Stability Manifolds of Varieties with Finite Albanese Morphisms

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§0: MOTIVATION

- $\mathrm{Stab}(X)$ = set of stability conditions on $D^b(X)$.

Th^m (Bridgeland '07): $\mathrm{Stab}(X)$ has the natural structure of a complex manifold

- Complete description only known in a few cases:
  - $\mathrm{Stab}(\mathbb{P}^1) \cong \mathbb{C}^2$
  - $\mathrm{Stab}(C) \cong \mathbb{C} \times \mathbb{H}$ where C is a smooth projective curve with $g \geq 1$.
  - Abelian surfaces
- Partial description: we know a principal connected component for:
  - K3 surfaces with $\rho = 1$
  - $\mathbb{P}^2$
  - Abelian 3-folds with $\rho = 1$

EXPECT: This will be the whole stability manifold.

- In each case, $\mathrm{Stab}(X)$ (or its known principal component) is: simply connected and contains an open subset of all geometric stability conditions.

Def^n A stability condition on $D^b(X)$ is geometric if all skyscraper sheaves are stable with the same phase.

§1: Background

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Def$^{\underline{n}}$ A slicing $\mathcal{P}$ on $D^b(X)$ is a collection of full additive subcategories $\mathcal{P}(\phi) \subset D^b(X)$ for each $\phi \in \mathbb{R}$, such that:

(1) $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi+1)$

(2) $F_1 \in \mathcal{P}(\phi_1), F_2 \in \mathcal{P}(\phi_2), \phi_1 > \phi_2 \Rightarrow \mathrm{Hom}(F_1, F_2) = 0$

(3) $\forall E \in D^b(X)$ there is a Harder-Narasimhan filtration:

i.e. real numbers $\phi_1 > \dots > \phi_m$, objects $E_i \in D^b(X)$, and a collection of distinguished triangles:

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{m-1} \arrow[r] & E_m \arrow[d] \\
& A_1 \arrow[ul, dashed] & A_2 \arrow[ul, dashed] & & & A_m \arrow[ul, dashed]
\end{tikzcd}$$

where $A_i \in \mathcal{P}(\phi_i)$ $\forall i$.

$$\phi^+_{\mathcal{P}}(E) := \phi_1, \quad \phi^-_{\mathcal{P}}(E) := \phi_m$$

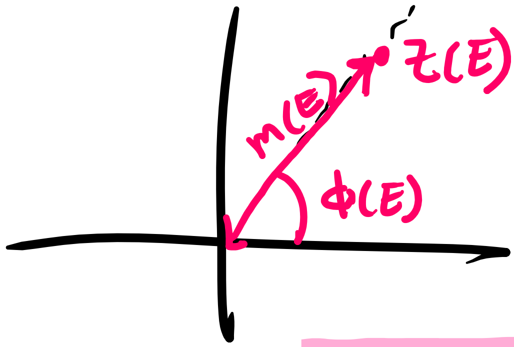
- Non-zero objects of $\mathcal{P}(\phi)$ are called: **semistable** of **phase ϕ**.
- Non-zero <u>simple</u> objects of $\mathcal{P}(\phi)$ are called: **stable** of **phase ϕ**.

Def^{n} A stability condition on $D^b(X)$ is a pair $\sigma = (P, Z)$ such that:

- P is a slicing
- $Z : K(X) \to \mathbb{C}$ homomorphism such that:

$$\forall E \neq 0,\ E \in P(\phi) \Rightarrow Z([E]) = m(E)e^{i\pi\phi}$$

$m(E) \in \mathbb{R}_{>0}$



- Z factors via a finite dim. lattice Λ:

$$Z : K(X) \xrightarrow{\lambda} \Lambda \to \mathbb{C}$$

- σ satisfies the support property

$$\inf \left\{ \frac{|Z(E)|}{\underbrace{\|[E]\|}_{\|\cdot\| \text{ is a fixed norm on } \Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}.}} : E \text{ is semistable} \right\} > 0$$

EQUIVALENTLY:

$$\sigma = (\underbrace{\overbrace{\mathcal{A}}^{\text{heart of a bdd. t-structure on } D^b(X)}, Z_{\mathcal{A}}}_{\text{stability function on } \mathcal{A}.})$$

This induces a slicing $\mathcal{P}$ where

$$\mathcal{A} = \mathcal{P}((0,1])$$

= extension closure of all $E \in D^b(X)$ s.t. $E \in \mathcal{P}(\phi)$, $\phi \in (0,1]$

$$E \to F \to G \to E[1]$$

Def^n A stability condition σ is **numerical** if λ factors via $K_{num}(X)$, i.e.

$$\begin{tikzcd} Z : K(X) \arrow[r] \arrow[rr, bend left, "\lambda"] & \underbrace{K_{num}(X)}_{\text{the numerical Grothendieck group of } X} \arrow[r] & \Lambda \arrow[r] & \mathbb{C} \end{tikzcd}$$

§2: Geometric Stability Conditions

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Prop^{n} 2.9 Let X be a smooth projective variety. Let σ be a numerical stability condition on $D^b(X)$ s.t. every skyscraper sheaf is σ-stable. Then all skyscraper sheaves are of the same phase, i.e. σ is geometric.

(We omit the proof, it is quite technical but fairly standard)

LEMMA 2.12 Let A be an abelian variety. Let $E \in D^b(A)$. Suppose $E \otimes L \simeq E \quad \forall L \in \mathrm{Pic}^0(A)$ $(= A^\vee)$, then E has finite support.

[Polischuk '03, 11.8]

Th^{m} 2.13 X sm. proj. variety such that its Albanese morphism is finite. Then skyscraper sheaves are all stable w.r.t. any numerical stability condition on $D^b(X)$

Recall

Albanese variety of X

$$a: X \longrightarrow \mathrm{Alb}(X)$$

Albanese morphism

abelian variety

s.t. every morphism $f: X \to A$ factors via a.

→ **Proof**

- Let $a : X \to \mathrm{Alb}(X)$ be the Albanese morphism of X.
- Fix a stability condition σ on X.
- Let $p \in X$ be a closed point.
- Let A_i be the HN factors of $\mathcal{O}_p$.
- Let E_i be the Jordan-Hölder factors in the HN filtration of $\mathcal{O}_p$

ie.
$$\begin{tikzcd} 0=F_0 \arrow[r] & F_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & F_{n-1} \arrow[r] & F_n = F \, "=" \, \mathcal{O}_p \arrow[d] \\ & A_1 \arrow[ul, dashed] & & & A_n \arrow[ul, dashed] \end{tikzcd}$$

Every $A_i \in P(\phi_i)$ has a filtration by stable objects B_{ij} of $P(\phi)$. These B_{ij} are called the Jordan-Hölder factors of F.

[Pol. '07, 3.5.2]: $\mathrm{Pic}^0(X) \subset \mathrm{Aut}(D^b(X))$ acts trivially on $\mathrm{Stab}(X)$.

Thus:

E_i & A_i are preserved by the action of $\mathrm{Pic}^0(X)$.

$\therefore\ \forall\, \mathcal{L} \in \mathrm{Pic}^0(X),\quad E_i \otimes \mathcal{L} \simeq E_i$

MOREOVER:

Let L be a degree 0 line bundle on $\mathrm{Alb}(X)$.

Then $\forall i$, $E_i \otimes \underbrace{\alpha^*(L)}_{\in \mathrm{Pic}^0(X)} \simeq E_i$

Projection Formula: $R\alpha_*(E_i) \otimes L \simeq R\alpha_*(E_i)$

$\overset{2.12}{\Longrightarrow}$ $R\alpha_*(E_i)$ has finite support.

α finite $\Rightarrow$ E_i has finite support.

Remains to prove:

LEMMA 2.14 Let X be a smooth projective variety. Let σ be a stability condition on $D^b(X)$.
Suppose the J.H. factors of $\mathcal{O}_p$ have finite support.
Then $\mathcal{O}_p$ is stable.

→ **Proof** Suppose $\mathcal{O}_p$ is not stable.
$\therefore \exists$ stable object $E \neq \mathcal{O}_p$ supported at p.
Let k, l: max. & min. non-vanishing cohomol. degrees of E.
Then we have maps:

$E \twoheadrightarrow \mathcal{O}_p$ ($\exists$ map whenever $p \in \mathrm{Supp}(E)$)

$\mathcal{O}_p \hookrightarrow E$ (E finite length $\therefore$ has filtration into simple length 1 sheaves. These can only be $\mathcal{O}_p$ itself)

This induces the following composition:

$$E \xrightarrow{can} \underbrace{H^k(E)[-k]}_{\text{supported at } p} \xrightarrow{i_1} \mathcal{O}_p[-k] \xrightarrow{i_2} H^l(E)[-k] \xrightarrow{can} E[l-k]$$

supported at p $\therefore$ we may let i_1 & $i_2 \neq 0$

$\therefore i_2 \circ i_1 \neq 0$

$\neq 0$: it induces a non-zero map from the term of E with maximal cohomological degree to the term of $E[l-k]$ with minimal cohomological degree.

i.e. $E \to E[\overbrace{l-k}^{\leq 0}]$ $\quad l-k \leq 0$

$\phi(E) \geq \phi(E[l-k]) = \phi(E) + (l-k)$

$P(\phi)[1] = P(\phi+1)$

$\Rightarrow l = k.$

& hence $E = H^k(E)[-k] = \mathcal{O}_p[-k]$

$\therefore \mathcal{O}_p$ is stable $\rightarrow\leftarrow$. $\blacksquare$

Corollary 2.15 Let X be a connected smooth projective variety with finite Albanese morphism. Then all numerical stability conditions on $D^b(X)$ are geometric.

→ Proof 2.9 + 2.13 $\blacksquare$

We will use this later to describe $\mathrm{Stab}(X)$.

APPLICATION OF 2.15:

Corollary 2.17 Let A be an abelian variety. Then all simple semi-homogeneous vector bundles are stable w.r.t. any numerical stability condition on $D^b(\hat{A})$

IDEA:

$$D^b(M) \xrightarrow{\text{equivalence.}} D^b(A)$$

skyscraper sheaves $\longmapsto$ simple semi-homogeneous vector bundles

§3: Surface Case

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Def^n Let (S,H) be a polarized smooth projective surface. We define:

$$\lambda_H : K(X) \twoheadrightarrow \Lambda_H \; (= \mathrm{im}\,\lambda_H \subseteq \mathbb{R}^3)$$

$$[E] \longmapsto (H^2 ch_0(E), H\,ch_1(E), ch_2(E))$$

[In general, if (X,H) is a polarized sm. variety of $\dim X = n$,

$$\lambda_H : [E] \longmapsto (H^n ch_0(E), H^{n-1} ch_1(E), \dots, ch_n(E))$$]

$$\mathrm{Stab}_H(S) := \{ \text{all stability conditions w.r.t. } \Lambda_H \}$$

NB: S surface with $\rho = 1$, $\mathrm{Stab}_H(S) = \mathrm{Stab}(S)$

Def^n Let $F \in \mathrm{Coh}(S)$. Then we define:

$$\mu_H(F) := \begin{cases} \dfrac{H\,ch_1(F)}{H^2 rk(F)} & rk(F) > 0 \\ +\infty & rk(F) = 0 \end{cases}$$

We say F is H-stable if:
(resp. -semistable)

$$\forall\, 0 \neq E \subsetneq F, \; \mu_H(E) < \mu_H(F/E)$$
(resp. $\leq$)

Def^n We define the Le Potier function $\Phi_{S,H} : \mathbb{R} \to \mathbb{R}$ as:

$$\Phi_{S,H}(x) := \limsup_{\mu \to x} \left\{ \frac{ch_2(F)}{H^2 rk(F)} \;\middle|\; \begin{array}{l} F \text{ is } H\text{-semistable with} \\ \mu_H(F) = \mu \end{array} \right\}.$$

TILT STABILITY

$\forall \beta \in \mathbb{R}$, define:

$\mathcal{T}_\beta := \{F \in \mathrm{Coh}(S) :$ any H-s.s. factor of F satisfies $\mu_H(F) > \beta\}$.

$\mathcal{F}_\beta := \{F \in \mathrm{Coh}(S) :$ any H-s.s. factor of F satisfies $\mu_H(F) \leq \beta\}$.

$(\mathcal{T}_\beta, \mathcal{F}_\beta)$ defines a torsion pair on $\mathrm{Coh}(S)$.

This means we can define:

$$\mathrm{Coh}^\beta(X) = \langle \mathcal{T}_\beta, \mathcal{F}_\beta[1] \rangle$$

(extension closure)

This is the heart of a bdd t-structure on $D^b(S)$)

Then $\forall \alpha \in \mathbb{R}$, define:

$$Z_{\alpha,\beta}(F) := \left(-ch_2(F) + \alpha H^2 rk(F)\right) + i\left(H ch_1(F) - \beta H^2 rk(F)\right).$$

Thm 3.4 For every $\alpha > \Phi_{S,H}(\beta)$, $\sigma_{\alpha,\beta} := (\mathrm{Coh}^\beta(S), Z_{\alpha,\beta})$ is a geometric stability condition (w.r.t. Λ_H) on S. MOREOVER:

$$\Sigma : \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > \Phi_{S,H}(\beta)\} \longrightarrow \mathrm{Stab}_H(S)$$
$$(\alpha,\beta) \longmapsto \sigma_{\alpha,\beta}$$

is a continuous embedding.

set of stab. cond.s w.r.t. Λ_H that are geometric.

Prop$^{\underline{n}}$ 3.6 Let $\sigma = (P, Z) \in \mathrm{Stab}_H^{\mathrm{Geo}}(S)$

Then $\sigma = \sigma_{\alpha,\beta} \cdot g$ for some $\alpha > \Phi_{S,H}(\beta)$, and $g \in \widetilde{GL}_2^+(\mathbb{R})$

(universal cover of $GL_2^+(\mathbb{R}) = \{2\times 2$ matrices with real entries & $\det > 0\}$

ROUGHLY: $\widetilde{GL}_2^+(\mathbb{R})$ acts on $\mathrm{Stab}_\Lambda(X)$ via:

- shifts: $P(\phi) \mapsto P(\phi + n) \ \forall \phi$
- translation in $\mathbb{C}^2$: $Z \mapsto A \circ Z$, $A \in GL_2^+(\mathbb{R})$

→ **SKETCH PROOF** (3.6)

- one can show $\exists g \in \widetilde{GL}_2^+(\mathbb{R})$ s.t.

$Z \longmapsto Z_{\alpha,\beta}$ and $\sigma = \sigma_{\alpha,\beta} \cdot g$.

(see e.g. Bridgeland's K3 paper, 2008 §10)

- Suppose $\alpha \leq \Phi_{S,H}(\beta)$.

Assume that $\Phi_{S,H} = \max\{\dots\}$ (rather than $\overline{\lim}\dots$)

Then by defn., $\exists$ H-s.s. sheaf F & $(\alpha_0, \beta_0) \in W$ s.t.

1. $H \cdot ch_1(F) = \beta_0 H^2 rk(F)$ i.e. $\mu_H(F) = \beta_0$

② $\underline{ch_2(F) > \alpha_0 H^2 rk(F)}$

• Let $\sigma_0 = (P_0, Z_0)$ in U, with

$$\ker Z_0 = (1, \beta_0, \alpha_0)\cdot \mathbb{R}$$

Then up to an element of $\widetilde{GL}_2^+(\mathbb{R})$ we may assume: $Z_0 = Z_{\alpha_0,\beta_0}$.

$$Z_{\alpha_0,\beta_0}(F) := \underbrace{\left(-ch_2(F) + \alpha_0 H^2 rk(F)\right)}_{<0 \text{ by } ②} + i\underbrace{\left(H ch_1(F) - \beta_0 H^2 rk(F)\right)}_{=0 \text{ by } ①}.$$

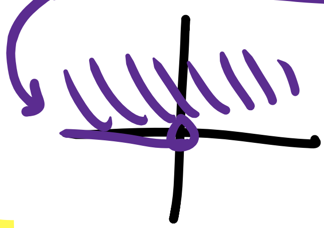
$\Rightarrow$ $Z_{\alpha_0,\beta_0}(F) \in \mathbb{R}_{<0}$

$\Rightarrow$ $Z_{\alpha_0,\beta_0}(F[1]) \in \mathbb{R}_{>0}$

BUT ALSO:

$\mu_H(F) = \beta_0$, hence $F \in \mathcal{F}_{\beta_0}$

$\therefore$ $F[1] \in Coh^{\beta_0}(S) = \underbrace{P_0((0,1])}$



i.e. $Z_{\alpha_0,\beta_0}(F[1]) \notin \mathbb{R}_{>0}$

$\Rightarrow$ $\alpha > \Phi_{S,H}(\beta)$ ∎

COROLLARY 3.7 Let (S,H) be a smooth polarized surface with finite Albanese morphism. Then $\mathrm{Stab}_H(S)$ is connected and contractible.

→ **Proof** 3.6 gives a homeomorphism:

$$\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha > \Phi_{S,H}(\beta)\} \times \widetilde{GL_2^+}(\mathbb{R}) \to \mathrm{Stab}_H^{Geo}(S)$$

$$((\alpha,\beta), g) \longmapsto \sigma_{\alpha,\beta} \cdot g$$

The product on the left hand side is connected and contractible, so it follows that $\mathrm{Stab}_H^{Geo}(S)$ is also.

Then by 2.15, $\mathrm{Stab}_H^{Geo}(S) = \mathrm{Stab}_H(S)$ ∎

Corollary 3.8 Let S be an irregular surface of Picard rank $\rho = 1$. Then $\mathrm{Stab}(S)$ is connected & contractible.

→ **proof**

- $a : S \to \mathrm{Alb}(S)$ is nontrivial ($\because$ S is irregular)
- a cannot contract any curves (else $\rho = 2$)

  $\therefore$ a is finite
- H ample divisor & $\rho = 1 \Rightarrow \mathrm{Stab}(S) = \mathrm{Stab}_H(S)$.

By 3.7, $\mathrm{Stab}(S)$ is connected & contractible.

∎

§4: Abelian Threefold Case and Open Questions

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Corollary 4.9 Let (A, H) be a polarised abelian threefold. Then: $\quad \mathrm{Stab}_H(A) = \widetilde{\mathcal{B}}$

In particular, if $\rho(A) = 1$, then:

$$\mathrm{Stab}_H(A) = \mathrm{Stab}(A) = \widetilde{\mathcal{B}}$$

and hence is connected & contractible.

What is $\widetilde{\mathcal{B}}$?

Recall in the surface case we had a map:

$$\{(\alpha,\beta)\in\mathbb{R}^2 : \alpha > \Phi_{S,H}(\beta)\} \longrightarrow \mathrm{Stab}_H(S)$$
$$(\alpha,\beta) \longmapsto \sigma_{\alpha,\beta}$$

In the threefold case we similarly have:

$$\{(a,b,\alpha,\beta)\in\mathbb{R}^4 : \alpha>0,\ a > \frac{\alpha^2}{6} + \frac{1}{2}|b|\alpha\} \xrightarrow{\Sigma} \mathrm{Stab}_H(A)$$

$$\Sigma : (a,b,\alpha,\beta) \longmapsto \sigma_{\alpha,\beta}^{a,b}$$

Then $\quad \widetilde{\mathcal{B}} := \widetilde{GL}_2^+(\mathbb{R}) \cdot (\operatorname{im}\Sigma)$

QUESTIONS

Q1 Let (X, H) be a smooth projective variety with finite Albanese morphism. Then is $\mathrm{Stab}_H(S)$ contractible?

Q2 Let X be a smooth projective variety with non-finite Albanese morphism. Then does there always exist a non-geo. stability condition?

- $\dim X = 1$ Yes! [Macrí 2007]
- $\dim X \geq 3$ Out of reach ☹
  (existence of stability conditions only known in a few cases)

$\rightsquigarrow$ $\dim X = 2$ MOST INTERESTING!

Conjecture Let (X, H) be a smooth surface with zero irregularity. Then $\Phi_{S,H}$ is <u>not</u> continuous at 0.