Counting Conics and Clemens' Conjecture

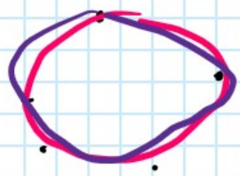
Friday, 5 February 2021 09:31

REFERENCE: Sheldon Katz – Enumerative Geometry & String Theory

The study of the number of solutions to geometric questions

⚠ $k = \mathbb{C}$ $(k = \bar{k})$

PROBLEM 1 How many nondegenerate conics pass through 5 points in general position in $\mathbb{P}^2$?



$$p_1, \dots, p_5 \in \mathbb{P}^2$$

$$C \subseteq \mathbb{P}^2 : \quad C = V(F)$$

$$F(X, Y, Z) = aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ$$

$$C \longleftrightarrow (a, b, c, d, e, f) \in \mathbb{C}^6$$

$$F = 0 \iff \lambda F = 0 \ \forall \lambda \in \mathbb{C}^*$$

$$C \longleftrightarrow [a:b:c:d:e:f] \in \mathbb{P}^5_{\mathbb{C}}$$

$\mathbb{P}^5_{\mathbb{C}}$ is a moduli space for conics in $\mathbb{P}^2$

$$p_1 \in C \iff F(p_1) = 0 \qquad p_1 = [q_1 : q_2 : q_3]$$

$$q_1^2 a + q_2^2 b + q_3^2 c + \dots = 0$$

LINEAR IN $(a, b, c, \dots, f)$

$$\rightsquigarrow H_1 \subseteq \mathbb{P}^5_{\mathbb{C}}$$

$P_1 \in C \iff [a:\dots:f] \in H_1$, so on

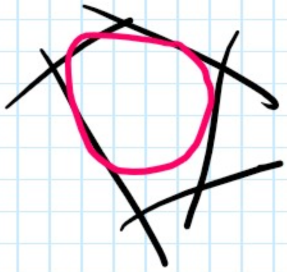
Bézout's Theorem Let $H_1, \dots, H_n$ be hypersurfaces in $\mathbb{P}^n$ with degrees $d_1, \dots, d_n$ respectively. If $H_1 \cap \dots \cap H_n$ is finite, then:

$$\#(H_1 \cap \dots \cap H_n) = d_1 \cdot \dots \cdot d_n.$$

$$\underbrace{H_1 \cap H_2 \cap \dots \cap H_5}_{\dim 0} \subseteq \mathbb{P}^5$$

$$\#(H_1 \cap \dots \cap H_5) = 1^5 = 1 \quad \underline{\text{conic}}$$

PROBLEM 2 How many nondegenerate conics are tangent to 5 lines in general position in $\mathbb{P}^2$?



<u>METHOD 1</u>

$$C \longleftrightarrow [a:\dots:f] \in \mathbb{P}^5_{\mathbb{C}}$$

WLOG: $L: (z=0) \subset \mathbb{P}^2$

$$L \cap C : \quad 0 = F(x, y, 0)$$

$$= ax^2 + by^2 + dxy$$

dehomogenise

$$0 = ax^2 + dx + b$$

L tangent to $C \iff d^2 - 4ab = 0$

$$Q = V(d^2 - 4ab) \subset \mathbb{P}^5$$

degree 2

Tangent to 5 lines $\longleftrightarrow$ $\underbrace{Q_1 \cap \dots \cap Q_5}_{\dim 0} \subseteq \mathbb{P}^5$

$$\#(Q_1 \cap \dots \cap Q_5) = 2^5 = \underline{32}$$

METHOD 2 : Projective duality

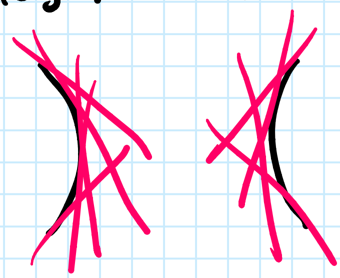
$$\mathbb{P}^2 \qquad (\mathbb{P}^2)^\vee$$

points $\longleftrightarrow$ lines

$$[p_1 : p_2 : p_3] \longleftrightarrow (p_1 X + p_2 Y + p_3 Z = 0) \subseteq \mathbb{P}^2$$

"$(\text{PROBLEM})^\vee$ = PROBLEM"

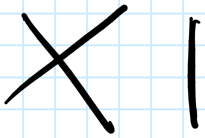
conics? $\longleftrightarrow$ conics



$(\text{PROBLEM 2})^\vee$ = PROBLEM

$\#$ conics $= 1$

⚠ Nondegeneracy.

 $\subseteq \mathbb{P}^5$

"31 extra"

EXCESS INTERSECTION

HOW TO GET ROUND THIS?

(a) Find a better moduli space

(b) Count the excess intersection contribution.

Counting Lines

05 February 2021 09:53

CLEMENS' CONJECTURE (1984) Let $X \subseteq \mathbb{P}^4$ be a quintic threefold. Then $\forall$ integer $d \geq 1$, $\exists$ finitely many irreducible rational curves of degree d on X.

$X \subseteq \mathbb{P}^4 : X = V(F)$, $\deg F = 5$

$d = 1$

Q: How many lines lie on X?

$L \subset \mathbb{P}^n$

Q_L = space of homogeneous linear polys on L

$L : (x_2 = \cdots x_n = 0)$, Q_L has basis: $\{x_0, x_1\}$.

$\therefore$ Q_L has dim $\underline{2}$

Lines in $\mathbb{P}^n \longleftrightarrow$ elements $[L] \in Gr(2, n+1)$

$$Gr(2,n+1) = \{\text{2-dim linear subspaces of } \mathbb{C}^{n+1}\} = \{\text{lines in } \mathbb{P}^n\}$$

$[L] \in Gr(2, n+1)$, we have Q_L

These fit together to form a rank 2 vector bundle, Q, on $Gr(2, n+1)$

section of Q: $\quad s(L) = x_0|_L$

Then $s(L) = 0 \iff L \subset (x_0 = 0) =: Z_0$

Cohomology of $Gr(2,n+1)$

Schubert cycles $\sigma_{a,b}$

$H \subseteq \mathbb{P}^n$ $\quad \sigma_{1,1}(H) = \{ [\ell] \in Gr(2,n+1) : \ell \subseteq H \}$

$\sigma_{1,1}$

$\mathbb{P}^3$: $\quad L_1, L_2 \quad p = L_1 \cap L_2$

ℓ intersect L_1 & L_2 $\iff$ $p \in \ell$ or $\ell \subset \langle L_1, L_2 \rangle$

$$\sigma_1^2 = \sigma_2 + \sigma_{1,1}$$

$s(\ell) = 0 \iff \ell \subset Z_0$

$\sigma_{1,1}(Z(x_0))$

$\sigma_{1,1} \in H^4(Gr(2,n+1))$

Chern classes:

X compact complex manifold

L line bundle

$s \neq 0$ section

$$Z(s) := \{ x \in X : s(x) = 0 \}$$

$\forall\, s, s' \neq 0 :$ $\quad [Z(s)] = [Z(s')] \in H^2(X)$

$$c_1(L) := [Z(s)]$$

⚠ $c_1(L \otimes L') = c_1(L) + c_1(L')$

$$c_1(L^{\otimes d}) = d\, c_1(L)$$

more generally:

E rank r bundle on X

$$c_r(E) = [Z(s)] \in H^{2r}(X)$$

$$c(E) = 1 + c_1(E) + \cdots + c_r(E)$$

⚠ $c(L_1 \oplus \cdots \oplus L_r) = (1 + c_1(L_1)) \cdots (1 + c_1(L_r))$

$$c_r(L_1 \oplus \cdots \oplus L_r) = c_1(L_1) \cdots c_1(L_r)$$

Q - rank 2 on $G = G(2, n+1)$

$$c_2(Q) = [Z(s)] \in H^4(G)$$

$$= \sigma_{1,1}$$

TURNS OUT: $c_1(Q) = \sigma_1$

$$c(Q) = 1 + \sigma_1 + \sigma_{1,1}$$

Back to $X = V(F) \subseteq \mathbb{P}^4$

$$l \subseteq X \iff F|_l = 0$$

F is in fact a section of $\mathrm{Sym}^5(Q)$

e.g.: $l = (x_2 = x_3 = x_4 = 0)$,

fibre of this at l has basis:

$\{x_0^5, x_0^4 x_1, \ldots, x_1^5\}$

$\therefore$ $\underline{\dim 6}$

$\therefore$ $Sym^5(Q)$ vector bundle on $Gr(2,5)$

$\underline{\text{rank } 6}$

$$\ell \subset X \iff c_6(\underbrace{Sym^5(Q)}_{[Z(s)]}) \in H^{12}(G)$$

$\underbrace{F|_\ell = 0}$

$\ell \subset Z(s)$

$\dim G(2,5) = 6$

real dim $\underline{12}$

$$n_1 = \int_{G(2,5)} c_6(Sym^5(Q))$$

$\int_{\mathbb{P}^n} H^n = 1$ $\qquad$ $\int : H^n(X) \to \mathbb{Z}$

HOW TO CALCULATE:

SPLITTING PRINCIPLE: $\quad$ "$Q = L_1 \oplus L_2$"

$$c(Q) = c(L_1) \cdot c(L_2)$$

Let $\quad \alpha = c_1(L_1) \quad \beta = c_1(L_2)$

$$\begin{aligned} c(Q) &= (1 + c_1(L_1))(1 + c_1(L_2)) \\ &= (1+\alpha)(1+\beta) \\ &= 1 + (\alpha+\beta) + \alpha\beta \end{aligned}$$

$$c(Q) = 1 + \sigma_1 + \sigma_{1,1}$$

$$\alpha\beta = \sigma_{1,1} \qquad \alpha + \beta = \sigma_1$$

$$\underset{(x+y)^5}{Sym^5(L_1 \oplus L_2)} \cong L_1^{\otimes 5} \oplus (L_1^{\otimes 4} \otimes L_2) \oplus \cdots \oplus L_2^{\otimes 5}$$

$$c(L_1 \oplus \cdots \oplus L_r) = c(L_1) \cdots c(L_r)$$

$$c(\mathrm{Sym}^5(Q)) = (1+5\alpha)(1+4\alpha+\beta)\cdots(1+5\beta)$$

$$\alpha\beta = \sigma_{1,1} \quad \cdots$$

$$= 2875\,\sigma_{3,3}$$

$$\int_{G(2,5)} \sigma_{3,3} = 1$$

$$n_1 = \int_{G(2,5)} 2875\,\sigma_{3,3} = 2875$$

Clemens' Conjecture

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CLEMENS' CONJECTURE (1984) Let $X \subseteq \mathbb{P}^4$ be a quintic threefold. Then $\forall$ integer $d \geq 1$, $\exists$ finitely many irreducible rational curves of degree d on X.

$$N_d := \int_{\overline{M}(\mathbb{P}^4, d)} c_{5d+1}(\overbrace{E_5}^{\text{rank } 5d+1})$$

stable maps of degree d.

$\hookrightarrow \in \mathbb{Q}$

$\hookrightarrow$ Gromov–Witten Invariant

$\quad\quad ||$

Topological string amplitude

$\hookrightarrow$ counts rational curves of degree d **PLUS** extra stuff

e.g. $d=2$

Theorem $\forall$ line $L \cong \mathbb{P}^1$ on X, the component $\overline{M}(\mathbb{P}^1, d)$ contributes $\frac{1}{d^3}$ to N_d

ie. 2875 lines on X

$\rightsquigarrow$ degree d multicovers of these lines contribute $\frac{2875}{d^3}$ to N_d

SIM degree k multicovers of smooth rational curves of degree $\frac{d}{k}$ $\rightsquigarrow$ $\frac{1}{k^3}$ to N_d

infinitesimally rigid

$\underline{n_d}$

$$N_d = \sum_{k \mid d} \frac{n_{d/k}}{k^3}$$

$n_1 = N_1 \quad \in \mathbb{Q}$

$n_2 = N_2 - \frac{N_1}{2^3} \quad \in \mathbb{Q}$

$\vdots$

$n_d \in \mathbb{Q}$

<u>CONJ</u>: (Gopakumar–Vafa) $n_d \in \mathbb{Z}$

?n_d?

1991: Candelas, de la Ossa, Green, & Parkes

X is C.Y. 3-fold

$X \longleftrightarrow X^{0}$ mirror manifold

$F_0 = 5 + \sum_{d=1}^{\infty} \underbrace{N_d}\, q^d$ Yukawa Coupling.

$\hookrightarrow n_d$

$d \leq 9 \quad n_d = n'_d$

PROBLEM 3 How many conics are tangent to 5 conics in general position in $\mathbb{P}^2$?

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