Interlude: Questions & exercises

⁎ Question / Clarification "$Z: K_0(\mathcal{D}) \to \Lambda \to \mathbb{C}$" in def. of BSC.
↳ what is Λ for $\mathcal{D} = D^b(X)$, X surface?

First some definitions:

- $K_0(\mathcal{D}) := \dfrac{\text{free group generated by ob}\,\mathcal{D}}{A \to B \to C \to A[1] \iff [B] = [A] + [C]}$ "Grothendieck group"

- Euler pairing on $K_0(\mathcal{D})$: (need $\mathcal{D}$: Ext-finite)

$$\chi(E,F) := \sum_i (-1)^i \dim \big(\underbrace{\mathrm{Ext}^i(E,F)}_{=\mathrm{Hom}_{\mathcal{D}}(E,\,F[i])}\big)$$

- $K_{num}(\mathcal{D}) := K_0(\mathcal{D}) / \underbrace{\mathrm{null}(\chi)}_{\{E \,:\, \chi(E,F)=0 \ \forall F \in \mathcal{D}\}}$ "numerical Grothendieck group"

  a B.S.C. $\sigma = (\mathcal{A}, Z)$ is numerical if Z factors via $K_{num}(\mathcal{D})$.

- Now fix $v: K_0(\mathcal{D}) \to \Lambda$.

  $\mathrm{Stab}_\Lambda(\mathcal{D}) :=$ all (numerical) stability conditions $\sigma = (\mathcal{A}, Z)$ s.t.
  or "$\mathrm{Stab}_{(\Lambda,v)}(\mathcal{D})$"
  $Z: K_0(\mathcal{D}) \xrightarrow{v} \Lambda \to \mathbb{C}$ (factors via v)

  and σ satisfies support property wrt Λ

  > $\exists$ quadratic form Q on $\Lambda \otimes \mathbb{R}$ s.t.
  > (a) $\ker Z \otimes \mathbb{R}$ is negative definite wrt Q
  > (b) $\forall$ σ-semistable object E
  > $Q(v(E)) \geq 0$

Now let X be a variety

$\mathrm{ch}: K_0(X) \longrightarrow CH^*(X)$ — ch: chern character; $CH^*(X)$: Chow group

⚠ $K_0(X)$ can be huge! e.g. $x, x' \in X$ not rationally equivalent
$\Rightarrow [\mathcal{O}_x] \neq [\mathcal{O}_{x'}]$

IN PARTIC., E elliptic curve $\Rightarrow K_0(E) \simeq \mathbb{Z} \oplus E$

But in $K_{num}(X)$, $[\mathcal{O}_x] = [\mathcal{O}_{x'}]$ always!

Consider

$$\begin{tikzcd} K_0(X) \arrow[r, "ch"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & CH^*(X) \arrow[d] \\ K_{num}(X) \arrow[r, "ch"] & CH^*_{num}(X) \end{tikzcd}$$

numerical Chow group

On a surface, $K_{num}(X) \xrightarrow[\sim]{ch} CH^*_{num}(X)$

$$CH^*_{num}(X) \cong \mathbb{Z} \oplus NS(X) \oplus \tfrac{1}{2}\mathbb{Z}$$

$$\ni (ch_0, ch_1, ch_2)$$

But we want a map to $\mathbb{C}$! and

$$K_{num}(X) \to \Lambda \to \mathbb{C}$$

Λ ← lattice

(!) $\forall D \in NS(X)_{\mathbb{R}}$, $D = c \cdot D'$, $D' \in NS(X)$ $c \in \mathbb{R}$

$$NS(X) \longrightarrow c \cdot \mathbb{Z}$$

$$B \longmapsto \deg(D \cdot B) = c \deg(\underbrace{D' \cdot B}))$$

in $CH^*(X)$ this is a sum of point classes, deg = count with multiplicity.

This works $\forall D \in \underline{NS(X)_{\mathbb{R}}}$

↑ rank $= \rho(X)$ Picard rk.

ie. choose an $\mathbb{R}$-basis $D_1, \dots, D_{\rho(X)}$ $(D_i = c_i D_i, D_i \in NS(X))$

$$NS(X) \longrightarrow (c_i \mathbb{Z})^{\oplus \dim \rho(X)}$$

$$B \longmapsto (D_1 \cdot B, \dots, D_{\rho(X)} \cdot B)$$

So let $\Lambda = \mathbb{Z} \oplus \mathbb{Z}^{\oplus \rho(X)} \oplus \frac{1}{2}\mathbb{Z}$. (for X: surface)

Remark To classify all geometric stability conditions, we look at possible $\Lambda \to \mathbb{C}$

* Exercise from lecture 1:

Lemma $\mathcal{O}_x$ is $\sigma_{H,D,\alpha,\beta}$-stable $\forall x \in X$. surface

Proof • $\mu_H(\mathcal{O}_x) = +\infty$ (+ no other quotients $\mathcal{O}_x \twoheadrightarrow Q$)
$\Rightarrow \mathcal{O}_x \in \mathcal{T}_{H,\beta}$

• **EXERCISE** $\mathcal{O}_x$ is simple $\triangle$ (= no subobjects) in $\mathrm{Coh}^{H,\beta}(X)$

Suppose $0 \to A \to \mathcal{O}_x \to B \to 0$ is a s.e.s. in $\mathrm{Coh}^{H,\beta}(X)$.
wrt. standard $\heartsuit$, i.e. $\mathrm{Coh}(X)$

Then taking cohomology we find $H^{-1}(A) = 0$ and:

$$(*) \quad 0 = H^{-1}(\mathcal{O}_x) \to \underset{\mathcal{F}_{H,\beta}}{\underset{\uparrow}{H^{-1}(B)}} \xrightarrow{f} \underset{\mathcal{T}_{H,\beta}}{\underset{\uparrow}{H^0(A)}} \to \underset{\mathcal{T}_{H,\beta}}{\underset{\uparrow}{\mathcal{O}_x}} \to \underset{\mathcal{T}_{H,\beta}}{\underset{\uparrow}{H^0(B)}} \to 0$$

This is exact in $\mathrm{Coh}(X)$

$Q := \operatorname{coker} f$ is a subsheaf of $\mathcal{O}_x$, so either

(i) $Q = 0 \Rightarrow H^{-1}(B) \simeq H^0(A) \Rightarrow$ both $= 0$ [$\because \mathrm{Hom}(\mathcal{T},\mathcal{F}) = 0$]

$\Rightarrow A = 0$ and $B \simeq \mathcal{O}_x$.

(ii) $Q = \mathcal{O}_x$. Now consider $0 \to H^{-1}(B) \to H^0(A) \to \mathcal{O}_x \to 0$.

$\mathrm{ch}(\mathcal{O}_x) = (0,0,1)$ & ch is additive $\Rightarrow \underset{\Rightarrow \mu_H \le \beta}{\underset{\uparrow \mathcal{F}}{\mu_H(H^{-1}(B))}} = \underset{\Rightarrow \mu_H > \beta}{\underset{\uparrow \mathcal{T}}{\mu_H(H^0(A))}}$.

$\Rightarrow H^{-1}(B) = 0 = H^0(A) \Rightarrow$ CLAIM $\square$

* Recap of
• def of $\mathcal{D}_G$

• Enriques surface example:

Running example
• $Y = X/\mathbb{Z}/2\mathbb{Z}$: Enriques surface
• X K3 surface
• $G = \mathbb{Z}/2\mathbb{Z} = \langle \iota \rangle$ where $\iota : X \xrightarrow{\sim} X$ involution

$\hat{G} \curvearrowright D^b_G(X)$, $\hat{G} = \mathbb{Z}/2\mathbb{Z} = \{1, -1\}$

Since $D^b_G(X) \simeq D^b(Y)$, this gives an action $\hat{G} \curvearrowright D^b(Y)$

CLAIM This is the action given by
• $\phi_1 = \mathrm{id}$
• $\phi_{-1} := - \otimes \underbrace{\omega_Y}_{\text{2-torsion line bdle!}}$

$\triangle$ in $\mathrm{Knum}(Y)$, $[\omega_Y] = \underset{0}{\underset{\parallel}{[\mathcal{O}_Y]}}$

EXERCISE Look at equivariant structure on $\pi^*\omega_Y$.