Problem – (P28.10)* A total amount of positive charge Q is spread onto a nonconducting, flat, circular annulus of inner radius a and outer radius b. The charge is distributed so that the charge density (charge per unit area) is given by $\sigma = k/r^3$, where r is the distance from the center of the annulus to any point on it. Show that (with $V = 0$ at infinity) the potential at the center of the annulus is given by

$$V = \frac{Q}{8\pi\epsilon_0}\left(\frac{a+b}{ab}\right).$$

Solution:
We know that $V = \int dV$. Let us first find Q knowing that $dq = \sigma\, dr$. So,

$$\begin{aligned} Q &= \int_a^b dq \\ &= \int_a^b \frac{k}{r^3} 2\pi r\, dr \\ &= 2\pi \int_a^b \frac{k}{r^2}\, dr \\ &= 2\pi k\left(\frac{1}{a} - \frac{1}{b}\right) \\ &= \frac{2\pi k(b-a)}{ab} \end{aligned}$$

So, we now find V with

$$\begin{aligned} V &= \int_a^b \frac{dq}{4\pi\epsilon_0 r} \\ &= \int_a^b \frac{k}{4\pi\epsilon_0 r^3} 2\pi r\, dr \\ &= \int_a^b \frac{k}{2\epsilon_0 r^2}\, dr \\ &= \frac{k}{4}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) \\ &= \frac{k(b-a)(a+b)}{4\epsilon_0 a^2 b^2} \\ &= \frac{2\pi k(b-a)(a+b)}{8\pi\epsilon_0 a^2 b^2} \\ &= \boxed{\frac{Q}{8\pi\epsilon_0}\left(\frac{a+b}{ab}\right).} && \text{(Using } Q = \tfrac{2\pi k(b-a)}{ab}.\text{)} \end{aligned}$$

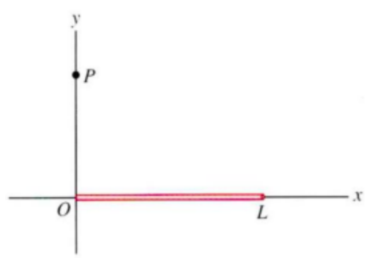
Problem – (P28.13) On a thin rod of length L lying along the x axis with one end at the origin ($x = 0$), as in the figure below, there is a charge per unit length given by $\lambda = kr$, where k is a constant and r is the distance from the origin.

a) Taking the electrostatic potential at infinity to be zero, find V at the point P on the y axis.

b) Determine the vertical component, E_y, of the electric field at P from the result of part (a) and also by direct calculation.

c) Why cannot E_x, the horizontal component of the electric field at P, be found using the result of part (a)?

d) At what distance from the rod along the y axis is the potential equal to one-half the value at the left end of the rod?



Solution:

a) Let us draw a diagram of how the charge "goes" to P, as well as trying to find an expression for dq and r_P.

We are given $\lambda = kr$. From our diagram, we see that $\lambda = \frac{dq}{dr}$, so $dq = kr\,dr$. Also from our diagram, we see that $r_P = \sqrt{r^2 + d^2}$. Knowing that $dV = \frac{dq}{4\pi\epsilon_0} r$, we can substitute this with what we know in this problem and get V.

So,

$$
\begin{aligned}
V &= \int_0^L \frac{dq}{4\pi\epsilon_0 r_P} \\
&= \int_0^L \frac{kr}{4\pi\epsilon_0\sqrt{r^2+d^2}}\,dr \\
&= \frac{k}{4\pi\epsilon_0}\int_0^L \frac{r}{\sqrt{r^2+d^2}}\,dr && (u = r^2+d^2,\ du = 2r\,dr.) \\
&= \frac{k}{4\pi\epsilon_0}\int_0^L \frac{du}{2\sqrt{u}} \\
&= \frac{k}{4\pi\epsilon_0}\left(\sqrt{r^2+d^2}\right)\Big|_0^L \\
V &= \boxed{\frac{k\sqrt{L^2+d^2}}{4\pi\epsilon_0} - \frac{kd}{4\pi\epsilon_0}.}
\end{aligned}
$$

b) Let us first find E_y through the result of part (a). We know the relationship $E = -\nabla V$. So, using d as our "y" we can get

$$
E_y = -\frac{\partial}{\partial y}V = -\frac{\partial}{\partial y}\left(\frac{k\sqrt{L^2+d^2}}{4\pi\epsilon_0} - \frac{kd}{4\pi\epsilon_0}\right) = \frac{k}{4\pi\epsilon_0} - \frac{kd}{4\pi\epsilon_0\sqrt{L^2+d^2}}\,\hat{\mathbf{y}} = \boxed{\frac{k}{4\pi\epsilon_0}\left[1 - \frac{d}{\sqrt{L^2+d^2}}\right]\hat{\mathbf{y}}.}
$$

Now let us find E_y using direct calculation. We know that $dE = \frac{dq}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{r}}$. We have expressions for all these except $\hat{\mathbf{r}}$. We know that we only want the y component of E, so from the diagram in the previous page, we find that this is equivalent to $\cos\theta$, which is $\cos\theta = \frac{d}{\sqrt{r^2+d^2}}$. So, we find

$$
\begin{aligned}
E_y &= \int_0^L \frac{kr}{4\pi\epsilon_0\left(r^2+d^2\right)}\frac{d}{\sqrt{r^2+d^2}}\,dr \\
&= \frac{kd}{4\pi\epsilon_0}\int_0^L \frac{r}{\left(r^2+d^2\right)^{3/2}}\,dr && (u = r^2+d^2,\ du = 2r\,dr.) \\
&= \frac{kd}{4\pi\epsilon_0}\left[-(r^2+d^2)^(1/2)\right]\Big|_0^L \\
&= \frac{kd}{4\pi\epsilon_0}\left[\frac{1}{d} - \frac{1}{\sqrt{L^2+d^2}}\right]\hat{\mathbf{y}} \\
&= \boxed{\frac{k}{4\pi\epsilon_0}\left[1 - \frac{d}{\sqrt{L^2+d^2}}\right]\hat{\mathbf{y}}.}
\end{aligned}
$$

c) Since P is only on the y-axis, we are really getting V when we set $x = 0$. This means that we cannot find E_x because we cannot find the gradient of a component of V, when we only know a single value of that component. It would be like asking to find $f'(7)$ if you are only given $f(7)$, you simply need more information.

d) At the left end of the rod, we note that $d = 0$, so we find that the potential V is

$$
V(0) = \frac{k\sqrt{L^2+d^2}}{4\pi\epsilon_0} - \frac{kd}{4\pi\epsilon_0} = \frac{k\sqrt{L^2+0^2}}{4\pi\epsilon_0} - \frac{k0}{4\pi\epsilon_0} = \frac{kL}{4\pi\epsilon_0}
$$

Now, we need to find at which value d do we get $\frac{1}{2}V(0)$. Let us set this up with

$$\begin{aligned}
V(d) &= \frac{1}{2}V(0) \\
\frac{k\sqrt{L^2+d^2}}{4\pi\epsilon_0} - \frac{kd}{4\pi\epsilon_0} &= \frac{1}{2}\frac{kL}{4\pi\epsilon_0} \\
\sqrt{L^2+d^2} - d &= \frac{1}{2}L \\
\sqrt{L^2+d^2} &= d + \frac{1}{2}L \\
L^2 + d^2 &= d^2 + dL + \frac{1}{4}L \\
L^2 &= dL + \frac{1}{4}L \\
L^2 - \frac{1}{4}L &= dL \\
d &= \frac{L^2 - \frac{1}{4}L}{L} \\
d &= L - \frac{1}{4} = \boxed{\frac{3}{4}L.}
\end{aligned}$$

Let us check this.

$$\begin{aligned}
V(d) &= V\left(L - \frac{1}{4}\right) \\
&= \frac{k\sqrt{L^2 + \left(\frac{3}{4}L\right)^2}}{4\pi\epsilon_0} - \frac{k\left(\frac{3}{4}L\right)}{4\pi\epsilon_0} \\
&= \frac{k\sqrt{L^2 + \frac{9}{16}L^2}}{4\pi\epsilon_0} - \frac{k\left(\frac{3}{4}L\right)}{4\pi\epsilon_0} \\
&= \frac{k\sqrt{\frac{25}{16}L^2}}{4\pi\epsilon_0} - \frac{k\left(\frac{3}{4}L\right)}{4\pi\epsilon_0} \\
&= \frac{k\frac{5}{4}L}{4\pi\epsilon_0} - \frac{k\left(\frac{3}{4}L\right)}{4\pi\epsilon_0} \\
&= \frac{k\frac{1}{2}L}{4\pi\epsilon_0}
\end{aligned}$$

Thus, we know that $V(d) = \frac{1}{2}V(0)$ when $d = \frac{3}{4}L$.

Problem – ¶3. In lecture, we calculated the amount of electrostatic energy contained in the interior electric field of a uniform sphere of radius R and charge density ρ. The result was,

$$U = \frac{2\pi\rho^2}{45\epsilon_0} R^5.$$

For this problem, complete the calculation by calculating the energy stored in the *exterior* electric field, and verify that the *total* stored energy matches the work done to assemble the sphere.

Solution:
From the previous homework, we found that the electric field outside a sphere of constant charge density ρ is

$$\mathbf{E}_{\text{out}} = \frac{\rho R^3}{3r^2\epsilon_0}.$$

So, we can find

$$u_E = \frac{1}{2}\epsilon_0 E^2 = \frac{\rho^2 R^6}{18 r^4 \epsilon_0}$$

So, we find

$$\begin{aligned}
U_{\text{out}} &= \int_R^\infty u_E \, dV \\
&= \int_R^\infty \frac{\rho^2 R^6}{18 r^4 \epsilon_0} 4\pi r^2 \, dr && (dV = 4\pi r^2 \, dr \text{ from lecture.}) \\
&= \frac{2\pi\rho^2 R^6}{9\epsilon_0} \int_R^\infty \frac{1}{r^2} dr \\
U_{\text{out}} &= \boxed{\frac{2\pi\rho^2 R^5}{9\epsilon_0}.}
\end{aligned}$$

From lecture, we found that the total work done was

$$W = \frac{4\pi\rho^2}{15\epsilon_0} R^5.$$

So, let us verify that the *total* stored energy matches the work done to assemble the sphere.

$$U_{\text{out}} + U_{\text{in}} = \frac{2\pi\rho^2 R^5}{9\epsilon_0} + \frac{2\pi\rho^2 R^5}{45\epsilon_0} = \frac{10\pi\rho^2 R^5}{45\epsilon_0} + \frac{2\pi\rho^2 R^5}{45\epsilon_0} = \frac{12\pi\rho^2 R^5}{45\epsilon_0} = \frac{4\pi\rho^2}{15\epsilon_0} R^5 = W.$$

Thus, we have verified that the *total* stored energy matches the work done to assemble the sphere.

Problem – 4. Consider an infinitely long line of charge with linear charge density λ . The line of charge is parallel to and a distance d above an infinite grounded conducting plane. Sketch the resulting electric field lines in the half-space above the plane, including any surface charges. Then using the concepts discussed in lecture, determine the electric field $E(x)$ at the surface of the plane, as a function of the horizontal distance x from the perpendicular between the plane and line.

Solution:
Let us draw sketch this out.

From this, we can imagine that under the grounded conducting plane, there is an identical infinitely long line of charge, but it has linear charge density $-\lambda$. So, we can imagine this sketched out as

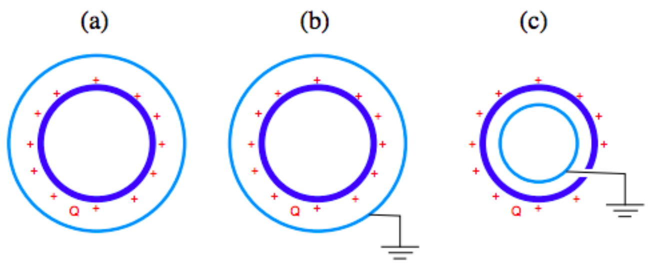
We know that, for a cylinder of radius r and length L, that

$$\mathbf{E}(r) = \frac{\lambda}{2\pi\epsilon_0 r}\hat{\mathbf{r}},$$

where $\lambda = Q/L$. From this, we can calculate $E(x)$ as

$$\begin{aligned}
E(x) &= 2\frac{\lambda}{2\pi\epsilon_0 r}\hat{\mathbf{r}} \\
&= \frac{\lambda}{\pi\epsilon_0\sqrt{x^2+d^2}}\cos\theta\,\hat{-\mathbf{y}} \\
&= -\frac{\lambda}{\pi\epsilon_0\sqrt{x^2+d^2}}\frac{d}{\sqrt{x^2+d^2}}\hat{\mathbf{y}} \\
&= \boxed{-\frac{\lambda d}{\pi\epsilon_0\left(x^2+d^2\right)}\hat{\mathbf{y}}.}
\end{aligned}$$

Problem – 5. Consider a hollow conducting sphere that carries a net positive charge Q. Next, a second initially uncharged concentric conducting sphere is brought into proximity with the first (without touching it). In scenario (a), the second sphere is placed outside the first and left *ungrounded.* In scenario (b), the second sphere is outside the first and *grounded.* In scenario (c), the second sphere is *inside* the first and also grounded.



For all three scenarios, describe the final arrangement of charge on the second sphere and the electric field (if any) outside the second sphere.

Solution:

a) In this configuration, we know that, since the spheres are both conductors there is an induced charge on the inner surface of the ungrounded conducting sphere. However, since it is ungrounded there will also be an induced charge on the outside of the sphere, so there will be an electric field of

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

outside the ungrounded conducting sphere. Here is a sketch:

b) In this configuration, we know that, since the spheres are both conductors there is an induced charge on the inner surface of the grounded conducting sphere. However, since it is grounded, there will NOT be an induced charge on the outside of the sphere, meaning there is no electric field outside the grounded conducting sphere. Here is a sketch:

c) In this configuration, since the spheres are conductors, there is an induced charge on the outside of the second sphere, and it will be negative. So, outside the second sphere, there will be a negative electric field, but outside the larger sphere, there will be a positive electric field. Here is a sketch:

Problem – ¶6. A cylindrical capacitor is made from two long thin concentric metal cylinders of length L and radii a and b. ($L \gg a$ and $a > b$)

a) Using the definition of capacitance $C = Q/V$, calculate the capacitance per unit length C/L of this configuration.

b) Repeat the capacitance calculation, this time using the stored energy $U = \frac{1}{2}\frac{Q^2}{C}$.

Solution:

a) We know that, for a cylinder of radius r and length L, that

$$\mathbf{E}(r) = \frac{\lambda}{2\pi\epsilon_0 r}\hat{\mathbf{r}},$$

where $\lambda = Q/L$. So, if we want to find capacitance C, then we must find the voltage V. So, we find V through $V = -\int_a^b E \cdot d\ell$, but $d\ell = dr$ and they are parallel, so we find.

$$V = -\int_a^b \frac{Q}{2\pi\epsilon_0 r L} dr = \frac{Q}{2\pi\epsilon_0 L}\int_b^a \frac{1}{r} dr = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{a}{b}\right).$$

So,

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{a}{b}\right)} = \boxed{\frac{2\pi\epsilon_0 L}{\ln\left(\frac{a}{b}\right)}}.$$

b) We know that $U = \int u_E\, dV$ and that $u_E = \frac{1}{2}\epsilon_0 E^2$. We know that $\mathbf{E}(r) = \frac{Q}{2\pi\epsilon_0 r L}\hat{\mathbf{r}}$, so

$$u_E = \frac{1}{2}\epsilon_0 \frac{Q^2}{4\pi^2\epsilon_0^2 r^2 L^2} = \frac{Q^2}{8\pi^2\epsilon_0 r^2 L^2}.$$

So,

$$\begin{aligned} U = \int_b^a u_E\, dV &= \int_b^a \frac{Q^2}{8\pi^2\epsilon_0 r^2 L^2} 2\pi r L\, dr \\ &= \int_b^a \frac{Q^2}{4\pi\epsilon_0 r L} dr \\ &= \frac{Q^2}{4\pi\epsilon_0 L}\int_b^a \frac{1}{r} dr \\ U &= \frac{Q^2}{4\pi\epsilon_0 L}\ln\left(\frac{a}{b}\right) \end{aligned}$$

Also, from the relationship from $U = \frac{1}{2}\frac{Q^2}{C}$, we know that $C = \frac{1}{2}\frac{Q^2}{U}$, so

$$C = \frac{1}{2}\frac{Q^2}{\frac{Q^2}{4\pi\epsilon_0 L}\ln\left(\frac{a}{b}\right)} = \boxed{\frac{2\pi\epsilon_0 L}{\ln\frac{a}{b}}}.$$