Units for 'pyprop8'

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The wave equation as stated in [1] is

$$T_{ij,j} + f_i = \rho \partial_t^2 u_i$$

with

$$\begin{aligned} T_{ij} &= \mu(u_{i,j} + u_{j,i}) + \lambda u_{k,k}\delta_{ij} \\ \mu &= \rho v_s^2 \end{aligned}$$

and f_i representing force *per unit volume* (see [2])

$$f_i = (F_i - M_{ij}\partial_j)\delta(\mathbf{x} - \mathbf{x}_s)H(t)$$

The purpose of this note is to derive the units of u_i given the units of all other quantites.

We assume that all input quantities are specified in arbitrary units that can nevertheless be related back to the SI system:

$$\begin{aligned} \text{distance} &\rightarrow \alpha\,[\,\mathrm{m}\,] \\ v_s, v_p &\rightarrow \alpha\,[\,\mathrm{m}\,]\,[\,\mathrm{s}\,]^{-1} \\ \rho &\rightarrow \beta\,[\,\mathrm{kg}\,]\,[\,\mathrm{m}\,]^{-3} \\ F_i &\rightarrow \gamma\,[\,\mathrm{kg}\,]\,[\,\mathrm{m}\,]\,[\,\mathrm{s}\,]^{-2} \\ M_{ij} &\rightarrow \epsilon\,[\,\mathrm{kg}\,]\,[\,\mathrm{m}\,]^2\,[\,\mathrm{s}\,]^{-2} \end{aligned}$$

For example, if velocities are expressed in km/s we would have $\alpha = 10^3$. The resulting seismogram has units

$$u_i \rightarrow \zeta\,[\,\mathrm{m}\,]$$

We are effectively seeking an expression for ζ in terms of α, β, γ and ϵ.

We now determine the units of derived quantities. Straightforwardly,

$$\lambda, \mu \rightarrow \alpha^2\beta\,[\,\mathrm{kg}\,]\,[\,\mathrm{m}\,]^{-1}\,[\,\mathrm{s}\,]^{-2}$$

while $u_{i,j}$ requires us to recognise that the coordinate system is expressed in units of distance, so that

$$u_{i,j} \to \frac{\zeta}{\alpha}$$

Thus we have

$$T_{ij} \to \alpha\beta\zeta\,[\,\mathrm{kg}\,]\,[\,\mathrm{m}\,]^{-1}\,[\,\mathrm{s}\,]^{-2}$$
$$T_{ij,j} \to \beta\zeta\,[\,\mathrm{kg}\,]\,[\,\mathrm{m}\,]^{-2}\,[\,\mathrm{s}\,]^{-2}$$

and straightforwardly

$$\partial_t^2 u_i \to \zeta\,[\,\mathrm{m}\,]\,[\,\mathrm{s}\,]^{-2}$$
$$\rho\partial_t^2 u_i \to \beta\zeta\,[\,\mathrm{kg}\,]\,[\,\mathrm{m}\,]^{-2}\,[\,\mathrm{s}\,]^{-2}$$

showing that $T_{ij,j}$ and $\rho\partial_t^2 u_i$ are dimensionally consistent, as required.

For the force term, we have

$$M_{ij}\delta_j \to \frac{\epsilon}{\alpha}\,[\,\mathrm{kg}\,]\,[\,\mathrm{m}\,]\,[\,\mathrm{s}\,]^{-2}$$

and we must have

$$\gamma = \frac{\epsilon}{\alpha}$$

for dimensional consistency. Noting that the delta-function acts as a volumetric density distribution, we have

$$\delta(\mathbf{x} - \mathbf{x}_s) \to \frac{1}{\alpha^3}\,[\,\mathrm{m}\,]^{-3}$$

and thus

$$f_i \to \frac{\epsilon}{\alpha^4}\,[\,\mathrm{kg}\,]\,[\,\mathrm{m}\,]^{-2}\,[\,\mathrm{s}\,]^{-2}$$

For this to be dimensionally-consistent with the other two terms, we require

$$\frac{\epsilon}{\alpha^4} = \beta\zeta$$

Thus we find

$$\zeta = \frac{\epsilon}{\alpha^4\beta}$$

and we have required $\gamma = \epsilon/\alpha$. We have also assumed that the spatial dimensions of the model, and the wave velocities, are expressed in the same system (i.e., α is shared by both).

If we express all model quantities in SI units (i.e. distances in metres, velocities in m/s, densities in kg/m^3, forces in N and moments in N m, then all constants are 1 and output is also in SI units of metres. However doing so may be numerically inconvenient.

One convenient alternative, with 'nice' numerical values for earth-like materials:

$$\begin{aligned} \text{distance} &\to \text{km} & &\Longrightarrow \alpha = 10^3 \\ v_p, v_s &\to \text{km/s} & & \\ \rho &\to \text{g/cm}^3 & &\Longrightarrow \beta = 10^3 \end{aligned}$$

Thus $\zeta = 10^{-15}\epsilon$. If we wish output to be in mm, which is convenient, then we require $\epsilon = 10^{12}$ i.e. moment should be expressed in TN m (or, equivalently, GN km). GCMT moment tensor values are quoted in units of dyne cm, or equivalently 10^{-7}N m. We therefore need to multiply the numerical value of catalogue source parameters by a factor of 10^{-19} in order to achieve output in mm[1]. For completeness, we note that this system implies $\gamma = 10^9$ i.e. any force term should be expressed in GN.

References

[1] T.B. O'Toole and J.H. Woodhouse. Numerically stable computation of complete synthetic seismograms including the static displacement in plane layered media. *Geophysical Journal International*, 187:1516–1536, 2011.

[2] J.H. Woodhouse and A.F. Deuss. Earth's free oscillations. In B. Romanowicz and A. Dziewoński, editors, *Seismology and structure of the Earth*, volume 1 of *Treatise on Geophysics*, chapter 2, pages 31–65. Elsevier, Amsterdam, 2007.

[1] Note that $x[\,\text{units}\,] = \frac{x}{\alpha}[\alpha\,\text{units}]$