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Extension of the Eulerian–Lagrangian description to nonlinear perturbations in an arbitrary inviscid flow

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ABSTRACT

A new theoretical formulation is proposed to describe the propagation of large amplitude perturbations in an arbitrary flow of inviscid fluid. This formulation relies on the mixed Eulerian–Lagrangian description, for which exact nonlinear equations are derived. Using properties of Hamiltonian formalism, generalized expressions are found for the acoustic pseudo-energy density and flux. They verify a conservative energy balance. All these expressions only depend on the reference flow variables and the displacement vector of the fluid particles due to the perturbation. When the small perturbation assumption is added, the linear expressions from the literature are retrieved.

1. Introduction

The aeroacoustic phenomena involved in aeronautical problems sometimes exhibit a nonlinear behaviour. For example, one of the most typical signatures of the noise radiated by a turbofan engine inlet during aircraft take-off and climb, the so-called "buzz-saw" noise, results from the nonlinear propagation of shock waves produced at the fan tip when the latter is rotating fast enough so that the relative air flow at the fan tip is locally supersonic. The nonlinear effects are crucial for this noise source since they are driving the energy transfer from the frequencies generated by the source (namely the fan) to their harmonics. In the literature, some models for the nonlinear propagation of buzz-saw noise are based on Burgers' equation or Kuznetsov's equation [1–3].

Generally speaking, most of the models for nonlinear acoustics are obtained by truncated expansions at orders greater than one (generally at second order) of fluid dynamics equations so that the nonlinear propagation can be represented by the evolution of a scalar quantity (e.g. velocity potential or pressure), ruled by a single nonlinear equation. Besides the previously mentioned equations, one can also cite, among many others, Aanonsen [4], Westervelt and Korteveg-de Vries equations [5]. Hence, despite their mathematical subtleness and elegance, these models involve approximations of the underlying physics.

Another difficulty of aeronautical problems is the complex nature of the flow, even when linear propagation is well suited to the problem. Under particular assumptions, either on the flow field (potential flow, parallel shear flow or slowly varying flow) or on the acoustic field (high-frequency wave), propagation equations can be derived. But in general cases, Euler or Navier–Stokes equations, full or linearized, have to be solved to describe the evolution of the perturbations [6,7].

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Nomenclature

Latin letters

$\mathcal{A}$	generic physical variable
A	volumic density of $\mathcal{A}$
c	sound speed
C_v	specific heat for constant volume
$\mathcal{D}$	material domain
e	total energy per unit mass
E	total energy perturbation
$\mathbf{F}$	deformation tensor
$\mathbf{G}$	inverse of $\mathbf{F}$
H	Hamiltonian energy density
$\mathbf{I}$	total energy flux vector
$\mathbf{Id}$	identity matrix
J	Jacobian determinant
$\mathcal{L}$	Lagrangian density
m	particle
M	current point in space $\mathcal{V}$
$\mathbf{N}$	matrix of $[\mathbf{grad}\ \tilde{\xi}]$ cofactors
O	origin of coordinates
p	pressure
$Q_{\mathcal{A}}$	source term for $\mathcal{A}$ balance equation
r	gas constant
s	entropy per unit mass
$\mathbf{S}$	Hamiltonian energy flux vector
$\mathcal{S}$	material surface
t	time
T	temperature
$\mathbf{T}$	matrix of $[\mathbf{F}]$ cofactors
u	internal energy per unit mass
$\mathbf{V}$	velocity vector
$\mathcal{V}$	material volume
x_1,x_2,x_3	Cartesian coordinates

Greek letters

γ	specific heat ratio
δ_{ij}	Kronecker's symbol
Θ_1	first invariant of a tensor (trace)
Θ_2	second invariant of a tensor
Θ_3	third invariant of a tensor (determinant)
$\mathbf{\Phi}_{\mathcal{A}}$	flux vector for $\mathcal{A}$ balance equation
ρ	density
ξ	displacement vector

Operators

$\otimes$	tensorial product: $[U_i]\otimes[V_j]=[U_iV_j]$
$\overline{\otimes}$ or $\cdot$	contracted tensorial product: $[A_{ij}]\overline{\otimes}[B_{ij}]=[A_{ik}B_{kj}]$
$\overline{\overline{\otimes}}$	twice contracted tensorial product: $[A_{ij}]\overline{\overline{\otimes}}[B_{ij}]=A_{ij}B_{ji}$
$\mathrm{d}/\mathrm{d}t$	convective derivative related to the reference flow

Subscripts

0	reference configuration
m	quantity per unit mass

Superscripts

$'$	Eulerian perturbation
$\tilde{}$	first kind Lagrangian perturbation
$\hat{}$	second kind Lagrangian perturbation
T	matrix transposition
$-T$	inverse matrix transposition

Whatever the relevant equations, the most common way to introduce the perturbation is based on the standard Eulerian description: the perturbation is defined as the difference between the perturbed flow and a so-called reference flow (mostly defined by the time-averaged flow field) *for a given position*. However, even within a linear framework and for an inviscid fluid, this Eulerian description makes very difficult the definition of a suitable acoustic energy balance. Indeed, the total energy fluctuation includes not only energy fluctuation due to acoustic waves, but also other kinds of energy fluctuation coming from non-acoustic waves (such as turbulence or hydrodynamic waves) and even from local nonwavy fluctuations. The problem of extracting the acoustic energy, related to acoustic waves only, from the total energy fluctuation is therefore yet an open problem. For small perturbations, several acoustic energy balance equations have been proposed for the last decades [8–14]. To obtain mathematically exact acoustic energy balance equations, an extra energy source term must often be added on the right-hand side. The physical meaning of this source term is not obvious: without any physical acoustic source inside a given volume of fluid, it does not cancel, except in particular cases where both reference and perturbed flows are potential, or when only high-frequency waves are concerned (geometric acoustics). A true energy conservation equation in Eulerian description was obtained by Möhring [15–17] but he had to introduce new variables like Clebsch potentials, making these expressions complex and difficult to use for practical purpose.

Up to now, the mixed Eulerian–Lagrangian description of flow perturbations has been found to be the most elegant method to deal with this problem, as far as small perturbations of an inviscid flow are concerned. This approach was first introduced by Galbrun [18] for acoustics. It has also been used in plasma physics [19]. Within this frame, each physical variable is perturbed *for a given fluid particle*, in a Lagrangian manner, rather than for a given position, as seen above for the Eulerian approach. A linearized propagation equation (called Galbrun's equation) can be derived for a single unknown vector which represents the displacement of the particle due to the small flow perturbation. The velocity, density and pressure perturbations can be expressed as functions of this displacement vector and eliminated from the propagation equation [20]. Due to the existence of a variational form for Galbrun's equation, numerical solutions with finite-volume methods have been undertaken [21–24]. Besides, Myers' acoustic boundary condition on a flexible wall with grazing flow

[25] takes a simple form when expressed with the mixed Eulerian–Lagrangian description [21]. The Galbrun equation is also well-suited to boundary conditions between a fluid and a solid [22] or between a fluid and a porous medium [24]. This comes from the fact that the displacement variable is the natural variable when wave propagation in elastic bodies is addressed.

Moreover, within this approach, an acoustic energy conservation theorem has been established for arbitrary steady flows [21,26]. The expressions of the acoustic pseudo-energy density and flux only depend on the displacement vector. No extra source term is necessary and this conservation law encompasses the previous acoustic energy balances of Cantrell and Hart or Myers when potential flows or high-frequency waves are considered [26]. Similar results, together with a reciprocity theorem, were also found independently by Godin [27,28]. The relevance of these results for linear acoustics was set out in a companion paper [29], together with physical interpretation and illustration on basic configurations.

The aim of the present work is to extend the above linearized results to describe the evolution of fully nonlinear perturbations in inviscid fluids within the frame of mixed Eulerian–Lagrangian description. No assumption will me made on the order of magnitude of the perturbations and therefore all the terms will be retained, without approximation. However, like in the linear case, the thermoviscous dissipation will not be taken into account, even though this dissipation may sometimes be an important mechanism when dealing with nonlinear waves.

2. Fluid thermodynamic model

In aeroacoustics problems, the medium considered is air. Most of the time, air is assumed to behave as an inviscid perfect gas, whose motion is described basically by Euler equations. Euler equations represent the mass, momentum and total energy balances, with the assumption of inviscid fluid. In conservative form and without body forces, they read:

$$\frac{\partial \rho}{\partial t}+\mathrm{div}\,(\rho \mathbf{V})=0 \tag{1}$$

$$\frac{\partial \rho \mathbf{V}}{\partial t}+\mathbf{div}(\rho \mathbf{V}\otimes \mathbf{V})=-\mathbf{grad}\,p \tag{2}$$

$$\frac{\partial \rho e}{\partial t}+\mathrm{div}\,(\rho e \mathbf{V})=\mathrm{div}\,(-p\mathbf{V}) \tag{3}$$

where ρ is the fluid density, $\mathbf{V}$ the velocity, p the pressure, $e=u+\frac{1}{2}\mathbf{V}^2$ the total energy per unit mass and u the internal (or potential) energy per unit mass.

The motion of an inviscid fluid is isentropic, since the entropy per unit mass s satisfies:

$$\frac{\partial \rho s}{\partial t}+\mathrm{div}\,(\rho s \mathbf{V})=0 \tag{4}$$

Finally, the fluid behaviour is described by the perfect gas law:

$$p=\rho r T \tag{5}$$

where r is the gas constant and T the temperature. For a calorically perfect gas, the specific heat at constant volume C_v is constant. With all these assumptions, some classical thermodynamic relations can be stated:

$$u=C_v T \tag{6}$$

$$s=C_v\,\ln\left(\frac{p}{\rho^\gamma}\right) \tag{7}$$

3. Mixed Eulerian–Lagrangian description

3.1. Flow perturbation

Acoustic waves can be seen as perturbations of the physical variables describing a given medium. In this way, each of these variables is expanded as the sum of a reference variable and a perturbation. The perturbation expansion can be achieved along two different ways: the first way relies on the Eulerian description, which is classically used in many books and studies [9,11,12,14,30]; the second way relies on a less familiar approach, called the mixed Eulerian–Lagrangian description [19,20,26,29,31]. The key idea of this description is that the perturbation of a physical variable should be described for a given fluid particle rather than for a given position. This idea can be explained with the scheme in Fig. 1. Let m be a fluid particle. Without perturbation of the fluid motion, m would move along the reference trajectory (dashed curve) and its position at time t would be M_0. As the fluid motion is perturbed, m actually moves along a slightly different trajectory (perturbed trajectory, plain curve). Its position at time t is M. All the perturbations are assumed to start at time $t=0$, *i.e.* they all cancel for $t\leq 0$. Both reference and perturbed flows verify the Euler equations set (1)–(3).

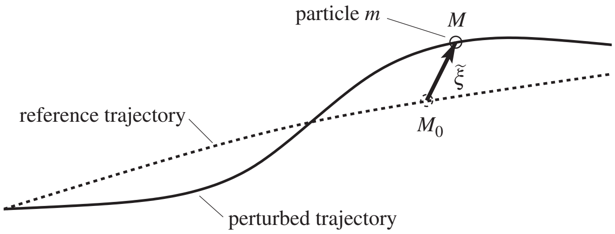

Fig. 1. Displacement vector definition.

3.2. Eulerian description of perturbations

Let $\mathcal{A}$ be any physical variable. It takes the value A_0 in the unperturbed (reference) flow, and A in the perturbed flow. Hence, the Eulerian perturbation A' is defined *for a given position M* and at time t as:

$$A'(M,t) = A(M,t) - A_0(M,t) \tag{8}$$

Once applied to the set of Eqs. (1)–(3), the perturbed Euler equations can be derived for the pressure, density and velocity.

3.3. Lagrangian description of perturbations

Again, let $\mathcal{A}$ be a physical quantity related to the particle m. The value of $\mathcal{A}$ would be A_0 along the reference trajectory while it is A along the perturbed trajectory. The Lagrangian perturbation $\widetilde{A}$ for the particle m at time t can be defined as:

$$\widetilde{A}(m,t) = A(m,t) - A_0(m,t) \tag{9}$$

The perturbation $\widetilde{A}$ can be interpreted as the perturbation of $\mathcal{A}$ for a given elementary mass.

3.4. Mixed Eulerian–Lagrangian description of perturbations

The mixed Eulerian–Lagrangian description [20,29] means that the perturbation at time t is described as a function of the particle position at the same time in the reference flow M_0. In this way, definition (9) becomes

$$\widetilde{A}(M_0,t) = A(M,t) - A_0(M_0,t) \tag{10}$$

It is possible to link the Eulerian and mixed Eulerian/Langrangian perturbations thanks to Eqs. (8) and (10):

$$A'(M,t) = \widetilde{A}(M_0,t) - \Big(A_0(M,t) - A_0(M_0,t)\Big) \tag{11}$$

This relation shows that the Eulerian and mixed Eulerian/Lagrangian representations are actually equivalent. If one can solve the perturbation problem in the mixed Eulerian/Lagrangian description, i.e. know $\widetilde{A}$ for all positions M_0 and how to move from M_0 to M at each time t, then A' can be deduced. It is worth noting that for an homogeneous reference field of $\mathcal{A}$, the last term on the right-hand side of Eq. (11) vanishes and then $A'(M,t) = \widetilde{A}(M_0,t)$. In this case, the perturbations have the same values at a given time t but they are actually assigned to different positions. This fact constitutes a difference with the linear case for small perturbations, where Eulerian and Lagrangian perturbations coincide at first order for a uniform reference flow [20,29].

4. Nonlinear formulation based on the mixed Eulerian–Lagrangian description

Contrary to the previous works involving mixed Eulerian–Lagrangian description, no hypothesis is done here about the order of magnitude of the perturbations. For this reason, the proposed nonlinear formulation is said to be exact, unlike the weakly nonlinear formulation based on second-order asymptotic expansions of the perturbations. Some general equations concerning the perturbations are now introduced.

4.1. Particle displacement and coordinates transformations

Most of the results presented in this section can be found in Continuum Mechanics textbooks. Therefore, the demonstrations will not be detailed.

A cartesian coordinates system (x_1,x_2,x_3) associated with an orthonormal basis constitutes the reference frame. With Lagrangian approach, the motion of particle m can be described as

$$\overrightarrow{OM_0} = \xi_0(m,t) \tag{12}$$

with the reference flow and

$$\overrightarrow{OM} = \xi(m,t) \tag{13}$$

with the perturbed flow. The displacement vector $\widetilde{\xi}$ due to the perturbation can be defined as:

$$\widetilde{\xi}(m,t) = \widetilde{\xi}(M_0,t) = \overrightarrow{M_0M} = \xi(m,t) - \xi_0(m,t) \tag{14}$$

Differentiation of relation (14) gives

$$\mathrm{d}\overrightarrow{\mathrm{OM}} = \mathbf{F} \cdot \mathrm{d}\overrightarrow{\mathrm{OM}_0} \quad \text{with} \quad \mathbf{F} = \mathbf{Id} + \mathbf{grad}\,\widetilde{\xi} \tag{15}$$

The second-order tensor $\mathbf{F}$ represents the local deformation due to the perturbation. Somewhat tedious but straightforward calculations give the following expression for the determinant of $\mathbf{F}$:

$$J = \det \mathbf{F} = 1 + \Theta_1(\mathbf{grad}\,\widetilde{\xi}) + \Theta_2(\mathbf{grad}\,\widetilde{\xi}) + \Theta_3(\mathbf{grad}\,\widetilde{\xi}) \tag{16}$$

where Θ_1, Θ_2 and Θ_3 are the tensor invariants:

$$\Theta_1(\mathbf{grad}\,\widetilde{\xi}) = \mathrm{Tr}\,(\mathbf{grad}\,\widetilde{\xi}) = \mathrm{div}\,\widetilde{\xi} \tag{17}$$

$$\Theta_2(\mathbf{grad}\,\widetilde{\xi}) = \tfrac{1}{2}[(\mathrm{div}\,\widetilde{\xi})^2 - \mathbf{grad}\,\widetilde{\xi}\,\overline{\overline{\otimes}}\,\mathbf{grad}\,\widetilde{\xi}] \tag{18}$$

$$\Theta_3(\mathbf{grad}\,\widetilde{\xi}) = \det(\mathbf{grad}\,\widetilde{\xi}) \tag{19}$$

The expression of $\mathbf{F}^{-1}$, inverse of $\mathbf{F}$, reads:

$$\mathbf{G} = \mathbf{F}^{-1} = \frac{1}{J}\mathbf{T}^T \tag{20}$$

with

$$\mathbf{T} = (1 + \mathrm{div}\,\widetilde{\xi})\mathbf{Id} - \mathbf{grad}^T\widetilde{\xi} + \mathbf{N} \tag{21}$$

and

$$\mathbf{N} = \Theta_3(\mathbf{grad}\,\widetilde{\xi})\;\mathbf{grad}^{-T}\widetilde{\xi} \tag{22}$$

A more convenient expression of $\mathbf{N}$ can be derived, starting with $\mathbf{F}\overline{\otimes}\mathbf{G} = \mathbf{Id}$ and taking into account (15), (16), (20) and (21) and $N^T\overline{\otimes}\mathbf{grad}\,\widetilde{\xi} = \Theta_3(\mathbf{grad}\,\widetilde{\xi})$:

$$\mathbf{N} = \Theta_2(\mathbf{grad}\,\widetilde{\xi})\mathbf{Id} - \mathrm{div}\,\widetilde{\xi}\;\mathbf{grad}^T\widetilde{\xi} + \left(\mathbf{grad}\,\widetilde{\xi}\,\overline{\otimes}\,\mathbf{grad}\,\widetilde{\xi}\right)^T \tag{23}$$

Moreover, it can be shown that:

$$\mathbf{divT} = \mathbf{divN} = 0$$

4.2. Derivatives transformations

If $\mathcal{V}$ is a material volume, the volume elements $\mathrm{d}\mathcal{V}$ and $\mathrm{d}\mathcal{V}_0$ are related by the formula:

$$\mathrm{d}\mathcal{V} = \mathrm{J}\,\mathrm{d}\mathcal{V}_0 \tag{24}$$

If $\mathcal{S}$ represents the outer surface of $\mathcal{V}$, the surface element vectors $\mathrm{d}\boldsymbol{\mathcal{S}}$ and $\mathrm{d}\boldsymbol{\mathcal{S}}_0$ are related by the formula:

$$\mathrm{d}\boldsymbol{\mathcal{S}} = \mathrm{J}\,\mathbf{G}^T \cdot \mathrm{d}\boldsymbol{\mathcal{S}}_0 = \mathbf{T} \cdot \mathrm{d}\boldsymbol{\mathcal{S}}_0 \tag{25}$$

In mixed Eulerian–Lagrangian description, using chain derivation rules, the Lagrangian time derivative of perturbation $\widetilde{A}$ can be expressed as:

$$\frac{\partial \widetilde{A}}{\partial t}(m,t) = \frac{\partial \widetilde{A}}{\partial t}(M_0,t) + \mathbf{grad}\,\widetilde{A}(M_0,t) \cdot \mathbf{V}_0(M_0,t) = \frac{\mathrm{d}\widetilde{\mathrm{A}}}{\mathrm{dt}}(M_0,t) \tag{26}$$

which can be read as a convective derivative related to the reference flow. In particular, the particle velocity is defined as

$$\mathbf{V} = \frac{\partial \xi}{\partial t}(m,t) = \frac{\partial \boldsymbol{\xi}_0}{\partial t}(m,t) + \frac{\partial \widetilde{\xi}}{\partial t}(m,t) = \mathbf{V}_0(M_0,t) + \widetilde{\mathbf{V}}(M_0,t) \tag{27}$$

Eqs. (26) and (27) entail

$$\widetilde{\mathbf{V}}(M_0,t) = \frac{\mathrm{d}\widetilde{\xi}}{\mathrm{dt}} \tag{28}$$

The velocity perturbation is thus equal to the convective derivative of the displacement due to the perturbation, like in the linear case [20,26,29].

Then, using chain derivation rules and the Green–Ostrogradsky theorem, the space derivatives of the perturbed field can be expressed with **G** and **T**:

$$\mathbf{grad}\;\mathbf{P}(M,t) = \mathbf{grad}\left(\mathbf{P}_0(M_0,t) + \widetilde{\mathbf{P}}(M_0,t)\right)\overline{\otimes}\mathbf{G}(M_0,t) \tag{29}$$

$$\mathbf{div}\;\mathbf{P}(M,t) = \frac{1}{J(M_0,t)}\mathbf{div}\left[\left(\mathbf{P}_0(M_0,t) + \widetilde{\mathbf{P}}(M_0,t)\right)\overline{\otimes}\mathbf{T}(M_0,t)\right] \tag{30}$$

where **P** is an arbitrary tensor.

4.3. Perturbation of a physical variable

Let $\mathcal{A}$ be a physical variable defined for any particle m of the material domain $\mathcal{D}$. Without loss of generality, $\mathcal{A}$ will be assumed to be scalar. An extension to vectors or tensors of order greater or equal to 2 can be done easily by applying the following results to each component. The value per unit volume of $\mathcal{A}$ is given by the function A in the perturbed configuration and by the function A_0 in the reference configuration. A first type of perturbation has been introduced in Section 3.4, Eq. (10).

Another kind of quantity perturbation will be defined now. The global values of the physical variable $\mathcal{A}$ for both configurations, *i.e.* the local values $A(M,t)$ and $A_0(M_0,t)$ integrated over the material domain $\mathcal{D}$, constitute the starting point of this definition. For a given time, integration over the material domain $\mathcal{D}$ is equivalent to integration over the set of particle positions, *i.e.* over volume $\mathcal{V}$ in the perturbed configuration and over volume $\mathcal{V}_0$ in the reference configuration. These integrals read

$$I_{\mathcal{A}}(t) = \int_{\mathcal{V}} A(M,t)\,\mathrm{d}\mathcal{V}$$

$$I_{\mathcal{A}_0}(t) = \int_{\mathcal{V}_0} A_0(M_0,t)\,\mathrm{d}\mathcal{V}_0$$

Let us calculate the difference between these integrals:

$$\widehat{I}_{\mathcal{A}}(t) = I_{\mathcal{A}}(t) - I_{\mathcal{A}_0}(t) \tag{31}$$

Using (24), relation (31) becomes:

$$\widehat{I}_{\mathcal{A}}(t) = \int_{\mathcal{V}_0}\left(A(M,t)J(M_0,t) - A_0(M_0,t)\right)\mathrm{d}\mathcal{V}_0 \tag{32}$$

The new perturbation is defined as the integrand of integral (32):

$$\widehat{A}(M_0,t) = A(M,t)\,J(M_0,t) - A_0(M_0,t) \tag{33}$$

According to Eq. (32), $\widehat{A}$ can be seen as the local perturbation of $\mathcal{A}$ per unit volume of the reference flow. Thus $\widehat{A}$ includes both the variation of $\mathcal{A}$ for a given particle and the change of the number of particles inside the volume due to the compressibility of the medium. On the other hand, the mixed Eulerian/Lagrangian perturbation $\widetilde{A}$ reflects only the variation of $\mathcal{A}$ for the same given particle, and can be therefore related to the variation of $\mathcal{A}$ per unit mass, since a given particle has a constant mass. Relations (10) and (33) yield a relation between “tilde” and “hat” perturbations:

$$\widehat{A}(M_0,t) = \widetilde{A}(M_0,t)\,J + (J-1)A_0(M_0,t) \tag{34}$$

where the arguments (M_0,t) of J have been dropped out. This second kind of perturbation will help to find general equations governing the evolution of perturbations as shown in the next section.

4.4. Evolution equations for the perturbation of a physical variable

4.4.1. General equations for $\widehat{A}$

The physical variable $\mathcal{A}$ (*e.g.* mass, momentum, total energy, etc.) is now supposed to satisfy a local balance equation:

$$\frac{\partial A}{\partial t} + \mathrm{div}\,(A\mathbf{V}) = Q_{\mathcal{A}} + \mathrm{div}\,\mathbf{\Phi}_{\mathcal{A}} \tag{35}$$

where $Q_{\mathcal{A}}$ and $\mathbf{\Phi}_{\mathcal{A}}$ represent respectively the source term and the flux density of $\mathcal{A}$. The definition of perturbation $\widehat{A}$ will prove to be useful. Indeed, time derivation of relation (31) gives:

$$\frac{\mathrm{d}\widehat{\mathrm{I}}_{\mathcal{A}}}{\mathrm{dt}}(t) = \frac{\mathrm{dI}_{\mathcal{A}}}{\mathrm{dt}}(t) - \frac{\mathrm{dI}_{\mathcal{A}_0}}{\mathrm{dt}}(t) \tag{36}$$

Using (35), it stems[1]:

$$\frac{\mathrm{dI}_{\mathcal{A}}}{\mathrm{dt}}(t)=\int_{\mathcal{V}}\left(Q_{\mathcal{A}}(M,t)+\mathrm{div}\ \mathbf{\Phi}_{\mathcal{A}}(M,t)\right)\mathrm{d}\mathcal{V} \tag{37}$$

$$\frac{\mathrm{dI}_{\mathcal{A}_0}}{\mathrm{dt}}(t)=\int_{\mathcal{V}_0}\left(Q_{\mathcal{A}_0}(M_0,t)+\mathrm{div}\ \mathbf{\Phi}_{\mathcal{A}_0}(M_0,t)\right)\mathrm{d}\mathcal{V}_0 \tag{38}$$

Moreover, the left-hand term of Eq. (36) can be written as:

$$\frac{\mathrm{d}\widehat{\mathrm{I}}_{\mathcal{A}}}{\mathrm{dt}}(t)=\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathcal{V}_0}\widehat{A}(M_0,t)\mathrm{d}\mathcal{V}_0=\int_{\mathcal{V}_0}\left(\frac{\partial\widehat{A}}{\partial t}(M_0,t)+\mathrm{div}\ (\widehat{A}\mathbf{V}_0)(M_0,t)\right)\mathrm{d}\mathcal{V}_0$$

Substituting (24) in Eq. (37), Eq. (36) yields the following local formulation:

$$\frac{\partial\widehat{A}}{\partial t}(M_0,t)+\mathrm{div}\ (\widehat{A}\mathbf{V}_0)(M_0,t)=\left[JQ_{\mathcal{A}}(M,t)-Q_{\mathcal{A}_0}(M_0,t)\right]+\left[J\mathrm{div}\ \mathbf{\Phi}_{\mathcal{A}}(M,t)-\mathrm{div}\ \mathbf{\Phi}_{\mathcal{A}_0}(M_0,t)\right]$$

Finally, the following equation holds almost everywhere[2]:

$$\frac{\partial\widehat{A}}{\partial t}(M_0,t)+\mathrm{div}\ (\widehat{A}\mathbf{V}_0)(M_0,t)=\widehat{Q}_{\mathcal{A}}(M_0,t)+\widehat{\mathrm{div}\ \mathbf{\Phi}}_{\mathcal{A}}(M_0,t) \tag{39}$$

with

$$\widehat{\mathrm{div}\ \mathbf{\Phi}}_{\mathcal{A}}(M_0,t)=J\ \mathrm{div}\ \mathbf{\Phi}_{\mathcal{A}}(M,t)-\mathrm{div}\ \mathbf{\Phi}_{\mathcal{A}_0}(M_0,t) \tag{40}$$

This expression can be transformed using Eq. (30):

$$\widehat{\mathrm{div}\ \mathbf{\Phi}}_{\mathcal{A}}(M_0,t)=\mathrm{div}\ [(\mathbf{\Phi}_{\mathcal{A}_0}+\widetilde{\mathbf{\Phi}}_{\mathcal{A}})\overline{\otimes}\mathbf{T}-\mathbf{\Phi}_{\mathcal{A}_0}] \tag{41}$$

Eq. (39) can also be rewritten as:

$$\frac{\mathrm{d}\widehat{\mathrm{A}}}{\mathrm{dt}}(M_0,t)+\widehat{A}(M_0,t)\mathrm{div}\ \mathbf{V}_0(M_0,t)=\widehat{Q}_{\mathcal{A}}(M_0,t)+\widehat{\mathrm{div}\ \mathbf{\Phi}}_{\mathcal{A}}(M_0,t) \tag{42}$$

For the sake of clarity, the argument (M_0,t) will be omitted in the following when no confusion is possible between the two sets of positions. The term div $\mathbf{V}_0$ in Eq. (42) can be transformed using the mass conservation Eq. (1) for the reference flow:

$$\frac{\mathrm{d}\widehat{\mathrm{A}}}{\mathrm{dt}}-\frac{\widehat{A}}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{dt}}=\widehat{Q}_{\mathcal{A}}+\widehat{\mathrm{div}\ \mathbf{\Phi}}_{\mathcal{A}}$$

Hence

$$\rho_0\ \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\widehat{A}}{\rho_0}\right)=\widehat{Q}_{\mathcal{A}}+\widehat{\mathrm{div}\ \mathbf{\Phi}}_{\mathcal{A}} \tag{43}$$

4.4.2. Application to density perturbation

A first result about density perturbation is now obtained thanks to Eq. (43). It will be useful for the next section. For this quantity, the right-hand term of Eq. (43) cancels because $Q_\rho=0$ and $\mathbf{\Phi}_\rho=0$. Indeed, mass conservation holds independently of the material domain configuration $\mathcal{D}$. So

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\widehat{\rho}}{\rho_0}\right)=0\Rightarrow\frac{\widehat{\rho}}{\rho_0}=\mathrm{constant}$$

Since the perturbation is equal to zero at instant $t\le 0$, the constant above is also zero. Hence

$$\widehat{\rho}(M_0,t)=0 \tag{44}$$

From (34), it stems:

$$\widetilde{\rho}=\frac{1-J}{J}\rho_0 \tag{45}$$

As J is a function of **grad** $\widetilde{\xi}$ defined by Eq. (16), the density perturbation $\widetilde{\rho}$ only depends on **grad** $\widetilde{\xi}$ and on the reference density field ρ_0.

[1] Discontinuities such as interfaces between media and shock waves are not considered in the present work but they can be introduced here.

[2] I.e. except on a limited set of discontinuities.

These results could have been derived in another way. Due to mass conservation, the mass of a material element dm is independent of the continuum configuration, *i.e.* $\mathrm{d}m=\rho \mathrm{d}\mathcal{V}=\rho_0 \mathrm{d}\mathcal{V}_0$. Since $\mathrm{d}\mathcal{V}=\mathrm{J}\ \mathrm{d}\mathcal{V}_0$, then $\rho J=\rho_0$. Substituting $\rho=\rho_0+\widetilde{\rho}$, Eq. (45) is found again.

4.4.3. Application to other variables

It is now considered that the value per unit volume of the physical variable $\mathcal{A}$ is written $A=\rho A_m$, where A_m is the value per unit mass of the physical variable $\mathcal{A}$. Substituting $J=\rho_0/\rho$ in Eq. (33) yields

$$\rho(A_0+\widehat{A})=\rho_0 A$$

then

$$\frac{\widehat{A}(M_0,t)}{\rho_0(M_0,t)}=\frac{A(M,t)}{\rho(M,t)}-\frac{A_0(M_0,t)}{\rho_0(M_0,t)}=\widetilde{\left(\frac{A}{\rho}\right)} \tag{46}$$

or more concisely

$$\widehat{A}=\rho_0\widetilde{A}_m \tag{47}$$

It stems

$$\widehat{I}_{\mathcal{A}}(t)=\int_{\mathcal{V}_0}\widehat{A}\ \mathrm{d}\mathcal{V}_0=\int_{\mathcal{D}}\widetilde{\mathrm{A}}_m\ \mathrm{dm}$$

This illuminates the relation between the two kinds of perturbation. The quantity $\widehat{A}$ appears as the perturbation of $\mathcal{A}$ per unit of the reference volume $\mathcal{V}_0$. Eq. (43) becomes

$$\rho_0\frac{\mathrm{d}\widetilde{\mathrm{A}}_m}{\mathrm{dt}}=\widehat{\mathrm{Q}}_{\mathcal{A}}+\mathrm{div}\ \widehat{\boldsymbol{\Phi}}_{\mathcal{A}} \tag{48}$$

Finally, the evolution equation for the perturbation $\widetilde{A}_m$ reads:

$$\rho_0\frac{\mathrm{d}\widetilde{\mathrm{A}}_m}{\mathrm{dt}}=\widehat{\mathrm{Q}}_{\mathcal{A}}+\mathrm{div}\ [(\boldsymbol{\Phi}_{\mathcal{A}_0}+\widetilde{\boldsymbol{\Phi}}_{\mathcal{A}})\overline{\otimes}\ \mathbf{T}-\boldsymbol{\Phi}_{\mathcal{A}_0}] \tag{49}$$

Or, in conservative form

$$\frac{\partial\rho_0\widetilde{A}_m}{\partial t}+\mathrm{div}\ (\rho_0\widetilde{A}_m\mathbf{V}_0)=\widehat{\mathrm{Q}}_{\mathcal{A}}+\mathrm{div}\ [(\boldsymbol{\Phi}_{\mathcal{A}_0}+\widetilde{\boldsymbol{\Phi}}_{\mathcal{A}})\overline{\otimes}\ \mathbf{T}-\boldsymbol{\Phi}_{\mathcal{A}_0}] \tag{50}$$

where $\mathbf{T}$ is defined by Eq. (21).

Eqs. (49) and (50), describing the evolution of perturbation $\widetilde{A}_m$, are functions of the reference flow, of source and flux perturbations for $\mathcal{A}$, and of the tensor **grad** $\widetilde{\xi}$. These equations, together with relations (28) and (45) linking the velocity and density perturbations to the displacement $\widetilde{\xi}$, are the most general set of equations describing the evolution of arbitrary perturbations in a material domain.

5. Application to an inviscid perfect gas

The equation set derived in Section 4 is now applied to the motion of inviscid fluids, leading to a generalization of the previously known linearized equations [20,26,29]. In a first part, equations for the displacement $\widetilde{\xi}$ and the pressure perturbation $\widetilde{p}$ are derived. In a second part, the pseudo-energy conservation law is dealt with.

5.1. Displacement and pressure perturbations

Considering the balance equations for momentum (2) and entropy (4), it stems:

$$Q_{\rho\mathbf{v}}=0 \quad \boldsymbol{\Phi}_{\rho\mathbf{v}}=-p\mathbf{Id}$$

and

$$Q_{\rho s}=0 \quad \boldsymbol{\Phi}_{\rho s}=\mathbf{0}$$

Hence, the sources perturbations are

$$\widetilde{Q}_{\rho\mathbf{v}}=0 \quad \widetilde{\boldsymbol{\Phi}}_{\rho\mathbf{v}}=-\widetilde{p}\mathbf{Id}$$

and

$$\widetilde{Q}_{\rho s}=0 \quad \widetilde{\boldsymbol{\Phi}}_{\rho s}=\mathbf{0}$$

Eq. (49) is first applied to entropy:

$$\frac{\mathrm{d}\tilde{s}}{\mathrm{d}t}=0$$

Since the perturbation cancels for time $t \le 0$, the above equation implies that $\tilde{s}=0$ for any time and any particle. Thus, the perturbation is isentropic. However, no hypothesis is necessary on the uniformity of the reference flow entropy s_0, like in the Eulerian description. This is a key feature of the mixed Eulerian/Lagrangian description. Thus, the coming equations are still valid for non-homentropic reference flows.

Moreover, the thermodynamic relation (7) yields

$$\tilde{s}=s-s_0=C_v \ln\left[\frac{p}{p_0}\left(\frac{\rho_0}{\rho}\right)^{\gamma}\right]=0$$

Hence

$$1+\frac{\tilde{p}}{p_0}=\left(1+\frac{\tilde{\rho}}{\rho_0}\right)^{\gamma}$$

Replacing Eq. (45) in the above equation yields

$$\tilde{p}=(J^{-\gamma}-1)p_0 \tag{51}$$

Like the mixed Eulerian/Lagrangian perturbation of density, the pressure perturbation is only a function of $\mathbf{grad}\,\tilde{\xi}$ and of the reference pressure field p_0.

Now, Eq. (49) is applied to the momentum. As $\tilde{\mathbf{V}}=\mathrm{d}\tilde{\xi}/\mathrm{d}t$, we obtain:

$$\rho_0\frac{\mathrm{d}\tilde{\mathbf{V}}}{\mathrm{d}t}=\rho_0\frac{\mathrm{d}^2\tilde{\xi}}{\mathrm{d}t^2}=\mathbf{div}[(-p_0-\tilde{p})\mathbf{Id}\overline{\otimes}\,\mathbf{T}+p_0\mathbf{Id}]$$

Since $\mathbf{div}\,\mathbf{T}=0$, the above equation becomes

$$\rho_0\frac{\mathrm{d}^2\tilde{\xi}}{\mathrm{d}t^2}=-\mathbf{T}\overline{\otimes}(\mathbf{grad}\,p_0+\mathbf{grad}\,\tilde{p})+\mathbf{grad}\,p_0 \tag{52}$$

Since $\tilde{p}$ and $\mathbf{T}$ are functions of $\mathbf{grad}\,\tilde{\xi}$ via relations (51) and (21), the only unknown of Eq. (52) is $\tilde{\xi}$. This equation can be recast in order to eliminate the tensor $\mathbf{T}$. Using the identity $\mathbf{G}\overline{\otimes}\mathbf{F}=\mathbf{Id}$, it can be shown that

$$\mathbf{T}^{-1}=\frac{1}{J}(\mathbf{Id}+\mathbf{grad}^T\tilde{\xi})$$

Calculating $\mathbf{T}^{-1}\overline{\otimes}$ (52) yields

$$\rho_0\frac{1}{J}(\mathbf{Id}+\mathbf{grad}^T\tilde{\xi})\overline{\otimes}\frac{\mathrm{d}^2\tilde{\xi}}{\mathrm{d}t^2}=-\mathbf{grad}\,\tilde{p}+\frac{\tilde{\rho}}{\rho_0}\mathbf{grad}\,p_0+\frac{1}{J}(\mathbf{grad}^T\tilde{\xi})\overline{\otimes}(\mathbf{grad}\,p_0) \tag{53}$$

where $\tilde{p}$ and $\tilde{\rho}$ can be eliminated with Eqs. (45) and (51). Finally, for a given reference field, $\tilde{\xi}$ remains the only unknown vector of Eqs. (52) and (53).

As a conclusion of this section, it can be stated that, for a given reference field ρ_0, $\mathbf{V}_0$ and p_0, all the perturbations $\tilde{\rho}$, $\tilde{\mathbf{V}}$ and $\tilde{p}$ are only functions of the displacement $\tilde{\xi}$, which is governed by the nonlinear Eq. (53). In this way, Eq. (52) or (53) represents a nonlinear extension of the Galbrun equation. Once the displacement is known from one of these two equations, then the mixed Eulerian/Lagrangian perturbations of pressure, density and velocity can be calculated, and finally the Eulerian perturbations can be deduced, as stated by relation (11). It can be also noted that Eq. (53) does not contain explicit gradients of the reference velocity field V_0, allowing to address discontinuous shear flows without Dirac distributions. It also greatly simplifies when the reference pressure p_0 is constant.

5.2. Energy conservation

Defining the acoustic energy in a moving medium is a tricky challenge, since it represents only a part of the total energy perturbation. For this reason, Godin [27,28] calls it “acoustic pseudo-energy”. Following the path used in the linear case [29], the derivation of a pseudo-acoustic energy balance will be processed in two steps. The first step consists in applying the nonlinear balance equation (50) to calculate the total energy perturbation. The second step relies on Hamiltonian formalism: splitting the total energy perturbation obtained in the first step into kinetic and potential parts, a Lagrangian density is built first, and then derived to obtain Hamiltonian density and flux which are found to be nonlinear extensions of Brazier’s expressions for pseudo-acoustic energy and flux for linear small perturbations. In the linear case, these Hamiltonian quantities have been found to be kindly related to the acoustic waves in nonuniform moving media, where non-acoustic perturbations also exist. Moreover, they reduce to the usual acoustic energy density and flux for a quiet homogeneous gas.

5.2.1. First step: balance equation for the total energy perturbation

The nonlinear balance Eq. (50) is applied to the total energy per unit volume, ρe. Eq. (3) yields

$$Q_{\rho e} = 0 \quad \mathbf{\Phi}_{\rho e} = -p\mathbf{V}$$

So the source terms perturbations are

$$\widetilde{Q}_{\rho e} = 0$$

$$\widetilde{\mathbf{\Phi}}_{\rho e} = -\widetilde{p\mathbf{V}} = -\widetilde{p}\widetilde{\mathbf{V}} - \widetilde{p}\mathbf{V}_0 - p_0\widetilde{\mathbf{V}}$$

As Eq. (50) is written in a conservative form, the perturbations of total energy density and flux E and $\mathbf{I}$ can be defined by

$$E = \widehat{\rho e} = \rho_0 \widetilde{e}$$

$$\mathbf{I} = E\mathbf{V}_0 - (\mathbf{\Phi}_{\rho_0 e_0} + \widetilde{\mathbf{\Phi}}_{\rho e}) \overline{\otimes}\, \mathbf{T} + \mathbf{\Phi}_{\rho_0 e_0}$$

E and $\mathbf{I}$ satisfy

$$\frac{\partial E}{\partial t} + \mathrm{div}\, \mathbf{I} = 0$$

Developing E, it stems

$$E = \rho_0 \widetilde{e} \tag{54}$$

$$E = \rho_0 (\widetilde{u} + \tfrac{1}{2}\widetilde{\mathbf{V}\cdot\mathbf{V}}) \tag{55}$$

$$E = \rho_0 \widetilde{u} + \rho_0 \mathbf{V}_0 \cdot \frac{\mathrm{d}\widetilde{\xi}}{\mathrm{dt}} + \frac{1}{2}\rho_0 \left(\frac{\mathrm{d}\widetilde{\xi}}{\mathrm{dt}}\right)^2 \tag{56}$$

The perturbation $\widetilde{u}$ reads:

$$\widetilde{u} = C_v \widetilde{T}$$

Using the perfect gas law (5), the following relation can be stated:

$$\widetilde{T} = \frac{1}{r}\widetilde{\left(\frac{p}{\rho}\right)}$$

Eq. (47), applied with $A = p/\rho$, yields $\widetilde{(p/\rho)} = \widehat{p}/\rho_0$, so

$$\widetilde{T} = \frac{1}{r}\frac{\widehat{p}}{\rho_0}$$

Moreover, $C_v = r/(\gamma - 1)$. Thus

$$\widetilde{u} = \frac{1}{\gamma - 1}\frac{\widehat{p}}{\rho_0}$$

Finally,

$$E = \frac{\widehat{p}}{\gamma - 1} + \rho_0 \mathbf{V}_0 \cdot \frac{\mathrm{d}\widetilde{\xi}}{\mathrm{dt}} + \frac{1}{2}\rho_0 \left(\frac{\mathrm{d}\widetilde{\xi}}{\mathrm{dt}}\right)^2 \tag{57}$$

where $\widehat{p}$ is obtained from Eqs. (51) and (34):

$$\widehat{p} = (J^{1-\gamma} - 1)p_0 \tag{58}$$

The flux $\mathbf{I}$ reads:

$$\mathbf{I} = E\mathbf{V}_0 + (p_0 + \widehat{p})(\mathbf{V}_0 + \widetilde{\mathbf{V}}) \overline{\otimes}\, \mathbf{T} - p_0 \mathbf{V}_0 \tag{59}$$

It is now possible to build a Lagrangian density based on the above expression of E and thus derive the expressions of the acoustic pseudo-energy density and flux via Hamiltonian formalism.

5.2.2. Second step: application of Hamiltonian formalism

Inspired by the approach followed by Élias [26] and Brazier [29] in the linear case, nonlinear expressions can be found out for the acoustic pseudo-energy density and flux thanks to properties resulting from Hamilton's theory [32]. The reference flow is now assumed to be steady. Actually, this assumption is not restrictive since the perturbations are still arbitrary and hence the unsteady character of the flow can be included into the perturbed part.

The Lagrangian $\mathcal{L}$ of a conservative system is the difference between the kinetic and potential parts of the total energy. In our case, the reference flow is given and we are only concerned with the perturbation. Therefore, the energy perturbation E is used to define the perturbation Lagrangian. In expression (55), the first term corresponds to the potential energy whereas the second one corresponds to the kinetic energy. Hence

$$\mathcal{L} = \rho_0(\widetilde{\tfrac{1}{2}\mathbf{V}\cdot\mathbf{V}} - \widetilde{u}) \tag{60}$$

or

$$\mathcal{L} = \frac{1}{2}\rho_0\left(\frac{\mathrm{d}\widetilde{\xi}}{\mathrm{dt}}\right)^2 + \rho_0\mathbf{V}_0\cdot\frac{\mathrm{d}\widetilde{\xi}}{\mathrm{dt}} - \frac{\widehat{p}}{\gamma-1} \tag{61}$$

One can check that the nonlinear propagation equation (52) can be retrieved from the Euler–Lagrange equation (see Appendix A for more details). Then, an energy per unit volume H and an energy flux $\mathbf{S}$ can be derived from the Lagrangian density $\mathcal{L}$ by

$$H = \frac{\partial\widetilde{\xi}}{\partial t}\cdot\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\widetilde{\xi}}{\partial t}\right)} - \mathcal{L} \tag{62}$$

$$\mathbf{S} = \frac{\partial\widetilde{\xi}}{\partial t}\cdot\frac{\partial\mathcal{L}}{\partial\,\mathbf{grad}\,\widetilde{\xi}} \tag{63}$$

If the Lagrangian density (61) does not depend on time except through unknown functions, *i.e.* for a steady reference flow, the stress-energy tensor has zero divergence [32] and then the above expressions verify[3]:

$$\frac{\partial H}{\partial t} + \mathrm{div}\,\mathbf{S} = 0 \tag{64}$$

In the present case, using the derivatives of the Lagrangian density from Appendix A, the expressions for H and $\mathbf{S}$ read

$$H = \rho_0\left(\mathbf{V}_0 + \frac{\mathrm{d}\widetilde{\xi}}{\mathrm{dt}}\right)\cdot\frac{\partial\widetilde{\xi}}{\partial t} - \frac{1}{2}\rho_0\left(\frac{\mathrm{d}\widetilde{\xi}}{\mathrm{dt}}\right)^2 - \rho_0\mathbf{V}_0\cdot\frac{\mathrm{d}\widetilde{\xi}}{\mathrm{dt}} + \frac{\widehat{p}}{\gamma-1} \tag{65}$$

$$\mathbf{S} = \rho_0\left(\frac{\partial\widetilde{\xi}}{\partial t}\cdot\frac{\mathrm{d}\widetilde{\xi}}{\mathrm{dt}}\right)\mathbf{V}_0 + \rho_0\left(\frac{\partial\widetilde{\xi}}{\partial t}\cdot\mathbf{V}_0\right)\mathbf{V}_0 + (p_0+\widetilde{p})\,\overline{\frac{\partial\widetilde{\xi}}{\partial t}\otimes\mathbf{T}} \tag{66}$$

Considering Eq. (64), for any arbitrary vector $\mathbf{w}$, then $H + \mathrm{div}\,\mathbf{w}$ and $\mathbf{S} - (\partial\mathbf{w}/\partial t)$ also satisfy the same conservation equation. Taking $\mathbf{w} = \rho_0(\widetilde{\xi}\cdot\mathbf{V}_0)\mathbf{V}_0 + p_0\widetilde{\xi}$, the linear and nonlinear terms of H and $\mathbf{S}$ can be separated:

$$H = H_1 + H_2$$

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$$

with

$$H_1 = -\mathrm{div}\,\mathbf{w} = \rho_0\mathbf{V}_0\cdot\frac{\partial\widetilde{\xi}}{\partial t} - \rho_0\mathbf{V}_0\cdot\frac{\mathrm{d}\widetilde{\xi}}{\mathrm{dt}} - p_0\,\mathrm{div}\,\widetilde{\xi} \tag{67}$$

$$\mathbf{S}_1 = \frac{\partial\mathbf{w}}{\partial t} = \rho_0\left(\frac{\partial\widetilde{\xi}}{\partial t}\cdot\mathbf{V}_0\right)\mathbf{V}_0 + p_0\frac{\partial\widetilde{\xi}}{\partial t} \tag{68}$$

and

$$H_2 = H + \mathrm{div}\,\mathbf{w} = \frac{\widehat{p}}{\gamma-1} + p_0\,\mathrm{div}\,\widetilde{\xi} + \rho_0\frac{\mathrm{d}\widetilde{\xi}}{\mathrm{dt}}\cdot\frac{\partial\widetilde{\xi}}{\partial t} - \frac{1}{2}\rho_0\left(\frac{\mathrm{d}\widetilde{\xi}}{\mathrm{dt}}\right)^2 \tag{69}$$

$$\mathbf{S}_2 = \mathbf{S} - \frac{\partial\mathbf{w}}{\partial t} = (p_0+\widetilde{p})\,\overline{\frac{\partial\widetilde{\xi}}{\partial t}\otimes\mathbf{T}} - p_0\frac{\partial\widetilde{\xi}}{\partial t} + \rho_0\left(\frac{\partial\widetilde{\xi}}{\partial t}\cdot\frac{\mathrm{d}\widetilde{\xi}}{\mathrm{dt}}\right)\mathbf{V}_0 \tag{70}$$

Indeed the linear second term $p_0\,\mathrm{div}\,\widetilde{\xi}$ in H_2 expression (69) compensates for a linear contribution hidden in the first term. In the same way, the linear second term $-p_0(\partial\widetilde{\xi}/\partial t)$ in $\mathbf{S}_2$ expression (70) compensates for the linear part of the first term. This will appear more clearly when these expressions are expanded for small perturbations in Section 5.3. The four expressions above verify the following conservation equations:

$$\frac{\partial H_1}{\partial t} + \mathrm{div}\,\mathbf{S}_1 = 0$$

[3] This property is also related to Noether's theorem.

$$\frac{\partial H_2}{\partial t}+\operatorname{div}\mathbf{S}_2=0$$

H_2 and $\mathbf{S}_2$ will be considered as the generalized expressions of acoustic pseudo-energy density and flux, as explained in the coming section.

5.3. Small perturbations

The nonlinear results derived up to now will be compared with the linear equations previously obtained for small perturbations [20,29]. Now, the perturbation is supposed to create small deformations and displacements. Keeping only terms which are $O(\|\mathbf{grad}\,\tilde{\xi}\|)$, the following first-order approximations are obtained:

$$J=1+\Theta_1(\mathbf{grad}\,\tilde{\xi})=1+\operatorname{div}\tilde{\xi} \tag{71}$$

$$\mathbf{T}=\mathbf{Id}-\mathbf{grad}^T\tilde{\xi}+\operatorname{div}\tilde{\xi}\,\mathbf{Id} \tag{72}$$

Indeed, $\Theta_2(\mathbf{grad}\,\tilde{\xi})=O(\|\mathbf{grad}\,\tilde{\xi}\|^2)$, $\Theta_3(\mathbf{grad}\,\tilde{\xi})=O(\|\mathbf{grad}\,\tilde{\xi}\|^3)$ and $\|\mathbf{N}\|=O(\|\mathbf{grad}\,\tilde{\xi}\|^2)$. Expression (45) for $\tilde{\rho}$ becomes:

$$\tilde{\rho}=-\rho_0\operatorname{div}\tilde{\xi}$$

The perturbation $\tilde{p}$, given by expressions (51) and (71), becomes

$$\tilde{p}=-\gamma p_0\operatorname{div}\tilde{\xi}$$

Introducing the speed of sound $c_0^2=\gamma(p_0/\rho_0)$, it stems:

$$\tilde{p}=-\rho_0 c_0^2\operatorname{div}\tilde{\xi}=c_0^2\tilde{\rho}$$

These expressions are identical to those obtained by Poirée and presented in Refs. [20,26,29]. Besides, the relation (11) between Eulerian and mixed Eulerian/Lagrangian perturbations becomes

$$A'(M_0,t)=\tilde{A}(M_0,t)-\mathbf{grad}\,A_0(M_0,t)\cdot\tilde{\xi}+O(\tilde{\xi}^2) \tag{73}$$

with A being p, ρ or any component of $\mathbf{V}$. As stated before in Section 3.4, Eulerian and Lagrangian linear perturbations coincide at first order for a uniform reference flow, which is no longer the case for nonlinear perturbations. Eq. (52) yields:

$$\rho_0\frac{\mathrm{d}^2\tilde{\xi}}{\mathrm{d}t^2}+\mathbf{grad}\,p_0\operatorname{div}\tilde{\xi}-\mathbf{grad}(\rho_0 c_0^2\operatorname{div}\tilde{\xi})-\mathbf{grad}^T\tilde{\xi}\cdot\mathbf{grad}\,p_0=0 \tag{74}$$

which is exactly Galbrun's equation.

The energetic expressions (67)–(70) are now dealt with. Since H_1 and $\mathbf{S}_1$ are linear expressions, they are left unchanged by the small perturbation hypothesis. Second-order terms must be retained now in J and $\hat{p}$ expansions:

$$\hat{p}=p_0(J^{1-\gamma}-1)=p_0\Big((1-\gamma)\operatorname{div}\tilde{\xi}+(1-\gamma)\Theta_2(\mathbf{grad}\,\tilde{\xi})-\tfrac{1}{2}(1-\gamma)\gamma(\operatorname{div}\tilde{\xi})^2\Big)$$

Then, the first two terms of H_2 become:

$$\frac{\hat{p}}{\gamma-1}+p_0\operatorname{div}\tilde{\xi}=-p_0\Theta_2(\mathbf{grad}\,\tilde{\xi})+\frac{\gamma}{2}p_0(\operatorname{div}\tilde{\xi})^2=\frac{p_0}{2}[(\gamma-1)(\operatorname{div}\tilde{\xi})^2+\mathbf{grad}\,\tilde{\xi}\,\overline{\overline{\otimes}}\,\mathbf{grad}\,\tilde{\xi}]$$

And finally

$$H_2=\rho_0\frac{\mathrm{d}\tilde{\xi}}{\mathrm{d}t}\cdot\frac{\partial\tilde{\xi}}{\partial t}-\frac{1}{2}\rho_0\left(\frac{\mathrm{d}\tilde{\xi}}{\mathrm{d}t}\right)^2-\frac{1}{2}(p_0-\rho_0\,c_0^2)(\operatorname{div}\tilde{\xi})^2+\frac{1}{2}p_0\,\mathbf{grad}\,\tilde{\xi}\,\overline{\overline{\otimes}}\,\mathbf{grad}\,\tilde{\xi} \tag{75}$$

$$\mathbf{S}_2=\rho_0\left(\frac{\mathrm{d}\tilde{\xi}}{\mathrm{d}t}\cdot\frac{\partial\tilde{\xi}}{\partial t}\right)\mathbf{V}_0+(p_0-\rho_0 c_0^2)\operatorname{div}\tilde{\xi}\,\frac{\partial\tilde{\xi}}{\partial t}-p_0\,\mathbf{grad}\,\tilde{\xi}\cdot\frac{\partial\tilde{\xi}}{\partial t} \tag{76}$$

It can be checked that as announced earlier, the above small-perturbation expressions for H_2 and $\mathbf{S}_2$ contain only second-order quadratic terms, the first-order linear terms being rejected in H_1 and $\mathbf{S}_1$. Since the variables of the reference flow are frequently time-averaged values, so that the time-average of the perturbation is equal to zero, the energy balance based on H_1 and $\mathbf{S}_1$ is also equal to zero in time-average. Thus, only H_2 and $\mathbf{S}_2$ are of practical interest. All these expressions agree with those found by Brazier [29] within a linearized approach. It was also shown in [29] that they differ from the expressions found by Élias and Godin by the fact that they include the power of the reference pressure on fluctuating volume, which is not included in Élias and Godin's expressions. Therefore, expressions (69) and (70) can be thought of as the generalized nonlinear expressions of the acoustic pseudo-energy density and flux derived by Brazier. Up to now, no nonlinear expressions have been found that correspond exactly for small perturbations to Élias and Godin's pseudo-energy density and flux.

6. Application: 1D nonlinear propagation in a quiet medium

We now consider one-dimensional propagation in an homogeneous quiet gas. Let x be the space coordinate. In one dimension, the vectors $\mathbf{V}_0$, $\widetilde{\boldsymbol{\xi}}$ and $\widetilde{\mathbf{V}}$ are replaced by the scalar quantities $V_0=0$, $\widetilde{\xi}$ and $\widetilde{V}$. Besides, $J=1+(\partial\widetilde{\xi}/\partial x)$ and the tensor $\mathbf{T}$ is reduced to a scalar equal to 1. The reference state being uniform, p_0 and ρ_0 are constant. The sound celerity in the reference flow is $c_0=\sqrt{\gamma p_0/\rho_0}$. Under these simplifications, Eqs. (28), (45) and (51) give

$$\widetilde{\mathbf{V}}=\frac{\mathrm{d}\widetilde{\xi}}{\mathrm{d}t}=\frac{\partial\widetilde{\xi}}{\partial t} \tag{77}$$

$$\frac{\widetilde{\rho}}{\rho_0}=-\frac{\frac{\partial\widetilde{\xi}}{\partial x}}{1+\frac{\partial\widetilde{\xi}}{\partial x}} \tag{78}$$

$$\frac{\widetilde{p}}{p_0}=\left(1+\frac{\partial\widetilde{\xi}}{\partial x}\right)^{-\gamma}-1 \tag{79}$$

The nonlinear Eq. (52) reduces to

$$\rho_0\frac{\partial^2\widetilde{\boldsymbol{\xi}}}{\partial t^2}+\frac{\partial\widetilde{p}}{\partial x}=0 \tag{80}$$

After short manipulations, it can be rewritten as:

$$\frac{\partial^2\widetilde{\boldsymbol{\xi}}}{\partial t^2}-c^2\frac{\partial^2\widetilde{\xi}}{\partial x^2}=0 \tag{81}$$

with

$$c=c_0\left(1+\frac{\partial\widetilde{\xi}}{\partial x}\right)^{-(\gamma+1)/2} \tag{82}$$

Eq. (81) is a classical wave equation with a local sound celerity which is a nonlinear function of $\widetilde{\xi}$. Its resolution is out of the scope of this paper. But a short discussion is proposed about the comparison between this exact nonlinear equation and the classical approach of Weakly NonLinear (WNL in formulae hereafter) waves, based on a truncation at second order. Using Eq. (79), $\partial\widetilde{\xi}/\partial x$ can be eliminated as a function of $\widetilde{p}/p_0$ and replaced in (82):

$$c=c_0\left(1+\frac{\widetilde{p}}{p_0}\right)^{(\gamma+1)/2\gamma} \tag{83}$$

Thus, thanks to the mixed Eulerian/Lagrangian description, a nonlinear sound speed can be very easily derived. It can be noticed that this sound celerity vanishes when $\widetilde{p}=-p_0$, i.e. when the local pressure $p_0+\widetilde{p}$ falls to zero. Assuming weakly nonlinear waves, the sound celerity can be expanded up to first order of $\widetilde{p}/p_0$:

$$c\simeq c_0\left(1+\frac{\gamma+1}{2\gamma}\frac{\widetilde{p}}{p_0}\right)=c_{\mathrm{WNL}} \tag{84}$$

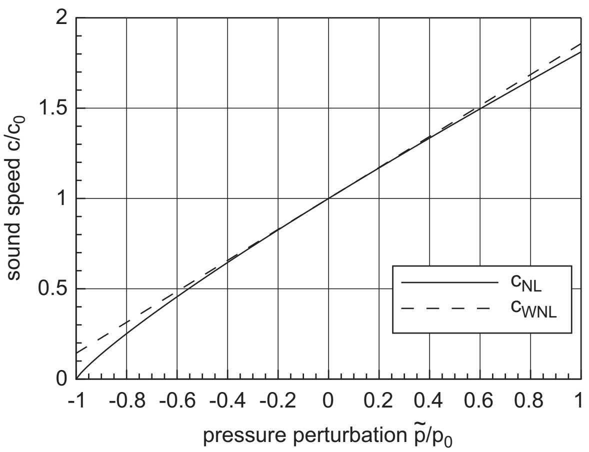

Fig. 2. Nonlinear and weakly nonlinear sound speeds.

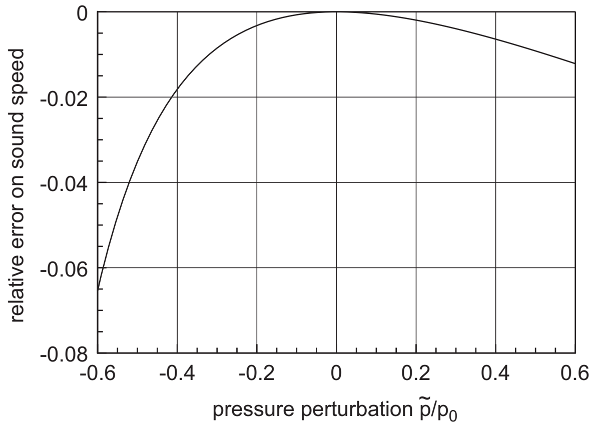

Fig. 3. Relative error on sound celerity due to the weakly nonlinear assumption.

The validity of the weakly nonlinear hypothesis can be evaluated by comparing expressions (83) and (84) of the sound celerity, which have been plotted in Fig. 2 as a function of the perturbation amplitude $\widetilde{p}/p_0$.

It appears that the weakly nonlinear expression constitutes a very good approximation of the fully nonlinear expression on most of the domain. The relative error is plotted in Fig. 3. It remains lower than 6% for a relative pressure perturbation lower than 0.6, that is for a sound level lower than 190 dB for $p_0 = 10^5$ Pa. So it might not be necessary to solve the exact nonlinear equation to cope with nonlinear propagation of typical aircraft noise sources. But it would be interesting to really compare the solutions obtained from the 1D exact nonlinear equation and the weakly nonlinear one on simple applications (e.g. sinus function as initial solution).

7. Conclusion

A theoretical formulation for nonlinear perturbations in an arbitrary reference flow of inviscid fluid has been developed in the frame of mixed Eulerian–Lagrangian description. In particular, no assumption was done concerning the order of magnitude of the perturbations. Balance equations in both integral and local forms were recast in order to provide perturbation equations for the physical variables of the continuum. Since the principle of mass conservation is written identically for any continuum, a general expression of the density perturbation has been found. For the other physical variables, the constitutive equations of the continuum had to be particularized. This was done in the present work for an inviscid perfect gas. An evolution equation was obtained where the displacement vector is the only unknown. This equation can be seen as a nonlinear extension of Galbrun's equation.

Using the same approach as in the linear theory, a Lagrangian density has been built from kinetic and potential energy perturbations. Thanks to this Lagrangian density, a conservation law could be derived for the acoustic pseudo-energy, using Hamiltonian formalism. When the small perturbation hypothesis is added, these new equations can be linearized. In this case, all the previous linear expressions are retrieved.

Although the resolution of the nonlinear equations might be tricky, their successful derivation highlights the relevance of the mixed Eulerian–Lagrangian description. Now, it would be interesting to apply this formulation to some simple nonlinear problems, as one-dimensional nonlinear $\mathcal{L}$propagation with uniform reference flow.

Appendix A. Derivatives of the Lagrangian $\mathcal{L}$

The calculation is performed in a cartesian frame of coordinates. The Lagrangian reads

$$\mathcal{L} = \frac{1}{2}\rho_0\left(\frac{\mathrm{d}\widetilde{\xi}}{\mathrm{d}t}\right)^2 + \rho_0\mathbf{V}_0\overline{\otimes}\frac{\mathrm{d}\widetilde{\xi}}{\mathrm{d}t} - \frac{1}{\gamma-1}\widehat{p} \tag{A.1}$$

with

$$\widehat{p} = (J^{1-\gamma}-1)p_0 \tag{A.2}$$

and J is given by (16). To derive the expressions of pseudo-acoustic energy density and flux, the partial derivatives of $\mathcal{L}$ with respect to $\partial\widetilde{\xi}/\partial t$ and $\mathbf{grad}\,\widetilde{\xi}$ must be calculated. In this process, $\partial\widetilde{\xi}/\partial t$ and $\mathbf{grad}\,\widetilde{\xi}$ are considered as independent variables. Reminding that

$$\frac{\mathrm{d}\widetilde{\xi}}{\mathrm{d}t} = \widetilde{\mathbf{V}} = \frac{\partial\widetilde{\xi}}{\partial t} + \mathbf{grad}\,\widetilde{\xi}\overline{\otimes}\mathbf{V}_0$$

it stems

$$\frac{\partial \widetilde{\mathbf{V}}}{\partial\left(\frac{\partial\widetilde{\xi}}{\partial t}\right)} = \mathbf{Id} \tag{A.3}$$

Since $\widehat{p}$ only depends on $\mathbf{grad}\,\widetilde{\xi}$,

$$\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\widetilde{\xi}}{\partial t}\right)} = \rho_0\frac{\mathrm{d}\widetilde{\xi}}{\mathrm{dt}} + \rho_0\mathbf{V}_0 \tag{A.4}$$

The calculation of the partial derivative relative to $\mathbf{grad}\,\widetilde{\xi}$ is carried out using index notation, with $g_{ij} = \partial\widetilde{\xi}_i/\partial x_j$:

$$\widetilde{V}_i = \frac{\partial\widetilde{\xi}_i}{\partial t} + g_{ij}V_{0j}$$

Hence

$$\frac{\partial}{\partial g_{ij}}\left(\frac{1}{2}\widetilde{\mathbf{V}}^2\right) = \frac{\partial}{\partial g_{ij}}\left(\frac{1}{2}\widetilde{V}_i\widetilde{V}_i\right) = \widetilde{V}_iV_{0j} \tag{A.5}$$

and

$$\frac{\partial\left(\frac{1}{2}\widetilde{\mathbf{V}}^2\right)}{\partial\mathbf{grad}\,\widetilde{\xi}} = \widetilde{\mathbf{V}}\otimes\mathbf{V}_0 \tag{A.6}$$

In the same way

$$\frac{\partial}{\partial g_{ij}}(\mathbf{V}_0\overline{\overline{\otimes}}\widetilde{\mathbf{V}}) = V_{0i}V_{0j}$$

and

$$\frac{\partial\mathbf{V}_0\overline{\overline{\otimes}}\widetilde{\mathbf{V}}}{\partial\mathbf{grad}\,\widetilde{\xi}} = \mathbf{V}_0\otimes\mathbf{V}_0 \tag{A.7}$$

From (A.2) and (51), we have

$$\frac{1}{\gamma-1}\frac{\partial\widehat{p}}{\partial g_{ij}} = \frac{1}{\gamma-1}p_0\frac{\partial}{\partial g_{ij}}(J^{1-\gamma}) \tag{A.8}$$

$$\frac{1}{\gamma-1}\frac{\partial\widehat{p}}{\partial g_{ij}} = -p_0J^{-\gamma}\frac{\partial J}{\partial g_{ij}} \tag{A.9}$$

$$\frac{1}{\gamma-1}\frac{\partial\widehat{p}}{\partial g_{ij}} = -(p_0+\widehat{p})\frac{\partial J}{\partial g_{ij}} \tag{A.10}$$

The Jacobian determinant was given by Eq. (16):

$$J = 1+\mathrm{div}\,\widetilde{\xi}+\tfrac{1}{2}\Big((\mathrm{div}\,\widetilde{\xi})^2-\mathbf{grad}\,\widetilde{\xi}\overline{\overline{\otimes}}\,\mathbf{grad}\,\widetilde{\xi}\Big)+\det(\mathbf{grad}\,\widetilde{\xi})$$

Since $\mathrm{div}\,\widetilde{\xi} = \partial\widetilde{\xi}_i/\partial x_i$, we have

$$\frac{\partial}{\partial g_{ij}}(\mathrm{div}\,\widetilde{\xi}) = \delta_{ij}$$

and therefore

$$\frac{\partial\mathrm{div}\,\widetilde{\xi}}{\partial\mathbf{grad}\,\widetilde{\xi}} = \mathbf{Id} \tag{A.11}$$

Then

$$\frac{\partial}{\partial g_{ij}}(\mathbf{grad}\,\widetilde{\xi}\overline{\overline{\otimes}}\mathbf{grad}\,\widetilde{\xi}) = \frac{\partial g_{kl}g_{lk}}{\partial g_{ij}} \tag{A.12}$$

$$\frac{\partial}{\partial g_{ij}}(\mathbf{grad}\,\widetilde{\xi}\overline{\overline{\otimes}}\mathbf{grad}\,\widetilde{\xi}) = \frac{\partial g_{kl}}{\partial g_{ij}}g_{lk}+g_{kl}\frac{\partial g_{lk}}{\partial g_{ij}} \tag{A.13}$$

$$\frac{\partial}{\partial g_{ij}}(\mathbf{grad}\ \tilde{\xi}\overline{\overline{\otimes}}\mathbf{grad}\ \tilde{\xi}) = \delta_{ik}\delta_{jl}g_{lk} + g_{kl}\delta_{il}\delta_{jk} \tag{A.14}$$

$$\frac{\partial}{\partial g_{ij}}(\mathbf{grad}\ \tilde{\xi}\overline{\overline{\otimes}}\mathbf{grad}\ \tilde{\xi}) = 2g_{ji} \tag{A.15}$$

So:

$$\frac{\partial}{\partial \mathbf{grad}\ \tilde{\xi}}(\mathbf{grad}\ \tilde{\xi}\overline{\overline{\otimes}}\mathbf{grad}\ \tilde{\xi}) = 2\mathbf{grad}^T\tilde{\xi} \tag{A.16}$$

The calculation of the partial derivative of det($\mathbf{grad}\ \tilde{\xi}$) can be performed term by term:

$$\det(\mathbf{grad}\ \tilde{\xi}) = g_{11}(g_{22}g_{33}-g_{23}g_{32}) - g_{21}(g_{12}g_{33}-g_{32}g_{13}) + g_{31}(g_{12}g_{23}-g_{22}g_{13})$$

Thus

$$\left[\frac{\partial \det(\mathbf{grad}\ \tilde{\xi})}{\partial\ \mathbf{grad}\ \tilde{\xi}}\right] = \begin{pmatrix} g_{22}g_{33}-g_{23}g_{32} & -(g_{21}g_{33}-g_{31}g_{23}) & g_{21}g_{32}-g_{31}g_{22} \\ -(g_{12}g_{33}-g_{32}g_{13}) & g_{11}g_{33}-g_{13}g_{31} & -(g_{32}g_{11}-g_{31}g_{12}) \\ g_{12}g_{23}-g_{22}g_{13} & -(g_{11}g_{23}-g_{21}g_{13}) & g_{11}g_{22}-g_{21}g_{12} \end{pmatrix} \tag{A.17}$$

which is the matrix $\mathbf{N}$ of the cofactors of [$\mathbf{grad}\ \tilde{\xi}$]. Hence

$$\frac{\partial \det(\mathbf{grad}\ \tilde{\xi})}{\partial \mathbf{grad}\ \tilde{\xi}} = \mathbf{N} \tag{A.18}$$

Thus, it stems

$$\frac{\partial J}{\partial \mathbf{grad}\ \tilde{\xi}} = (1+\mathrm{div}\ \tilde{\xi})\mathbf{Id} - \mathbf{grad}^T\tilde{\xi} + \mathbf{N} = \mathbf{T} \tag{A.19}$$

and therefore

$$\frac{1}{\gamma-1}\frac{\partial \hat{p}}{\partial \mathbf{grad}\ \tilde{\xi}} = -(p_0+\tilde{p})\mathbf{T} \tag{A.20}$$

Finally,

$$\frac{\partial \mathcal{L}}{\partial \mathbf{grad}\ \tilde{\xi}} = \rho_0\frac{\mathrm{d}\tilde{\xi}}{\mathrm{d}t}\otimes \mathbf{V}_0 + \rho_0\mathbf{V}_0\otimes\mathbf{V}_0 + (p_0+\tilde{p})\mathbf{T} \tag{A.21}$$

It can be checked that Eq. (52) can be retrieved from the Euler–Lagrange equation which reads

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \tilde{\xi}}{\partial t}\right)}\right) + \mathbf{div}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{grad}\ \tilde{\xi}}\right) = \frac{\partial \mathcal{L}}{\partial \tilde{\xi}} \tag{A.22}$$

Clearly

$$\mathcal{L} = \mathcal{L}\left(\frac{\partial \tilde{\xi}}{\partial t}, \mathbf{grad}\ \tilde{\xi}\right)$$

since $\hat{p} = \hat{p}(\mathbf{grad}\ \tilde{\xi})$ according to Eqs. (58) and (16). So $\partial\mathcal{L}/\partial\tilde{\xi} = 0$ and the right-hand-side of Eq. (A.22) is actually zero. Substituting expressions (A.4) and (A.21) into Eq. (A.22) and using mass and momentum equations for the reference flow (1) and (2), the propagation Eq. (52) can be retrieved.

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