UiO : **Department of Informatics**
University of Oslo

Mandatory exercise 2

High resolution beamforming on farfield monochromatic signals, MATLAB version

Thomas Aussaguès

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Table of contents

Problem statement [1]

Geometry:

- We consider $N_s = 2$ monochromatic farfield sources
- We use a linear array with $M = 10$ isotropic sensors with element distance $d = 0.5\lambda$ and $\lambda = 1$ m
- We recorded $N = 100$ time samples with sampling frequency $\dfrac{1}{T}$

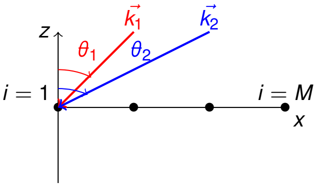

Figure: Linear array geometry

Signal model:

- Phase matrix: $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ \vdots & \vdots \\ e^{-jMkd\sin\theta_1} & e^{-jMkd\sin\theta_2} \end{pmatrix} \in \mathbb{C}^{M\times 2}$
- Signal matrix: $\mathbf{S}(t) = \begin{pmatrix} a_1 e^{j2\pi\Phi_{11}} & \dots & a_1 e^{j2\pi\Phi_{1N}} e^{jN\omega_1 T} \\ a_1 e^{j2\pi\Phi_{21}} & \dots & a_2 e^{j2\pi\Phi_{2N}} e^{jN\omega_2 T} \end{pmatrix} \in \mathbb{C}^{2\times N}$ where Φ_1 and Φ_2 are two Gaussian random vectors ($\mathcal{N}(0,1)$)
- Noise vector: $\mathbf{N}(t)$, we assume AWGN
- Received signal vector: $\mathbf{Y}(t)$

$$\mathbf{Y}(t) = \mathbf{AS}(t) + \mathbf{N}(t) \tag{1}$$

- Weights vector[1]: $\mathbf{w} = \begin{pmatrix} w_1 \dots w_M \end{pmatrix}^H$
- Steering vector: $\mathbf{a}(\theta) = \begin{pmatrix} 1 & \dots & e^{-jMkd\sin\theta} \end{pmatrix}^H$

[1] Why is there an ^{H} ? To be coherent with the the output signal calculation: $z(t) = \langle \mathbf{w} | \mathbf{Y}(t) \rangle = \sum_{i=1}^{M} w_i y_i(t)$

Spatial correlation matrix estimation [1, 3, 4]

- In this first part, we consider incoherent sources: $\omega_1 \neq \omega_2$
- One of beamforming challenges is to **estimate** the spatial correlation matrix $\mathbf{R} = \mathbb{E}\{\mathbf{Y}\mathbf{Y}^H\}$ where $r_{ij} = \mathbb{E}\{y_i(t)y_j(t)^*\}$
- What estimator should we use? First, we will use $\hat{\mathbf{R}} = \dfrac{1}{N}\mathbf{Y}\mathbf{Y}^H$ (note that we assume W.S.S and ergodicity)
- Let's check some properties of $\hat{\mathbf{R}}$
- $\hat{\mathbf{R}}$ amplitude is symmetric, its phase is anti-symmetric $\implies$ $\hat{\mathbf{R}}$ is **hermitian** : $\hat{\mathbf{R}} = \hat{\mathbf{R}}^H$

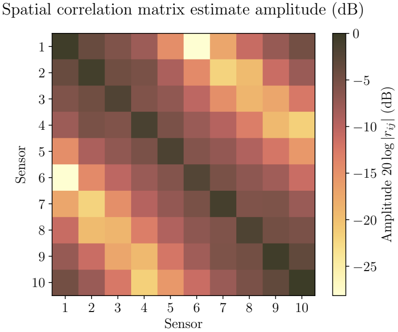

Figure: Spatial correlation matrix amplitude estimate $|\hat{\mathbf{R}}|$, incoherent signals

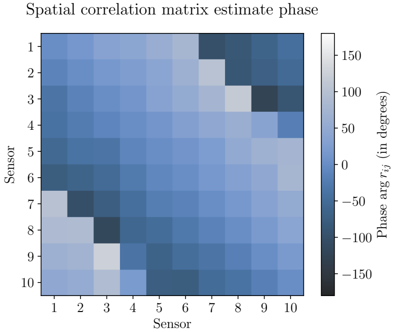

Figure: Spatial correlation matrix phase estimate $\arg \hat{\mathbf{R}}$, incoherent signals

- Reduced correlation outside the diagonal (brighter colours[2]): incoherent signals

[2]Using Fabio CRAMERI's colormaps [2]

DAS power estimate [1, 3, 4]

- For slides 4 to 10, we use the following spatial correlation matrix estimate: $\hat{\mathbf{R}} = \frac{1}{N}\mathbf{Y}\mathbf{Y}^H$
- Through DAS beamforming, we can get the 'easiest' power estimate

$$P(\mathbf{a}(\theta))_{DAS} = \mathbf{a}(\theta)^H \mathbf{R}\mathbf{a}(\theta) \tag{2}$$

- Is this a 'good' estimate? $\longleftrightarrow$ Is $P(\mathbf{a}(\theta))_{DAS}$ **unbiased**, **consistent**[3]?

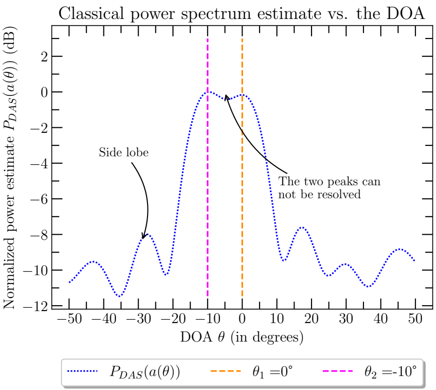

Figure: Classical DAS beamforming power estimate, incoherent signal

- Obviously, DAS beamforming is not a good choice here!
- The constructed array is unable to distinguish the two signals sources. . . The ideal spectrum should be two DIRACs @ 0 ° and −10°
- The two sources are merged into one source
- Some side lobes (level ≈ −6 dB) around the (large) main lobe
- It only requires two matrix multiplication (complexity $\mathcal{O}(N^{2.4})$)
- Can we do better?

[3] To estimate the variance, we need to run this experiment several times

CAPON's/minimum-variance beamformer [1, 3, 4]

- Adaptive beamforming: let the weights be a function of the incoming signal
- Idea: minimizing the noise power while conserving a unity gain in the steering direction

$$\begin{cases} \min P_{MV}(\mathbf{w}) = \mathbf{w}^H \mathbf{R} \mathbf{w} \\ \text{s.t } \mathbf{w}^H \mathbf{a}(\theta) = 1 \end{cases} \implies \mathbf{w} = \frac{\mathbf{a}(\theta)^H \mathbf{R}^{-1}}{\mathbf{a}(\theta)^H \mathbf{R}^{-1} \mathbf{a}(\theta)} \text{ and } P_{MV}(\mathbf{a}(\theta)) = \frac{1}{\mathbf{a}(\theta)^H \mathbf{R}^{-1} \mathbf{a}(\theta)}$$

- This constrained optimization problem has a closed solution! Which requires a matrix inversion...

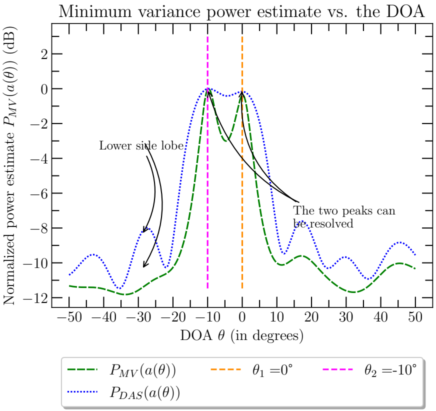

Figure: Minimum variance power estimate, incoherent signal

- Some improvements: we have two distinguishable peaks @ 0 ° and -10°
- Lower side lobes levels
- Lower ML widths
- Lower side lobes
- The beamformer can distinguish the two signals
- But only 10 dB between the peaks and the noise level
- Inverting a matrix is costly (Complexity: $\mathcal{O}(N^3)$)!
- Here, matrix inversion and two matrix multiplication
- Can we do better?

Eigenvalues distribution [1, 3, 4]

- Main idea: 'Find a new base such that **R** is a diagonal matrix, get rid of some data and compute its inverse. Then, go back to the standard base.'
- **R** being **hermitian** and **positive semi-definite**, it is diagonalizable and it has a positive spectrum Sp(**R**) $\subset \mathbb{R}^+$

$$\mathbf{R} = \mathbf{V}^H \mathbf{\Lambda} \mathbf{V} = \underbrace{\mathbf{V_s}^H \mathbf{\Lambda_s} \mathbf{V_s}}_{\text{signal + noise}} + \underbrace{\mathbf{V_n}^H \mathbf{\Lambda_n} \mathbf{V_n}}_{\text{noise}} \quad (3)$$

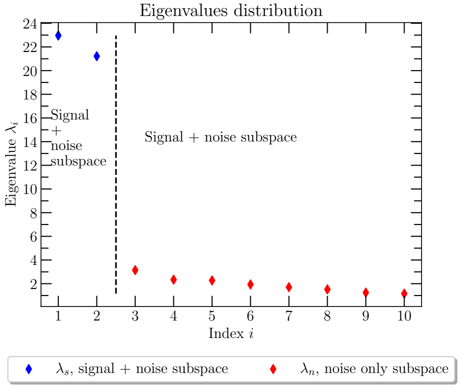

Figure: Eigenvalues distribution for two incoherent sources

- Two eigenvalues are larger than the other ($\lambda_1 = 26$ and $\lambda_2 = 20$) : we can identify both the signal+noise and noise only subspaces
- Moreover, we get the number of sources $N_s = 2$
- Same number of sources as in the simulation parameters ✔
- Note that if we have *a priori* knowledge about the noise power (σ_n^2), we can easily estimate the number of sources: $\lambda_{i \leq N_S} > \sigma_n^2 = 1$ and $\lambda_{i > N_S} = \sigma_n^2 = 1$

MUSIC algorithm [1, 3, 4]

- We estimate the beamformer power output using only the **normalized** noise part in the previous eigendecomposition

$$\mathbf{R} = \mathbf{V}^H \boldsymbol{\Lambda} \mathbf{V} = \underbrace{\mathbf{V_s}^H \boldsymbol{\Lambda}_\mathbf{s} \mathbf{V_s}}_{\text{signal + noise}} + \underbrace{\mathbf{V_n}^H \boldsymbol{\Lambda}_\mathbf{n} \mathbf{V_n}}_{\text{noise}} \approx \mathbf{V_n}^H \boldsymbol{\Lambda}_\mathbf{n} \mathbf{V_n} \implies \underbrace{\mathbf{R}^{-1} \approx \mathbf{V_n} \boldsymbol{\Lambda}_\mathbf{n}^{-1} \mathbf{V_n}^H \approx \mathbf{V_n} \mathbf{V_n}^H}_{\text{normalized } \boldsymbol{\Lambda}_\mathbf{n}^{-1} = I_n} \tag{4}$$

- Why is it efficient? $\boldsymbol{\Lambda}$ is a diagonal matrix so its inverse is easy to compute

$$P(\mathbf{a}(\theta))_{MUSIC} = \frac{1}{\mathbf{a}(\theta) \mathbf{V_n} \mathbf{V_n}^H \mathbf{a}(\theta)^H} \tag{5}$$

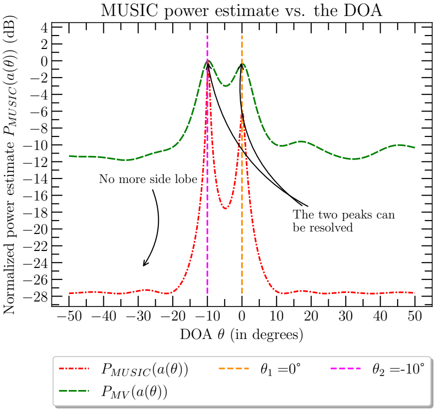

Figure: MUSIC power estimate

- Thinner −3 dB main lobes widths
- Lower ML widths than the MV beamformer!
- We can clearly distinguish the two sources $\implies$ DOA estimation ✔
- No more side lobes, outside the peaks the spectrum is flat with a mean level of −25 dB ✔
- Altough having the same SNR, the two sources do not have the same level in the spectrum
- Not a reliable power (level) estimate ✘

Eigenvector method [1, 3, 4]

- We estimate the beamformer power output using only the **non-normalized** noise part in the previous eigendecomposition

$$\mathbf{R} = \mathbf{V}^H \mathbf{\Lambda} \mathbf{V} = \underbrace{\mathbf{V_s}^H \mathbf{\Lambda_s} \mathbf{V_s}}_{\text{signal + noise}} + \underbrace{\mathbf{V_n}^H \mathbf{\Lambda_n} \mathbf{V_n}}_{\text{noise}} \approx \mathbf{V_n}^H \mathbf{\Lambda_n} \mathbf{V_n} \implies \mathbf{R}^{-1} \approx \mathbf{V_n} \mathbf{\Lambda_n}^{-1} \mathbf{V_n}^H \tag{6}$$

- Why is it efficient? $\mathbf{\Lambda}$ is a diagonal matrix so its inverse is easy to compute

$$P(\mathbf{a}(\theta))_{EV} = \frac{1}{\mathbf{a}(\theta)\mathbf{V_n}\mathbf{\Lambda_n}^{-1}\mathbf{V_n}^H \mathbf{a}(\theta)^H} \tag{7}$$

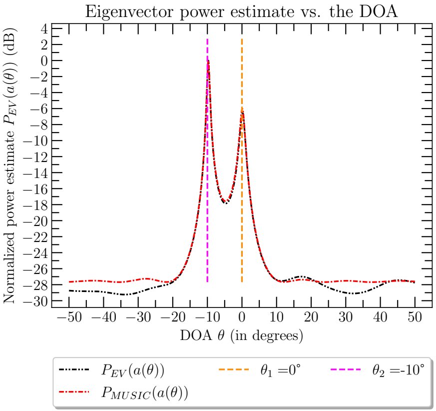

Figure: EV method power estimate

- Both MUSIC and EV power estimates have low ML widths ✓
- They suffer from the same problem on power peak level estimation ✗
- The MUSIC algorithm gives thinner peaks (@−10 dB)
- The MUSIC power estimate seems more flat outside the peaks region

Incorrect number of sources N_s

- Using MUSIC and EV algorithm provides better results but a the cost of some *a priori* knowledge : **the number of sources N_s must be known!**
- What happens when using an incorrect number of sources?

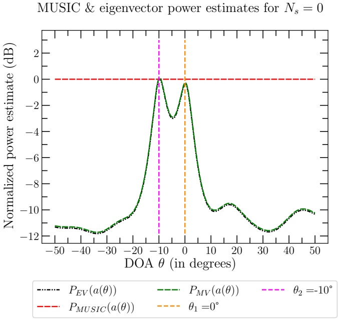

Figure: MUSIC & EV power estimates, $N_s = 0$

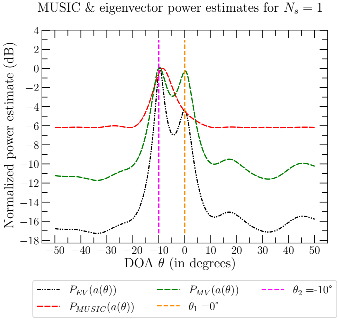

Figure: MUSIC & EV power estimates, $N_s = 1$

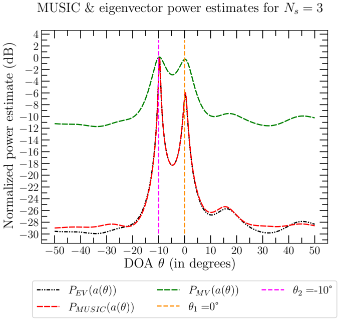

Figure: MUSIC & EV power estimates, $N_s = 3$

- Note that for $N_s = 0$, the EV power estimate perfectly fits the MV estimate:

$$P_{MUSIC}(\mathbf{a}(\theta)) = \left(\sum_{i=N_s+1}^{M} \|\mathbf{a}(\theta)^H \mathbf{v_i}\|^2 \right)^{-1} = \left(\sum_{i=1+0}^{M} \|\mathbf{a}(\theta)^H \mathbf{v_i}\|^2 \right)^{-1} = P_{MV}(\mathbf{a}(\theta)) \tag{8}$$

- $N_s < 2$: we take into account noise and noise + signal subspaces, larger eigenvalues dominate the others
- $N_s > 2$: we take into account only the noise subspace so we 'respect' the philosophy of eigendecomposition methods
- It seems that the EV method is more robust than the MUSIC one for wrong N_s values

Summary

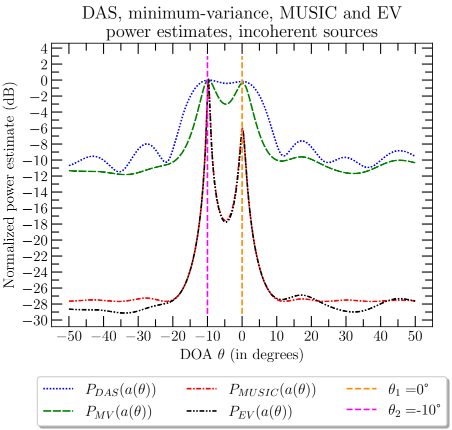

Figure: DAS, MV, MUSIC & EV power estimates

Method	Power estimate	Side lobes	DOA
DAS	✗	✗	✗
MV	✓	✗	∼
MUSIC	✗	✓	✓
EV	✗	✓	✓

Table: Characteristics of beamforming algorithms

- We assumed that sources were incoherent $\longleftrightarrow$ anechoic chamber, without reflections $\longleftrightarrow$ **anechoic signal propagation model**
- To make our simulation more realistic, we can use **coherent** sources
- Similar to a multipath propagation environment: **echoic signal propagation model**

Table of contents

Spatial correlation matrix for coherent sources [1, 3, 4]

- Same problem as before but two sources with the same frequencies and phase terms
- We also made the angle between the two sources smaller: $\theta_1 = 0$ ° and $\theta_2 = 10$ °

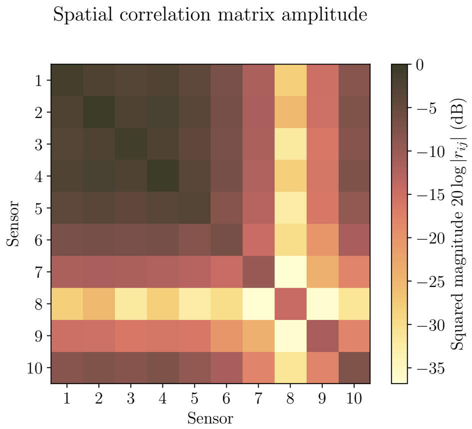

Figure: Spatial correlation matrix amplitude estimate $|\hat{\mathbf{R}}|$, coherent signals

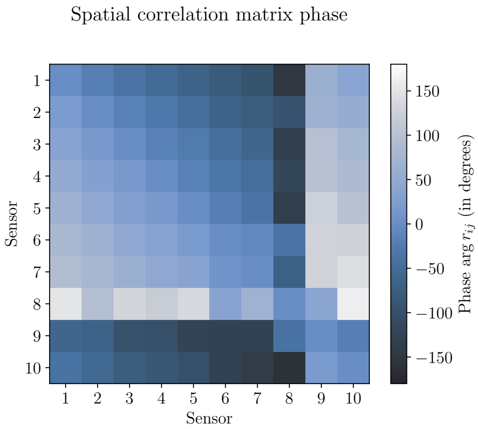

Figure: Spatial correlation matrix phase estimate $\arg \hat{\mathbf{R}}$, coherent signals

EV and MUSIC algorithms for correlated sources [1, 3, 4]

- Let's try both EV and MUSIC algorithm on correlated sources

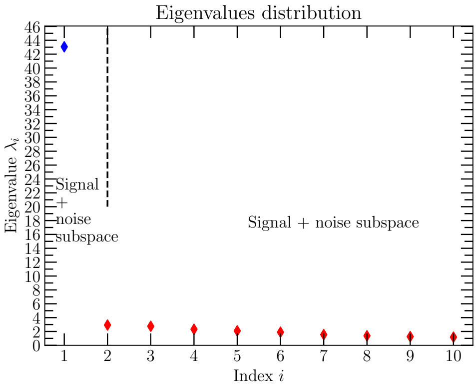

Figure: Eigenvalues distribution, coherent sources

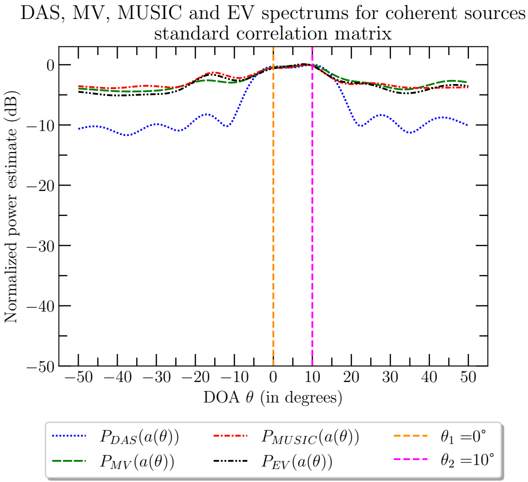

Figure: DAS, MV, MUSIC and EV spectrums for coherent sources

- Only 1 eigenvalue for the signal + noise subspace $\Longrightarrow$ wrong number of sources…

How to improve performances

- We can apply some transformations to **R** to suppress or reduce the correlation between sources
  1. Spatial averaging or smoothing
  2. Forward-backward transformation
  3. Both the previous transformations
- Spatial averaging: compute the correlation matrices of all subarrays with size L ($L = 5$ here). It reduces the cross-correlation terms but also reduces the number of sensors: smaller aperture $\implies$ poorer resolution (similar to WELCH's periodogram)

$$\mathbf{R} = \begin{pmatrix} \boxed{\begin{matrix} r_{11} & r_{12} \\ r_{12}^* & r_{22} \end{matrix}} & \cdots & r_{1M-1} & r_{1M} \\ & \cdots & r_{2M-1} & r_{2M} \\ \vdots \quad \vdots & \boxed{\ddots} & \vdots & \vdots \\ r_{1M-1}^* \quad r_{2M-1}^* & \cdots & \boxed{\begin{matrix} r_{M-1M-1} & r_{M-1M} \\ r_{M-1M}^* & r_{MM} \end{matrix}} & \\ r_{1M}^* \quad r_{2M}^* & \cdots & & \end{pmatrix} \implies \mathbf{R_S} = \underbrace{\frac{1}{M+L-1}}_{=K} \sum_{i=1}^{K} \mathbf{R_i} \text{ where } \mathbf{R_i} = \mathbf{Y}[i : L*i]\,\mathbf{Y}[i : L*i]^H \tag{9}$$

- Forward-backward: let $\mathbf{J} = \begin{pmatrix} 0 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 0 \end{pmatrix}$ be the exchange matrix. Then, the forward-backward spatial correlation matrix is defined as:

$$\mathbf{R_{FB}} = \frac{1}{2}\left(\mathbf{R} + \mathbf{J}\mathbf{R}^*\mathbf{J}\right) \tag{10}$$

What transformation should we apply to R? [1, 3, 4]

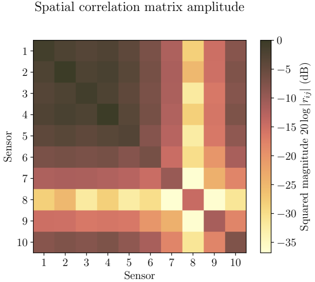

Figure: Standard **R**

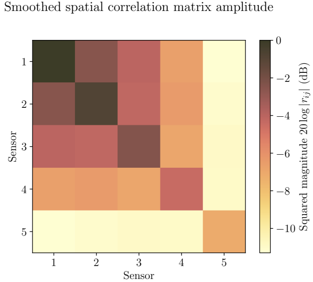

Figure: Smoothed: $\mathbf{R_S}$

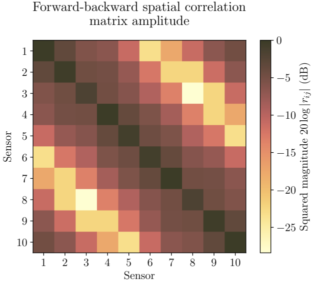

Figure: Forward-backward: $\mathbf{R_{FB}}$

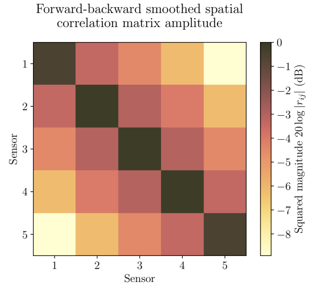

Figure: FB + smoothed: $\mathbf{R_{FBS}}$

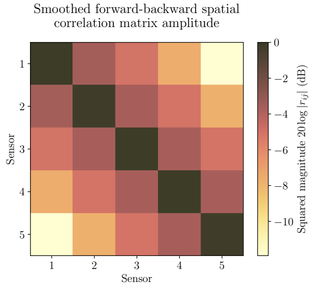

Figure: Smoothed + FB: $\mathbf{R_{SFB}}$

- Spatial averaging: reduces correlation outside the diagonal but lower number of sensors $\Longrightarrow$ lower resolution
- Lower aperture size
- What is the best choice here?

What transformation should we apply to R?

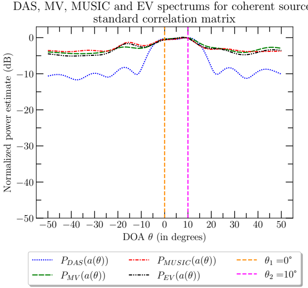

Figure: Standard **R**

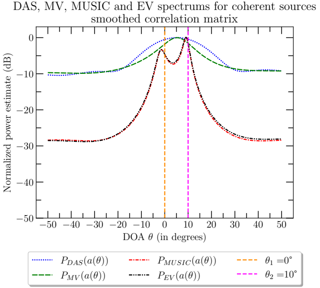

Figure: Smoothed: $\mathbf{R_S}$

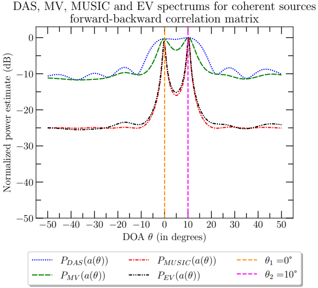

Figure: Forward-backward: $\mathbf{R_{FB}}$

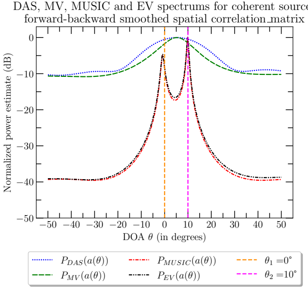

Figure: FB + smoothed: $\mathbf{R_{FBS}}$

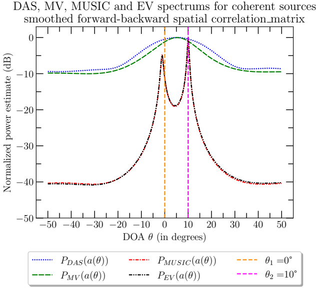

Figure: Smoothed + FB: $\mathbf{R_{SFB}}$

- FB, FB + smoothed and smoothed FB techniques provide the best results here
- Acceptable DOA estimates and ML width
- We can not judge the resolution. . .
- To do it, we will to need run the previous experiment for multiple DOA θ and see what is the smallest angle difference such that two sources can be resolved

Bias and ML width

- We have seen that using EV/MUSIC algorithms combined with a forward-backward smoothed (or smoothed forward-backward) spatial correlation matrix improved the 'quality' of the power estimate
- Since both EV and MUSIC methods have similar results, we focus on assessing MUSIC performances
- What is the 'right' choice for the correlation matrix estimate? Let's compare the bias and the ML width

R estimate	θ_1 (°)	ML width $\Delta\theta_1$ (°)	θ_2 (°)	ML width $\Delta\theta_2$ (°)	Min level (dB)
Standard	-14.390	26.770	-0.090	27.600	-3.136
Smoothed	-2.850	1.985	7.820	7.925	-25.319
FB	-0.120	0.620	9.770	0.650	-23.244
FB + smoothed	-1.260	0.310	8.880	0.415	-41.971
Smoothed + FB	-1.220	0.375	9.020	0.900	-40.113

Table: DOA estimation, resolution and minimum power level for the MUSIC algorithm

R estimate	Mean error (°)	Mean ML width (°)
Standard	12.240	27.185
Smoothed	2.515	4.955
FB	0.175	0.635
FB + smoothed	1.190	0.362
Smoothed + FB	1.100	0.637

Table: Mean DOA estimation error and mean resolution for the MUSIC algorithm

- Standard and smoothed **R** estimates provide poor results...
- According to these tables, FB, FB + smoothed and smoothed + FB correlation matrices give the best results
- FB provides an excellent DOA estimation but with a acceptable ML width whereas FB + smoothed and smoothed + FB give an higher bias but better ML width
- Trade-off... Here, we focus on the FB **R** estimate

SNR

- How are performances affected when decreasing the SNR?

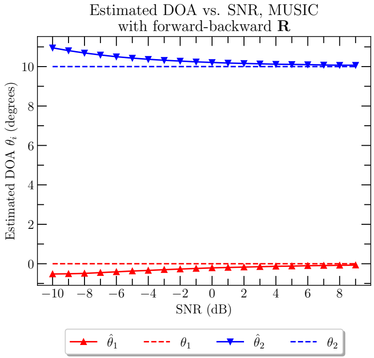

Figure: Estimated DOAs vs. SNR, FB + smoothed

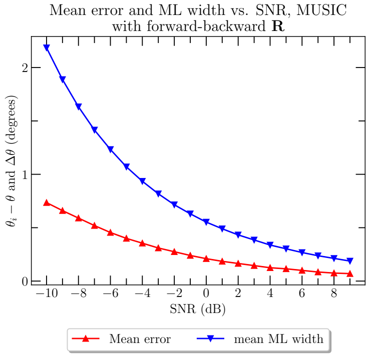

Figure: Mean error and ML width vs. SNR, FB + smoothed

- Decreasing the SNR leads to poor performances as expected (better to use other methods such as root MUSIC)
- Optimal SNR value: 6 dB

Stationarity and ergodicity

- We assume that **Y** was a W.S.S ergodic stochastic process. Is that really the case?
- One way to answer this question is to plot **Y** as a 2D-image[4]

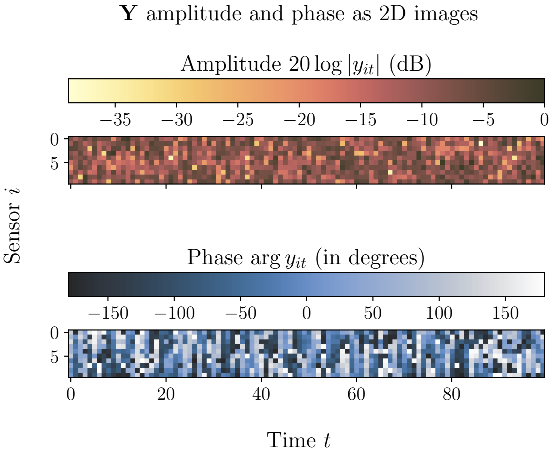

Figure: **Y** amplitude and phase as 2D-images

Stationarity

- If **Y** is an stationary process then its statistical properties are constant with time
- So if we look at two vertical cuts of both images at two time instants, then the mean along these cuts should be the same
- **Stationarity** ✓

Ergodicity

- If **Y** is an ergodic process then its time average is equal to its ensemble average
- So if we take two horizontal cuts of both images at two time instants, then the mean along these cuts should be the same
- **Ergodicity** ✓

[4]Note that this is only a first order test...

Diagonal loading and rotary averaging

- Diagonal loading: $\mathbf{R}_{DL} = \mathbf{R}(1 - \delta \mathbf{I_M}), \delta \in \mathbb{R}$
- Rotary averaging: $\mathbf{R_{RA}} = \dfrac{1}{4}\left(\mathbf{R} + \mathbf{J}\mathbf{R}^T + \mathbf{J}\mathbf{R} + \mathbf{R}\mathbf{J}\right)$

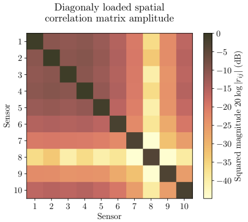

Figure: $\mathbf{R_{DL}}$ squared magnitude

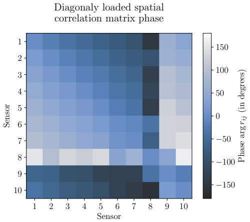

Figure: $\mathbf{R_{DL}}$ phase

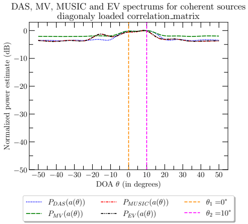

Figure: Power estimates, $\mathbf{R_{DL}}$

- Does not work. . .

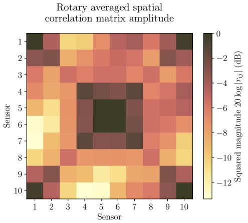

Figure: $\mathbf{R_{RA}}$ squared magnitude

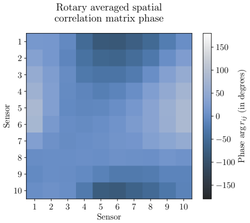

Figure: $\mathbf{R_{RA}}$ phase

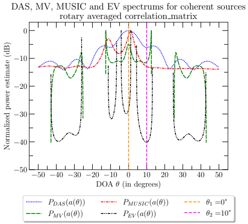

Figure: Power estimates, $\mathbf{R_{RA}}$

- Does not work. . .

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UiO : Department of Informatics
University of Oslo

Thomas Aussaguès

Mandatory exercise 2
High resolution beamforming on farfield monochromatic signals, MATLAB version