UiO : Department of Informatics
University of Oslo

Mandatory exercise 1

Angular spectrum approach and array pattern

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GIBBS phenomenon 1

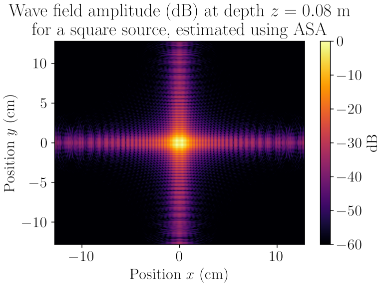

Figure: Wave field at $z = 0.08$ m

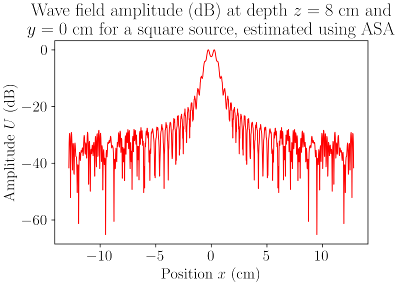

Figure: Center cut of the wave field at $z = 0.08$ m

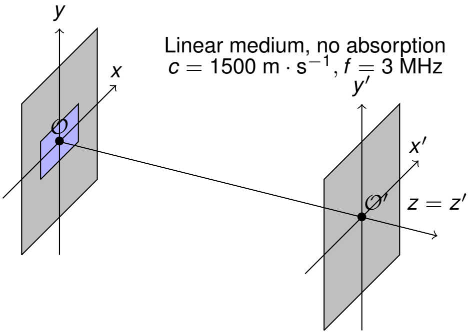

Figure: Geometry

- Energy located at the wave field center and along the x and y-axis
- Square aperture U ($15 \times 15\lambda^2$)
- Very low side lobes !
- Here, **energy outside** the wave field center... Why ?
- GIBBS phenomenon : we are trying to approximate a **discontinuous** function by a **finite** sum of continuous functions
- $\Longrightarrow$ oscillations and artefacts, energy is spread outside the center
- **How to make the aperture continuous ?**

GIBBS phenomenon 2

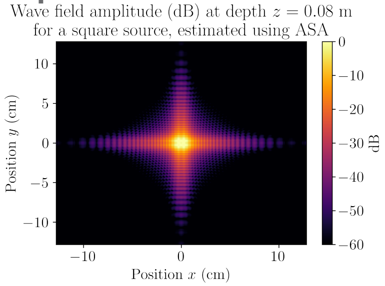

Figure: Wave field at $z = 0.08$ m (smoothed aperture)

- How to make the jump from 0 to 1 at $x, y = r$ **continuous** ?
- HANNING window $\mathcal{H} \longleftrightarrow$ filter : new aperture $U_{mod} = \mathcal{H} \star U$
- The new aperture is **continuous** !
- Energy located at the wave field center
- Lower side lobes : -20 dB $\implies$ reduced artefacts
- Smoother curve
- Same remarks with a circular aperture (except that the observed image should be a BESSEL function J_1 in the far field)

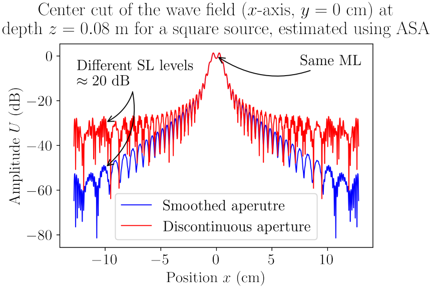

Figure: Center cut of the wave field at $z = 0.08$ m

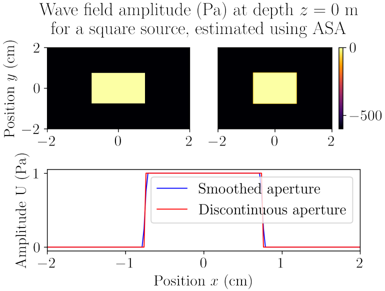

Figure: Effect of the HANNING window

Near field vs. far field 1

- **Near field**[1] : $\lambda \leq \frac{D^2}{4\lambda} = \frac{(2\times15\lambda)^2}{4\lambda} = 225\lambda = $ 0.11m, **superposition** of an infinite number of spherical waves with different phases : interferences
- **Far field** : $\lambda \geq \frac{D^2}{\lambda} = 900\lambda = 0.45$ m : **superposition** of an infinite number of plane waves with same phases : constructive interferences
- Limit ? Check if the amplitude is varying along the x-axis

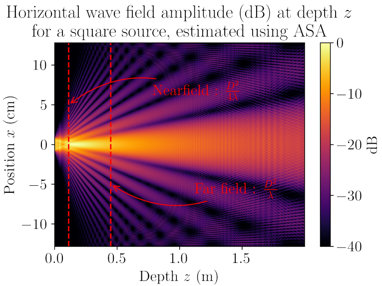

Figure: 2D image of the predicted wave field for $y = 0$ m

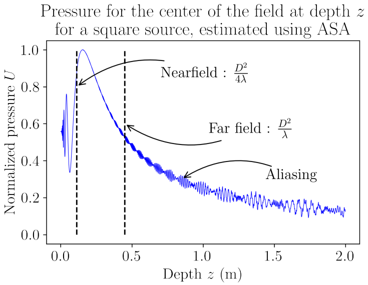

Figure: Pressure at the wave field center

- **Near field** : oscillations for the amplitude (transitional regime)
- **Far field** : amplitude uniform inside a "small" plane xy, approximation of a spherical surface by a plane, amplitude $\propto \frac{1}{r}$, coherent with the spherical wave equation solution $U(\vec{r}, t) = U_0 \frac{\exp(j(\omega t - \vec{k}\cdot\vec{r}))}{||\vec{r}||}$
- **High frequency patterns : aliasing**

[1] For a circular source, or we have a square source here. . .

Near field vs. far field 2

- We can define a direction $\theta = \arctan \dfrac{x}{2z}$
- **Near field** : wave field varies along the x-axis, energy focused in the direction $\theta = 0$, low side lobes
- **Far field** : wave field "uniform" along the x-axis, for small values of θ, paraxial approximation + aliasing[2]

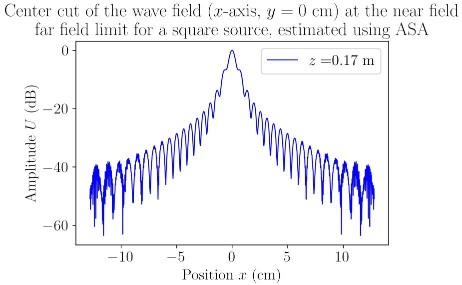

Figure: Wave field center cut at the near field far field limit

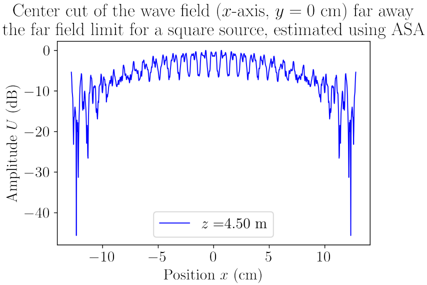

Figure: Wave field center cut at a range much larger than the near field far field limit

[2] Due to the source replicas

Focused ASA simulation

- **Goal** : each wave has the same phase term at $z = r \implies$ delayed source $U_{0,delayed}(x, y) = U_0(x, y)\exp(-j\alpha(x, y))$
- Distance source point $(x, y) \leftrightarrow$ focusing point at $z = r$: $\Delta = \sqrt{x^2 + y^2 + r^2} - r$
- **Phase delay** $\alpha = 2\pi\left(\frac{\Delta}{\lambda} - \lfloor\frac{\Delta}{\lambda}\rfloor\right)$

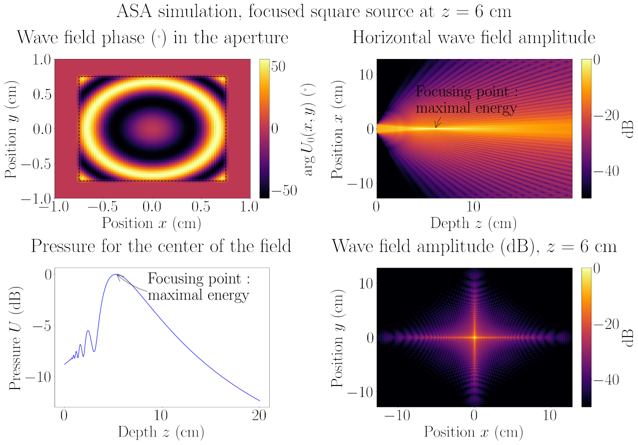

Figure: ASA simulation for a focused square source

- **Steered source** : constant phase over circles (same distance to the focusing point $\implies$ same delay)
- All the energy is focused in one point...
- But this point is not really at 6 cm
- Influence of the $\frac{1}{r}$ factor!
- Near the focusing point, we have the same pattern for the wave field amplitude as in the far field (here $\frac{\sin x}{x}$)

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Grating lobes

- We consider a ULA with $M = 24$ sensors
- Array pattern for different element spacings $d = \frac{\lambda}{4}, \frac{\lambda}{2}, \lambda$ and 2λ

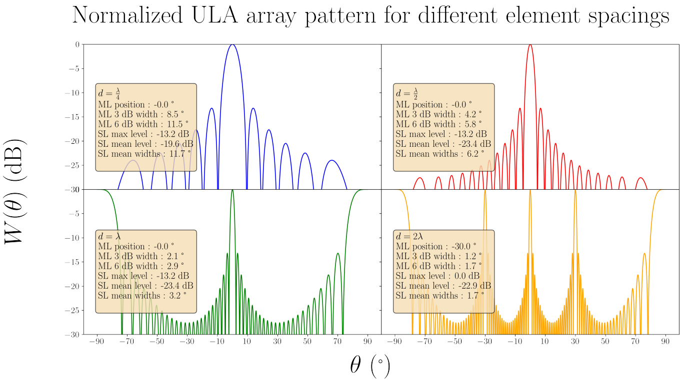

Figure: Array patterns for multiple element spacings

Equivalent results when plotting $W(k_x)$, the plot is just compressed because $k_x = \frac{2\pi}{\lambda} \sin\theta$ is not the identity function.

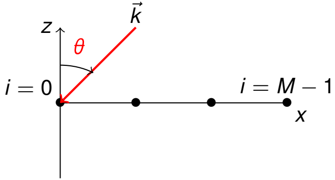

Figure: ULA geometry

- ML width $\searrow$ with d
- SL width $\searrow$ with d
- SL mean level $\searrow$ with d (-3 dB $\Longleftrightarrow$ divided by 2)
- For $d = \frac{\lambda}{4}$ and $d = \frac{\lambda}{2}$: **no grating lobes but larger SL** for $d = \frac{\lambda}{4}$
- $d = \lambda$ and $d = 2\lambda$: **grating lobes** because we don't verify the **ULA sampling criterion** + aliasing in the reconstructed signal
- **More grating lobes** for $d = 2\lambda$: the period of the function is reduced $\Longrightarrow$ more grating lobes in the window $\sin\theta \in [-1, 1]$

Element weighting and element spacing : KAISER window

- We apply a KAISER window $\Longrightarrow$ new weights : $\vec{w}_{\text{KAISER}} = \mathcal{K} \cdot \vec{w}$

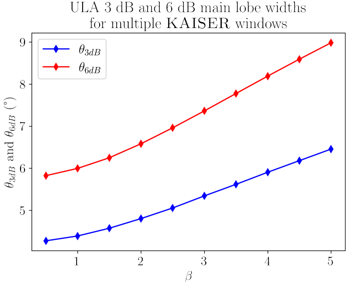

Figure: Influence of the KAISER window on the main lobe width

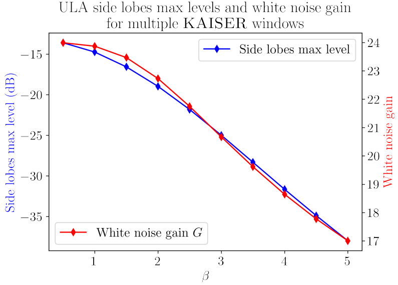

Figure: Influence of the KAISER window on the side lobes max level and white noise gain

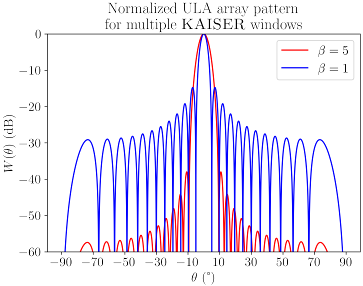

Figure: Influence of the KAISER window on the array pattern

- Larger SL but with lower levels
- Larger ML
- Same aperture size
- **Spatial resolution decreases**!
- **Lower white noise gain** $\Longrightarrow$ we decrease the noise level...
- **Trade-off**
- Other windows : HANNING, CHEBYSHEV...

Array from [1]

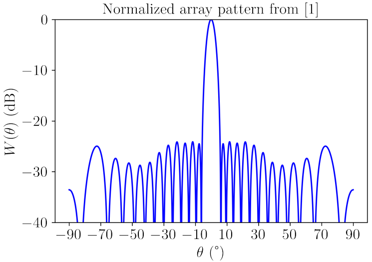

Figure: Array pattern from [1]

$d = \lambda/2$
ML position : -0.0 °
ML 3 dB width : 4.9 °
ML 6 dB width : 6.8 °
SL max level : -24.1 dB
SL mean level : -26.0 dB
SL mean widths : 7.0 °

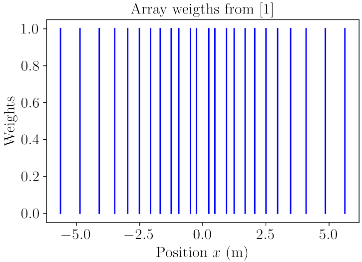

Figure: Array weights from [1]

We compare multiple arrays with same : number of sensors $M = 24$, same element spacing $d = \dfrac{\lambda}{2} \implies$ same size

Array	ML width °		SL			Grating	White
	3 dB	6 dB	Mean width (°)	Mean level (dB)	Max level (dB)	lobes	noise gain
Standard ULA	4.2	11.5	11.7	-19.6	-13.2	✓	24
ULA + KAISER ($\beta = 1$)	4.4	6.0	6.1	-25.0	-14.7	✓	23.5
ULA + KAISER ($\beta = 5$)	6.5	9.0	5.7	-52.0	-38.0	✓	17.0
Array from [1]	**4.9**	**6.8**	**7.0**	**-26.0**	**-24.1**	✓	24

Table: Comparison of multiples arrays

- Array from [1] : **same white noise gain** as a ULA but with **lower narrower SL** (-6 dB)

Steering

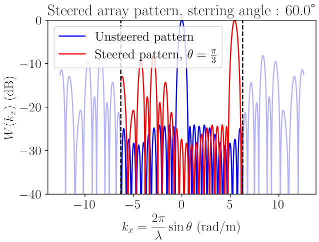

Figure: Steered array pattern [1]

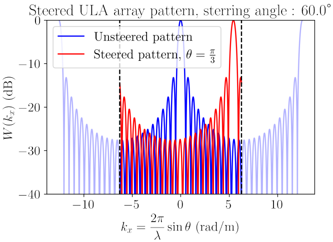

Figure: Steered ULA array pattern

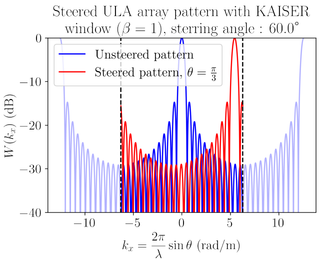

Figure: Steered ULA array pattern with KAISER window ($\beta = 1$)

- Array from [1] : very high side lobes $\implies$ **grating lobes**. . .
- Why? Grating lobes are extremely close to the visible region!
- Steered ULA and ULA with KAISER window arrays patterns are free form grating lobes!
- 👉 For very low sterring angles, the pattern might be free from grating lobes

Steering

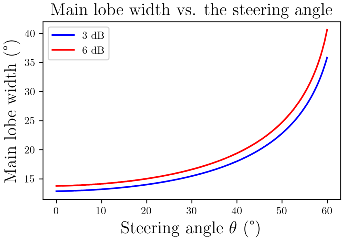

Figure: Main lobe width w.r.t θ

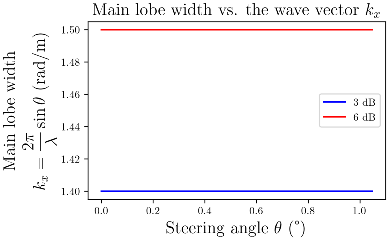

Figure: Main lobes widths w.r.t k_x

- Main lobe width w.r.t θ increases but is constant w.r.t k_x... Why ?
- Steering : $W(k_x) \rightarrow W(k_x - k_x^\circ)$ and $W(\theta) = W\left(\frac{2\pi}{\lambda}\sin\theta\right) \rightarrow W\left(\frac{2\pi}{\lambda}\sin\theta - \frac{2\pi}{\lambda}\sin\theta^\circ\right)$
- The array pattern w.r.t $\mathbf{k_x^\circ}$ is only translated from a constant factor $\implies$ constant main lobe width
- The array pattern w.r.t θ is **shifted and stretched**
- Why ? Because the beam pattern is a function of $\sin\theta$**!**
- Non-linear dependence
- Stretching effect varies with θ
- For high steering angles, we have large ML
- **Better to tilt the array rather than to steer it...**

Thinning : uniform distribution [2]

- Uniform distribution
- Results for active elements number $N_{pos} = 25, 50$ and 75

Array	ML 3 dB width(°)		Mean SL level (dB)		Max SL level (dB)	
	Mean	Std	Mean	Std	Mean	Std
$N_{pos} = 25$	1.02	7.0×10^{-2}	-14.1	4.6×10^{-1}	-7.66	7.6×10^{-1}
$N_{pos} = 50$	1.01	6.0×10^{-2}	-18.3	5.8×10^{-1}	-11.53	1.21
$N_{pos} = 75$	1.01	3.0×10^{-2}	-22.5	3.3×10^{-1}	-13.77	1.35

Table: Thinned arrays characteristics (uniform distribution)

Array	ML 3 dB width (°)	Mean SL level (dB)	Max SL level (dB)
ULA 101 elements	1.00	-69.3	-26.5
Fig.3 [2] ($N_{pos} = 75$)	1.19	-13.7	-12.1
Fig.4 [2] ($N_{pos} = 75$)	1.46	-14.2	-12.7

Table: Arrays characteristics

Analysis (other arrays):

- ULA and thinned arrays have the **same ML width**
- But **higher SL** for thinned arrays
- Thinned arrays have **lower SL** than the arrays presented in [2]

Analysis (N_{pos}):

- ML width remains **constant** when $N_{pos} \nearrow$
- Both SL max and mean level $\searrow$
- Variance is decreasing
- Thinned array $\xrightarrow[N_{pos} \to M]{}$ ULA
- **Reduced sensitivity to signal coming from directions $\theta \neq 0$**

- Remark : using thinned arrays reduces the white Gaussian noise! We are only using N_{pos} elements! $\left(\|w\|^{-2} = N_{pos}\right)$

Thinning : Gaussian distribution [2]

- Gaussian distribution
- Results for active elements number $N_{pos} = 25, 50$ and 75
- Standard deviation : $\frac{N_{els}}{6} \approx 16.83$

Array	ML 3 dB width(°)		Mean SL level (dB)		Max SL level (dB)	
	Mean	Std	Mean	Std	Mean	Std
$N_{pos} = 25$	1.43	0.14	-15.07	0.74	-8.47	0.86
$N_{pos} = 50$	1.42	0.07	-20.45	0.67	-13.82	0.87
$N_{pos} = 75$	1.22	0.03	-24.54	0.59	-18.17	1.05

Table: Thinned arrays characteristics (Gaussian distribution, std $\approx$ 16.8 elements)

Array	ML 3 dB width (°)	Mean SL level (dB)	Max SL level (dB)
ULA 101 elements	1.0	-69.3	-26.5
Fig.3 [2] ($N_{pos} = 75$)	1.19	-13.7	-12.1
Fig.4 [2] ($N_{pos} = 75$)	1.46	-14.2	-12.7

Table: Arrays characteristics

Analysis vs. other arrays :

- Wider ML and lower SL than the thinned array with a uniform distibution
- **Larger ML and higher SL** than the ULA
- **Lower SL** than the arrays presented in [2]...
- Using a gaussian distribution for the elements position is equivalent to use a Gaussian distibution for the weights : spatial tapering $\Longleftrightarrow$ amplitude tapering

Analysis (N_{pos}) :

- ML width decreases when $N_{pos} \nearrow$
- Both SL max and mean level $\searrow$

Binned sparse array [3]

How to design a binned spare array?

- We consider an array with $M = 101$ sensors with element spacing $d = \dfrac{\lambda}{2}$
- The size of the array is $L = (M - 1)d$
- We divide it into N bins of size $w = \dfrac{L}{N}$
- The position of the m-th element is given by $x_m = -\dfrac{L}{2} + m \cdot w + y_m$ where y_m is drawn from $\mathcal{U}_{[0,1]}$

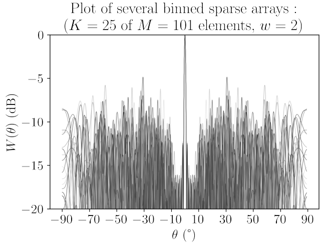

Figure: Binned spare arrays

Array	ML 3 dB width(°)		Mean SL level (dB)		Max SL level (dB)	
	Mean	Std	Mean	Std	Mean	Std
$N_{pos} = 25$ (from [1])	1.0	7.0×10^{-2}	-14.1	4.6×10^{-1}	-7.66	7.6×10^{-1}
$K = 25$ (from [3])	1.0	1.0×10^{-2}	-14.8	5.0×10^{-1}	-6.73	1.01

Table: Thinned array with uniform distribution vs. binned spare array

Analysis

- Same ML width but **lower variance** for the binned spare array
- "This means that the binned array has a much larger region around the steered direction where the effect of the randomness is small" [3]
- SL mean and max levels are **similar**
- **Different aperture** : for the binned spare array, the $\Longrightarrow$ K first elements are **jittered**; for the thinned array, outer elements are conserved

Element directivity

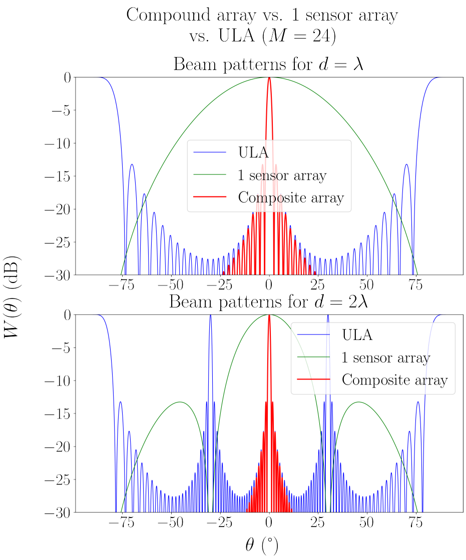

Figure: Composite arrays for $d = \lambda$ and $d = 2\lambda$

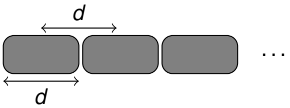

Figure: Composite array

- Composite array : M sensors of size d with element spacing d, unity weights
- $d > \frac{\lambda}{2}$: **ULA sampling criterion** (and also NYQUIST's sampling theorem) is not verified ✗ $\Longrightarrow$ grating lobes + aliasing for the ULA
- Composite arrays : low SL levels ✓
- And no grating lobes
- Composite array beam pattern perfectly fits the ULA ML for both element spacings!
- So, using composite arrays reduces the sensitivity in directions $\theta \neq 0$ while keeping a good resolution. But what about steering?

Steered composite array

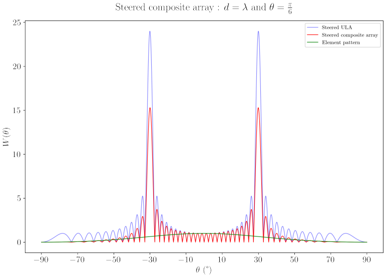

Figure: Steered compound array

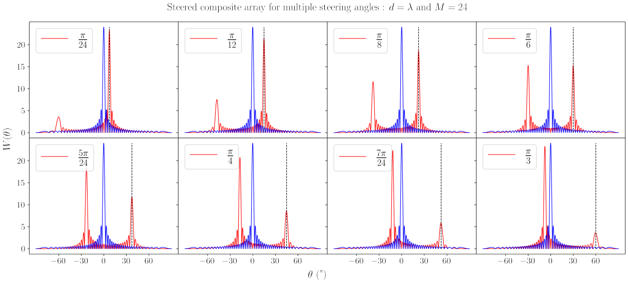

Figure: Steered composite array pattern for multiple steering angles

- As the steering angle increases :
- A **grating lobe** appears
- And the **side lobes** (w.r.t the main lobe level) levels **increase**
- For high value of the steering angle, the grating lobe **replaces** the main lobe
- This composite array is unsuitable for high steering angle values!

Linear array design

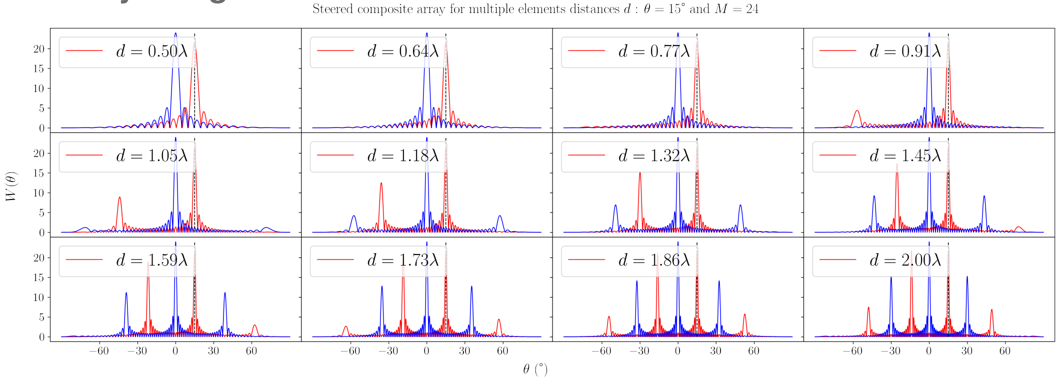

Figure: Steered composite array pattern for multiple elements distances

- Acceptable grating lobes levels?
- **Should be less or equal to the half main lobe level**
- For $d \geq 1.32\lambda$: the grating lobe level is too high
- $d = 1.05\lambda$ and $d = 1.18\lambda$ seem to be good trade-off between aperture and grating lobes levels

References & additional figures

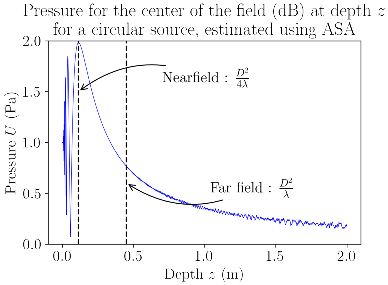

Figure: Pressure at the wave field center (circular source)

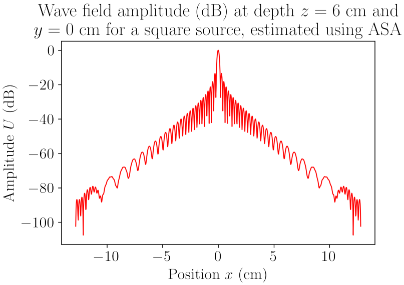

Figure: Wave field center cut at the focusing point (square source)

[1] F. Hodjat and S. Hovanessian. 'Nonuniformly spaced linear and planar array antennas for sidelobe reduction'. In: *IEEE Trans. on Antennas and Propagation 26.2 198-204* (1978).

[2] J. F. Hopperstad and S. Holm. 'Optimization of sparse arrays by an improved simulated annealing algorithm'. In: *Proc. Int. Workshop on Sampling Theory and Applications, pp. 91-95* ().

[3] Holm et. al. 'Sampling theory and practice, Chapter 19'. In: ().

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