FRE-GY 6083 Final Group Project Report
Delta Hedging Optimized by Crank-Nicholson

Thomas Shen, tshen595@gmail.com
Sylvia Ma, ym3378@nyu.edu
Yina Chen, yc4628@nyu.edu

December 2024

Contents

1 Introduction

Risk management requires accurately pricing options to prevent arbitrage opportunities. Accurately-priced options allow risk managers to correctly derive Greeks and as a result take the correct offsetting positions, or optimal hedge, in their portfolio to maintain the desired risk spread.

The Crank-Nicholson (CN) scheme is a widely used finite-difference method for pricing American and Exotic options. It operates on a two-dimensional grid, with one dimension representing time and the other representing the underlying asset price. By using it to solve the Black-Scholes PDE, it provides as table and accurate options pricing framework.

This project proposes a delta hedging stategy that derives the optimal delta hedge by pricing American options with the Crank-Nicolson scheme. We then compares the results of delta hedging with the Crank-Nicolson scheme to the binomial model.

We begin by explaining the Crank-Nicolson framework, followed by an introduction to the concepts of delta and delta hedging. Next, we discuss the Black-Scholes Merton model and its application in delta hedging, as well as the binomial model. Afterwards, we discuss stability and convergence in both models. We then demonstrate how the Crank-Nicolson framework can be utilized to price options using SPX data spanning four months. Subsequently, we present the results of our delta hedging strategy based on the Crank-Nicolson model and compare them with the outcomes from the binomial model. Finally, we address the limitations of delta hedging and propose potential solutions to overcome these challenges.

2 The Crank-Nicolson Framework

The CN scheme averages the explicit and implicit finite difference methods:

$$AV_j^{n+1} = BV_j^n, \tag{1}$$

where A and B are tridiagonal matrices defined as:

$$\begin{aligned} A &= \left(1 + \frac{\Delta t}{2}\left[r + \sigma_i^2 S_i^2\right]\right) \quad \text{(implicit part)}, \\ B &= \left(1 - \frac{\Delta t}{2}\left[r + \sigma_i^2 S_i^2\right]\right) \quad \text{(explicit part)}. \end{aligned}$$

The matrices incorporate second derivatives using central differences:

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{V_{i+1} - 2V_i + V_{i-1}}{\Delta S^2}.$$

Boundary conditions are applied:

- At $S = 0$, $V(0,t) = 0$.
- As $S \to \infty$, $V(S,t) \to S - Ke^{-r(T-t)}$ for a call option.

The CN method solves the Black-Scholes PDE by discretizing both time and the underlying asset price dimensions. It approximates the derivatives using central differences, leading to a weighted average of explicit (forward-time) and implicit (backward-time) schemes:

$$\frac{V_i^{n+1} - V_i^n}{\Delta t} = \frac{1}{2}\left[\mathscr{L}V_i^n + \mathscr{L}V_i^{n+1}\right], \tag{2}$$

where V_i^n represents the option value at grid point i and time step n, Δt is the time step size, and $\mathscr{L}$ is the Black-Scholes operator:

$$\mathscr{L}V = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV. \tag{3}$$

This semi-implicit formulation ensures stability while maintaining accuracy, making it suitable for American and exotic options where precise results are essential.

3 Delta: A Key Measure in Options Pricing and Risk Management

In financial derivatives, **Delta** is one of the most widely used "Greeks," which are measures of risk sensitivity in options trading. Delta quantifies the relationship between changes in the price of an option and changes in the price of the underlying asset. It plays a central role in hedging strategies and portfolio management, as it helps traders understand and manage the directional exposure of their positions.

Mathematically, delta is expressed as the first partial derivative of the option price (V) with respect to the underlying asset price (S):

$$\Delta = \frac{\partial V}{\partial S}.$$

3.1 Definition and Key Properties

- **Definition:** Delta is the rate of change of an option's price with respect to changes in the price of the underlying asset. It represents the sensitivity of the option value to small price movements in the underlying asset.
- **Delta Range:**
  - For a **Call Option**: Delta ranges between 0 and 1. A positive delta indicates that the option's price increases as the underlying asset's price rises.
  - For a **Put Option**: Delta ranges between -1 and 0. A negative delta indicates that the option's price decreases as the underlying asset's price rises.
- **Delta and Moneyness:**
  - *In-the-Money Options*: Delta approaches 1 for deep in-the-money call options and -1 for deep in-the-money put options.
  - *At-the-Money Options*: Delta is approximately 0.5 for call options and -0.5 for put options.
  - *Out-of-the-Money Options*: Delta approaches 0 as the option becomes increasingly out-of-the-money.
- **Practical Interpretation:**
  - A delta of 0.5 for a call option implies that for every \$1 increase in the underlying asset price, the option price increases by \$0.50.
  - A delta of -0.5 for a put option implies that for every \$1 increase in the underlying asset price, the option price decreases by \$0.50.

4 Delta as a Hedge Ratio:

Delta Hedging is a fundamental risk management strategy used in options trading to mitigate the directional risk associated with movements in the price of the underlying asset. By maintaining a *delta-neutral* portfolio, traders aim to minimize the impact of small changes in the underlying asset price on the overall portfolio value.

The essence of delta hedging lies in using the **delta** of an option, which measures the sensitivity of the option price to changes in the underlying asset price. By dynamically adjusting the position in the underlying asset, the portfolio's net delta is kept close to zero, reducing risk.

4.1 Key Concepts and Formula

- **Delta-Neutral Portfolio:** A portfolio is delta-neutral when the total delta of all positions equals zero:
$$\Delta_{\text{portfolio}} = \sum_i \Delta_i \cdot N_i = 0,$$
where Δ_i is the delta of the i-th position, and N_i is the number of units held in that position.

- **Hedging Example:** If a trader holds 1,000 call options with a delta of 0.5, the total delta of the option position is:
$$\Delta_{\text{options}} = 1,000 \cdot 0.5 = 500.$$
To neutralize this delta, the trader would short 500 shares of the underlying asset.

- **Dynamic Hedging:** Since delta changes with the price of the underlying asset, volatility, and time to expiration, delta hedging is a *dynamic* process requiring frequent adjustments to the hedge position. The adjusted hedge is calculated using the updated delta.

- **Cost of Delta Hedging:** While delta hedging reduces risk, it incurs transaction costs due to frequent rebalancing. These costs must be carefully managed, especially in volatile markets.

4.2 Advantages and Limitations

- **Advantages:**
  - Reduces directional risk associated with underlying price movements.
  - Helps stabilize portfolio value in the short term.
  - Provides a foundation for advanced hedging strategies.
- **Limitations:**
  - High transaction costs due to frequent rebalancing.
  - Assumes continuous trading, which is not always feasible.
  - Does not protect against large, sudden price movements (gap risk).

5 Black-Scholes Model(BSM)

The **Black-Scholes Model** (BSM) is a mathematical framework used to price European options. It assumes that the underlying asset price follows geometric Brownian motion with constant volatility and no dividends. The model provides a closed-form solution for the price of an option and plays a critical role in Delta Hedging by providing the mathematical tools to calculate the option's delta.

The Black-Scholes Partial Differential Equation (PDE) is expressed as:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

where:

- V: Option price,
- S: Underlying asset price,
- t: Time,
- σ: Volatility of the underlying asset,
- r: Risk-free interest rate.

5.1 Delta in the Black-Scholes Model

The delta of an option (Δ) can be derived from the Black-Scholes formula as:

$$\Delta = \frac{\partial V}{\partial S}.$$

For a call option, delta is given by:

$$\Delta_{\text{call}} = \Phi(d_1),$$

and for a put option:

$$\Delta_{\text{put}} = \Phi(d_1) - 1,$$

where:

$$d_1 = \frac{\ln(S/K) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},$$

- $\Phi(\cdot)$: Cumulative distribution function of the standard normal distribution,
- K: Strike price,
- T: Time to maturity.

5.2 Black-Scholes in Delta Hedging

The Black-Scholes Model provides a robust framework for delta hedging by:

1. **Calculating Delta:** The BSM allows precise calculation of delta for any given option using the formulas above. Traders can determine how much of the underlying asset to hold or short to achieve a delta-neutral position.

2. **Dynamic Adjustments:** Since delta changes as S, T, or σ changes, the BSM enables continuous recalibration of the hedge position by recalculating delta in real-time.

3. **Risk Management:** The BSM helps quantify and mitigate directional risk, allowing traders to focus on other risks (e.g., gamma, theta, or vega).

5.3 Example of Delta Hedging Using the Black-Scholes Model

Suppose a trader holds 1,000 call options on a stock with the following parameters:

- Current stock price (S) = \$100,
- Strike price (K) = \$105,
- Time to maturity (T) = 1 year,
- Volatility (σ) = 20%,
- Risk-free rate (r) = 5%.

Using the Black-Scholes formula, the trader calculates d_1 and Δ_{call}:

$$d_1 = \frac{\ln(100/105) + \left(0.05 + \frac{0.2^2}{2}\right) \cdot 1}{0.2\sqrt{1}} \approx -0.175.$$

$$\Delta_{\text{call}} = \Phi(-0.175) \approx 0.43.$$

The delta of the portfolio is:

$$\Delta_{\text{portfolio}} = 1,000 \cdot 0.43 = 430.$$

To achieve a delta-neutral portfolio, the trader shorts 430 shares of the underlying stock.

5.4 Advantages of Using the Black-Scholes Model in Delta Hedging

- **Analytical Precision:** Provides an accurate delta value for any market condition.
- **Dynamic Recalibration:** Enables adjustments as market conditions (e.g., S, σ) change.
- **Risk Quantification:** Helps traders quantify and hedge directional exposure systematically.

6 Binomial Model

The **Binomial Model** is a numerical method for pricing options and computing risk measures, such as delta. It approximates the price evolution of the underlying asset using a discrete-time framework, where the asset price can move up or down by specific factors at each time step. This framework allows for straightforward calculation of delta and provides a practical approach to implementing delta hedging.

In each time step, the underlying price S either increases by a factor u or decreases by a factor d. The corresponding risk-neutral probabilities (p) are used to calculate the expected value of the option.

- S: Current price of the underlying asset.

- u: Upward movement factor ($u > 1$).
- d: Downward movement factor ($0 < d < 1$).
- p: Risk-neutral probability of an upward movement:

$$p = \frac{e^{r\Delta t} - d}{u - d},$$

where r is the risk-free interest rate and Δt is the length of each time step.

- V_u, V_d: Option values in the up and down states.
- Δ: Delta of the option:

$$\Delta = \frac{V_u - V_d}{S_u - S_d}.$$

6.1 Delta Calculation in the Binomial Model

Delta is calculated at each node of the binomial tree as the ratio of the change in the option price to the change in the underlying asset price:

$$\Delta = \frac{V_u - V_d}{S_u - S_d},$$

where:

- $S_u = S \cdot u$, the price of the underlying asset after an upward movement.
- $S_d = S \cdot d$, the price of the underlying asset after a downward movement.
- V_u and V_d are the corresponding option prices in the up and down states.

6.2 Binomial Model in Delta Hedging

The binomial model provides a step-by-step method for implementing delta hedging:

1. Construct a binomial tree for the underlying asset prices over the option's life.
2. Compute the option prices at each node of the tree, starting from the terminal nodes and working backward to the present.
3. Calculate delta at each node using:

$$\Delta = \frac{V_u - V_d}{S_u - S_d}.$$

4. Use delta to determine the number of shares of the underlying asset required to hedge the option position.
5. Rebalance the hedge dynamically as the underlying price evolves.

6.3 Example of Delta Hedging Using the Binomial Model

Suppose an investor holds a call option with the following parameters:

- Current stock price (S) = \$100,
- Strike price (K) = \$105,
- Time to maturity (T) = 1 year,
- Volatility (σ) = 20%,
- Risk-free rate (r) = 5%,
- Number of steps (N) = 2.

The parameters for the binomial tree are calculated as:

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}, \quad \Delta t = \frac{T}{N}.$$

For $N = 2$, $\Delta t = 0.5$, and $\sigma = 0.2$:

$$u = e^{0.2\sqrt{0.5}} \approx 1.151, \quad d = e^{-0.2\sqrt{0.5}} \approx 0.869.$$

The risk-neutral probability is:

$$p = \frac{e^{0.05 \cdot 0.5} - d}{u - d} \approx 0.524.$$

Step 1: Terminal Option Values At $T = 1$:

$$S_u = S \cdot u = 100 \cdot 1.151 = 115.1, \quad S_d = S \cdot d = 100 \cdot 0.869 = 86.9.$$

The terminal option values are:

$$V_u = \max(S_u - K, 0) = \max(115.1 - 105, 0) = 10.1,$$

$$V_d = \max(S_d - K, 0) = \max(86.9 - 105, 0) = 0.$$

Step 2: Delta Calculation At $T = 0.5$:

$$\Delta = \frac{V_u - V_d}{S_u - S_d} = \frac{10.1 - 0}{115.1 - 86.9} \approx 0.35.$$

The hedge ratio is 0.35, meaning the investor should buy 0.35 shares of the underlying asset per option.

6.4 Advantages and Limitations of the Binomial Model in Delta Hedging

- **Advantages:**
  - Intuitive and straightforward for small numbers of steps.
  - Provides a clear framework for calculating delta at each point in time.
  - Flexible enough to handle American options and early exercise.
- **Limitations:**
  - Computationally intensive for large numbers of steps.
  - Assumes discrete-time trading, which may not fully reflect real-world conditions.

7 Comparison Tables

Dimension	**Black-Scholes Model (BSM)**	**Binomial Model (BM)**
Type	Analytical model with a continuous-time framework	Numerical model with a discrete-time framework
Underlying Assumptions	Assumes constant volatility, interest rate, and no dividends; underlying follows geometric Brownian motion	Approximates price movements using up and down factors at discrete time intervals
Delta Calculation	Delta is derived analytically as the first derivative of the option price with respect to the underlying price: $$\Delta = \frac{\partial V}{\partial S}$$	Delta is calculated numerically at each node as the ratio of the change in option price to the change in the underlying price: $$\Delta = \frac{V_u - V_d}{S_u - S_d}$$
Flexibility	Less flexible for handling American options or options with complex features	More flexible; handles American options and features like early exercise
Computational Intensity	Requires minimal computational resources due to closed-form solutions	Computationally intensive for a large number of time steps
Accuracy	Highly accurate for European options but relies on assumptions that may not hold in all markets	Accuracy improves with more time steps but may face computational limits
Application in Delta Hedging	Provides continuous delta for dynamic hedging; suitable for markets with frequent trading	Provides stepwise delta; suitable for less frequent rebalancing or discrete trading intervals
Real-World Applicability	Better suited for liquid markets with stable assumptions (e.g., constant volatility)	More practical for complex markets, options with early exercise, or where assumptions deviate from Black-Scholes

Dimension	Crank-Nicholson Method	Binomial Model
Type	Numerical method for solving partial differential equations (PDEs)	Numerical model with a discrete-time framework
Purpose	Solves PDEs such as the Black-Scholes equation numerically	Approximates option prices based on possible price movements over discrete time intervals
Input	Black-Scholes PDE, boundary conditions, and initial conditions	Initial asset price, up and down factors, and probabilities
Output	Numerical approximation of option prices	Option prices at different nodes and a final result after back-propagation
Mathematical Framework	Finite difference method combining explicit and implicit schemes: $$\frac{V^{n+1}-V^n}{\Delta t}=\frac{1}{2}\left[AV^n+AV^{n+1}\right]$$ where V^n and V^{n+1} are option prices at t_n and t_{n+1}	Uses recursive backward induction: $$\Delta=\frac{V_u-V_d}{S_u-S_d}$$ for delta and risk-neutral pricing for node valuation
Flexibility	Suitable for exotic options, complex boundary conditions, and non-linear problems	Handles American options and early exercise features efficiently
Computational Intensity	Computationally intensive for high-dimensional problems or real-time applications	Computational complexity grows with finer time steps
Advantages	Combines stability and accuracy, suitable for a wide range of PDEs	Simple to implement and intuitive for pricing derivatives
Limitations	Not suitable for real-time applications in high dimensions due to high computational demand	Limited by computational feasibility for large time steps and complex problems
Real-World Applicability	Useful for markets requiring precision in pricing exotic derivatives or options with complex features	Practical for standard options or cases where high flexibility (e.g., early exercise) is needed

8 Data

For this project, we use real market data from S&P 500 options to evaluate and implement the Crank-Nicholson method for pricing and hedging strategies. Specifically, we focus on call options with a time to maturity of approximately four months. The dataset includes call options with a range of strike prices, spanning from 5700 to 6200.

The choice of this dataset allows us to analyze how the Crank-Nicholson method performs under varying moneyness levels, from deep out-of-the-money to deep in-the-money options. By incorporating strike prices across this range, we aim to ensure that the results are robust and applicable across different market conditions.

The four-month time to maturity aligns well with typical trading horizons and allows us to capture both short-term market fluctuations and longer-term trends. This maturity period also provides sufficient time steps for the Crank-Nicholson discretization to yield meaningful and accurate results without excessive computational burden.

Using real-world data ensures that our findings are grounded in practical scenarios, making the conclusions more relevant for financial practitioners. By focusing on the S&P 500, a widely followed index representing the largest companies in the U.S. market, the analysis is both broadly applicable and reflective of market realities.

	S	Dividend	C_mkt	R	TTM	Moneyness
count	85.000000	85.000000	85.000000	85.00000	85.000000	85.000000
mean	5718.486941	1.345927	315.221176	4.80564	168.870588	18.458824
std	201.911555	0.047218	95.263399	0.10632	35.091305	201.966260
min	5186.330000	1.276500	111.850000	4.59072	108.000000	-514.000000
25%	5597.120000	1.311000	257.450000	4.76095	140.000000	-103.000000
50%	5728.800000	1.336400	311.800000	4.77997	169.000000	29.000000
75%	5853.980000	1.372100	383.750000	4.88116	198.000000	154.000000
max	6049.880000	1.481600	487.300000	5.01499	228.000000	350.000000

	K
count	85.0
mean	5700.0
std	0.0
min	5700.0
25%	5700.0
50%	5700.0
75%	5700.0
max	5700.0

Figure 1: data from code

9 Stability and Convergence

9.1 Crank-Nicolson Method

Stability: The Crank-Nicolson method is unconditionally stable for solving parabolic partial differential equations (PDEs) such as the Black-Scholes equation. Stability in this context means that the numerical solution does not diverge regardless of the choice of time-step size (Δt) or spatial grid density (Δx). This can be expressed mathematically as:

$$\|V^{n+1}\| \leq \|V^n\|$$

for all time levels n, where V^n represents the numerical solution at time t_n.

Convergence: The Crank-Nicolson method exhibits second-order convergence in both time (Δt) and space (Δx), which means the error in the solution decreases quadratically with grid refinement. [2]Specifically, for sufficiently small Δt and Δx, the error E satisfies:

$$E = \mathcal{O}((\Delta t)^2) + \mathcal{O}((\Delta x)^2).$$

This can be verified by:

- Ensuring the error decreases as the grid density increases.
- Checking that the rate of error reduction matches the theoretical $\mathcal{O}((\Delta t)^2 + (\Delta x)^2)$ property.

Discussion: **Figure 2** shows a convergent implementation of the CN scheme, while **figure 3** shows a non convergent implementation. During our experiments, we found that convergence depended on boundary conditions. Boundary conditions that may not allow the scheme to capture full market dynamics will result in errors. For example, the scheme is non convergent when the max boundary is not large enough to take into account the probability that the price of the underlying asset will go up. Likewise, it is non convergent when the minimum boundary is not equal to 0. The last point is likely because the probability that a given stock price goes down to 0 is much higher than the probability that it will go to infinity.

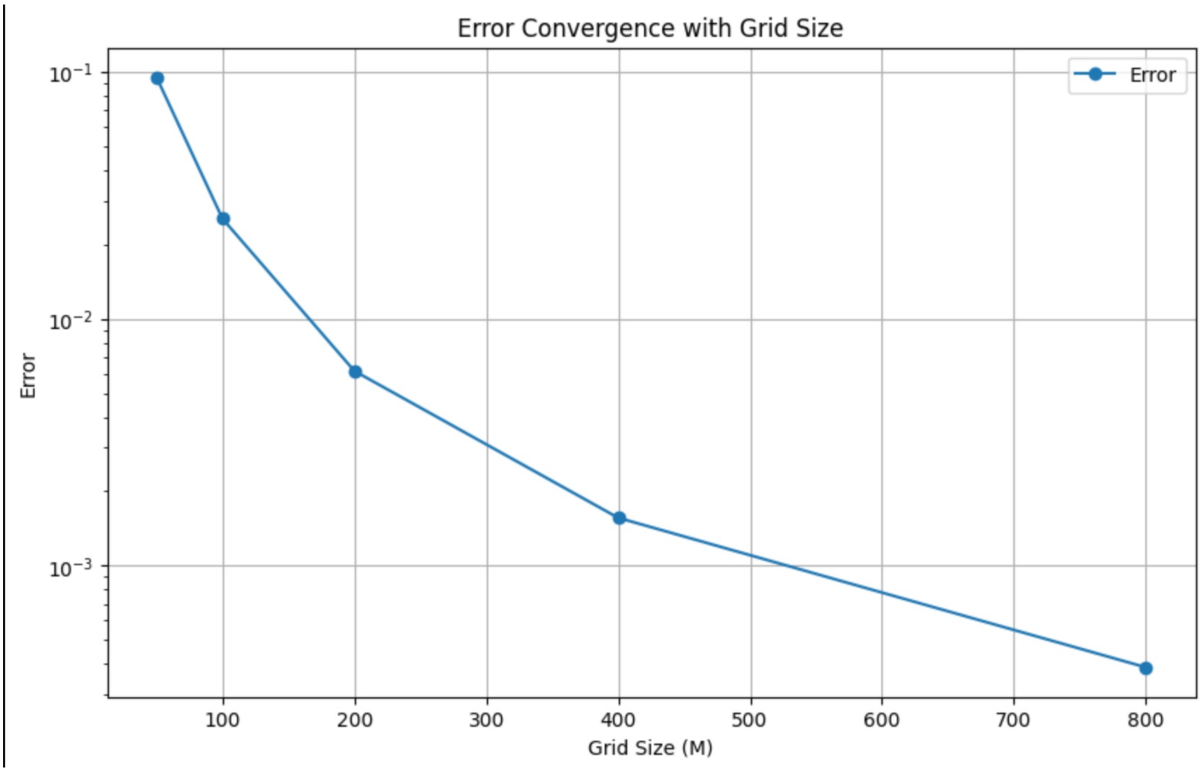

Figure 2: Error Convergence with Grid Size

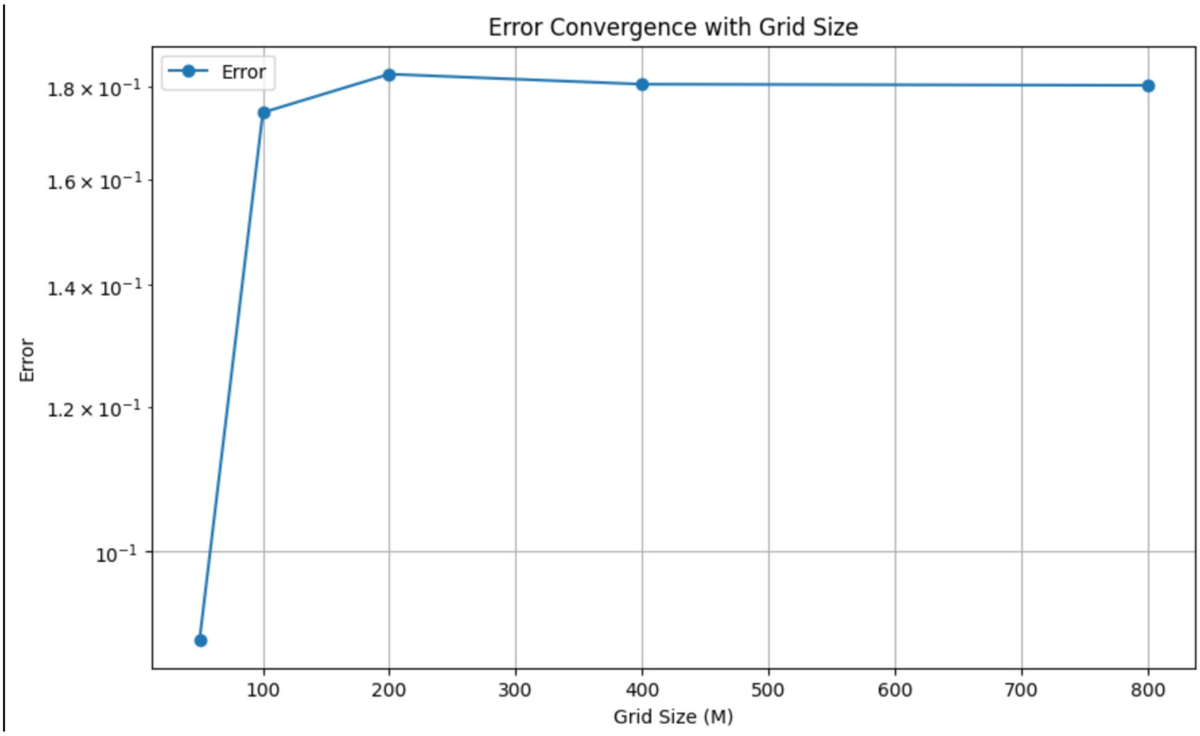

Figure 3: Error Convergence with Grid Size

9.2 Binomial Model

Stability: The stability of the Binomial Model depends on the choice of time-step size (Δt) and the up and down factors (u, d). To ensure stability, the model must satisfy the following no-arbitrage condition[1]:

$$d < e^{r\Delta t} < u,$$

where r is the risk-free rate. Violating this condition can lead to unrealistic or divergent option prices.

Convergence: The Binomial Model converges to the true option price as the number of time steps N increases. For large N, the Binomial Model approximates the continuous-time Black-Scholes model. The error E decreases as:

$$E = \mathcal{O}\left(\frac{1}{N}\right),$$

where $N = T/\Delta t$ is the number of time steps. This first-order convergence in time can be observed by:

- Calculating option prices for increasing N.
- Observing the error decreases linearly with finer time steps.

Discussion: The binomial tree will not converge when volatility is particularly high or payoff functions are discontinuous. Additionally, rounding off may introduce error, particularly for longer binomial tree models.

9.3 Comparison

- **Crank-Nicolson**: Higher order of convergence ($\mathscr{O}((\Delta t)^2 + (\Delta x)^2)$) compared to the Binomial Model ($\mathscr{O}(1/N)$).
- **Binomial Model**: Simpler implementation but requires a significantly larger N for comparable accuracy.

10 Using Crank Nicolson to Price American Options using BSM

The Crank-Nicolson (CN) scheme provides a numerical approach to solve partial differential equations (PDEs) that arise in option pricing models such as the Black-Scholes equation:

Using the CN method to approximate the time evolution of the Black-Scholes PDE, we discretize the time and price dimensions, resulting in:

$$\frac{V_i^{n+1} - V_i^n}{\Delta t} = \frac{1}{2}\left[\mathscr{L}V_i^n + \mathscr{L}V_i^{n+1}\right], \tag{4}$$

where $\mathscr{L}$ is the Black-Scholes operator:

$$\mathscr{L}V = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV. \tag{5}$$

The CN method results in a tridiagonal matrix system that can be solved efficiently. Given observed market prices for options (V_{market}), we solve the PDE iteratively to minimize the difference between the numerical option price V and the observed price V_{market}. This iterative process adjusts σ until convergence:

$$\text{Objective:} \quad \min_{\sigma} \left|V(\sigma) - V_{\text{market}}\right|. \tag{6}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{7}$$

where V is the option price, S is the asset price, r is the risk-free rate, and σ is the volatility.

As we can see, the Crank-Nicolson (CN) framework is one method for solving the Black-Scholes-Merton (BSM) equation. Its iterative nature provides flexibility at each time step, making it particularly well-suited for pricing more complex financial instruments such as American or exotic options. Unlike other methods, the CN framework allows for adjustments at each step, accommodating the unique features of these options, such as early exercise rights in American options or path dependencies in exotic options. For American options, we would apply this function at each time step:

$$V_j^n = max(V_j^n, S^n - K)$$

Where:

- S_j^n: the stock price at the grid point j at time n,
- K: the strike price,
- V_j^n: represents the option price of stock j at time n

Crank-Nicholson Method

- **Role**: The CN method solves the Black-Scholes PDE numerically to calculate option prices, including the predicted option price and Delta.
- **Delta Calculation**: CN is used to compute the option value for slightly higher and lower prices of the underlying asset. The difference is used to calculate Delta:

3. Workflow

4. Code

```
def crank_nicholson(S, K, r, T, sigma, N=100, M=100):
    dt = T / N  # Time step size
    S_max = S * np.exp((r - (sigma**2)/2) * T + 3 * sigma * np.sqrt(T))
    S_min = S * np.exp((r - (sigma**2)/2) * T - 3 * sigma * np.sqrt(T))
    dS = (S_max - S_min)/M + S_min

    # Initialize the price grid
    S_grid = np.linspace(S_min, S_max, M + 1)
    V = np.maximum(S_grid - K, 0)  # Option value at maturity

    # Coefficinents for Crank-Nicolson
    alpha = 0.25 * dt * ((sigma**2 * S_grid**2) / (dS**2) - r * S_grid / dS)
    beta = -0.5 * dt * (sigma**2 * S_grid**2 / (dS**2) + r)
    gamma = 0.25 * dt * ((sigma**2 * S_grid**2) / (dS**2) + r * S_grid / dS)

    # Implicit matrix
    A = np.zeros((M-1, M-1))
    B = np.zeros((M-1, M-1))
    for i in range(1, M):
        if i > 1:
            A[i-1, i-2] = -alpha[i]
            B[i-1, i-2] = alpha[i]
        A[i-1, i-1] = 1 - beta[i]
        B[i-1, i-1] = 1 + beta[i]
        if i < M-1:
            A[i-1, i] = -gamma[i]
            B[i-1, i] = gamma[i]

    print(f"A: {A}, B: {B}")

    # Time stepping
    for _ in range(N):
        V_inner = V[1:M]
        V_inner = np.linalg.solve(A, B @ V_inner)
        V[1:M] = V_inner
```

```
        V[0] = 0  # Boundary condition at S = 0
        V[-1] = S_max - K * np.exp(-r * (T - _ * dt))  # Boundary condition at S -> infinity

    print(f"S: {S}, S_grid: {S_grid}, V: {V}")

    return np.interp(S, S_grid, V)
```

11 Conclusion

Delta hedging is a way to minimize portfolio sensitivity to underlying asset price movements. This can be done by deriving delta from the Black-Scholes model. However, for path-dependent options such as American Options or Exotic options, the Black-Scholes method is not suitable.

The Crank-Nicolson scheme is a powerful method augmenting the Black-Scholes method for pricing American or Exotic options. Same goes for the Binomial Tree.

We are still trying to figure out how to correctly implement the Crank-Nicolson method in option pricing and delta hedging. The Crank-Nicolson method is a solution to the BSM that allows for accurate pricing of American and Exotic Options. The Binomial tree seems to be accurate, a little too accurate in our implementation. We still have more work to do.

References

[1] Tomasz R. Bielecki and Marek Rutkowski. A review of the binomial and trinomial models for option pricing and their convergence to the black-scholes model result. *Operations Research and Decisions*, 2020:45–57, 2020.

[2] Wei Qu and Yong Liang. Stability and convergence of the crank-nicolson scheme for a class of variable-coefficient tempered fractional diffusion equations. *Advances in Difference Equations*, 2017:1–19, 2017.