Geometric stability conditions & group actions

Notes: hannahdell.me/cimpa2025

§ 0 (Short) motivation & overview

The Bridgeland stability machine:

Geometry $\rightsquigarrow$ Triangulated Category $\xrightarrow{\mathrm{Stab}(-)}$ Complex Manifold

$X \mapsto D^b\mathrm{Coh}X$

Algebra $\rightsquigarrow$ Triangulated Category

$A \mapsto D^b\mathrm{Mod}A$

(!) Bridgeland Conj: $\mathrm{Stab}(X)$ connected & contractible

(Q1) How does the geometry of $\mathrm{Stab}(X)$ relate to X?

(Q2) How do invariants (of X or ✱) behave under (finite) group actions?

GOAL OF THESE LECTURES: answer both by studying "free quotients" and see lots of examples along the way!

PLAN
1: Geometric stability conditions on surfaces] independent, some overlap with XZ
2: Equivariant categories
3: Equivariant stability conditions] some quivers!
4: Free quotients (put together 1–3 to answer Q1 & Q2)
5: Further applications & questions

§ 1 Geometric stability conditions on surfaces

– some overlap with Cristian L2

"Long motivation / survey"

SETUP X : smooth projective surface / $\mathbb{C}$

($K_0(X)$: Grothendieck group $= \dfrac{\mathbb{Z}\cdot\langle [E] : E \in \mathrm{Coh}(X)\rangle}{[A]+[C]=[B] \Leftrightarrow 0\to A\to B\to C\to 0}$

Chern character $\mathrm{ch}: K_{num}(X) \to \mathbb{Z}\oplus NS(X)\oplus\mathbb{Z}$ iso over $\mathbb{Q}$ $\leftarrow CH^*_{num}(X)$)

$NS(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ Neron – Severi group

$NS_{\mathbb{R}}(X) := NS(X)\otimes\mathbb{R}$

$\cup$

$\mathrm{Amp}_{\mathbb{R}}(X)$: ample cone (ie $H \in NS_{\mathbb{R}}(X)$ s.t. $H^2>0$)

Last week we saw:

Def^n Let $0 \neq E \in \mathrm{Coh}(X)$ and $H \in \mathrm{Amp}_{\mathbb{R}}(X)$.

(1) The H-slope of E is $\mu_H(E) := \begin{cases} +\infty & \mathrm{ch}_0(E)=0 \\ \dfrac{H\cdot \mathrm{ch}_1(E)}{H^2\,\mathrm{ch}_0(E)} & \text{otherwise} \end{cases}$

[INTERSECTION PRODUCT: $A, B \in NS_{\mathbb{R}}(X) \Rightarrow A\cdot B \in CH^0(X)$, $A\cdot B := \deg(A\cdot B)$]

(2) E is H-(semi)stable if

$0 \neq F \subsetneq E \Rightarrow \mu_H(F) \underset{(=)}{<} \mu_H(E/F)$

$\mu_H(F)<\mu_H(E)$ if E tors-free

In Crishna's lectures, we learnt that all line bundles are H-stable, and in general their direct sums are not (only if degrees equal)

EX $\mathrm{ch}_0(E) = 0 \Rightarrow E$ is H-semistable
If E is also simple (i.e. no subobjects) then E is H-stable

e.g. $E = \mathcal{O}_x \ \forall x \in X$. Note $\mathrm{ch}(\mathcal{O}_x) = (0,0,1)$

Problem: $Z_H = -H\cdot \mathrm{ch}_1 + iH^2\mathrm{ch}_0$ is not a stability function for $\mathrm{Coh}(X)$ since $Z_H(\mathcal{O}_x) = 0$. In fact, $\mathrm{Coh}(X)$ does not admit a stability function that factors through ch. [Toda]

Same for any tors sheaf with dim supp $T = 0$

Solution: Change the heart!

Def^n $\sigma = (\mathcal{A}, Z)$ is a (numerical) B.S.C. for $D^b(X)$
(i) $\mathcal{A} \subset D^b(X)$ heart of a bdd t-structure
i.e. abelian & "generates"
(ii) $Z : K_0(X) \to \mathbb{C}$ homomorphism s.t.
- $0 \neq E \in \mathcal{A} \Rightarrow Z(E) \in \mathbb{H}$
- HN filtrations exist
- factors via $K_{num}(X)$

(iii) support property

Fix
- $H \in \mathrm{Amp}_{\mathbb{R}}(X)$
- $\beta \in \mathbb{R}$

Define $\mathcal{T}_{H,\beta} := \{ E \in \mathrm{Coh}(X) : E \twoheadrightarrow Q \neq 0 \Rightarrow \mu_H(Q) > \beta \}$

$\mathcal{F}_{H,\beta} := \{ E \in \mathrm{Coh}(X) : 0 \neq F \subsetneq E \Rightarrow \mu_H(F) \leq \beta \}$

[Q: why is "$\mu_H(E) > \beta$" not enough : let $0 \to A \to E \to B \to 0$ be HN filtr. (wrt μ) so $\mu(A) > \mu(B)$. Suppose $\mu(E) > \beta > \mu(B)$
But $\mathcal{T} = \{F : \mu(F) > \beta\} \ni E$, $\mathcal{F} = \{F : \mu(F) \leq \beta\} \ni B$ & $\mathrm{Hom}(E,B) \neq 0 \Rightarrow (\mathcal{T}, \mathcal{F})$ not a torsion pair]

EXERCISE: This is a torsion pair on $\mathrm{Coh}(X)$ HINT: use HN filtration

Def^n/Prop^n
$$\mathrm{Coh}^{H,\beta}(X) := \left\{ E \in D^b(X) : \begin{array}{l} \mathcal{H}^0(E) \in \mathcal{T}_{H,\beta} \\ \mathcal{H}^{-1}(E) \in \mathcal{F}_{H,\beta} \\ \mathcal{H}^i(E) = 0 \text{ otherwise} \end{array} \right\} = \langle \mathcal{F}_{H,\beta}[1], \mathcal{T}_{H,\beta} \rangle$$
is a heart of a bounded t-structure on $D^b(X)$.

EX $[E^{-1} \xrightarrow{d} E^0] \in \mathrm{Coh}^{H,\beta}(X)$ if $\ker d \in \mathcal{F}_{H,\beta}$ and $\mathrm{coker}\, d \in \mathcal{T}_{H,\beta}$

Fix
- $D \in NS_{\mathbb{R}}(X)$
- $\alpha \in \mathbb{R}$

Define $Z_{H,D,\alpha,\beta} := (\alpha + i\beta) H^2 \mathrm{ch}_0 + (D + iH)\cdot \mathrm{ch}_1 - \mathrm{ch}_2$

⚠ $E \in D^b(X) \Rightarrow \mathrm{ch}(E) = \sum_i (-1)^i \mathrm{ch}(\mathcal{H}^i(E))$

[$\rightsquigarrow$ new slope $\mu_Z := \dfrac{-\mathrm{Re}\, Z}{\mathrm{Im}\, Z}$ (object in ♡ is stable if any sub has < slope)]

Th^m 1 [Bridgeland '08, Arcara–Bertram '13, Macrì–Schmidt '17] nice proof!

For $\alpha >> 0$, $\mathbb{C} \cdot \{ \sigma_{H,D,\alpha,\beta} = (\mathrm{Coh}^{H,\beta}(X), Z_{H,D,\alpha,\beta}) \}$
$\alpha > \frac{1}{2}\left[\left(\beta - \frac{H\cdot B}{H^2}\right)^2 - \frac{B^2}{H^2}\right]$
is a continuous family of Bridgeland stability conditions.
(as H, D, α, β vary)

(Q3) Is this all of $\mathrm{Stab}(X)$?

Lemma $\mathcal{O}_x$ is $\sigma_{H,D,\alpha,\beta}$-stable $\forall x \in X$.

Proof • $\mu_H(\mathcal{O}_x) = +\infty$ (+ no other quotients $\mathcal{O}_x \twoheadrightarrow Q$)
$\Rightarrow \mathcal{O}_x \in \mathcal{T}_{H,\beta}$

• EXERCISE (proof in notes) $\mathcal{O}_x$ is simple ⚠ (= no subobjects) in $\mathrm{Coh}^{H,\beta}(X)$

Suppose $0 \to A \to \mathcal{O}_x \to B \to 0$ is a s.e.s. in $\mathrm{Coh}^{H,\beta}(X)$.

Then taking cohomology we find $H^{-1}(A) = 0$ and:

$$(*) \quad 0 = H^{-1}(\mathcal{O}_x) \to \underset{\mathcal{F}}{\underset{\cap}{H^{-1}(B)}} \xrightarrow{f} \underset{\mathcal{T}}{\underset{\cap}{H^0(A)}} \to \underset{\mathcal{T}}{\underset{\cap}{\mathcal{O}_x}} \to \underset{\mathcal{T}}{\underset{\cap}{H^0(B)}} \to 0$$

$Q := \mathrm{coker} f$ is a subsheaf of $\mathcal{O}_x$, so either

(i) $Q = 0 \Rightarrow H^{-1}(B) \simeq H^0(A) \Rightarrow$ both $= 0$ $(\mathcal{T} \cap \mathcal{F} = \{0\})$

$\Rightarrow A = 0$ and $B \simeq \mathcal{O}_x$.

(ii) $Q = \mathcal{O}_x$. Now consider $0 \to H^{-1}(B) \to H^0(A) \to \mathcal{O}_x \to 0$.

$\mathrm{ch}(\mathcal{O}_x) = (0,0,1)$ & ch_i additive $\Rightarrow \mu_H(\underset{\mathcal{F} \Rightarrow \mu_H \le \beta}{\underset{\cap}{H^{-1}(B)}}) = \mu_H(\underset{\mathcal{T} \Rightarrow \mu_H > \beta}{\underset{\cap}{H^0(A)}})$.

$\Rightarrow H^{-1}(B) = 0 = H^0(A) \Rightarrow$ CLAIM □

Def^n X sm. proj. variety. $\sigma \in \mathrm{Stab}(X)$ is *geometric* if $\forall x \in X$, $\mathcal{O}_x$ is σ-stable. Write $\mathrm{Stab}^{geo}(X)$ for the subset of geometric stability conditions.

Now we can refine (Q3) (on a surface):

(Q3.1) Does Th^m 1 describe all geometric stability conditions? (on surfaces)

(Q3.2) Do there exist nongeometric stability conditions? (on surfaces)

§1.1 The Le Potier function and (Q3.1) (skipped!)

Def^n X surface, $(H,D) \in \mathrm{Amp}_{\mathbb{R}}(X) \times NS_{\mathbb{R}}(X)$. The *Le Potier function twisted by D*, $\Phi_{X,H,D} : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is:

$$\Phi_{X,H,D}(x) := \cancel{\max}\ \limsup_{\mu \to x} \left\{ \underbrace{\frac{\mathrm{ch}_2(E) - D.\mathrm{ch}_1(E)}{H^2 \mathrm{ch}_0(E)}}_{\nu_{H,D}} : \begin{array}{l} E \text{ is } H\text{-semistable} \\ \text{with } \mu_H(E) = \cancel{x}\ \mu \end{array} \right\}$$

SLOGAN: Fix $\mu_H(E) = \mu$, how big can this get?

Th^m (Bogomolov–Gieseker inequality) X surface, $E \in \mathrm{Coh}(X)$

H-semistable and torsion free $\Rightarrow$ $ch_1(E)^2 - 2\,ch_0(E)\,ch_2(E) \geq 0$

To ease notation, let $D=0$

Corollary $\Phi_{X,H,0}(x) \leq \frac{x^2}{2}$

Proof **exercise!** HINT use the following consequence of the Hodge Index Th^m

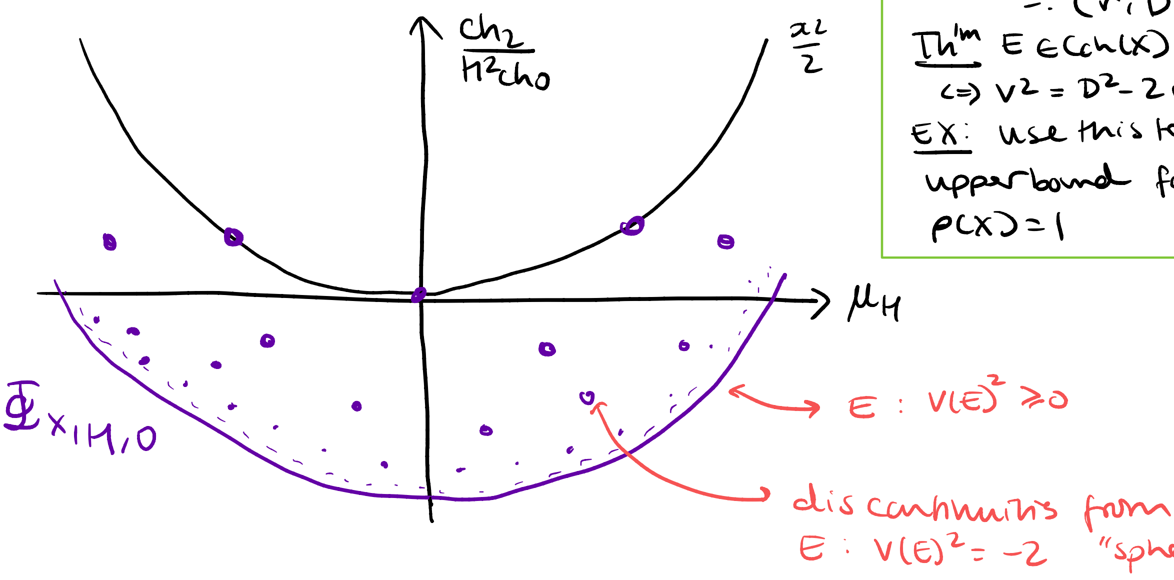
$\left[\text{Th}^{m}\ (H,D) \in \mathrm{Amp}_{\mathbb{R}}(X) \times NS_{\mathbb{R}}(X) \Rightarrow H^2 \cdot D^2 \leq (H \cdot D)^2\right]$

EX ① X abelian surface

[Mukai] $\forall\, \mu \in \mathbb{Q}\ \exists$ H-semistable vector bundle E_μ with $\mu_H(E_\mu) = \mu$ and $ch(E_\mu) = ch_0(E_\mu)\cdot(1, \mu H, \frac{1}{2}\mu^2 H^2)$

These attain the upperbound, i.e. $\Phi_{X,H,0}(x) = \frac{x^2}{2}$

② X K3 surface



EXERCISE:
$v(E) = (ch_0, ch_1, ch_0 + ch_2)$
$=: (r, D, s)$
Th^m $E \in \mathrm{Coh}(X)$ H-s.s.
$\Leftrightarrow v^2 = D^2 - 2rs \geq -2$
EX: use this to find an upperbound for $\Phi_{X,H,0}$ when $\rho(X)=1$

Th^m [Lie Fu – Chunyi Li – Xiaolei Zhao '21, $\rho(X)=1$; D '23] X surface

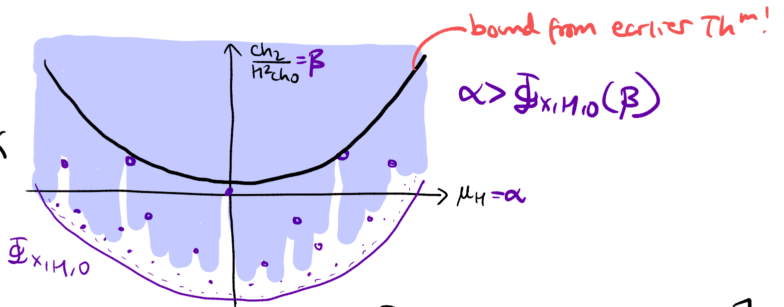
$$\mathrm{Stab}^{geo}(X) \simeq \mathbb{C} \times \left\{(H, D, \alpha, \beta) : \Phi_{X,H,D}(\alpha) > \beta\right\}$$
$$\cap$$
$$\mathbb{C} \times \mathrm{Amp}_{\mathbb{R}}(X) \times NS_{\mathbb{R}}(X) \times \mathbb{R}^2$$

$\sigma_{H,D,\alpha,\beta} \longleftarrow (H, D, \alpha, \beta)$

In particular, $\mathrm{Stab}^{geo}(X)$ is connected.

Th^m [Rekuski '23] X surface $\Rightarrow$ $\mathrm{Stab}^{geo}(X)$ contractible.

EX X K3, $\rho(X)=1$,

$\mathrm{Stab}^{geo}(X) \simeq \widetilde{GL_2^+(\mathbb{R})} \times$ 

IN FACT, $\mathrm{Stab}(X)$ is the universal cover of this [Bayer–Bridgeland]

So indeed X has nongeometric stability conditions!

Rmk The above theorem can be used to investigate Q3.2 by studying

i.e. study the boundary $\overline{\mathrm{Stab}^{geo}(X)} \setminus \mathrm{Stab}^{geo}(X)$.

(Q3.2) Do there exist nongeometric stability conditions (in general)?

- $\dim X = 1$: no except for $\mathbb{P}^1$
- $\dim X \geq 3$: little known, yes if X has a full exceptional collection
- $\dim X = 2$:

(A3.2) no for ① abelian surfaces

IN FACT: Thm 1 gives everything $\Rightarrow$ $\mathrm{Stab}(X)$ connected & contractible

(A3.2) yes for ② $\mathbb{P}^2$, e.g. using quiver representations! (F.E.C.)
③ K3 surfaces

[Bridgeland '08]:

In ③, there are two (known) phenomena for nongeometric stability conditions

(i) $\exists$ $\sigma = (\mathcal{A}, Z) \in \mathrm{Stab}(X)$ s.t. $\boxed{\text{all } \mathcal{O}_x \text{ are } \sigma\text{-unstable}}$

i.e. $\mathcal{O}_x \in \mathcal{A}$ and $\exists$ $A^{\oplus r} \hookrightarrow \mathcal{O}_x$ in $\mathcal{A}$ s.t. $\mu_Z(E) > \mu_Z(\mathcal{O}_x)$

[A is a rank r spherical vector bundle, i.e. $\mathrm{Hom}^i_X(E,E) = \begin{cases} \mathbb{C} & i=0,2 \\ 0 & \text{o/w} \end{cases}$]

Similar construction with exceptional bundles on rational surfaces!

(ii) if $X \supset C$ rational curve s.t. $C^2 < 0$

$\exists$ $\sigma = (\mathcal{A}, Z)$ s.t. $\boxed{\mathcal{O}_x \text{ is } \sigma\text{-unstable} \quad \forall x \in C}$

This uses: $0 \to \mathcal{O}_C \to \mathcal{O}_x \to \mathcal{O}_C(-1)[1] \to 0$ in $\mathcal{A}$

Same construction works $\forall$ $X \supset C \simeq \mathbb{P}^1$ s.t. $C^2 < 0$

(We might look at more details of this in Lecture 3)

TAKEAWAY: $\mathrm{Stab}(X)$ sees something about how ①–④ are different!

But what is it precisely? Here's the first general answer.

Thm 2 [Lie Fu – Chunyi Li – Xiaolei Zhao '21] (any dim!)

X sm. proj. variety / $\mathbb{C}$

X has finite Albanese morphism, alb_X $\Rightarrow$ $\mathrm{Stab}(X) = \mathrm{Stab}^{geo}(X)$

Remark (Albanese morphisms)

think: $(\mathbb{C}^\times)^n$

* Any variety X has an associated abelian variety, called the Albanese variety $\mathrm{Alb}(X) := \mathrm{Pic}^0(\mathrm{Pic}^0(X))$ (!) also $\exists$ $\mathbb{C}$-AG-def!

deg 0 line bdles (not just conn. comp.)

and $\mathrm{alb}_X : X \longrightarrow \mathrm{Alb}(X)$

s.t. $\forall$

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "\mathrm{alb}_X"'] & A \\ & \mathrm{Alb}(X) \arrow[u, dashed, "\exists"'] \end{tikzcd}$$

A abelian variety, $\exists$ (unique up to translation)

e.g. C : curve, $C \xrightarrow{\mathrm{alb}_C} J(C) \simeq \mathrm{Alb}(C)$ Abel–Jacobi map

$x \longmapsto \mathcal{O}(x - x_0)$

* $\dim \mathrm{Alb}(X) = h^1(X, \mathcal{O}_X)$

e.g. C curve, $h^1(C, \mathcal{O}_C) = g$, and alb_C is $\begin{cases} \text{trivial} & g = 0 \\ \text{finite} & g > 0 \end{cases}$

* alb_X finite $\Leftrightarrow$ $\exists$ finite morphism $X \longrightarrow A$ abelian variety

SLOGAN: X is close to being abelian

Proof sketch (5–10 mins)

- let $\sigma \in \mathrm{Stab}(X)$
- let $E_1, \dots, E_k$ be the Jordan–Hölder factors of $\mathcal{O}_x$ wrt σ

ASIDE: JH FACTORS: (w.r.t. $\sigma \in \mathrm{Stab}(\mathcal{D})$)

EXERCISE: $P(\phi)$ are abelian & finite length = Noetherian & Artinian

HINT: show support property $\Rightarrow$ [$E \in P(\phi)$ has only finitely many classes of subobjects in $P(\phi)$]

Hence $A \in P(\phi) \Rightarrow \exists$ filtration $0 = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = A$ in $P(\phi)$
s.t. each quotient $B_i := A_{i+1}/A_i$ is stable "JH factor"

(!) this filtration may not be unique (exercise!)
but the JH factors are unique up to permutation

JH factors of $E \in \mathcal{D}$:= {JH factors of HN factors of E}
By construction, they are all stable!

* $\mathcal{L} \in \mathrm{Pic}^0(X)^+$ $\Rightarrow$ [Polishchuk* '07] $-\otimes\mathcal{L}$ does not change stability

* original paper contains a mistake

$\mathrm{Pic}^0(X)$ = connected component containing the identity

IN PARTIC. HN filtration & JH factors preserved

$\Rightarrow E_i \otimes \mathcal{L} \simeq E_i$ ⚠ $\mathrm{Hom}(E_i, E_j) = 0$ for $i \neq j$ ∵ simple

* $E_i \otimes \mathrm{alb}^*(\mathcal{L}) \simeq E_i$ $\overset{\text{proj. formula}}{\Rightarrow}$ $\mathrm{alb}_*(E_i) \otimes \mathcal{L} \simeq \mathrm{alb}_*(E_i)$

($\mathrm{alb} := \mathrm{alb}_X$)

[Polishchuk '03] ⊛

$\Rightarrow \mathrm{alb}_*(E_i)$ finite support

⊛ A abelian variety $\Rightarrow F \in D^b(A)$ "rarely invariant under $-\otimes\mathcal{L}$":
$\forall \xi \in A^\vee \rightsquigarrow \mathcal{L}_\xi$ associated line bundle on A

Th^m [Polishchuk '03] $F \otimes \mathcal{L}_\xi \simeq F \ \forall \xi \Rightarrow F$ has finite support

alb finite

$\Rightarrow E_i$ finite support

CLAIM

$\Rightarrow \mathcal{O}_x$ is σ-stable $\square$

Proof of CLAIM:

Assume $\mathcal{O}_p$ not stable $\Rightarrow \exists$ stable $E \neq \mathcal{O}_p$ supported at p.

* $k = \max\{i : \mathcal{H}^i(E) \neq 0\}$
* $l = \min\{ \quad '' \quad \}$

$$E \xrightarrow{\text{can}} \mathcal{H}^k(E)[-k] \xrightarrow{i_1} \mathcal{O}_p[-k] \rightarrow \mathcal{H}^l(E)[-k] \xrightarrow{\text{can}} E[l-k]$$

* "can" = canonical maps from cohomology filtration
* may assume $i_1, i_2 \neq 0$ (∵ $\mathcal{H}^k, \mathcal{H}^l$ are sheaves supported at p) $\Rightarrow i_2 \circ i_1 \neq 0$
* composition is $\neq 0$

E stable, $k \geq l$, $\mathrm{Hom}(E, E[l-k]) \neq 0 \Rightarrow k = l$

$\Rightarrow E = \mathcal{H}^k(E)[-k] = \mathcal{O}_p[-k]$ $\rightarrow\leftarrow$ $\square$

(Q4) alb_X not finite $\Rightarrow \exists$ nongeometric stability conditions ?

This will be our focus for the rest of the course!

* True for $\mathbb{P}^n$, K3, ...

IDEA Investigate examples which arise as free quotients

ie

$$\begin{array}{c} G \overset{\text{free}}{\curvearrowright} X \\ \downarrow \\ Y = X/G \end{array}$$

G finite

alb_X finite

alb_Y not finite

eg. Beauville-type and bielliptic surfaces.

Aim of §2 & §3

Compare $\mathrm{Stab}(X)$ and $\mathrm{Stab}(X/G)$.