★ Good Reference: Weibel – Introduction to Homological Algebra

LAST TIME:

$\mathcal{A}$ – abelian category ($=\underline{\text{mod}}-R$)

$C_\bullet$ – chain complex in $\mathcal{A}$:

$$\cdots \xrightarrow{d} C_{n+1} \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \cdots \qquad H_n(C) := \ker d_n / \operatorname{im} d_{n+1}$$

$$\boxed{d^2=0}$$

DUALLY WE HAVE:

$C^\bullet$ – cochain complex in $\mathcal{A}$:

$$\cdots \xrightarrow{d} C^{n-1} \xrightarrow{d} C^n \xrightarrow{d} C^{n+1} \xrightarrow{d} \cdots \qquad H^n(C) := \ker d_n / \operatorname{im} d_{n-1}$$

$$\boxed{d^2=0}$$

CONVENTION: stick to cochain complexes

$\underline{\mathrm{Ch}}(\mathcal{A}) :=$ (cochain complexes in $\mathcal{A}$, chain maps)

We say that cochain maps $f, g : C \to D$ are chain homotopic if $\exists$ cochain map $h : C \to D$ such that $h^i : C^i \to D^{i-1}$ and:

$$f^i - g^i = d^{i-1} \circ h^i + h^{i+1} \circ d^i$$

WE THEN DEFINE:

$\underline{K}(\mathcal{A}) :=$ (cochains in $\mathcal{A}$, cochain maps up to homotopy)

A chain map $f : C \to D$ is a quasi-isomorphism if the induced maps on cohomology: $\quad H^i(f) : H^i(C) \to H^i(D)$

are all <u>isomorphisms</u>.

FINALLY WE DEFINE:

$\underline{D}(A)$ = localisation of $\underline{K}(A)$ such that:

quasi-isomorphisms "=" isomorphisms

The morphisms in $\underline{D}(A)$ are "roofs":

quasi-iso.

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[dr, "g"] & \\ A \arrow[rr, dashed, "\text{"} g \circ f^{-1} \text{"}"'] & & B \end{tikzcd}$$

LECTURE 2: TRIANGLES, EXACTNESS, & DERIVED FUNCTORS:

By passing to $\underline{K}(\mathcal{A})$ we lost the ability to say that a sequence of morphisms is exact (no kernels & cokernels). But there is a form of exactness that is still preserved: EXACT TRIANGLES

RECALL If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a s.e.s. of cochain complexes, there are natural maps

$$\partial : H^n(C) \to H^{n+1}(A)$$ "shift function"

and a long exact sequence:

$$\cdots \xrightarrow{g} H^{n-1}(C) \xrightarrow{\partial} H^n(A) \xrightarrow{f} H^n(B) \xrightarrow{g} H^n(C) \xrightarrow{\partial} H^{n+1}(A) \xrightarrow{f}$$

We can reorganise this data as follows:

$$\begin{tikzcd}
H^*(A) \arrow[rr, "f"] & & H^*(B) \arrow[dl, "g"] \\
& H^*(C) \arrow[ul, "\partial"] &
\end{tikzcd}$$

$H^{n+1}(A)$ $H^n(C)$

⚠ This is not a commutative diagram.

This is an exact triangle

We write: $\partial : H^*(C) \to H^*(A)[-1]$

NOTATION C (chain) cochain complex, $p \in \mathbb{Z}$, then $C[p]$ is the complex:

$$C[p]^n = C^{n-p} \quad ((C[p]_n = C_{n+p}))$$

This has differential: $\underline{(-1)^p \cdot d}$

We then have: $H^n(C[p]) = H^n(C^{n-p})$

And for f a cochain map: $f^n[p] = f^{n-p}$.

AIM: Generalise this notion of an exact triangle.

IDEA: Construct some distinguished triangles. Then say anything isomorphic to these is an exact triangle.

CONE CONSTRUCTION

Let $f: A \to B$ be a map of cochain complexes.
Define $\mathrm{cone}(f) = C$ to be the complex:

$$C^i = \underline{A^{i+1}} \oplus \underline{B^i}$$

i.e. $C = A[-1] \oplus B$

& differential:

$$d(a,b) = (-d(a), d(b) - f(a))$$

$$d^i_C(a,b) = \begin{pmatrix} -d^i_A & 0 \\ -f^{i+1} & d^i_B \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \qquad \boxed{d^2 = 0}$$

We then define chain maps:

$g: B \to C$ $\qquad$ $h: C \to \underline{A[-1]}$

$g^i: b \mapsto (0,b)$ $\qquad$ $h^i: (a,b) \mapsto a$

$\in A^{i+1}$

These give us a short exact sequence in $\underline{Ch}(\mathcal{A})$:

$$0 \to B \xrightarrow{g} \mathrm{cone}(f) \xrightarrow{h} A[-1] \to 0$$

We define the shift triangle on f to be the triple (f, g, h) of maps in $\underline{K}(A)$. This data is written as:

$$\begin{tikzcd} A \arrow[rr, "f"] & & B \arrow[dl, "g"] \\ & \mathrm{cone}(f) \arrow[ul, "h"] & \end{tikzcd}$$

DEFINITION Let A, B, C be cochain complexes. Then given any three maps: $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[-1]$ in $\underline{K}(A)$, we say (f, g, h) is an exact triangle on (A, B, C) if it is isomorphic to a shift triangle (f', g', h') on $f' : A' \to B'$

i.e.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[r, "g"] \arrow[d, "v"'] & C \arrow[r, "h"] \arrow[d, "w"'] & A[-1] \arrow[d, "u[-1]"'] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & \mathrm{cone}(f') \arrow[r, "h'"'] & A' \end{tikzcd}$$

(u, v, w) isomorphisms s.t. this commutes. (in $\underline{K}(A)$)

This endows $\underline{K}(A)$ with the structure of a triangulated category.

(a category with a shift functor & a collection of exact triangles that satisfy axioms (TR1–TR4))

(see Weibel: 10.2.1)

EXERCISE 2.1 Check if $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ s.e.s. in $\underline{Ch}(\mathcal{A})$, then:

$$\begin{tikzcd} A \arrow[rr, "f"] & & B \arrow[dl, "g"] \\ & C \arrow[ul, "\exists h"] & \end{tikzcd}$$

is an exact triangle in $\underline{D}(\mathcal{A})$
(HINT: used cone(f), $\underline{1.5.8}$) Weibel

⚠ exact triangles in $\underline{D}(\mathcal{A})$ are not neccesarily exact in $\underline{K}(\mathcal{A})$.

e.g. $0 \to \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{1} \mathbb{Z}/2 \to 0$

INJECTIVE RESOLUTIONS & DERIVED FUNCTORS

We are interested in functors on $D(\mathcal{A})$, e.g. when working with sheaves we often work with: f_*, Hom, f^*, $\otimes$, $\Gamma(X,-)$ etc.

Now that we have introduced a notion of exactness in $D(\mathcal{A})$, we want to study when functors preserve exact triangles.

As we saw last time, in general functors do not do this, so we want a notion of derived functors that "make" our functors exact.

⚠ $D(\mathcal{A})$ is not abelian, so we will need to do a bit more work to come up with a well defined notion of derived functors.

RECALL An object I in an abelian category $\mathcal{A}$ is **injective** if whenever we have a diagram of this form:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[dl, dashed, "\exists \beta"] & \text{(exact)} \\ & I & & \end{tikzcd}$$

s.t. $\alpha = \beta \circ f$

If A is an object of $\mathcal{A}$, an **injective resolution** of A is a cochain complex of injectives $I^\bullet$ w/ $I^i = 0 \ \forall i < 0$ & a map $A \to I^0$ s.t.

$$0 \to A \to I^0 \xrightarrow{d} I^1 \xrightarrow{d} I^2 \xrightarrow{d} \cdots$$

is exact

We say $\mathcal{A}$ has **enough injectives** if any object of $\mathcal{A}$ has an injective resolution.

AIM: define right derived functors on $D(A)$

F left exact $\rightsquigarrow$ enough injectives $\longrightarrow$ $\underline{R}F$

F right exact $\rightsquigarrow$ enough projectives $\longrightarrow$ $\underline{L}F$

FOR SIMPLICITY: we will only work with bdd complexes

ie, $A^\cdot$ s.t. $H^i(A^\cdot)=0$ $\forall |i| >> 0$.

These form a full subcategory in $\underline{K}(A)$, called

$\underline{K}^b(A)$

The corresp. localisation of quasi-iso's is denoted

$\underline{D}^b(A)$.

FIRST we need some results on injective resolutions.

PROOFS ARE LEFT AS AN EXERCISE!

PROPOSITION 2.3 In $\underline{K}^b(A)$, let $f: A \to B$ be a quasi-iso. of bdd below cochain complexes of injectives. Then f is invertible (ie. $\exists g: B \to A$ s.t. $gf \simeq \mathrm{id}_A$, $fg \simeq \mathrm{id}_B$)

PROPOSITION 2.5 Assume A has enough injectives, then every bdd below complex of $\underline{K}^b(A)$ is quasi-isomorphic to a complex of injectives.

COROLLARY 2.7 The composition of the following sequence of functors is an equivalence of categories:

$$\underline{K}^b(\mathrm{inj}(\mathcal{A})) \xrightarrow{i} \underline{K}^b(\mathcal{A}) \xrightarrow{Q} \underline{D}^b(\mathcal{A})$$

complexes of injectives — inclusion — localisation functor.

PROP 2.8 Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact functor, between abelian categories. Then F induces an exact functor $\bar{F} : \underline{K}^-(\mathrm{inj}(\mathcal{A})) \to \underline{K}^-(\mathcal{B})$

obtained by directly applying F to a complex of injectives in $\mathcal{A}$.

DEF 2.10 Assume $\mathcal{A}$ has enough injectives, & that $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor. Fix an inverse Q^{-1} to the equivalence of categories in 2.7, & define the right derived functor of F, $\underline{R}F$, to be the composite:

$$\underline{D}^b(\mathcal{A}) \xrightarrow{Q^{-1}} \underline{K}^b(\mathrm{inj}(\mathcal{A})) \xrightarrow{\bar{F}} \underline{K}^b(\mathcal{B}) \xrightarrow{Q} \underline{D}^b(\mathcal{B})$$

NOTE: For A a single object of $\mathcal{A}$, $I^\bullet = Q^{-1}(A)$ is just the injective resolution of A. We apply F to this to get $\underline{R}F(A)$. Taking homology:

$$H^i(\underline{R}F(A)) = H^i(F(I^\cdot)) \overset{\text{classical def}}{=} \underline{R^i F(A)}$$

What have we gained? $\underline{R}F(A^\cdot)$ is a complex.
(not just homology!)

What if F is right exact?

⚠ need "enough projectives" - but this is almost never true in sheaf theory.

INSTEAD: use resolutions of acyclic objects.

↳ we say a class of objects is acyclic for a right exact functor F if $H^n(FX) = 0 \ \forall n$, $\forall X$ in our class

$\rightsquigarrow \underline{L}F$ (via similar construction to $\underline{R}F$)

EXAMPLES / APPLICATIONS

We can define derived functors:

resulting complexes compute "usual derived functors"

$\underline{R}\Gamma(X,-) \longrightarrow H^*(X,-)$

$\underline{R}\,\mathcal{H}om(-,-) \longrightarrow \mathrm{Ext}^*_X(-,-)$

$\underline{R}f_* \longrightarrow R^* f_*$

"GREAT TECHNICAL ADVANTAGE":

under mild hypotheses, we have:

$$\underline{R}(F \circ G) \cong \underline{R}(F) \circ \underline{R}(G)$$

MILD HYPOTHESES: (10.8.3 Weibel)

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories such that:

- $\mathcal{A}, \mathcal{B}$ have enough injectives
- $G: \mathcal{A} \to \mathcal{B}$, $F: \mathcal{B} \to \mathcal{C}$ left exact functors

$$\begin{tikzcd} \mathcal{A} \arrow[r, "G"] & \mathcal{B} \arrow[dl, "F"] \\ \mathcal{C} \arrow[u, "F \circ G"] & \end{tikzcd}$$

$H^i(X) = 0 \ \forall i$ $\qquad$ $H^i(FX) = 0 \ \forall i$

- G sends injective objects of $\mathcal{A}$ to F-acyclic objects of $\mathcal{B}$
- G sends acyclic complexes to F-acyclic objects
- F & G have finite cohomological dimension.

Then: $\quad \underline{R}(FG): \underline{D}(\mathcal{A}) \to \underline{D}(\mathcal{B})$ exists &

$$\underline{R}(FG) \cong \underline{R}(F) \circ \underline{R}(G). \quad (+)$$

This relation normally requires the use of spectral sequences.

e.g. $f: X \to Y$ morphism, sheaf $\mathcal{F}$ on X, the Leray spectral sequence:

$$H^i(Y, \underline{R}^j f_* \mathcal{F}) \Rightarrow H^{i+j}(X, \mathcal{F})$$

is an application of $(+)$ for the composition of left exact functors:

$$\Gamma(X, -) = \Gamma(Y, f_*(-))$$

We can do this because derived functors yield complexes which we can use to build the spectral sequences.

Another advantage: many results that only hold for locally free sheaves in the usual sense now hold in general.

e.g. $D^b_{coh}(X)$ bdd derived cat. of coh. sheaves on X. Then

$\mathcal{E}, \mathcal{F}, \mathcal{G} \in \underline{D}^b_{coh}(X)$, then:

$$(\ast)\quad \underline{R}\,\mathcal{H}om_X(\mathcal{E}, \mathcal{F} \overset{L}{\otimes} \mathcal{G}) \cong \underline{R}\,\mathcal{H}om_X(\mathcal{E} \overset{L}{\otimes} \mathcal{F}^\vee, \mathcal{G})$$

where $\mathcal{F}^\vee = \underline{R}\,\mathcal{H}om_X(\mathcal{F}, \mathcal{O}_X)$

⚠ We have skimmed over some technical issues.

- unbounded complexes
- smoothness

  e.g. X scheme not smooth

  $\rightsquigarrow$ problems w/ $(\ast)$

FROM NOW ON WE WILL WORK WITH:

- D^b_{Coh}
- X smooth projective scheme.

ASIDE:

X smooth proj. variety

Any complex in $D^b(X)$ is quasi-iso. to a bounded complex of vector bundles.

(another advantage of using derived categories)