LAST TIME:

(Q5) alb_X not finite $\Rightarrow$ $\exists$ nongeometric stability conditions?

This will be our focus for the rest of the course!

IDEA Investigate examples which arise as free quotients

ie

$$\begin{array}{c} \underset{\text{finite}}{G} \overset{\text{free}}{\curvearrowright} X \\ \downarrow \\ Y = X/G \end{array}$$

alb_X finite

alb_Y not

eg. Beauville-type and bielliptic surfaces.

> **Aim of §2 & §3**
>
> Compare $\mathrm{Stab}(X)$ and $\mathrm{Stab}(X/G)$.

§2 Equivariant categories

Finite group actions on categories

G : finite group

$\mathcal{D}$: k-linear category, where $k = \bar{k}$ and $(|G|, \mathrm{char}\, k) = 1$.

i.e. Hom spaces are k-vector spaces $\Rightarrow$ additive

Def^n (Group action) [Deligne '97]
("strict")

A (right) action of G on $\mathcal{D}$ (write $G \curvearrowright \mathcal{D}$) is the data of

1. $\forall g \in G$, a functor $\phi_g : \mathcal{D} \to \mathcal{D}$
2. $\forall g, h \in G$, a natural isomorphism $\varepsilon_{g,h} : \phi_g \circ \phi_h \xrightarrow{\sim} \phi_{hg}$

such that

$$\begin{tikzcd} \phi_f \circ \phi_g \circ \phi_h \arrow[r, "\varepsilon_{g,h}"] \arrow[d, "\varepsilon_{f,g}"'] \arrow[dr, phantom, "\copyright"] & \phi_f \circ \phi_{hg} \arrow[d, "\varepsilon_{f,hg}"] \\ \phi_{gf} \circ \phi_h \arrow[r, "\varepsilon_{gf,h}"] & \phi_{hgf} \end{tikzcd}$$

commutes $\forall f, g, h \in G$.

Remark This is more than a group homomorphism

$G \to \mathrm{Aut}(\mathcal{D})$ — group of iso classes of autoequivalences. Indeed given $G \to \mathrm{Aut}(\mathcal{D})$, $\forall E \in \mathcal{D}$,
(+ associativity as above)

$\phi_g \circ \phi_h(E) \simeq \phi_{gh}(E)$, but may not come from a natural isomorphism

of functors $\phi_g \circ \phi_h \Rightarrow \phi_{gh}$. Over $\mathbb{C}$, the obstruction lives in $H^3(G, \mathbb{C}^*)$

see [Beckmann–Oberdieck: On equivariant derived categories].

EX ① Let X be a scheme. Suppose $G \leq \mathrm{Aut}(X)$ (write $G \curvearrowright X$)

$\forall g \in G$ define $\phi_g := g^* : \mathrm{Coh}(X) \xrightarrow{\sim} \mathrm{Coh}(X)$.

Then $\forall g, h$ there are canonical isomorphisms:

$$\phi_g \circ \phi_h = g^* \circ h^* \xrightarrow{\sim} (hg)^* = \phi_{hg} \quad \text{(property of pullbacks)}$$

This also lifts to an action on $D^b(X)$.

Running example
- $Y = X/\mathbb{Z}/2\mathbb{Z}$: Enriques surface, i.e.
- X K3 surface
- $G = \mathbb{Z}/2\mathbb{Z} = \langle \iota \rangle$ where $\iota : X \xrightarrow{\sim} X$ involution

e.g. $\iota^*(\mathcal{O}_x) = \mathcal{O}_{\iota^{-1}(x)} \quad \forall x \in X$

② Let Q be an acyclic quiver. $G \leq \underbrace{\mathrm{Aut}^+(Q)}$ automorphisms of the underlying graph that preserve orientation

Then $G \curvearrowright \mathrm{Rep}\, Q$ via "reordering":

$$(M_i, \varphi_\alpha) \mapsto (M_{g(i)}, \varphi_{g(\alpha)})$$

e.g.

$$A_{2n+1} \quad \begin{tikzcd} & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \\ \bullet \arrow[ur] \arrow[dr] & & & \\ & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \end{tikzcd} \qquad G = \mathbb{Z}/2\mathbb{Z} \curvearrowright \mathrm{Rep}(A_{2n+1})$$

(with the two branches of length n)

$$\begin{tikzcd}[column sep=small, row sep=small] & k \arrow[r] & 0 \cdots & 0 \\ k \arrow[ur] \arrow[dr] & & & \\ & 0 \arrow[r] & 0 \cdots & 0 \end{tikzcd} \mapsto \begin{tikzcd}[column sep=small, row sep=small] & 0 \arrow[r] & 0 \cdots \arrow[r] & 0 \\ k \arrow[ur] \arrow[dr] & & & \\ & k \arrow[r] & 0 \cdots \arrow[r] & 0 \end{tikzcd} \qquad \left(\begin{tikzcd}[column sep=small, row sep=small] & k \\ k \arrow[ur] \arrow[dr] & \\ & k \end{tikzcd} \text{ invariant} \right)$$

Equivariantisation

upshot of Deligne's definition: we can build...

Def^n Suppose $G \curvearrowright \mathcal{D}$. A G-equivariant object is a pair $(E, \{\lambda_g\}_{g \in G})$ where

- $E \in \mathcal{D}$
- $\lambda_g : E \xrightarrow{\sim} \phi_g(E)$ choice of isomorphism $\forall g \in G$

s.t.

$$\begin{tikzcd} E \arrow[r, "\lambda_g"] \arrow[d] \arrow[dr, phantom, "\copyright"] & \phi_g(E) \arrow[d, "\phi_h(\lambda_h)"] \\ \phi_{hg}(E) & \phi_g(\phi_h(E)) \arrow[l, "\varepsilon_{g,h}"'] \end{tikzcd}$$

(strongly) commutes $\forall g, h \in G$. ⊛

"more than G-invariant"

A morphism of G-equivariant objects $(E, \{\lambda_g\}) \to (E', \{\lambda'_g\})$ is $F \in \mathrm{Hom}_{\mathcal{D}}(E, E')$ s.t.

$$\begin{tikzcd} E \arrow[r, "\lambda_g"] \arrow[d, "F"'] \arrow[dr, phantom, "\copyright"] & \phi_g(E) \arrow[d, "\phi_g(F)"] \\ E' \arrow[r, "\lambda'_g"] & \phi_g(E') \end{tikzcd}$$

commutes $\forall g \in G$

"G-invariant morphisms"

Together these form a category $\mathcal{D}_G$ called the (sometimes $\mathcal{D}^G$) G-equivariant category, or G-equivariantisation.

Remark Note this uses the isomorphisms $\varepsilon_{g,h}$ [all morphisms in (#) are isos, but we want equality when we compose]

EX (1) $G \curvearrowright X$, G-equivariant objects are G-equivariant coherent sheaves, $(\mathrm{Coh}(X))_G =: \mathrm{Coh}_G(X)$.

In fact, $\mathrm{Coh}_G(X) \simeq \mathrm{Coh}(\underbrace{[X/G]}_{\text{quotient stack!}})$

Also $(D^b(X))_G \simeq D^b(\mathrm{Coh}_G(X)) =: D^b_G(X)$.

If G acts freely, $\pi : X \to X/G$ quotient ($\Rightarrow$ principal bundle)

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "\pi"'] & \copyright & X \arrow[dl, "\pi"] \\ & X/G & \end{tikzcd}$$

then $\mathrm{Coh}(X/G) \simeq \mathrm{Coh}_G(X)$ via:

$$E \longmapsto (\pi^* E, \{\lambda_g\}_G)$$

where $\lambda_g : \pi^* E \xrightarrow{\copyright} (\pi \circ g)^* E \overset{\text{pullback}}{=} g^*(\pi^* E)$ so $\pi^* E$ has a natural equivariant structure

Running example

- $Y = X / \mathbb{Z}/2\mathbb{Z}$: Enriques surface
- X K3 surface
- $G = \mathbb{Z}/2\mathbb{Z} = \langle i \rangle$ where $i : X \xrightarrow{\sim} X$ involution

e.g. $i^*(\mathcal{O}_x) = \mathcal{O}_{i^{-1}(x)} \quad \forall x \in X$

- $\pi : X \to Y = X/G$

$y \in Y$, $\pi^{-1}(y) = \{x, x' = i^{-1}(x)\}$

Then $\pi^*(\mathcal{O}_y) = \mathcal{O}_x \oplus \mathcal{O}_{x'}$.

What is $\lambda_g : \pi^*(\mathcal{O}_y) \to g^*(\pi^*(\mathcal{O}_y))$?

$$\lambda_{id} : \begin{matrix} \mathcal{O}_x & \xrightarrow{id} & \mathcal{O}_x \\ \oplus & & \oplus \\ \mathcal{O}_{x'} & \xrightarrow{id} & \mathcal{O}_{x'} \end{matrix} \qquad \lambda_i : \begin{matrix} \mathcal{O}_x & \xrightarrow{i^*} & \mathcal{O}_{x'} \\ \oplus & & \oplus \\ \mathcal{O}_{x'} & \xrightarrow{i^*} & \mathcal{O}_x \end{matrix}$$

Rmk In general $(D^b(X))_G \simeq D^b(\mathrm{Coh}_G(X)) =: D^b_G(X)$.

Let Y be a "nice resolution" of X/G. Then sometimes $D^b([X/G]) \simeq D^b(Y)$! See e.g. [Bridgeland–King–Reid]

(2) **Th^{m}** [Demonet '10] $G \leq \mathrm{Aut}^+(Q)$ $\Rightarrow \exists$ quiver Q_G s.t.

quiver

$$\mathrm{Rep}(Q_G) \simeq \mathrm{Rep}(Q)_G$$

e.g.

$$A_{2n+1} \qquad \begin{tikzcd}[row sep=small] & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \\ \bullet \arrow[ur] \arrow[dr] & & & \\ & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \end{tikzcd} \qquad G = \mathbb{Z}/2\mathbb{Z} \curvearrowright \mathrm{Rep}(A_{2n+1})$$

$$\mathrm{Rep}(A_{2n+1})_{\mathbb{Z}/2\mathbb{Z}} \simeq \mathrm{Rep}(\underbrace{D_{n+2}}_{Q_G})$$

$$D_{n+2} \qquad \begin{tikzcd}[row sep=small] \bullet \arrow[dr] & & & \\ & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \\ \bullet \arrow[ur] & & & \end{tikzcd}$$

n

SLOGAN: stabilizer splits

(3) G: abelian. Then $\hat{G} := \mathrm{Hom}(G, k^*)$ acts on $\mathcal{D}_G$

by: $$\underset{\hat{G}}{\phi_{\chi}}((E, \{\lambda_g\}_G)) := (E, \{\lambda_g\}_G) \otimes \chi$$
$$= (E, \{\lambda_g \cdot \underbrace{\chi(g)}_{\in k^*}\}_G)$$

(and ϕ_χ is the identity on morphisms, and $\varepsilon_{\chi,\psi} = \mathrm{id}$)

Running example
- $Y = X/\mathbb{Z}/2\mathbb{Z}$: Enriques surface
- X K3 surface
- $G = \mathbb{Z}/2\mathbb{Z} = \langle i \rangle$ where $i: X \xrightarrow{\sim} X$ involution

$$\hat{G} \curvearrowright D^b_G(X), \quad \hat{G} = \mathbb{Z}/2\mathbb{Z} = \{1, -1\}$$

e.g.
$$\phi_{-1}\left(\pi^*(\mathcal{O}_y), \lambda_{id}: \begin{matrix} \mathcal{O}_x \xrightarrow{id} \mathcal{O}_x \\ \oplus \qquad \oplus \\ \mathcal{O}_{x'} \xrightarrow{id} \mathcal{O}_{x'} \end{matrix}, \lambda_i: \begin{matrix} \mathcal{O}_x \xrightarrow{i^*} \mathcal{O}_{x'} \\ \oplus \qquad \oplus \\ \mathcal{O}_{x'} \xrightarrow{i^*} \mathcal{O}_x \end{matrix}\right) = \left(\pi^*(\mathcal{O}_y), \lambda_{id}: \begin{matrix} \mathcal{O}_x \xrightarrow{id} \mathcal{O}_x \\ \oplus \qquad \oplus \\ \mathcal{O}_{x'} \xrightarrow{id} \mathcal{O}_{x'} \end{matrix}, \lambda_i: \begin{matrix} \mathcal{O}_x \xrightarrow{-i^*} \mathcal{O}_{x'} \\ \oplus \qquad \oplus \\ \mathcal{O}_{x'} \xrightarrow{-i^*} \mathcal{O}_x \end{matrix}\right)$$

Exercise: check this is an isomorphism of equivariant objects

Since $D^b_G(X) \simeq D^b(Y)$, this gives an action $\hat{G} \curvearrowright D^b(Y)$

IN FACT under this equivalence, $\phi_{-1} = - \otimes \underbrace{\omega_Y}_{\text{2-torsion line bdle!}}$

(Skipped)

e.g. $G \curvearrowright pt$ trivially. $\chi_0, \dots, \chi_r$ irreps of G

CLAIM $D^b([pt/G]) = \langle D^b(pt) \otimes \chi_0, \dots, D^b(pt) \otimes \chi_r \rangle$, where $[pt/G]$ is BG

completely orthogonal decomposition

This holds more generally for trivial actions on categories, see [Kuznetsov–Perry '17]

⚠ There are few examples of actions on categories that do not arise from $G \curvearrowright X$. Example ③ gives one!

Th'm [Elagin '15]. G abelian $\curvearrowright \mathcal{D}$. Then

$$(\mathcal{D}_G)_{\hat{G}} \simeq \mathcal{D}$$

GOAL OF §3: $\mathcal{D} =$ , compare $\mathrm{Stab}(\mathcal{D})$ & $\mathrm{Stab}(\mathcal{D}_G)$, and hence $\mathrm{Stab}(X)$ & $\mathrm{Stab}(X/G)$

⚠ This will be the most technical section, o.k. to skip to §4.

EXTRA: Going between $\mathcal{D}$ and $\mathcal{D}_G$ (we may discuss in lecture 3/4)

There are two natural functors:

Def^n Suppose $G \curvearrowright \mathcal{D}$, define

- (forgetful functor) $\mathrm{Forg}_G : \mathcal{D}_G \to \mathcal{D}$ by $(F, \{\lambda_g\}_G) \mapsto F$
- (inflation functor) $\mathrm{Inf}_G : \mathcal{D} \to \mathcal{D}_G$, where

$$\mathrm{Inf}_G(E) := \Big(\bigoplus_G \phi_g(E), \{\lambda_g^{nat}\}_G \Big)$$

and λ_g^{nat} is the composition:

$$\bigoplus_{h \in G} \phi_h(E) \xrightarrow[\sim]{\text{reorder}} \bigoplus_{h \in G} \phi_{hg}(E) \xrightarrow[\sim]{\bigoplus_{h \in G} \varepsilon_{g,h}^{-1}} \bigoplus_{g \in G} \phi_g \phi_h(F) \overset{!}{=} \phi_g\Big(\bigoplus_{h \in G} \phi_h(F)\Big)$$

(!) ϕ_g additive functor

Exercise Forg_G is faithful and Inf_G is the left and right adjoint. ⚠ $\mathrm{Forg}_G \circ \mathrm{Inf}_G(E) = \bigoplus_{g \in G} \phi_g(E)$

EX ① $G \curvearrowright X$ free, $\pi : X \to X/G$

Consider the (derived) functors

$$\begin{tikzcd} D^b(X) \arrow[r, shift left, "\pi_*"] \arrow[d, shift right, "\mathrm{Inf}_G"'] & D^b(X/G) \arrow[l, shift left, "\pi^*"] \arrow[dl, bend left, "\sim"', "\Psi"] \\ D^b_G(X) \arrow[u, shift right, "\mathrm{Forg}_G"'] & \end{tikzcd}$$

CLAIM $\mathrm{Inf}_G \simeq \Psi \circ \pi_*$ and $\mathrm{Forg}_G \simeq \pi^* \circ \Psi^{-1}$

e.g. $x \in X$, $\mathrm{Forg}_G \circ \mathrm{Inf}_G(\mathcal{O}_x) = \bigoplus_{g \in G} \mathcal{O}_{g^{-1}x} = \pi^* \circ \pi_*(\mathcal{O}_x)$

Th^m [Elagin '15] G abelian, $G \curvearrowright \mathcal{D}$ (so $\hat{G} \curvearrowright \mathcal{D}_G$). Then there is an equivalence $\Omega : (\mathcal{D}_G)_{\hat{G}} \simeq \mathcal{D}$

and $\mathrm{Forg}_G \simeq \Omega \circ \mathrm{Inf}_{\hat{G}}$, $\mathrm{Inf}_G \simeq \mathrm{Forg}_{\hat{G}} \circ \Omega^{-1}$

(the proof uses monads)

Q What happens when G is not abelian?

SLOGAN: $\hat{G}$ is too small, so we need a replacement!

e.g. G alternating group $\Rightarrow \hat{G} = \phi$

First, let's reformulate the definition of a group action

Def^n $\mathrm{Cat}(G)$ is a monoidal category such that:

- objects : group elements
- morphisms : identities
- $- \otimes -$: group multiplication

Def^n (Group action version 2) An action of G on $\mathcal{D}$ is an additive monoidal functor $\phi : \mathrm{Cat}(G) \to \mathrm{Aut}(\mathcal{D})$.

EXERCISE This is equivalent to Deligne's definition.

(Exercise in definition of monoidal categories)

When G is abelian, all irreducible representations are 1-dim. IN FACT: $\mathrm{Cat}(\widehat{G}) = \mathrm{irr}(G)$

This suggests we want something like "$\mathrm{irr}(G) \curvearrowright \mathcal{D}_G$"

IN FACT $\mathcal{D}_G$ has a natural "$\mathrm{Rep}(G)$-action"...

Def^n An action of $\mathrm{rep}(G)$ on $\mathcal{C}$ is an additive monoidal functor $\mathrm{Rep}(G) \to \mathrm{End}(\mathcal{C})$

monoidal cat. of endofunctors (may not be invertible)

$$\rho \longmapsto - \otimes \rho$$

... given by:

$$(E, \{\lambda g\}_G) \otimes \rho := (E \otimes V, \{\lambda_g \otimes \rho(g)\}_G)$$

$$\rho = (V, \{\rho_g\}_{g\in G})$$

vector space (V), linear maps (ρ_g)

Remark ① If $\dim(\rho) = 1$ (i.e. $\dim V = 1$), this is exactly the action of $\widehat{G}$ on $\mathcal{D}_G$ we saw in Ex ③ earlier!

② $\mathcal{D}$ can also be recovered from $\mathcal{D}_G$ using the $\mathrm{Rep}(G)$ action, [Drinfeld – Gelaki – Nikshych – Ostrik '10] but (as far as I know) there is no version for triangulated categories in the literature.

③ In the above definition, we can replace $\mathrm{Rep}(G)$ with any fusion category "=" monoidal category + strong finiteness assumptions