rightarrow \mathcal{O}_y \in \mathcal{P}'(\phi) \overset{(*)}{\Longleftrightarrow} \pi^*\mathcal{O}_y \in \mathcal{P}(\phi)$

$\Rightarrow \mathcal{O}_x \in \mathcal{P}(\phi)$ ($\mathcal{P}(\phi)$ closed under direct summands)

Then $E \hookrightarrow \mathcal{O}_x \twoheadrightarrow F$ in $\mathcal{P}(\phi)$

$\Rightarrow \pi_* E \hookrightarrow \mathcal{O}_y \twoheadrightarrow \pi_* F$ in $\mathcal{P}'(\phi)$

$\mathcal{O}_y$ σ'-stable $\Rightarrow \pi_* E$ or $\pi_* F = 0$

$\Rightarrow E$ or $F = 0$ (π finite $\Rightarrow \pi_*$ conservative) $\square$

EXERCISE find $\sigma \in (\mathrm{Stab}^{geo}(X))^G$ for X a surface ($Y = X/G$)

HINT: $\mathrm{DENS}_{\mathbb{R}}(Y) \Rightarrow \pi^* D$ G-invariant divisor class

Thm 6 [D'23, DML: non abelian version]

- X surface, alb_X finite
- G finite abelian group
- $Y = X/G$ free quotient

Then $\mathrm{Stab}^{geo}(Y) = \mathrm{Stab}(Y)^{\hat{G}} \simeq \mathrm{Stab}(X)^G$ and this is a contractible connected component of $\mathrm{Stab}(Y)$.

open, closed, connected

Remark ① Version in higher dim.

$\mathrm{Stab}(Y)^{\hat{G}} \subseteq \mathrm{Stab}^{Geo}(Y)$

and this is a union of connected components

② In [DML'23] we extend this to non abelian G

Proof

- alb_X finite $\overset{[FLZ]}{\Rightarrow} \mathrm{Stab}(X) = \mathrm{Stab}^{geo}(X)$
- Thm 5 + Lemma $\Rightarrow \mathrm{Stab}(Y)^{\hat{G}} \subseteq \mathrm{Stab}^{geo}(Y)$
- $\mathrm{Stab}(Y)^{\hat{G}}$ is open & closed

$\Rightarrow$ version in any dim.

- [D., Rekuski '23] $\mathrm{Stab}^{geo}(Y)$ connected & contractible $\Rightarrow \square$

(see Lecture 1 notes, §1.1)

Examples

① $C_i \subset \mathbb{P}^2 : x^5 + y^5 + z^5 = 0$ Fermat quintic curve

$$G = (\mathbb{Z}/5\mathbb{Z})^2 \overset{\text{free}}{\curvearrowright} X = C_1 \times C_2 \xrightarrow{\text{alb}_X} J(C_1) \times J(C_2)$$

finite ☺ $g(C_i) = 6 > 0$

$C_i/G \cong \mathbb{P}^1$

$$\downarrow$$

$Y = X/G$ **Beauville surface**

$\dim \mathrm{Alb}(Y) = h^1(Y, \mathcal{O}_Y) = 0 \Rightarrow \mathrm{alb}_Y$ not finite

This construction generalises to

$$Y = (C_1 \times C_2)/G$$

- $g(C_i) > 1$: 17 families with $h^1(\mathcal{O}_Y) = h^2(\mathcal{O}_Y) = 0$ called **Beauville-type surfaces**

  [G abelian for 5 families $G \cong (\mathbb{Z}/2\mathbb{Z})^3$, $(\mathbb{Z}/2\mathbb{Z})^4$, $(\mathbb{Z}/3\mathbb{Z})^2$, $(\mathbb{Z}/5\mathbb{Z})^2$]

  ↳ alb_Y trivial

- $g(C_i) = 1$: 7 families with $h^1(\mathcal{O}_Y) = 1$ called **bielliptic surfaces**, alb_Y is an elliptic fibration!

By Th^m 6 either
- $\mathrm{Stab}(Y) = \mathrm{Stab}^{geo}(Y)$ so (A4) is no!
- $\mathrm{Stab}(Y)$ disconnected

both surprising!

Th^m 7 [D - Gebhard Martin, work in progress]

Y bielliptic surface $\Rightarrow \mathrm{Stab}^{geo}(Y) = \mathrm{Stab}(Y)$

ie. (A4) no!

② (Non example: Enriques surface) $Y = X/G$ ← K3, G ← $\mathbb{Z}/2\mathbb{Z}$

$\mathrm{Stab}(X) \setminus \mathrm{Stab}^{geo}(X) \neq \emptyset$

Lemma $\Rightarrow$ $\mathrm{Stab}(Y) \setminus \mathrm{Stab}^{geo}(Y) \neq \emptyset$.

EXERCISE Is $\sigma_{H,D,\alpha,\beta} \in \mathrm{Stab}^{geo}(Y)$ invariant under $- \otimes \omega_Y$?

Higher dim. examples:

3 (Calabi-Yau threefolds of abelian-type)

X abelian 3fold, $Y = X/G$ free quotient s.t.

$\omega_Y \simeq \mathcal{O}_Y$ and $H^1(Y, \mathbb{C}) = 0$ ($G = \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ or D_4 [Oguiso-Sakurai])

$\exists\, \mathcal{B} \subset \mathrm{Stab}(X)$: connected component of geometric stability conditions [Bayer-Macrì-Stellari, Oberdieck-Piyaratne-Toda]

Thm 6

$\Rightarrow$ $\mathrm{Stab}(Y)$ has a connected component of geometric stability conditions

4 (Generalised hyperelliptic varieties)

$Y = A/G$

- A abelian
- $G \curvearrowright A$ finite free, contains no translations.

Y is Kähler, Kodaira dim. 0.

[Catanese-Demleitner, Aguiló Vidal] construct examples:

$Y = E^{2n} \times E' / D_{4n}$ $\quad E, E'$ elliptic curves, alb$_Y$ not finite.

§5 Where next?

Future directions & applications

Q What geometric condition characterises $\mathrm{Stab}(X) \neq \mathrm{Stab}^{geo}(X)$?

$\dim X$	$\exists$ geo?	$\exists$ non-geo? (c) = contractible, (cc) = (known) connected comp. contractible
1 (c)	$\mathrm{Stab}^{geo}(X) \simeq \mathbb{C} \times \mathbb{H}$ [Bridgeland '07], [Macrì '07]	$g(X) \geq 1$ no [Macrì '07] $\mathrm{Stab}(\mathbb{P}^1) \simeq \mathbb{C}$ yes [Okada '06] uses: $D^b \mathrm{Coh}\,\mathbb{P}^1 \simeq D^b \mathrm{Rep}$
2	yes! see §1, $\mathrm{Stab}^{geo}(X)$ described in §1.1 (c) [Rekuski]	abelian surface no (c) [Bridgeland '08, Huybrechts-Macrì-Stellari '08] $\mathbb{P}^2$ yes (cc) [Chunyi Li '17] Rep K3 yes Picard rk 1: (cc) [Bridgeland '08] [Bayer-Bridgeland '17] $S \supset C$, $C^2 < 0$ yes [Tramel-Xia '22] finite Albanese & free quotients: no [FLZ, DHL]
3	yes for some threefolds	abelian 3folds, Picard rk 1 no (cc) [Maciocia-Piyaratne '13+'15, Bayer-Macrì-Stellari '16] $\mathbb{P}^3$ yes finite Albanese & free quotients no: [FLZ, DHL]
≥ 4	??? $\mathbb{P}^n$ [Dapeng Mu '21, Petković '22]	$\mathbb{P}^n$ (or full strong exc. coll'n) yes [Macrì '07] finite Albanese & free quotients no: [FLZ, DHL]

Q What happens for non-free quotients?

Th^m 4 $\Rightarrow$ $\mathrm{Stab}_\Lambda(X)^G \simeq \mathrm{Stab}_\Lambda([X/G])^{\hat{G}}$

(or non abelian version)

e.g.

* $D^b([pt/G])$: ⚠ being $\hat{G}$-invariant is very restrictive
* ramified covers (I have some ideas for curves!)
* Crepant resolutions

$$\begin{tikzcd} X \arrow[dr] & & Y \arrow[dl] \\ & X/G & \end{tikzcd}$$

e.g.

(1) Kummer K3 surface : Y [Bridgeland-King-Reid]

$G=\{\pm 1\} \curvearrowright X = A$ abelian surface, $D^b(Y) \simeq D^b_G(X)$

Q What is $\mathrm{Stab}(Y)^{\hat{G}}$?

(2) [Perry-Shah] $X = C_1 \times \cdots \times C_n$ C_i curves, $\sigma \in \mathrm{Stab}_\Lambda(X)$ [Yucheng Liu] $\emptyset$ #

$G = \mathbb{Z}/2\mathbb{Z} \rightsquigarrow \mathrm{Stab}_\Lambda(Y) \neq \emptyset$, Y strict Calabi-Yau

warning: lattice Λ for $\mathrm{Stab}(X)$ may be too small for $\mathrm{Stab}(Y)$.

* Application to DT theory :

[Bridgeland-Del Monte-Giovenzana] $X = \mathrm{Tot}(\omega_{\mathbb{P}^1\times\mathbb{P}^1})$

Use Th^m 4 to study $\mathrm{Stab}(X)$ using (1) resolved conifold (2) quiver rep-s

and compute DT invariants ("count stable objects")

* Applications in representation theory

(1) In [DHL] we also applied Th^m 4 to

$\mathcal{D} = D^b(\mathrm{Rep}\ Q)$ for some wild quivers. relate to:

(2) [HL'24] use actions of fusion categories to study

Stab (CY2 category assoc. to Q), and use this to investigate

$K(\pi,1)$ conjecture for non-simply laced Artin groups

Q Do you have other examples of $G \curvearrowright \mathcal{D}$?

I'd like to hear about them!