Bridgeland Stability Conditions & Group Actions

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§1: Geometric Stability on Surfaces

- X: smooth projective surface $/\mathbb{C}$
- $H \in \mathrm{Amp}_{\mathbb{R}}(X)$

$$0 \neq E \in \mathrm{Coh}(X) \rightsquigarrow \mu_H(E) := \begin{cases} +\infty & \mathrm{rk}(E)=0 \\ \dfrac{H \cdot ch_1(E)}{H^2 \cdot \mathrm{rk}(E)} & \text{otherwise} \end{cases}$$

Defn E is H-(semi)stable if

$$\forall\, 0 \neq F \subsetneq E \Rightarrow \mu_H(F) \underset{(-)}{<} \mu_H(E)$$

- $\beta \in \mathbb{R}$

$$\mathcal{T}_{H,\beta} := \{E \in \mathrm{Coh}(X) : \forall E \twoheadrightarrow Q \neq 0,\ \mu_H(Q) > \beta\}$$

$$\mathcal{F}_{H,\beta} := \{E \in \mathrm{Coh}(X) : \forall\, 0 \neq F \subset E,\ \mu_H(F) \leq \beta\}$$

$$\underbrace{\mathrm{Coh}^{H,\beta}(X)}_{\text{"heart"}} := \left\{ E \in D^b(X) : \begin{array}{l} \mathcal{H}^0(E) \in \mathcal{T}_{H,\beta} \\ \mathcal{H}^{-1}(E) \in \mathcal{F}_{H,\beta} \\ \mathcal{H}^i(E) = 0 \text{ otherwise} \end{array} \right\}$$

- $B \in NS_{\mathbb{R}}(X)$
- $\alpha \in \mathbb{R}$

$$\left.\underbrace{Z_{H,B,\alpha,\beta}(E)}_{\text{"central charge"}} = \begin{array}{l}(\alpha - i\beta) H^2 \mathrm{rk}(E) \\ + (B + iH) \cdot ch_1(E) - ch_2(E)\end{array}\right] \in \mathbb{C}$$

$\rightsquigarrow$ new slope $\mu_Z := \dfrac{-\mathrm{Re}\, Z}{\mathrm{Im}\, Z}$

$\underline{\text{Th}^{\underline{m}}}$ [Bridgeland '08,] For $\alpha >> 0$,
$\{\sigma_{H,B,\alpha,\beta} = (\mathrm{Coh}^{H,\beta}(X), Z_{H,B,\alpha,\beta})\}$ is a continuous family of Bridgeland stability conditions.

⚠ $\forall x \in X$, $\mathcal{O}_x$ is simple in $\mathrm{Coh}^{H,\beta}(X) \Rightarrow \mathcal{O}_x$ is $\sigma_{H,B,\alpha,\beta}$-stable

$\underline{\text{Def}^{\underline{n}}}$ $\sigma \in \underbrace{\mathrm{Stab}(X)}_{\text{all stability conditions}}$ is **geometric** if $\mathcal{O}_x$ is σ-stable $\forall x \in X$

(Q1): $\exists$ nongeometric stability conditions?

(A1) Surfaces:
① abelian surfaces: no
yes: ② K3 surfaces] $\mathcal{O}_x$ destabilized by rigid bundles
③ rational surfaces
④ $X \supset C$ rational curve s.t. $C^2 < 0$
$\forall x \in C$, $\mathcal{O}_x$ destab. by $\mathcal{O}_C(k)$

$\underline{\text{Th}^{\underline{m}}}$ [Lie Fu – Chunyi Li – Xiaolei Zhao '21]
X finite Albanese morphism, alb_X $\Rightarrow$ $\mathrm{Stab}(X) = \underbrace{\mathrm{Stab}^{\mathrm{Geo}}(X)}_{\text{all geos}}$
$\Updownarrow$
has finite map to abelian variety

(Q2) [FLZ, Q 4.11] non finite Albanese morphism $\overset{??}{\Rightarrow}$ $\exists$ nongeometric stability conditions?

§2 The Le Potier Function

X : surface

<u>Conj</u> [FLZ] X minimal, $h^1(\mathcal{O}_X)=0$ ($\Leftrightarrow \mathrm{alb}_X$ trivial)

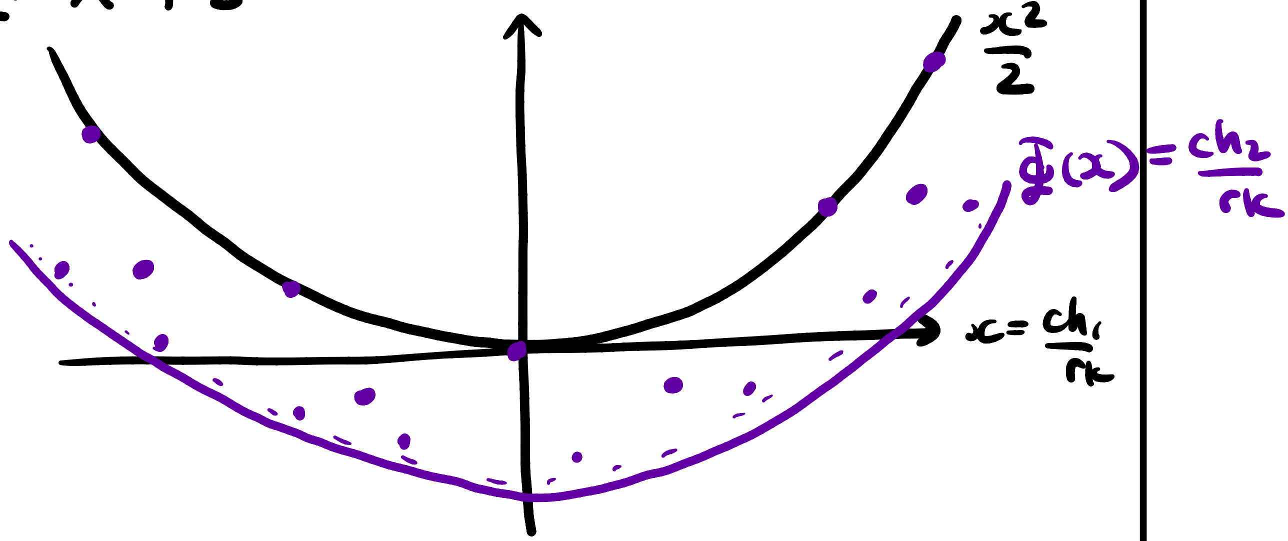
$\Rightarrow \Phi_{X,H,0}$ discontinuous at 0

↪ Beauville surfaces give a counterexample

<u>Def</u>^{n} $(H,B) \in \mathrm{Amp}_{\mathbb{R}}(X) \times NS_{\mathbb{R}}(X)$. The Le Potier function twisted by B, $\Phi_{X,H,B} : \mathbb{R} \to \mathbb{R}$, is:

$$\Phi_{X,H,B}(x) := \cancel{\max}\ \limsup_{\mu \to x} \left\{ \frac{ch_2(E) + B \cdot ch_1(E)}{H^2\, rk(E)} : \begin{array}{l} E \in Coh(X) \\ H\text{-semistable} \\ \mu_H(E) = \cancel{x}\ \mu \end{array} \right\}$$

<u>Ex</u> X K3



<u>Th</u>^{m} A [$\rho(X)=1$ FLZ, $\rho>1$ D.]

$$Stab^{Geo}(X) \simeq \mathbb{C} \times \left\{ (H,B,\alpha,\beta) \in \mathrm{Amp} \times NS \times \mathbb{R}^2 : \alpha > \Phi_{X,H,B}(\beta) \right\}$$

<u>Rmk</u> can use this to answer Q1

§3 Free Quotients

KEY IDEA: finite (abelian) $G \overset{\text{free}}{\curvearrowright} X$ ab'l$_X$ finite

$$\begin{array}{c} X \\ \downarrow \pi \\ Y := X/G \end{array}$$

F.Q. $Y := X/G$ ab'l$_Y$ not finite

$\oplus$

GOAL: compare $\mathrm{Stab}(X)$ & $\mathrm{Stab}(X/G)$

G: finite group, $\mathfrak{D}$: K-linear category, $(|G|, \mathrm{char}\, K) = 1$ $\quad K = \bar{K}$

$\mathrm{Cat}(G)$: monoidal category

- objects : group elements
- morphisms : identities
- $\otimes$: group multiplication

Def^{n} An action of G on $\mathfrak{D}$ is an additive monoidal functor: $\phi : \mathrm{Cat}(G) \to \mathrm{Aut}(\mathfrak{D})$

i.e.. $\phi(g) =: \phi_g$

$\cdot\ \forall g, h$, $\varepsilon_{g,h} : \phi_g \phi_h \xrightarrow{\sim} \phi_{hg}$

EX ① $G \curvearrowright X$, $\phi_g = g^* : D^b(X) \to D^b(X)$,

$g^* h^* \xrightarrow{\sim} (hg)^*$

Recall $\mathrm{Coh}_G(X)$: G-equivariant sheaves,

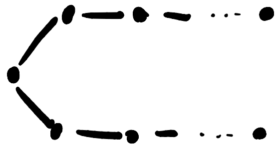
objects: $(E, \{\lambda_g\}_G)$ $\quad E \in \mathrm{Coh}(X)$

$\lambda_g : E \xrightarrow{\sim} g^* E$

$G \curvearrowright \mathfrak{D} \rightsquigarrow \mathfrak{D}_G$: category of G-equivariant objects

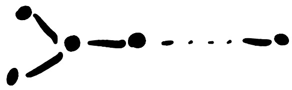
EX ① $(D^b(X))_G = D^b(\mathrm{Coh}_G(X)) \simeq D^b([X/G])$

if G acts freely, $\simeq D^b(X/G)$

② A_{2n+1} 

[Demonet '10]

$\mathrm{rep}(A_{2n+1})_{\mathbb{Z}/2\mathbb{Z}} \simeq \mathrm{rep}(D_{n+2})$



$\mathcal{D}_G$ has a natural $\mathrm{rep}(G)$ "action"

Def$^{\underline{n}}$ An action of $\mathrm{rep}(G)$ on $\mathcal{D}$ is an additive monoidal functor $\mathrm{rep}(G) \longrightarrow \mathrm{End}(\mathcal{D})$

$$\rho \longmapsto -\otimes\rho$$

given by $(E,\{\lambda_g\})\otimes\rho = (E\otimes V, \{\lambda_g\otimes\rho_g\})$

$\rho = (V, \{\rho_g\}_{g\in G})$

NOW: $\mathcal{D}$ , $G \curvearrowright \mathcal{D}$ s.t. ϕ_g all exact

$\Rightarrow G \curvearrowright \mathrm{Stab}(\mathcal{D})$ via $\phi_g\cdot(\mathcal{A},Z) = (\phi_g(\mathcal{A}), Z\circ\phi_g)$

Thm B [Polischchuk '07, Macrì–Mehrotra–Stellari '09, DHL '23]

There is an analytic isomorphism

$$(\mathrm{Stab}(\mathcal{D}))^G \cong \mathrm{Stab}_{\mathrm{rep}(G)}(\mathcal{D}_G)$$

$(\mathrm{Stab}(\mathcal{D}))^G = \{\phi_g\cdot\sigma=\sigma \ \forall g\in G\}$

$\mathrm{Stab}_{\mathrm{rep}(G)}(\mathcal{D}_G) = \left\{\sigma=(\mathcal{A},Z) : \begin{array}{l}\bullet\ \mathcal{A}\otimes\rho\subseteq\mathcal{A}\\ \bullet\ Z(E\otimes\rho)=\dim\rho\cdot Z(E)\ \forall\rho\end{array}\right\}$

Rmk

G abelian $\Rightarrow \hat{G} = \mathrm{Hom}(G,k^*) \curvearrowright \mathcal{D}_G$

$\Rightarrow \mathrm{Stab}_{\mathrm{rep}(G)}(\mathcal{D}_G) = (\mathrm{Stab}(\mathcal{D}_G))^{\hat{G}}$

Now $X \xrightarrow{\pi} Y = X/G$ F.Q.

$(*)$: $\left(\mathrm{Stab}(X)\right)^G \simeq \left(\mathrm{Stab}_{\mathrm{rep}(G)}(Y)\right)$

$\sigma_X \longmapsto \sigma_Y$

where: • $E \in D^b(Y)$ σ_Y-semistable $\iff$ $\pi^*(E)$ σ_X-semistable

• $\pi_* \mathcal{O}_X = \bigoplus_{\rho \in \mathrm{irr}(G)} \underbrace{E_\rho}_{\text{vector bundles}}^{\otimes \dim \rho}$

• $\mathrm{rep}(G) \curvearrowright D^b(Y)$ by $- \otimes E_\rho$

LEMMA $(*)$ preserves geometric stability conditions

Th$^{\underline{m}}$ C [DMC] X surface, alb_X finite, $Y = X/G$ F.Q.

Then $\left(\mathrm{Stab}_{\mathrm{rep}(G)}(Y)\right) = \mathrm{Stab}^{\mathrm{Geo}}(Y)$

& this is a connected component of $\mathrm{Stab}(Y)$

APPLICATIONS: **EX**: $C_i \subset \mathbb{P}^2$: $x^5 + y^5 + z^5 = 0$

$X = C_1 \times C_2 \xrightarrow[\text{finite}]{\mathrm{alb}_X} J(C_1) \times J(C_2)$

$G = (\mathbb{Z}/5\mathbb{Z})^2$, $Y = X/G$ Beauville surface

$h^1(\mathcal{O}_Y) = 0 \Rightarrow \mathrm{alb}_Y$ not finite

This generalises: • $g(C_i) > 1$: Beauville-type surfaces $h^1(\mathcal{O}_Y) = 0$

• $g(C_i) = 1$: bielliptic surfaces, $h^1(\mathcal{O}_Y) = 1$

By Th^{m}C either $\begin{cases} \mathrm{Stab}(Y) = \mathrm{Stab}^{\mathrm{Geo}}(Y) \text{ so (A2) no} \\ \mathrm{Stab}(Y) \text{ disconnected} \end{cases}$

Main ingredient to prove Thm C

CLAIM $\mathrm{Stab}_{\mathrm{rep}(G)}(Y)$ open.

STEP 1

under the equivalence $\Psi : D^b(Y) \xrightarrow{\sim} D^b(X)_G$

$$\pi_* \mathcal{O}_X \mapsto (\mathcal{O}_X, \mathrm{id}) \otimes \mathbb{C}[G]$$

STEP 2 WTS $\mathrm{ch}(E_\rho) = (\dim \rho, 0 \cdots 0)$

use:

$$\begin{aligned}\Psi(\pi_* \mathcal{O}_X \otimes E_\rho) &= (\mathcal{O}_X, \mathrm{id}) \otimes \mathbb{C}[G] \otimes \rho \\ &\cong (\mathcal{O}_X, \mathrm{id}) \otimes \mathbb{C}[G]^{\oplus \dim \rho} \\ &= \dim \rho \cdot \Psi(\pi_* \mathcal{O}_X)\end{aligned}$$

$\Rightarrow$ In $K_0(Y)$, $[F \otimes E_\rho] = \dim \rho\, [F]$ $] + (\mathrm{I}) \Rightarrow$ **CLAIM**

$\Rightarrow Z$ always $\mathrm{rep}(G)$ - invariant

$= (ii)$

defo. prop. for $\mathrm{Stab}_{\mathcal{C}}(\mathcal{D})$