NOTE

Simple Equations for Linear Partially Elastic Collisions

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Although most everyday collisions are partially elastic, only perfectly elastic or perfectly inelastic collisions are treated in most popular introductory physics textbooks, an exception being the text by Crummett and Western.[1] Perhaps this approach is understandable, given the complexity of the equations that are obtained for the kinematics of collisions in the laboratory reference frame. Many students encounter the problem of analyzing partially elastic collisions between air-track gliders or dynamics carts, since it is very difficult if not impossible to produce perfectly elastic collisions. To clarify the physics and simplify the analysis of these partially elastic collisions, I have developed a set of simple equations for motion and energy based on the center-of-mass reference frame.

Kinematics

Consider two objects of mass m_1 and m_2 colliding head-on with initial velocities v_{1i} and v_{2i}, and final velocities v_{1f} and v_{2f}, respectively. The equation for the conservation of momentum is:

$$m_1 v_{1f} + m_2 v_{2f} = m_1 v_{1i} + m_2 v_{2i} \tag{1}$$

and the velocity of the center of mass is:

$$V = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2} = \frac{m_1 v_{1f} + m_2 v_{2f}}{m_1 + m_2} \tag{2}$$

Equation (1) can be recast in the center-of-mass reference frame by using the transformation:

$$v = V + u \tag{3}$$

where u is the velocity of the objects relative to the center of mass. By multiplying Eq. (2) by m1 + m2 and applying Eq. (3), it can be shown that the momentum of the system of two objects relative to the center of mass vanishes at all times; i.e.,

$$m_1 u_{1f} + m_2 u_{2f} = m_1 u_{1i} + m_2 u_{2i} = 0 \tag{4}$$

Partially inelastic collisions are characterized by the coefficient of restitution, which is the ratio of the speed of separation to the speed of approach of the objects:

$$\varepsilon = \frac{v_{2f} - v_{1f}}{v_{1i} - v_{2i}} = \frac{u_{2f} - u_{1f}}{u_{1i} - u_{2i}} \tag{5}$$

A relationship between the initial and final velocities of each object can be found by first obtaining the relationship between the velocities of both objects both before and after the collision in the center-of-mass frame. From Eq. (4),

$$u_2 = -\frac{m_1}{m_2} u_1 \tag{6}$$

for both the initial and final velocities. This result implies that the velocities of the objects relative to the center of mass are inversely proportional to their masses and in opposite directions, as we would intuitively expect. By applying Eq. (6) to Eq. (5) it can be shown that

$$u_{1f} = -\varepsilon u_{1i} \tag{7}$$

Since an identical equation for object 2 can be obtained, it follows that the speeds of both objects in the center-of-mass frame are reduced by the factor ε after the collision. Then by applying the transformation, Eq. (3), to both u_{1i} and u_{1f},

$$v_{1f} = (1 + \varepsilon)V - \varepsilon v_{1i} .$$

Since an identical equation can be obtained for the second object, this result can be written as

$$v_f = (1 + \varepsilon)V - \varepsilon v_i \tag{8}$$

Thus, given the masses and initial velocities of the two objects and the coefficient of restitution, only Eqs. (2) and (8) are needed to obtain the velocities of both objects after the collision.

Special Cases

For perfectly elastic collisions, $\varepsilon = 1$, and Eq. (8) becomes

$$v_f = 2V - v_i \tag{9}$$

This result is identical to that obtained recently by Millet.[2] At the opposite extreme, $\varepsilon = 0$ for perfectly inelastic collisions, and Eq. (8) becomes

$$v_f = V \tag{10}$$

as expected. Figure 1 illustrates a more interesting case where $m_1 = 2m$, $m_2 = m$, $v_{1i} = 2V$, $v_{2i} = -V$ and $\varepsilon = 0.5$.

Energetics

Following Crummett and Western,[3] the kinetic energy of a sys-

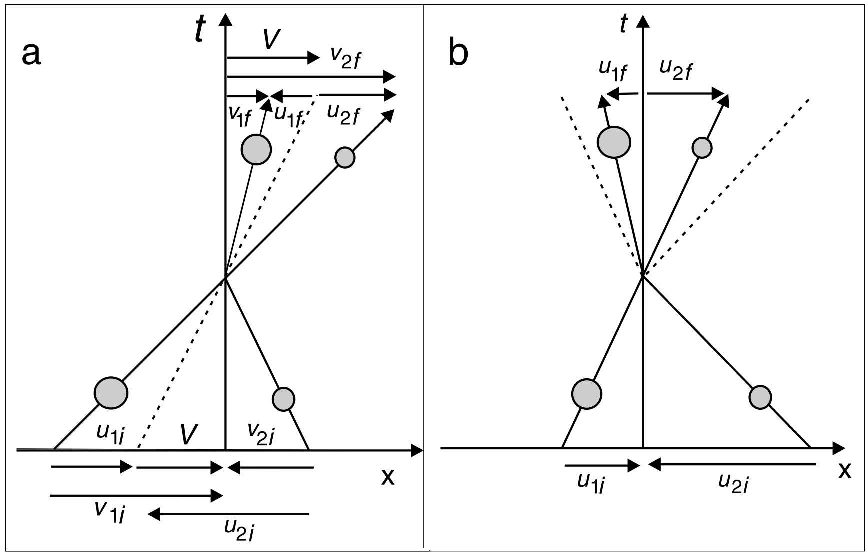

Fig. 1. a) Motion of colliding objects in the laboratory frame as described in the text. Dashed line indicates motion of the center of mass. b) Motion of objects in the center-of-mass frame. Dashed lines indicate motion the objects would have for a perfectly elastic collision.

tem of particles can be partitioned into the kinetic energy of the center of mass and the kinetic energy of the particles *relative* to the center of mass:

$$K = K_C + K_R \tag{11}$$

where

$$K_C = ½MV^2 \tag{12}$$

and

$$K_R = \sum_{j=1}^{n} ½m_j u_j^2 \tag{13}$$

This result is a more general form of the equation for the kinetic energy of an object that is both translating and rotating; i.e., rolling, given in most introductory physics texts. For collisions

$$K_{Ri} = ½m_1 u_{1i}^2 + ½m_2 u_{2i}^2 \tag{14}$$

and

$$K_{Rf} = ½m_1 u_{1f}^2 + ½m_2 u_{2f}^2 \tag{15}$$

Using Eq. (7), the latter equation becomes

$$K_{Rf} = \varepsilon^2(½m_1 u_{1i}^2 + ½m_2 u_{2i}^2) \tag{16}$$

so,

$$K_{Rf} = \varepsilon^2 K_{Ri} \tag{17}$$

By applying this result to Eq. (11), the final total kinetic energy of the system becomes:

$$K_f = K_C + \varepsilon^2 K_{Ri} \tag{18}$$

For perfectly elastic collisions,

$$K_f = K_C + K_{Ri} = K_i \tag{19}$$

which is the expected condition for the conservation of kinetic energy for these collisions. On the other hand, for perfectly inelastic collisions,

$$K_f = K_C \tag{20}$$

and all of the kinetic energy relative to the center of mass is lost. In general, Eqs. (11), (12), (14), and (18) will be needed to calculate the kinetic energies of the system before and after the collision. In an experimental context, Eq. (12) can be calculated at times before and after the collision to show that the center-of-mass kinetic energy is conserved. Then Eq. (13) can be calculated before and after the collision to obtain the total relative kinetic energies, and in conjunction with a calculation of Eq. (5) these results can be used to show the validity of Eq. (17); i.e., that the ratio of the final to the initial relative kinetic energies is equal to the square of the coefficient of restitution.

Summary

These simple equations for partially elastic collisions have been used successfully in my introductory physics classes for health science majors since the fall semester of 1997, both for solving problems and for analyzing laboratory data. This classroom experience lends support to Millet's sentiment that the traditional equations currently used in introductory physics courses are unnecessarily complicated and that the simpler equations derived in the center-of-mass frame should be used instead. Since most texts include discussions of both collisions and the motion of the center of mass in quick succession and in that order, it is recommended that the order of instruction of these topics be reversed so that the simpler equations can be more easily introduced. Doing so has the added advantage of making the usefulness of the concept of the center of mass more obvious. Furthermore, the center-of-mass frame is often used in the description of particle scattering in modern physics. What better way can we begin to prepare our students for an introduction to this topic?

Acknowledgments

The author would like to thank Michael C. Kennedy for allowing me the freedom to explore new ideas in teaching physics and also Roger and Sandra Cowley and N. John DiNardo for valuable suggestions.

References

1. W. P. Crummett and A. B. Western, *University Physics: Models and Applications* (Wm. C. Brown Publishers, 1994).
2. L. Edward Millet, *Phys. Teach.* **36**, 186 (March 1998).
3. Ref. 1, p. 263.