Online Convex Optimization
Home Assignment

This homework should be uploaded on Moodle by **Friday, January 9, 2026** in a PDF file separate from the code. The code can be written in any language (Python or R) but should be in a single file that runs immediately when executed in an environment containing the dataset `mnist.csv`.

Part 1. The "Rock Paper Scissors" game

We consider the sequential version of a repeated two-player zero-sum game between a player and an adversary, whose setting is detailed below. Let $L \in \{-1, 1\}^{M \times N}$ be a loss matrix known by the players. At each round $t = 1, \ldots, T$:

1. The player chooses a distribution $p_t \in \Delta_M := \{p \in [0, 1]^M, \sum_{i=1}^M p_i = 1\}$.
2. The adversary chooses a distribution $q_t \in \Delta_N$.
3. The actions of both players are sampled $I_t \sim p_t$ and $J_t \sim q_t$.
4. The player incurs the loss L_{J_t, I_t} and the adversary the loss $-L_{I_t, J_t}$.

1. Rock Paper Scissors matrix

Recall M, N and a loss matrix $L \in \{-1, 1\}^{M \times N}$ that corresponds to the game "Rock Paper Scissors"[1].

2. Implementation of EWA

(a) Implement the function `rand_weighted` that takes as input a probability vector $p \in \Delta_M$ and return $I = 1, \ldots, M$ satisfying $\mathbb{P}(I = i) = p_i$.

(b) Define a function `EWA_update` that takes as input a vector $p_t \in \Delta_M$ and a loss vector $\ell_t \in \{-1, 1\}^M$ and returns the updated vector $p_{t+1} \in \Delta_M$

[1] This is a common game where two players choose one of 3 options: (Rock, Paper, Scissors). The winner is decided according to the following: Rock crushes scissors, Paper covers Rock, Scissors cuts Paper.

defined for all $i = 1, \dots, M$ by

$$p_{t+1}(i) = \frac{p_t(i) \exp\left(-\eta \ell_t(i)\right)}{\sum_{j=1}^{M} p_t(j) \exp\left(-\eta \ell_t(j)\right)}$$

and starting from the uniform weights $p_0(i) = 1/M$ for $i = 1, \dots, M$.

3. Simulation against a fixed adversary

Consider the game "Rock Paper Scissors" and assume that the adversary chooses $q_t = (1/6, 1/3, 1/2)$ and samples $J_t \sim q_t$ for all rounds $t \geq 1$.

(a) What is the loss $\ell_t(i)$ incurred by the player if she chooses action i at time t? Simulate an instance of the game for $t = 1, \dots, T = 100$ for $\eta = 1$.

(b) Plot the evolution of the weight vectors $p_1, p_2, \dots, p_T$. What seems to be the best strategy here?

(c) Repeat the simulation $n = 200$ times and plot the average loss $\bar{\ell}_t = \frac{1}{t} \sum_{s=1}^{t} \ell(i_s, j_s)$ as a function of t (averaged over the experiments).

(d) Repeat one simulation for different values of learning rates $\eta \in \{0.01, 0.03, 0.1, 0.3, 1\}$ and plot the average loss as a function of η. What are the best η in practice and in theory?

4. Implementation of OGD

Consider a version of OGD as an adversary with decision set $q_t = \Delta_N$ that chooses action $J_t \sim q_t$ and suffers the expected loss $\ell_t(q_t) = \sum_{j=1}^{N} q_t(j) L_{I_t,j}$ against the player's action I_t at round t. The update of OGD in this case is

$$q_{t+1} = \mathrm{Proj}_{\Delta_N}\left(q_t - \eta_t \nabla \ell_t(q_t)\right),$$

with $\eta_t = 1/\sqrt{t}$.

(a) Define the Euclidean projection over the simplex `proj_simplex` that takes $x \in \mathbb{R}^N$ and returns its projection $\mathrm{Proj}_{\Delta_N}(x) \in \Delta_N$. See Alg. 2 in the lecture notes `https://wintenberger.fr/cours/OCO/OCO2022.pdf`.

(b) Define a function `OGD_update` that takes as input a vector q_t, the loss vector $L_{I_t,\cdot}$, and returns $q_{t+1} \in \Delta_N$.

5. EWA vs OGD

Run an experiment of the game "Rock Paper Scissors" in which EWA is the player and OGD the adversary. Define $\bar{p}_t = \frac{1}{t} \sum_{s=1}^{t} p_s$. Plot in log log scale $\|\bar{p}_t - (1/3, 1/3, 1/3)\|_2$ as a function of $t = 1, \dots, 10000$.

6. Hedge vs OGD

Recall that the loss function considered by Hedge is

$$\ell_t(p) = \sum_{i=1}^{M} \sum_{j=1}^{N} q_t(j) L(j,i) p_t(i).$$

(a) Define a function `Hedge_update` that takes as input a vector p_t, the adversary's strategy q_t, and returns $p_{t+1} \in \Delta_N$.

(b) Run an experiment of the game "Rock Paper Scissors" in which Hedge is the player and OGD the adversary. Define $\bar{p}_t = \frac{1}{t} \sum_{s=1}^{t} p_s$. Plot in log log scale $\|\bar{p}_t - (1/3, 1/3, 1/3)\|_2$ as a function of $t = 1, \dots, 10000$. Compare with the results obtained in Question 5.

Part II. MNIST

We consider a problem of image classification on the MNIST dataset available at

"https://www.kaggle.com/datasets/hojjatk/mnist-dataset.

We have images of digits "0", "1", "2",... and the objective is to build a classifier. The goal is to classify if an image $X_t \in \mathbb{R}^d$ corresponds to the digit "0" in grayscale: the label is $y_t = 1$ if image t is "0" and $y_t = -1$ otherwise. The data is composed of two subfolders, one set of 10000 images. Given a region of search $\mathcal{K} \subset \mathbb{R}^d$, we propose to estimate sequentially $\theta_t \in \mathcal{K}$ to predict the label y_t with $\hat{y}_t = \text{sign}(\theta_t^\top X_t)$.

1. Dimension and Region of Search

What is the value of d here when, as usual, one adds an intercept (a fictitious pixel = 1) after standardizing X so that the grayscale of the pixels fits to $[0, 1]$? Propose a possible region of search $\mathcal{K}$.

2. Loss Functions

We define the the logistic loss function $\ell(y_t, \theta^\top X_t) = \log(1 + \exp(-y_t \theta^\top X_t))$.

(a) Implement the loss function `loss_logistic`. Prove it is exp-concave.

(b) Compute the gradient and the Hessian of the logistic loss. Write the associated functions `gradient_logistic`, `hessian_logistic`.

3. Algorithms

We consider $\mathcal{K} = \{\theta \in \mathbb{R}^d : \|\theta\|_2 \le D\}$ for some $D > 0$.

(a) Implement OGD and AdaGrad in this setting.

(b) Run the algorithms for a choice of D and a choice of the gradient step sizes η_t. Motivate theses choices.

(c) How often do your algorithms leave the convex set $\mathcal{K}$? Is your choice of D restrictive?

(d) From the sequence $\theta_1, \ldots, \theta_T$ obtained for each algorithm, plot the cumulative loss $S_t = \sum_{s=1}^t \ell(y_s, \theta_s^\top X_s)$ as a function of t. Compare the different algorithms. Plot also the cumulative $0-1$ loss. Interpret.

4. Regret

Recall that the regret is defined by

$$Regret_t = \sum_{s=1}^{t} \left(\ell(y_s, \theta_s^\top X_s) - \ell(y_s, \theta^{\star\top} X_s) \right) .$$

(a) Propose an estimate of $\theta^\star \in \arg\min_{\theta \in \mathcal{K}} \sum_{s=1}^t \ell(y_s, \theta^\top X_s)$.

(b) Plot the evolution of the regret of the algorithms defined in 3(a) and 3(b). Interpret with respect to the regret bounds seen in class.

Bonus

Propose an alternative sequential algorithm for the problem considered by modifying the convex set $\mathcal{K}$.