RECAP:
- G: finite group,
- $\mathcal{D}$: k linear category, $k=\bar{k}$, $(\operatorname{char} k, |G|)=1$

- $G \curvearrowright \mathcal{D}$ $\left[\text{i.e. } \forall g,h\in G,\ \phi_g : \mathcal{D} \xrightarrow{\sim} \mathcal{D},\ \varepsilon_{g,h}: \phi_g\circ\phi_h \xrightarrow{\sim} \phi_{hg} \text{ + compatibility}\right]$

⚠ There is another "cleaner" definition via monoidal categories in Lecture 2 notes. **Exercise**: Show they are equivalent.

- $\mathcal{D}_G$: G-equivariantisation
  $\ni (E, \{\lambda_g\}_{g\in G})$: $E\in\mathcal{D}$, $\lambda_g : E \xrightarrow{\sim} \phi_g(E)$ (+ compatibility)

Going between $\mathcal{D}$ and $\mathcal{D}_G$

There are two natural functors:

Def^{n} Suppose $G\curvearrowright\mathcal{D}$, define

- (forgetful functor) $\mathrm{Forg}_G : \mathcal{D}_G \to \mathcal{D}$ by $(F, \{\lambda_g\}_G) \mapsto F$
- (inflation functor) $\mathrm{Inf}_G : \mathcal{D} \to \mathcal{D}_G$, where

$$\mathrm{Inf}_G(E) := \Big(\bigoplus_G \phi_g(E), \{\lambda_g^{nat}\}_G\Big)$$

and λ_g^{nat} is the composition:

$$\bigoplus_{h\in G}\phi_h(E) \xrightarrow[\sim]{\text{reorder}} \bigoplus_{h\in G}\phi_{hg}(E) \xrightarrow[\sim]{\oplus_{h\in G}\varepsilon_{g,h}^{-1}} \bigoplus_{h\in G}\phi_g\phi_h(E) \overset{①}{=} \phi_g\Big(\bigoplus_{h\in G}\phi_h(E)\Big)$$

① ϕ_g additive functor

Exercise Forg_G is faithful and Inf_G is the left and right adjoint. ⚠ $\mathrm{Forg}_G\circ\mathrm{Inf}_G(E) = \bigoplus_{g\in G}\phi_g(E)$

EX ① $G\curvearrowright X$ free, $\pi: X\to X/G$

Consider the (derived) functors

$$\begin{tikzcd}
D^b(X) \arrow[r, shift left, "\pi_*"] \arrow[d, shift right, "\mathrm{Inf}_G"'] & D^b(X/G) \arrow[l, shift left, "\pi^*"] \arrow[dl, bend left, "\sim", "\Psi"'] \\
D^b_G(X) \arrow[u, shift right, "\mathrm{Forg}_G"'] &
\end{tikzcd}$$

Q: why equivalent?
A: (Brion) uses faithfully flat descent

Exercise $\mathrm{Inf}_G \simeq \Phi \circ \pi_*$ and $\mathrm{Forg}_G \simeq \pi^* \circ \Phi^{-1}$

e.g. $x \in X$, $\mathrm{Forg}_G \circ \mathrm{Inf}_G(\mathcal{O}_x) = \bigoplus_{g \in G} \mathcal{O}_{g^{-1}x} = \pi^* \circ \pi_*(\mathcal{O}_x)$

saw in lecture 2:

Th'm | [Elagin '15] G abelian, $G \curvearrowright \mathcal{D}$ (so $\hat{G} \curvearrowright \mathcal{D}_G$). Then

$$\Omega : (\mathcal{D}_G)_{\hat{G}} \xrightarrow{\sim} \mathcal{D}$$

and $\mathrm{Forg}_G \simeq \Omega \circ \mathrm{Inf}_{\hat{G}}$, $\mathrm{Inf}_G \simeq \mathrm{Forg}_{\hat{G}} \circ \Omega^{-1}$

(the proof uses monads)

§ 3 Equivariant stability conditions

(⚠ technical! ok to "black box" and skip to § 4)

GOAL OF § 3 : Compare $\mathrm{Stab}(\mathcal{D})$ and $\mathrm{Stab}(\mathcal{D}_G)$

[and hence $\mathrm{Stab}(X)$ and $\mathrm{Stab}(X/G)$]

SETUP : • $\mathcal{D}$: 

- G finite group $G \curvearrowright \mathcal{D}$ is exact ie $g^* := \Phi_g \in \mathrm{Aut}_{\mathrm{exact}}(\mathcal{D})$
- $\mathcal{D}_G$  s.t. Forg_G - exact (eg if $\mathcal{D}$ is DG-enhanced [Elagin])
- Fix $v : \underbrace{K_0(\mathcal{D})}_{} \twoheadrightarrow \Lambda \simeq \mathbb{Z}^r$

Grothendieck group $= \dfrac{\text{free group generated by ob}\,\mathcal{D}}{A \to B \to C \to A[1] \Leftrightarrow [B] = [A] + [C]}$

We will consider $\mathrm{Stab}_\Lambda(\mathcal{D})$, ie.

Def^n $\sigma = (P, Z) \in \mathrm{Stab}_\Lambda(\mathcal{D})$ iff

(i) $P = (P(\phi))_{\phi \in \mathbb{R}}$ slicing

ie. a coll'n of additive subcats
s.t. (a) $E \in P(\phi_1)$, $F \in P(\phi_2)$, $\phi_1 > \phi_2 \Rightarrow \mathrm{Hom}(E, F) = 0$
(b) $P(\phi)[1] = P(\phi + 1)$
(c) $E \in \mathcal{D}$ has a unique HN filtration by stable objects [= simple in $P(\phi)$]

$$\begin{tikzcd}[column sep=small] 0 = E_0 \arrow[rr] & & E_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\ & A_1 \in P(\phi_1) \arrow[ul, dashed] & & & & A_n \in P(\phi_n) \arrow[ul, dashed] & \end{tikzcd}$$

$\phi_1 > \cdots > \phi_n$

$0 \neq E \in P(\phi)$ called σ-semistable (of phase ϕ)

(ii) $Z : K_0(\mathcal{D}) \xrightarrow{v} \Lambda \longrightarrow \mathbb{C}$ group homomorphism

s.t. $0 \neq E \in P(\phi) \Rightarrow Z(E) = \underbrace{m(E)}_{\in \mathbb{R}_{>0}} \cdot e^{i\pi\phi}$

(iii) (support property) $\exists$ quadratic form Q on $\Lambda \otimes \mathbb{R}$ s.t.
(a) $\ker Z \otimes \mathbb{R}$ is negative definite w.r.t. Q
(b) $\forall$ σ-semistable object E
$Q(v(E)) \geq 0$

↑ feel free to ignore today

Then G acts on $\mathrm{Stab}_\Lambda(\mathcal{D})$ via:

$$g\cdot(P,Z) = \left(\left(g^*(P(\phi))\right)_{\phi\in\mathbb{R}},\ Z\circ g^* \right)$$

and $\mathrm{Stab}_\Lambda(\mathcal{D})^G = \{ \sigma\in\mathrm{Stab}_\Lambda(\mathcal{D}) : g\cdot\sigma=\sigma\ \forall g\in G\}$

closed submanifold

we call such σ "G-invariant"

(⚠ sometimes called "G-fixed" in the literature)

Prop'n 2 [Polishchuk '07, Macri-Mehrotra-Stellari '09]

$$\sigma\in\mathrm{Stab}_\Lambda(\mathcal{D})^G \Rightarrow \underbrace{\mathrm{Forg}_G^{-1}(\sigma)}_{=(P_G,\,Z_G)} \in \mathrm{Stab}_\Lambda(\mathcal{D}_G)$$

where $P_G(\phi) := \{ E\in\mathcal{D}_G : \mathrm{Forg}_G(E)\in P(\phi)\}$

$$Z_G := Z\circ\mathrm{Forg}_G$$

⚠ We use the same lattice Λ

$$\left[\begin{array}{ccccc} Z: K_0(\mathcal{D}) & \to & \Lambda & \to & \mathbb{C} \\ \uparrow & \nearrow & & & \\ K_0(\mathcal{D}_G) & & & & \end{array}\right]$$

Proof

CLAIM $E\in P(\phi) \Rightarrow \underbrace{\mathrm{Forg}_G(\mathrm{Inf}_G(E))}_{=\bigoplus_{g\in G} g^*(E)} \in P(\phi)$

Proof $\sigma\in\mathrm{Stab}_\Lambda(\mathcal{D})^G \Rightarrow g^*(E)\in P(\phi)\ \forall g\in G \Rightarrow \square$ ($P(\phi)$ additive)

$\Rightarrow \mathrm{Forg}_G\circ\mathrm{Inf}_G$ right t-exact [i.e. $FG(\mathcal{D}^{\leq 0})\subseteq\mathcal{D}^{\leq 0}$, where $\mathcal{D}^{\leq 0} = P(\leq 0)) := \left\{\begin{array}{l} E\in D^b(X) \text{ s.t.} \\ \phi^+(E)\leq 0\end{array}\right\}$]

$\Rightarrow$ can apply criterion* to induce (pre-) stability conditions seen in Xiaolei's 3rd Lecture [Polishchuk '07]

[* also need a list of assumps on $\mathcal{D}, \mathcal{D}_G$ that will hold where we want to apply it. e.g. $\mathcal{D}$: DG-enhanced is sufficient.

under these, $\sigma\in\mathrm{Stab}(\mathcal{D})$, $F\circ G(P(\phi))\subseteq P(\phi\geq 0)$
$\Rightarrow F^{-1}(\sigma=(P,Z))\in\mathrm{Stab}(\mathcal{D})$]

EXERCISE: check support property $\square$

(Q) What is the image of $\mathrm{Forg}_{\hat{G}}^{-1}$?

Prop'n 3 G abelian, $\sigma \in \mathrm{Stab}_\Lambda(\mathcal{D})^G$
$\Rightarrow \mathrm{Forg}_G^{-1}(\sigma)$ is $\hat{G}$-invariant

Proof EXERCISE [HINT: $\mathrm{Forg}_G((E,\lambda)\otimes\chi) = \mathrm{Forg}_G(E,\lambda)$]

Th'm 4 [Polishchuk '07, Macrì–Mehrotra–Stellari '09, Perry–Pertusi–Zhao '23, D'23] Suppose G is abelian. There are biholomorphisms of (closed) complex submanifolds:

$$\mathrm{Forg}_G^{-1} : \mathrm{Stab}_\Lambda(\mathcal{D})^G \overset{\sim}{\rightleftarrows} \mathrm{Stab}_\Lambda(\mathcal{D}_G)^{\hat{G}} : \mathrm{Forg}_{\hat{G}}^{-1}$$

which are mutually inverse up to rescaling the central charge by $|G|$.

Proof
- $\sigma \in \mathrm{Stab}_\Lambda(\mathcal{D})^G$
- $\sigma' = (P', Z') := \mathrm{Forg}_{\hat{G}}^{-1}\left(\mathrm{Forg}_G^{-1}(\sigma)\right) \in \mathrm{Stab}_\Lambda\left((\mathcal{D}_G)_{\hat{G}}\right)$

$$P'(\phi) = \{ E \in \mathcal{D} : \mathrm{Forg}_G(\mathrm{Forg}_{\hat{G}}(E)) \in P(\phi)\}$$

Under $\mathcal{D} \simeq (\mathcal{D}_G)_{\hat{G}}$ (Th'm 3), this becomes:

$$P'(\phi) = \{ E \in \mathcal{D} : \underbrace{\mathrm{Forg}_G(\mathrm{Inf}_G(E))}_{= \bigoplus_{g\in G} g^*E} \in P(\phi)\}$$

- CLAIM $P(\phi) = P'(\phi)$.
  - $\subseteq$: closed under direct summands
  - $\supseteq$: same argument as Prop 1.
- $\underset{\mathcal{D}}{Z'(E)} = Z\left(\sum_{g\in G} g^*(E)\right)$
  $= |G|\, Z(E)$ (σ G-invariant)

- Forg_G^{-1} is continuous [Macrì–Mehrotra–Stellari] and $\mathbb{C}$-linear on $\mathrm{Hom}(\Lambda, \mathbb{C})$
- $\mathrm{Forg}_{\hat{G}}^{-1} \circ \mathrm{Forg}_G$ is a $\mathbb{C}$-linear isomorphism on $\mathrm{Hom}(\Lambda, \mathbb{C})$

$\Rightarrow \mathrm{Forg}_{\hat{G}}^{-1}$ is a biholomorphism

The same argument applies for $\mathrm{Forg}_{\hat{G}}^{-1}$ $\Rightarrow$ $\square$

Remark In [D. – Edmund Heng – Anthony Licata '23] we generalised this to non abelian groups.

Key: use Inf_G^{-1} instead of $\mathrm{Forg}_{\hat{G}}^{-1}$.

Hard part: What is the image of Forg_G^{-1}? For this we need a generalisation of group actions to actions of "fusion categories". monoidal categories + strong finiteness assumptions

(Lecture 2 notes) $\rightsquigarrow$ $\mathrm{Rep}(G) \circlearrowright \mathcal{D}_G$.

We show:

① $$\mathrm{Stab}_{\mathrm{Rep}(G)}(\mathcal{D}_G) := \left\{ \sigma = (P, Z) : \begin{array}{l} \forall \rho \in \mathrm{irr}(G) \\ \cdot\ P(\phi) \otimes \rho \subseteq P(\phi) \\ \cdot\ Z(E \otimes \rho) = \dim(\rho) Z(E) \end{array} \right\}$$

is a complex submanifold. [in fact can replace $\mathrm{Rep}(G)$ with any "fusion category"]

② $$\mathrm{Stab}(\mathcal{D})^G \simeq \mathrm{Stab}_{\mathrm{Rep}(G)}(\mathcal{D}_G)$$

Next time:
- Apply Th^m 4 to $\mathcal{D} = D^b(X)$, compare $\mathrm{Stab}(X)$ & $\mathrm{Stab}(X/G)$
- answer Q4 alb X not finite $\overset{?}{\Rightarrow}$ $\exists$ nongeometric B.S.C.-s