ME-467: Turbulence April 2025
Tobias Schneider
Omid Ashtari
Savyaraj Deshmukh
Jean-Clément Ringenbach

Course Project: Turbulence at finite Reynolds number and the skeleton of chaos.

This 11-page document contains the instructions for the project of the course **ME-467: Turbulence**, which determines the final grade. This project consists of two independent parts: In the first part you will be analyzing data from a modern wind tunnel experiment, and relating the findings to predictions as well as limitations of Kolmogorov's turbulence theory (hereafter called K41). In the second part you will follow a modern dynamical systems approach to turbulence and characterize chaos in a simplified version of Navier-Stokes.

Material

To complete the assignment, you need the following files:

- `instructions.pdf`: This instruction sheet.
- `report.tex` (with files `titlepic.sty`, `fancyhdr.sty`, `figures/EPFL_LOGO.jpg`): Please use this template to prepare your report. Set your name in the `MyName` variable, so that it is displayed on every page.
- Three data files `veldata1.txt`, `veldata2.txt` and `veldata3.txt`: The ascii datafiles (29.5MB, around 3.3 million data points each) that contain all velocity data for the data analysis.
- Simulation codes are available via Github.

Rules / Honor Code

- **Individual report:** Each student has to write and submit her/his own report.
- **Collaborations:** We allow and encourage discussions and collaborations with your classmates. However, write the report individually, and at the end of the report cite all sources including a list of the people with whom you collaborated. Please limit collaborations to groups of at most 4 people.
- **Resources:** You are free to use any sources that you find useful, including books, scientific papers and internet resources. **The only exceptions are exercises/exams of ME-467 from previous years, which may <u>not</u> be consulted.** Note that beyond expecting you to follow the Honor Code out of respect for yourself and your fellow students, your submission might be checked against submissions from previous years.
- **Citing:** Cite all literature and other resources used.
- **Page limits:** The report should be only as long as is needed to convey the message clearly. No extended introductions or lengthy final discussions are needed!

- **Software:** For the data analysis, please use `MATLAB`. Note that you have to submit readable and commented scripts / code and the quality of the code is part of the evaluation criteria.

Submission

Please electronically submit a `zip`-archive by the project deadline **May 30, 2025, 5 pm**. The `zip`-archive must be named
`<Lastname>_<Firstname>_<SciperNo>.zip` (e.g. `Frisch_Uriel_123456.zip`)
and include

- your report, as `.pdf` and `.tex` source code;
- the analysis scripts (readable ASCII source code with comments - no `MATLAB` live scripts or Jupyter notebooks or similar) you used for data analysis and production of figures;
- sources you have used to complete the assignment, if they are not available through the usual library channels;
- signed last page of the report certifying that you followed the Honor Code.

Please use the Moodle upload function to submit these files. Do **not** include the velocity datafiles in your upload. **The upload closes at 5h00 PM sharp. Late submissions will not be accepted.**

Tasks

In Part I 'Finite Reynolds Number Effects in Turbulence' you will work on experimental data obtained in a modern wind tunnel using state-of-the-art instrumentation. Specifically, you will

- Process and analyze experimental data using `MATLAB`.
- Interpret the data in view of K41 turbulence theory and present, discuss and interpret the findings.

In Part II 'Chaos, its skeleton, and quantitative characterization' you will study deterministic chaos — the phenomenon necessitating a statistical description of turbulence despite its emergence from a deterministic evolution equation — for a simplified model system. Specifically, you will

- Numerically (using again `MATLAB`) investigate properties of the model system.
- Explain and discuss your findings.

Part I: Finite Reynolds Number Effects in Turbulence

The `MATLAB` code required for this part can be downloaded from the linked repository. The datasets needed to run these scripts should be downloaded from Moodle and stored in the `/data/` directory. Use the template `report.tex` to prepare your report.

1 Data Analysis

The files `veldata1.txt`, `veldata2.txt` and `veldata3.txt` contain a time series of the downstream velocity from a wind tunnel experiment in the Warhaft Wind and Turbulence Tunnelat Cornell University[1], in units of m/s for different Reynolds numbers. The data were acquired in turbulent air at ambient pressure, using a hot wire of length 1 mm, positioned a few meters downstream of an array of randomly rotating paddles about 100 mm in size. The first three lines of each file report the number of data blocks acquired, the number of datapoints per block and the sampling frequency in Hz, as shown in the example below.

200	number of data blocks
18292	number of datapoints per block
12400	data sampling frequency
4.845328	start of downstream velocity data
4.912931	
...	

To investigate the influence of the Reynolds number, carry out all analysis steps for all provided datasets. Where appropriate, combine the data for different flow-speeds in a single plot to simplify comparisons.

1.1 Velocity Signal in the Spatial Domain

The script to be used and modified for this subpart is `/programs/PART1_velocity_signal.m`. The statistical turbulence theory studied in this project is formulated mainly in terms of the spatial structure of the flow fields. Thus, interpret the time series as a spatial measurement of the total streamwise velocity signal, $u_{\text{tot}}(x)$, via Taylor's frozen flow hypothesis, with x being the streamwise coordinate. For this:

1.1.1 Plot u_{tot} against x for each dataset (Plot A). The plotting ranges must be chosen such that similarities and differences between the datasets can be identified visually.

1.1.2 Calculate the mean velocity, U, for each dataset, and fill the relevant row in the results table (see the report document).

1.1.3 Calculate the turbulence intensity,

$$I \equiv \frac{\sqrt{\langle u^2 \rangle}}{U}, \tag{1}$$

for each dataset, where u is the deviation of the total velocity form its mean value:

$$u \equiv u_{\text{tot}} - U. \tag{2}$$

Fill the relevant row in the results table.

1.1.4 To what extent are spatial interpretations of the time series appropriate? [Extra: Can you quantify possible errors?]

[1] Yoon and Warhaft, J. Fluid Mech. 215, 601–38 (1990), access

1.2 Correlation Length of the Velocity Signal

The script to be used and modified for this subpart is `/programs/PART2_velocity_signal.m`. For short distances the velocity signal is correlated, whereas for larger distances velocity values can be considered statistically independent. For the velocity fluctuations, $u(x)$, define the autocorrelation, $C(l)$, as

$$C(l) \equiv \frac{\langle u(x+l)u(x)\rangle}{\langle u^2(x)\rangle}. \tag{3}$$

The correlation length, L_C, is the length over which the fluctuations are correlated. Define L_C as the length at which $C(l)$ has dropped to $1/e$.

1.2.1 Use the provided `/functions/autocorrelation.m` function to plot $C(l)$ against l for each dataset, and mark the corresponding correlation length on each plot (Plot B). The provided autocorrelation function is not normalized, you have to properly normalize it. Fill in the relevant row in the results table.

1.2.2 Comment on the observed variation of L_C between the three datasets. Explain whether or not this trend is consistent with your intuition about the physics of the flow. The correlation length approximates the integral scale given by the *integral* (hence the name)

$$L_{\text{int}} \equiv \int_0^\infty C(l)\mathrm{d}l. \tag{4}$$

1.2.3 Compute L_{int} for each dataset, and compare to the estimates L_C. Fill in the relevant row in the results table.

From here on, only use L_C as estimate for the integral scale.

1.3 Energy Spectrum of the Flow

The script to be used and modified for this subpart is `/programs/PART3_energy_spectrum.m`. Essential information about the turbulent flow can be extracted by analyzing the spectral energy density $E(k)$. In the experiment, velocity data is not available over an infinitely extended spatial domain, but only for a finite downstream distance L. The spectral energy density is then

$$\tilde{E}(k) \equiv \frac{1}{2}\left|\frac{1}{\sqrt{2\pi L}}\int_0^L u(x)e^{-ikx}\mathrm{d}x\right|^2 ; \qquad k \in \mathbb{R}. \tag{5}$$

1.3.1 In the provided `/functions/spectral_energy_density.m` function, MATLAB's implementation of the Fast Fourier Transform is called to perform an efficient Fourier transform of the spatial velocity signal $u(x)$. However, its normalization is not the same as the one used in equation (5). Using MATLAB's fft documentation, calculate the required prefactor to properly normalize the function's output, and report it in the results table.

1.3.2 Calculate the energy spectrum for $k > 0$, which is $E(k) \equiv \tilde{E}(k) + \tilde{E}(-k)$. Plot $E(k)$ against k in log-log scale for all three datasets <u>in a single figure</u> (Plot C). Use appropriate smoothing to minimize the noise.

1.3.3 In order to confirm that your normalization is correct, use Parseval's theorem,

$$\frac{1}{2}\left\langle u^2\right\rangle = \int_0^\infty E(k)\mathrm{d}k, \tag{6}$$

and report the relative error for each dataset in the results table.

1.3.4 Indicate predictions of the K41 theory on Plot C. Does the data follow the theory predictions?

1.3.5 Estimate the integral length scale, $L_{\mathrm{int},E}$, and Kolmogorov length scale, η_E, by inspecting each spectrum. Mark the corresponding length scale estimates on each curve in Plot C, and report the estimated values in the relevant rows in the results table.

1.4 The Dissipation Rate and Different Reynolds Numbers

The script to be used and modified for this subpart is `/programs/PART4_dissipation_rate.m`. The Taylor Reynolds number is

$$Re_\lambda \equiv \frac{\sqrt{\langle u^2 \rangle}\lambda}{\nu}, \tag{7}$$

where ν and λ denote the kinematic viscosity of the fluid and the Taylor length scale, respectively. $\lambda = \sqrt{15\nu \langle u^2 \rangle / \epsilon}$ depends on the energy dissipation rate ϵ that can be estimated from the velocity fluctuations at the integral scale where energy is injected into the energy cascade:

$$\epsilon = \frac{1}{2} \frac{\sqrt{\langle u^2 \rangle^3}}{L_C}. \tag{8}$$

1.4.1 Based on the analysis above estimate the energy dissipation rate ϵ for each dataset, and fill the relevant row in the results table.

1.4.2 Estimate the Taylor Reynolds number, Re_λ, for each dataset, and fill the relevant row in the results table.

1.4.3 Estimate the (outer-scale) Reynolds number, Re, of the flow for each dataset, and fill the relevant row in the results table.

1.5 Velocity Increments

Consider the longitudinal velocity increment

$$\delta u_{||}(x, l) \equiv u(x + l) - u(x). \tag{9}$$

1.5.1 Complete the function `/functions/increment.m` that computes the velocity increments for a given l value.

1.5.2 For $l \in \{1\,mm,\ 1\,cm,\ 10\,cm,\ 10\,m\}$, plot $\delta u_{||}$ against x for suitable x- and $\delta u_{||}$-ranges (Plot D) (use the script `/programs/PART5_velocity_increments.m`).

1.5.3 Where do these l-values lie relative to the turbulent length scales you determined from the energy spectrum?

1.5.4 Describe the differences in the signal between small and large length scales. Explain whether or not your observations can reveal the trend in the autocorrelation curve you plotted in Section 1.2.

1.6 Statistics of Velocity Increments

For each l, the probability distribution of $\delta u_{||}$ has a bell-shaped curve.

1.6.1 Complete the function `/functions/fit_gaussian.m` that returns a Gaussian distribution fit for a given arbitrary probability density function (PDF).

1.6.2 Plot the PDF of $\delta u_{||}$ for $l \in \{1\,mm,\ 1\,cm,\ 10\,cm,\ 10\,m\}$ (Plot E) and compare it to a Gaussian distribution (use the script `/programs/PART6_statistics.m`).

1.6.3 Discuss how the distribution changes with l.

1.6.4 Discuss in what way the l-dependence of the PDF supports or contradicts the assumption of self-similarity, on which K41 is based. Comment on the different Reynolds number data.

1.7 Structure Functions and Energy Dissipation

The n^{th}-order structure function is defined as

$$S_n(l) \equiv \left\langle \delta u_{||}^n(x,l) \right\rangle \; ; \qquad n \in \mathbb{N}. \tag{10}$$

1.7.1 Complete the function `/functions/structure_function.m` which computes the n^{th}-order structure function for a series of l values.

1.7.2 Plot $S_2(l)$ and $S_3(l)$ versus l for the three datasets in a log-log scale using the script `/programs/PART7_1_structure_functions.m` (Plots F and G).

1.7.3 K41 predicts a scaling of S_2 as a function of l. How is the scaling of $S_2(l)$ related to scaling of the spectral energy spectrum $E(k)$? Fill the expected slope value in the script.

1.7.4 According to K41, the third order structure function follows the four-fifth law. Fill the predicted slope value in the script and discuss to which extent your data supports the K41 prediction.

1.7.5 Update the script to include integral scale (L_C) and the Taylor scale (λ) of each dataset and discuss the relevance of these scales for the K41 predictions of S_2 and S_3.

1.7.6 K41 theory predictions for the structure function scalings are expected to hold true in the $l \to 0$ limit. Does the data support this prediction? Explain the trend for increasing Reynold's number.

1.7.7 How can you estimate the energy dissipation rate ϵ from $S_2(l)$ and $S_3(l)$? Complete the corresponding program `/programs/PART7_2_dissipation_estimate.m`. Discuss how the estimates compare to the estimates based on integral-scale quantities in Section 1.4 and fill in the values in the corresponding table (Note: According to K. Sreenivasan, the prefactor of the second order structure function is $C_2 \approx 2.1$ with $S_p(l) = C_p \epsilon^{p/3} l^{p/3}$).

1.8 Flatness of Velocity Increments

The flatness of the velocity increment signal $\delta u_{||}(x,l)$ is defined as

$$f(l) \equiv \frac{\left\langle \delta u_{||}^4(x,l) \right\rangle}{\left\langle \delta u_{||}^2(x,l) \right\rangle^2}. \tag{11}$$

For this:

1.8.1 Plot the flatness as a function of l using the script `/programs/PART8_flatness.m` (Plot H) and mark the relevant turbulent length scales.

1.8.2 Relate the behavior in flatness to the shape of the PDFs of $\delta u_{||}$.

1.8.3 Explain the convergence behavior of $f(l)$ for large l. Mark the corresponding value in the plot.

1.8.4 Do the data support the validity of the assumption of self-similarity (H2) of K41? (Comment briefly)

2 Interpretation and Discussion

(1/2 page max.) Based on the data analysis, to which extent does the K41 theory derived for $Re \to \infty$ limit describe the "real" data at finite Reynold's numbers?

Hints / Suggestions

The following ideas might be helpful:

- The purpose of this project is to develop deeper understanding of turbulence and not to assess your programming skills. Therefore, you are allowed (and encouraged) to employ built-in functions for calculating statistical quantities whenever such a function or library is available, but be careful! You always need to consult the documentation of the function or library you use to make sure it returns exactly the intended quantity.
- When developing your analysis code, use only part of the full data (to speed up the computation).
- When giving the results of calculations, whenever possible include the uncertainty in the result.
- When discussing data compared to theoretical predictions, add the predictions to the plot.
- If you make an assumption, check its validity to the extent possible.

Part II: Chaos, its skeleton, and quantitative characterization

Kuramoto–Sivashinsky equation

The one-dimensional (1D) Kuramoto–Sivashinsky equation (KSE) is believed to be the simplest partial differential equation exhibiting spatiotemporal chaos. Despite its simplicity, the KSE displays features similar to the Navier–Stokes equations. Therefore, it is commonly employed as a sandbox model in the development of new concepts and methods for chaotic fluid flows before their application to the 3D Navier–Stokes equations. Here, we specifically explore the description of 'turbulence' in terms of invariant solutions of the governing equations.

The 1D KSE reads

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^4}, \tag{12}$$

where $u(x,t)$ is a real-valued field on the spatial domain $x \in [-L/2, L/2)$, subject to periodic boundary conditions, and varying with time $t \in \mathbb{R}$. The dynamics of the KSE is controlled by the domain length L. In this project, we study the KSE for $L = 38.6$, where the system exhibits spatiotemporal chaos. We consider the KSE within the inversion-symmetric subspace of functions, where

$$u(x,t) = -u(-x,t). \tag{13}$$

Notation

- We define the energy content of $u(x,t)$ as

  $$E(t) := \int_{-L/2}^{L/2} \frac{1}{2} u^2(x,t)\,\mathrm{d}x. \tag{14}$$

  Given this definition of the energy, we define the energy production rate as

  $$P(t) := \int_{-L/2}^{L/2} \left(\frac{\partial u(x,t)}{\partial x}\right)^2 \mathrm{d}x, \tag{15}$$

  and the energy dissipation rate as

  $$D(t) := \int_{-L/2}^{L/2} \left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^2 \mathrm{d}x, \tag{16}$$

  as it can be shown that $\mathrm{d}E(t)/\mathrm{d}t = P(t) - D(t)$. We will use these quantities to create a 3D visualization of the state space.

- Throughout this document, the minimal vector of state variables associated with $u(x,t)$ will be denoted by $\mathbf{v}(t) \in \mathbb{R}^n$, and the standard Euclidean L_2-norm in $\mathbb{R}^n$ will be denoted by $\|\cdot\|$.

Computational tools

Computational tools required for this project are available in the linked repository. You can find the '`Download ZIP`' button under the '`Code`' tab to download the folder containing script files. Refer to the instructions provided therein for information on how to use the scripts.

Report

As for the first part of the project, use the provided template to prepare your report. Discuss your results and analysis in the corresponding subsections. No extended introduction or final discussion is needed.

3 Chaotic behavior of the KSE

3.1 Qualitative investigation

We aim to investigate error amplification due to chaos by perturbing an arbitrary snapshot along a chaotic trajectory and visualizing the deviation of the perturbed trajectory from the reference one. To this end, complete `/programs/chaos_task1.m` based on the following steps:

- Step 1: Start with the initial condition

$$u(x, 0) = \sin(2\pi x/L). \tag{17}$$

  Advance the KSE in time for 1,000 time units to ensure transient effects have vanished and the trajectory has settled onto the chaotic attractor.

- Step 2: Advance the final state from step (1) for 250 time units. Plot the space–time contour of the resulting evolution, $u_1(x, t)$ (plot A, panel 1).

- Step 3: Perturb the state vector obtained at the end of step (1), $\mathbf{v}(1000)$, by adding a random vector $\mathbf{r}$ constructed as follows:

$$\mathbf{r}_k = \epsilon \cdot (2\eta - 1) \cdot \mathbf{v}_k(1000), \quad k = 1, 2, \ldots, n, \tag{18}$$

  where $\epsilon = 10^{-2}$ is the relative amplitude of component perturbations, η is a random number in the interval $(0, 1)$, and k is the component index of the vector. Advance the perturbed state, $\mathbf{v}(1000) + \mathbf{r}$, for 250 time units. Plot the space–time contour of the resulting evolution, $u_2(x, t)$ (plot A, panel 2).

- Step 4: Plot the absolute difference $|u_1(x, t) - u_2(x, t)|$ (plot A, panel 3).

- Discuss your observations: How long does it typically take for the two evolutions to begin to deviate qualitatively? Due to the random generation of the perturbations, you may wish to repeat the observation, but demonstrate only one of the cases in plot A.

3.2 Quantitative characterization

We aim to compute a characteristic time scale for the separation of nearby trajectories based on the leading Lyapunov exponent of the dynamics, and to investigate whether it is consistent with our visual inspection in Sec. 3.1.

- Complete the function `/functions/Jacobian.m` that constructs the Jacobian matrix

$$\mathbf{J}^t_{\mathbf{v}(0)} := \frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}\mathbf{v}(0)}. \tag{19}$$

  As the state vectors are $n \times 1$ column vectors, the Jacobian is an $n \times n$ matrix, with

$$\left(\mathbf{J}^t_{\mathbf{v}(0)}\right)_{ij} = \frac{\partial \mathbf{v}_i(t)}{\partial \mathbf{v}_j(0)}, \quad i, j = 1, 2, \ldots, n. \tag{20}$$

  Use central differences to numerically construct the Jacobian matrix.

- Compute the 10 leading Lyapunov exponents, χ_1 to χ_{10}, of the system using the so-called standard algorithm (see the lecture slides):
  - Complete the script `/programs/chaos_task2.m`. Consider a total duration of 10,000 time units, including $M = 5{,}000$ uniformly distributed re-normalizations, i.e., every $\tau = 2$ time units. Use Eq. (17) as the initial condition.

  - Plot the convergence of $\{\chi_i\}_{i=1}^{10}$ against time (plot B). Report the final values in the relevant results table.

- Discuss your results. Make sure the following points are addressed in a short (!!) discussion.
  - How many zero values did you expect in the spectrum of Lyapunov exponents? Are the obtained values sufficiently separated to distinguish which of the computed exponents correspond to the expected zero value(s)?
  - Was the suggested duration of 10,000 time units sufficiently long for the Lyapunov exponents to practically converge to their asymptotic values?
  - Compute the Lyapunov time,
$$t_L^* := \frac{1}{\chi_1}, \tag{21}$$
    where χ_1 is the leading Lyapunov exponent. How does t_L^* compare to the typical time you determined in Sec. 3.1 for a complete loss of temporal correlation?

4 Invariant sets embedded in the chaotic attractor

We aim to compute equilibria and periodic orbits embedded within the chaotic attractor of the KSE, and to investigate their relevance to the dynamics. Due to their instability, these simple solutions cannot be found naturally by time-marching the chaotic system. Therefore, we must first make an initial guess and then try to converge the guess to an exact invariant solution using an appropriate numerical algorithm. We extract initial guesses for both equilibria and periodic orbits from a time series of length 2,500 time units, using Eq. (17) as the initial condition. For the computations related to this section, complete `/programs/chaos_task3.m`.

4.1 Equilibrium solutions

Make rough initial guesses and attempt to compute equilibrium solutions as follows:

- Select six arbitrary snapshots from the trajectory, and use each one as input to the convergence algorithm `/functions/search4EQ.m`. Report the time instant of the selected snapshots and the corresponding search result. For each successful search, plot the initial guess and the corresponding converged solution in physical state (Plot C, one panel for each guess–solution pair). Report the values of E and $P = D$ in the results table. Use the function `/functions/projection.m` to perform the projection.

- Initialize your search with the following additional six guesses:
$$u(x) = \sin\left(k(2\pi x/L)\right), \quad k = 1, \ldots, 6. \tag{22}$$
  Do these guesses result in any additional solution? For each successful search, plot the initial guess and the corresponding converged solution in physical state (Plot D, one panel for each guess–solution pair). Also report the corresponding values of E and $P = D$.

- Visualize the chaotic attractor by projecting the chaotic trajectory onto E, P, and D (plot E). Overlay the computed equilibrium solutions as markers in the same plot.

- Compare the relevance of the equilibrium solutions you have found to the dynamics. Succinctly explain the evidence supporting your conclusion.

4.2 Periodic orbits

A chaotic attractor is expected to be populated with unstable periodic orbits (UPOs). At any given time, the chaotic trajectory is shadowing some UPO. A UPO is detectable if it is followed for at least one cycle, as this leads to close recurrences in the time series. Similar to equilibria, once a UPO is guessed, it needs to be computed exactly using an appropriate convergence algorithm. We aim to compute some UPOs and investigate their relevance to the dynamics.

- Compute and plot the recurrence indicator defined as follows (plot F):

$$r(t,T) := \frac{\|\mathbf{v}(t+T)-\mathbf{v}(t)\|}{\|\mathbf{v}(t)\|}, \quad T>0. \tag{23}$$

  Hint: Plotting $\ln(r(t,T))$ can help make the minima with lower values easier to identify.

- Based on visual inspection, select five pairs of (t,T) corresponding to a local minimum of r. Select the pairs such that various values of $T<120$ are included. Pass $\mathbf{v}(t)$ (as the guessed periodic state) and T (as the guessed period) to the convergence algorithm `/functions/search4PO.m`. Report the search results in the relevant results table. If five guesses were not enough, try additional points until you find at least three UPOs.

- For each UPO, time-march the KSE for one full cycle starting from the converged state. Plot, again, the chaotic attractor in the 3D projection onto E, P, and D, and overlay the computed UPOs (plot G).

- Determine the stability of the computed UPOs based on the Floquet multipliers λ_i, that are the eigenvalues of the Jacobian $\mathbf{J}^T_{\mathbf{v}_p}$. Here, $\mathbf{v}_p$ represents the converged state vector, and T is the period of the UPO. Report the modulus of the leading Floquet multiplier in the results table.

- Compute a time scale for divergence of nearby trajectories from each UPO:

$$t_F^* = \frac{T}{\ln|\lambda_1|}, \tag{24}$$

  where T is the period of the UPO and λ_1 is the Floquet multiplier with the largest modulus. Report t_F^* in the relevant results table.

- Discuss your results. Make sure the following questions are addressed in the discussion.
  - Based on the 3D projection of the chaotic attractor and the UPOs, do you find all of the computed UPOs relevant in supporting the chaotic dynamics of the KSE?
  - How do the values of t_F^* compare to the value of t_L^*? Discuss whether this result was expected, and briefly explain your reasoning.
  - The period T of a UPO is the shortest nonzero time duration after which the state of the system repeats itself. However, the definition of UPO is satisfied for any integer multiple of T, and the convergence algorithm cannot distinguish whether it has found the fundamental period T or a multiple of it, mT, $m=2,3,\ldots$. Using properties of the Jacobian, demonstrate that in the latter case, t_F^* would remain unchanged, while $|\lambda_1|$ would be raised to the power of m. Based on this analysis, is any of your converged UPOs actually a multiple-cycle version of another one?
  - Based on your observations in this section and those in Sec. 4.1, do you think that UPOs are more relevant, less relevant, or equally relevant compared to equilibria for representing features and properties of the chaotic dynamics? Briefly (!!) explain your reasoning.