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Selfish unsplittable flows☆

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Abstract

What is the price of anarchy when unsplittable demands are routed selfishly in general networks with load-dependent edge delays? Motivated by this question we generalize the model of Koutsoupias and Papadimitriou (Worst-case equilibria, in: Proc. of the 16th Annual Symp. on Theoretical Aspects of Computer Science (STACS '99), Lecture Notes in Computer Science, Vol. 1563, Springer, Berlin, 1999, pp. 404–413) to the case of *weighted congestion games*. We show that varying demands of users crucially affect the nature of these games, which are no longer isomorphic to exact potential games, even for very simple instances. Indeed we construct examples where even a single-commodity (weighted) network congestion game may have no pure Nash equilibrium.

On the other hand, we prove that any weighted network congestion game with linear edge delays admits a pure Nash equilibrium that can be found in pseudo-polynomial time. Finally, we consider the family of *ℓ-layered networks* and give a surprising answer to the question above: the price of anarchy of any weighted congestion game in a ℓ-layered network with m edges and edge delays equal to the loads is $\Theta(\log m / \log \log m)$.

Keywords: Pure Nash equilibria; Network congestion games; Unsplittable flows; Price of anarchy; Quadratic programming

1. Introduction

Consider a model where selfish users having varying demands compete for some shared resources. The quality of service provided by a resource decreases with its *congestion*, i.e., the amount of demands of the users willing to be served by it. Each user may reveal its actual (unique) choice (called a *pure strategy*) among the resources available to it, or it may reveal a probability distribution for choosing one of its candidate resources (a *mixed strategy*). The users determine their actual behavior based on other users' behavior, but they do not cooperate. We are interested in situations where the users have reached some kind of equilibrium. The most popular notion of equilibrium in non-cooperative game theory is the *Nash equilibrium*: a "stable point" among the users, from which no user is willing to deviate unilaterally. In [14] the notion of the *coordination ratio* or *price of anarchy* was introduced, as a means for measuring the performance degradation due to lack of users' coordination when sharing common goods.

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A realistic scenario for the above model is when unsplittable demands are routed selfishly in general networks with load-dependent edge delays. When the underlying network consists of two nodes and parallel links between them, there has been an extensive study on the existence and computability of equilibria, as well as on the price of anarchy. Motivated by the work of [14], we generalize their concept to the *weighted congestion games* in a non-trivial way. When users have identical demands, such a game is indeed isomorphic to an *exact potential game* [19] and thus always possesses a pure Nash equilibrium, i.e., an equilibrium where each user adopts a pure strategy. We show that varying demands of users crucially affect the nature of these games, which are no longer isomorphic to exact potential games. Indeed, we construct examples where even a single-commodity (weighted) network congestion game may have no pure Nash equilibrium at all.

On the other hand, we prove that any weighted multi-commodity network congestion game with linear resource delays admits a pure Nash equilibrium. We also propose a pseudo-polynomial time algorithm for constructing one. Finally, we study the price of anarchy for weighted single-commodity network congestion games on *ℓ-layered networks*. We come to a rather surprising conclusion: within constant factors, the worst case instance (wrt the price of anarchy) among weighted ℓ-layered network congestion games with m edges and edge delays equal to the loads is the parallel links game introduced in [14].

2. The model

Consider having a set of resources E in a system. For each $e \in E$, let $d_e(\cdot)$ be the delay per user that requests its service, as a function of the total usage of this resource by all the users. Each such function is considered to be non-decreasing in the total usage of the corresponding resource. Each resource may be represented by a pair of points: an entry point to the resource and an exit point from it. So, we represent each resource by an arc from its entry point to its exit point and we associate with this arc the cost (e.g., the delay as a function of the load of this resource) that each user has to pay if she is served by this resource. The entry/exit points of the resources need not be unique; they may coincide in order to express the possibility of offering joint service to users, that consists of a sequence of resources. We denote by V the set of all entry/exit points of the resources in the system. Any non-empty collection of resources corresponding to a directed path in $G \equiv (V, E)$ comprises an *action* in the system.

Let $N \equiv [n]$ be a set of users, each willing to adopt some action in the system. $\forall i \in N$, let w_i denote user i's *demand* (e.g., the flow rate from a source node to a destination node), while $\Pi_i \subseteq 2^E \setminus \emptyset$ is the collection of actions, any of which would satisfy user i (e.g., alternative routes from a source to a destination node, if G represents a communication network). The collection Π_i is called the *action set* of user i and each of its elements contains at least one resource. Any vector $\mathbf{r} = (r_1, \ldots, r_n) \in \Pi \equiv \times_{i=1}^{n} \Pi_i$ is a *pure strategies profile*, or a *configuration* of the users. Any vector of real functions $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ s.t. $\forall i \in [n]$, $p_i : \Pi_i \to [0, 1]$ is a probability distribution over the set of allowable actions for user i $\left(\text{i.e., } \sum_{r_i \in \Pi_i} p_i(r_i) = 1\right)$ is called a *mixed strategies profile* for the n users.

A congestion model typically deals with users of identical demands, and thus, user cost function depending on the *number* of users adopting each action [22,19,7]. In this work we consider the more general case, where a *weighted congestion model* is the tuple $((w_i)_{i \in N}, (\Pi_i)_{i \in N}, (d_e)_{e \in E})$. That is, we allow the users to have different demands for service from the whole system, and thus affect the resource delay functions in a different way, depending on their own weights. A *weighted congestion game* associated with this model is a game in strategic form with the set of users N and user demands $(w_i)_{i \in N}$, the action sets $(\Pi_i)_{i \in N}$ and cost functions $(\lambda^i_{r_i})_{i \in N, r_i \in \Pi_i}$ defined as follows: for any configuration $\mathbf{r} \in \Pi$ and $\forall e \in E$, let $\Lambda_e(\mathbf{r}) = \{i \in N : e \in r_i\}$ be the set of users exploiting resource e according to $\mathbf{r}$ (called the *view* of resource e wrt configuration $\mathbf{r}$). The *cost* $\lambda^i(\mathbf{r})$ *of user i for adopting strategy* $r_i \in \Pi_i$ in a given configuration $\mathbf{r}$ is equal to the cumulative *delay* $\lambda_{r_i}(\mathbf{r})$ along this path:

$$\lambda^i(\mathbf{r}) = \lambda_{r_i}(\mathbf{r}) = \sum_{e \in r_i} d_e(\theta_e(\mathbf{r})), \tag{1}$$

where $\forall e \in E,\ \theta_e(\mathbf{r}) \equiv \sum_{i \in \Lambda_e(\mathbf{r})} w_i$ is the load on resource e wrt the configuration $\mathbf{r}$.

On the other hand, for a mixed strategies profile $\mathbf{p}$, the *expected cost of user i for adopting strategy* $r_i \in \Pi_i$ is

$$\lambda^i_{r_i}(\mathbf{p}) = \sum_{\mathbf{r}^{-\mathbf{i}} \in \Pi^{-i}} P\left(\mathbf{p}^{-\mathbf{i}}, \mathbf{r}^{-\mathbf{i}}\right) \cdot \sum_{e \in r_i} d_e\left(\theta_e(\mathbf{r}^{-\mathbf{i}} \oplus r_i)\right), \tag{2}$$

where $\mathbf{r^{-i}}$ is a configuration of all the users except for user i, $\mathbf{p^{-i}}$ is the mixed strategies profile of all users except for i, $\mathbf{r^{-i}} \oplus r_i$ is the new configuration with user i choosing strategy r_i, and $P\left(\mathbf{p^{-i}}, \mathbf{r^{-i}}\right) \equiv \prod_{j \in N \setminus \{i\}} p_j(r_j)$ is the occurrence probability of $\mathbf{r^{-i}}$.

Remark 1. We abuse notation a little bit and consider the user costs $\lambda^i_{r_i}$ as functions whose exact definition depends on the other users' strategies: in the general case of a mixed strategies profile $\mathbf{p}$, Eq. (2) is valid and expresses the expected cost of user i wrt $\mathbf{p}$, conditioned on the event that i chooses path r_i. If the other users adopt a pure strategies profile $\mathbf{r^{-i}}$, we get the special form of Eq. (1) that expresses the exact cost of user i choosing action r_i.

A congestion game in which all users are indistinguishable (i.e., they have the same user cost functions) and have the same action set is called *symmetric*. When each user's action set Π_i consists of sets of resources that comprise (simple) paths between a unique origin–destination pair of nodes (s_i, t_i) in an network $G = (V, E)$, we refer to a *network congestion game*. If additionally all origin–destination pairs of the users coincide with a unique pair (s, t) we have a *single commodity network congestion game* and then all users share exactly the same action set. Observe that a single-commodity network congestion game is not necessarily symmetric because the users may have different demands and thus their cost functions will also differ.

Selfish behavior: Fix an arbitrary (mixed in general) strategies profile $\mathbf{p}$ for a congestion game $((w_i)_{i\in N}, (\Pi_i)_{i\in N}, (d_e)_{e\in E})$. We say that $\mathbf{p}$ is a *Nash equilibrium* (*NE*) if and only if $\forall i \in N, \forall r_i, \pi_i \in \Pi_i, p_i(r_i) > 0 \Rightarrow \lambda^i_{r_i}(\mathbf{p}) \leqslant \lambda^i_{\pi_i}(\mathbf{p})$. A configuration $\mathbf{r} \in \Pi$ is a *pure Nash equilibrium* (*PNE*) if and only if $\forall i \in N, \forall \pi_i \in \Pi_i, \lambda_{r_i}(\mathbf{r}) \leqslant \lambda_{\pi_i}(\mathbf{r^{-i}} \oplus \pi_i)$ where $\mathbf{r^{-i}} \oplus \pi_i$ is the same configuration with $\mathbf{r}$ except for user i that now chooses action π_i. The *social cost* $SC(\mathbf{p})$ in this congestion game is the expected maximum cost among all the users:

$$SC(\mathbf{p}) = \sum_{\mathbf{r}\in\Pi} P(\mathbf{p}, \mathbf{r}) \cdot \max_{i\in N}\{\lambda_{r_i}(\mathbf{r})\}, \tag{3}$$

where $P(\mathbf{p}, \mathbf{r}) \equiv \prod_{i=1}^{n} p_i(r_i)$ is the probability of configuration $\mathbf{r}$ occurring, wrt the mixed strategies profile $\mathbf{p}$. The *social optimum* of this game is defined as the minimum (over all possible configurations) maximum (over all users) user cost

$$OPT = \min_{\mathbf{r}\in\Pi}\left\{\max_{i\in N}[\lambda_{r_i}(\mathbf{r})]\right\}. \tag{4}$$

The *price of anarchy* for this game is then defined as follows:

$$\mathcal{R} = \max_{\mathbf{p}\text{ is a NE}}\left\{\frac{SC(\mathbf{p})}{OPT}\right\}. \tag{5}$$

We consider *atomic* assignments of users to actions, i.e., each user $i \in N$ requires all its demand w_i from exactly one allowable action $r_i \in \Pi_i$. Nevertheless, we allow users to adopt mixed strategies. Our focus in this paper is two-fold: we are interested in families of resource delay functions for which a weighted network congestion game possesses a PNE, and we are also interested in the price of anarchy for a special case of this problem where G has the form of an ℓ-layered network and the delay functions are identical to the loads of the resources.

3. Related work

Existence and tractability of PNE: It is already known that the class of unweighted (atomic) congestion games (i.e., users have the same demands and thus, the same affection on the resource delay functions) is guaranteed to possess at least one PNE: actually, Rosenthal [22] proved that any potential game has at least one PNE and it is easy to write any unweighted congestion game as an exact potential game using Rosenthal's potential function [1]

[1] For more details on Potential Games, see [19].

(e.g., [7, Theorem 1]). In [7] it is proved that a PNE for any unweighted single-commodity network congestion game [2] (no matter what resource delay functions are considered, so long as they are non-decreasing with loads) can be constructed in polynomial time, by computing the optimum of Rosenthal's potential function, through a nice reduction to an instance of the min-cost flow problem. On the other hand, it is shown in the same work that even for a symmetric congestion game or an unweighted multi-commodity network congestion game, it is PLS-complete to construct a PNE (though it certainly exists).

The special case of single-commodity, parallel-edges network congestion game where the resources are considered to behave as parallel machines, has been extensively studied in recent literature. In [9] it was shown that for the case of users with varying demands and uniformly related parallel machines, there is always a PNE which can be constructed in polynomial time. It was also shown that it is NP-hard to construct the best or the worst PNE. In [10] it was proved that the fully mixed NE (FMNE), introduced and thoroughly studied in [17], is worse than any PNE, and any NE is at most $(6 + \varepsilon)$ times worse than the FMNE, for varying users and identical parallel machines. In [16] it was shown that the FMNE is the worst possible for the case of two related machines and tasks of the same size. In [15] it was proved that the FMNE is the worst possible when the global objective is the sum of squares of loads.

Feldman et al. [8] studied the problem of constructing a PNE from any initial configuration, of social cost at most equal to that of the initial configuration. This immediately implies the existence of a PTAS for computing a PNE of minimum social cost: first compute a configuration of social cost at most $(1 + \varepsilon)$ times the social optimum [11], and consequently transform it into a PNE of at most the same social cost. In [6] it is also shown that even for the unrelated parallel machines case a PNE always exists, and a potential-based argument proves a convergence time (in case of integer demands) from arbitrary initial configuration to a PNE in time $\mathrm{O}(m W_{\mathrm{tot}} + 4^{W_{\mathrm{tot}}/m + w_{\max}})$ where $W_{\mathrm{tot}} = \sum_{i \in N} w_i$ and $w_{\max} = \max_{i \in N}\{w_i\}$.

Milchtaich [18] studies the problem of weighted parallel-edges network congestion games with user-specific costs: each allowable action of a user consists of a single resource and each user has its own private cost function for each resource. It is shown that: (1) weighted (parallel-edges network) congestion games involving only two users, or only two possible actions for all the users, or equal delay functions (and thus, equal weights), always possess a PNE; (2) even a single-commodity, 3-user, 3-actions, weighted (parallel-edges network) congestion game may not possess a PNE (using 3-wise linear delay functions).

Price of anarchy in congestion games: In the seminal paper [14] the notion of coordination ratio, or price of anarchy, was introduced as a means for measuring the performance degradation due to lack of users' coordination when sharing common resources. In this work it was proved that the price of anarchy is $3/2$ for two related parallel machines, while for m machines and users of varying demands, $\mathcal{R} = \Omega(\log m / \log\log m)$ and $\mathcal{R} = \mathrm{O}(\sqrt{m \log m})$. For m identical parallel machines, [17] proved that $\mathcal{R} = \Theta(\log m / \log\log m)$ for the FMNE, while for the case of m identical parallel machines and users of varying demands it was shown in [13] that $\mathcal{R} = \Theta(\log m / \log\log m)$. In [4] it was finally shown that $\mathcal{R} = \Theta(\log m / \log\log\log m)$ for the general case of related machines and users of varying demands. Czumaj et al. [3] present a thorough study of the case of general, monotone delay functions on parallel machines, with emphasis on delay functions from queuing theory. Unlike the case of linear cost functions, they show that the price of anarchy for non-linear delay functions in general is far worse and often even unbounded.

In [23] the price of anarchy in a multi-commodity network congestion game among infinitely many users, each of negligible demand is studied. The social cost in this case is expressed by the total delay paid by the whole flow in the system. For linear resource delays, the price of anarchy is at most $4/3$. For general, continuous, non-decreasing resource delay functions, the total delay of any Nash flow is at most equal to the total delay of an optimal flow for double flow demands. Roughgarden [24] proves that for this setting, it is actually the class of allowable latency functions and not the specific topology of a network that determines the price of anarchy.

4. Our contribution

In this paper, we generalize the model of [14] (KP-model) to the weighted congestion games. We also define a special class of networks, the ℓ-layered networks, which demonstrate a rather surprising behavior: their worst instance wrt the

[2] Since [7] only considers unit-demand users, this is also a symmetric network congestion game.

price of anarchy is (within constant factors) the parallel links network introduced in [14]. More specifically, we prove the following results:

- Weighted congestion games are *not* isomorphic to potential games. We show the existence of weighted single-commodity network congestion games with resource delays being either linear or 2-wise linear functions of the loads, for which a PNE cannot exist (Lemma 6).
- There exist weighted single-commodity network congestion games which admit no exact potential function, even when the resource delays are identical to their loads (Lemma 3).
- Any weighted (multi-commodity) network congestion game with linear resource delays admits a PNE which can be constructed in pseudo-polynomial time (Theorem 1).
- The price of anarchy of any weighted ℓ-layered network congestion game with m resources (edges) and resource delays equal to their loads is at most $8e(\log m/\log\log m + 1)$, where e is the basis of the natural logarithm (Theorem 7). To our knowledge this is the first time that the KP-model is studied in non-trivial networks (other than the parallel links).

5. Preliminaries

Configuration paths and dynamics graph: For a congestion game $\Gamma = ((w_i)_{i\in N}, (\Pi_i)_{i\in N}, (d_e)_{e\in E})$, a *path* in $\Pi = \times_{i\in N}\Pi_i$ is a sequence of configurations $\gamma = (\mathbf{r}(0), \mathbf{r}(1), \ldots, \mathbf{r}(\mathbf{k}))$ s.t. $\forall j \in [k]$, $\mathbf{r}(\mathbf{j}) = \mathbf{r}(\mathbf{j-1})^{-\mathbf{i}} \oplus \pi_i$, for some $i \in N$ and $\pi_i \in \Pi_i$. γ is a *closed path* if $\mathbf{r}(\mathbf{0}) = \mathbf{r}(\mathbf{k})$. It is a *simple path* if no configuration is contained in it more than once. γ is an *improvement path* wrt Γ, if $\forall j \in [k]$, $\lambda^{i_j}(\mathbf{r}(\mathbf{j})) < \lambda^{i_j}(\mathbf{r}(\mathbf{j-1}))$ where i_j is the unique user differing in its strategy between $\mathbf{r}(\mathbf{j})$ and $\mathbf{r}(\mathbf{j-1})$. That is, the unique defector of the jth move in γ is actually willing to make the move because this move improves its own cost. The *Dynamics Graph* of Γ is a directed graph whose vertices are configurations and there is an arc from a configuration $\mathbf{r}$ to a configuration $\mathbf{r}^{-\mathbf{i}} \oplus \pi_i$ for some $\pi_i \in \Pi_i$ if and only if $\lambda^i(\mathbf{r}) > \lambda^i(\mathbf{r}^{-\mathbf{i}} \oplus \pi_i)$.

Layered networks: We consider a special family of networks whose behavior wrt the price of anarchy is (as we shall prove here) asymptotically equivalent to that of the parallel links model of [14], which is actually a 1-layered network: let $\ell \geqslant 1$ be an integer. A directed network $G = (V, E)$ with a distinguished source–destination pair (s, t), $s, t \in V$ is *ℓ-layered* if every directed $s - t$ path has length exactly ℓ and each node lies on a directed $s - t$ path. In a layered network there are no directed cycles and all directed paths are simple. In the following, we always use m to denote the number $|E|$ of edges in an ℓ-layered network $G = (V, E)$. We use $\mathcal{P}$ to denote the set of all directed $s - t$ paths in G.

6. Pure Nash equilibria

In this section we deal with the existence and tractability of PNE in weighted network congestion games. First, we show that it is not always the case that a PNE exists, even for a weighted single-commodity network congestion game with only linear and 2-wise linear (e.g., the maximum of two linear functions) resource delays. In contrast, it is well known [22,7] that any unweighted (not necessarily single-commodity, or even network) congestion game has a PNE, for any kind of non-decreasing delays.

Lemma 1. *There exist instances of weighted single-commodity network congestion games with resource delays being either linear or* 2*-wise linear functions of the loads, for which there is no PNE.*

Proof. We demonstrate this by the example shown in Fig. 1. In this example there are exactly two users of demands $w_1 = 1$ and $w_2 = 2$, from node s to node t. The possible paths that the two users may follow are labeled in the figure. The resource delay functions are indicated by the three possible values they may take, given the weights of the two users. Observe now that this example has no PNE: there is a simple closed path $\gamma = ((P3, P2), (P3, P4), (P1, P4), (P1, P2), (P3, P2))$ of length 4 that is an improvement path (actually, each defecting user moves to its new best choice) and additionally, any other configuration not belonging in γ is either one, or two best-choice moves away from some of these nodes. Therefore, there is no sink in the Dynamics Graph of the game and thus there exists no PNE. Observe that the delay functions are not user-specific in our example, as was the case in [18]. □

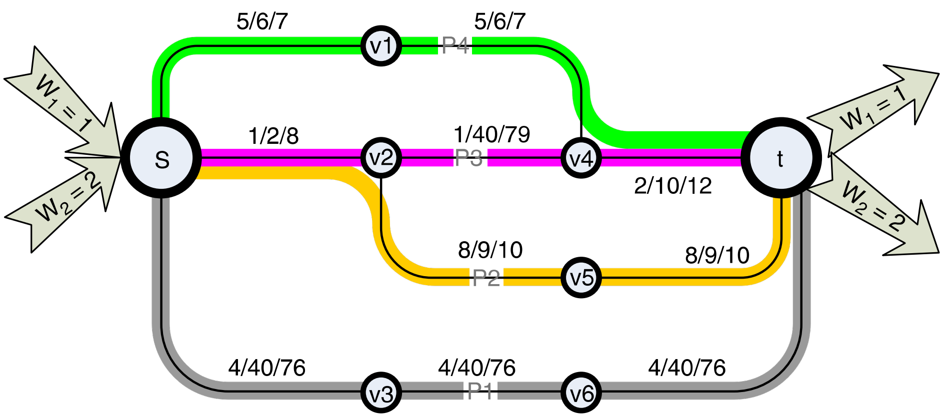

Fig. 1. A weighted single-commodity network congestion game that may have no PNE. Consider two players with demands $w_1 = 1$ and $w_2 = 2$. The notation $a/b/c$ means that a load of 1 has delay a, a load of 2 has delay b and a load of 3 has delay c.

Consequently, we show that there may exist no exact potential function[3] for a weighted single-commodity network congestion game, even when the resource delays are identical to their loads. The next argument shows that Theorem 3.1 of [19] does not hold anymore even in this simplest case of weighted congestion games. It should be mentioned that the same result has been independently proved also in [25].

Lemma 2. *There exist weighted single-commodity network congestion games which are not exact potential games, even when the resource delays are identical to their loads.*

Proof. Let $\Gamma = ((w_i)_{i\in N}, (\Pi_i)_{i\in N}, (d_e)_{e\in E})$ denote a weighted single-commodity network congestion game with $d_e(x) = x, \forall e \in E$. Recall the definition of users' costs for a configuration (Eq. (1)). Let us now define the quantity

$$I(\gamma, \lambda) \equiv \sum_{k=1}^{r} [\lambda^{i_k}(\mathbf{r(k)}) - \lambda^{i_k}(\mathbf{r(k-1)})],$$

where i_k is the unique user in which the configurations $\mathbf{r(k)}$ and $\mathbf{r(k-1)}$ differ. Our proof is based on the fact that Γ is an (exact) potential game if and only if every simple closed path γ of length 4 has $I(\gamma, \lambda) = 0$ [19, Theorem 2.8].

For the sake of contradiction, assume that every closed simple path γ of length 4 for Γ has $I(\gamma, \lambda) = 0$, fix arbitrary configuration $\mathbf{r}$, and consider the path

$$\gamma = (\mathbf{r},\ \mathbf{x} = \mathbf{r}^{-1} \oplus \pi_1,\ \mathbf{y} = \mathbf{r}^{-(1,2)} \oplus (\pi_1, \pi_2),\ \mathbf{z} = \mathbf{r}^{-2} \oplus \pi_2,\ \mathbf{r})$$

for some paths $\pi_1 \neq r_1$ and $\pi_2 \neq r_2$. We shall demonstrate that $I(\gamma, \lambda)$ cannot be identically 0 when there are at least two users of different demands. So, consider that the first two users have different demands: $w_1 \neq w_2$. We observe that

$$\lambda^1(\mathbf{x}) - \lambda^1(\mathbf{r}) = \sum_{e\in\pi_1} \theta_e(\mathbf{x}) - \sum_{e\in r_1} \theta_e(\mathbf{r}) = |\pi_1 \setminus r_1| \cdot w_1 + \sum_{e\in\pi_1\setminus r_1} \theta_e(\mathbf{r}) - \sum_{e\in r_1\setminus \pi_1} \theta_e(\mathbf{r})$$

since the resources in $r_1 \cap \pi_1$ retain their initial loads. Similarly we have

$$\lambda^2(\mathbf{y}) - \lambda^2(\mathbf{x}) = \sum_{e\in\pi_2\setminus r_2} [\theta_e(\mathbf{x}) + w_2] - \sum_{e\in r_2\setminus \pi_2} \theta_e(\mathbf{x}) = |\pi_2 \setminus r_2| \cdot w_2 + \sum_{e\in\pi_2\setminus r_2} \theta_e(\mathbf{x}) - \sum_{e\in r_2\setminus \pi_2} \theta_e(\mathbf{x}),$$

$$\lambda^1(\mathbf{z}) - \lambda^1(\mathbf{y}) = \sum_{e\in r_1\setminus \pi_1} \theta_e(\mathbf{z}) - \sum_{e\in\pi_1\setminus r_1} [\theta_e(\mathbf{z}) + w_1] = \sum_{e\in r_1\setminus \pi_1} \theta_e(\mathbf{z}) - \sum_{e\in\pi_1\setminus r_1} \theta_e(\mathbf{z}) - |\pi_1 \setminus r_1| \cdot w_1,$$

$$\lambda^2(\mathbf{r}) - \lambda^2(\mathbf{z}) = \sum_{e\in r_2\setminus \pi_2} \theta_e(\mathbf{r}) - \sum_{e\in\pi_2\setminus r_2} \theta_e(\mathbf{r}) - |\pi_2 \setminus r_2| \cdot w_2.$$

[3] Fix some vector $\mathbf{b} \in \mathbb{R}^n_{>0}$. A function $F : \times_{i\in N} \Pi_i \to \mathbb{R}$ is a $\mathbf{b}$-*potential* for a weighted congestion game $\Gamma = ((w_i)_{i\in N}, (\Pi_i)_{i\in N}, (d_e)_{e\in E})$, if $\forall \mathbf{r} \in \times_{i\in N} \Pi_i, \forall i \in N, \forall \pi_i \in \Pi_i, \lambda^i(\mathbf{r}) - \lambda^i(\mathbf{r^{-i}} \oplus \pi_i) = b_i \cdot [F(\mathbf{r}) - F(\mathbf{r^{-i}} \oplus \pi_i)]$. It is an *exact potential* for Γ, if $\mathbf{b} = \mathbf{1}$.

Thus, since $I \equiv I(\gamma, \lambda) = \lambda^1(\mathbf{x}) - \lambda^1(\mathbf{r}) + \lambda^2(\mathbf{y}) - \lambda^2(\mathbf{x}) + \lambda^1(\mathbf{z}) - \lambda^1(\mathbf{y}) + \lambda^2(\mathbf{r}) - \lambda^2(\mathbf{z})$, we get

$$I = \sum_{e\in\pi_1\setminus r_1} [\theta_e(\mathbf{r}) - \theta_e(\mathbf{z})] + \sum_{e\in\pi_2\setminus r_2} [\theta_e(\mathbf{x}) - \theta_e(\mathbf{r})] + \sum_{e\in r_1\setminus\pi_1} [\theta_e(\mathbf{z}) - \theta_e(\mathbf{r})] + \sum_{e\in r_2\setminus\pi_2} [\theta_e(\mathbf{r}) - \theta_e(\mathbf{x})].$$

Observe now that [4]

$$\begin{aligned}
\forall\, e \in \pi_1 \setminus r_1, &\quad \theta_e(\mathbf{r}) - \theta_e(\mathbf{z}) = \theta_e(\mathbf{r}) - \theta_e(\mathbf{r}^{-2} \oplus \pi_2) = w_2 \cdot (\mathbb{I}_{[e\in r_2\setminus\pi_2]} - \mathbb{I}_{[e\in\pi_2\setminus r_2]}),\\
\forall\, e \in \pi_2 \setminus r_2, &\quad \theta_e(\mathbf{x}) - \theta_e(\mathbf{r}) = \theta_e(\mathbf{r}^{-1} \oplus \pi_1) - \theta_e(\mathbf{r}) = w_1 \cdot (\mathbb{I}_{[e\in\pi_1\setminus r_1]} - \mathbb{I}_{[e\in r_1\setminus\pi_1]}),\\
\forall\, e \in r_1 \setminus \pi_1, &\quad \theta_e(\mathbf{z}) - \theta_e(\mathbf{r}) = \theta_e(\mathbf{r}^{-2} \oplus \pi_2) - \theta_e(\mathbf{r}) = w_2 \cdot (\mathbb{I}_{[e\in\pi_2\setminus r_2]} - \mathbb{I}_{[e\in r_2\setminus\pi_2]}),\\
\forall\, e \in r_2 \setminus \pi_2, &\quad \theta_e(\mathbf{r}) - \theta_e(\mathbf{x}) = \theta_e(\mathbf{r}) - \theta_e(\mathbf{r}^{-1} \oplus \pi_1) = w_1 \cdot (\mathbb{I}_{[e\in r_1\setminus\pi_1]} - \mathbb{I}_{[e\in\pi_1\setminus r_1]}).
\end{aligned}$$

Then,

$$\begin{aligned}
I = (w_1 - w_2) \cdot [&|(\pi_1 \setminus r_1) \cap (\pi_2 \setminus r_2)| + |(r_1 \setminus \pi_1) \cap (r_2 \setminus \pi_2)|\\
&-|(r_1 \setminus \pi_1) \cap (\pi_2 \setminus r_2)| - |(\pi_1 \setminus r_1) \cap (r_2 \setminus \pi_2)|],
\end{aligned}$$

which is typically not equal to zero for a single-commodity network. It should be noted that the second parameter, which is network dependent, can be non-zero even for some cycle of a very simple network. For example, in the network of Fig. 1 (which is a simple 2-layered network) the simple closed path

$$(\mathbf{r(0)} = (P1, P3),\ \mathbf{r(1)} = (P2, P3),\ \mathbf{r(2)} = (P2, P1),\ \mathbf{r(3)} = (P1, P1),\ \mathbf{r(4)} = (P1, P3))$$

has this quantity equal to -4 and thus, no weighted single-commodity network congestion game on this network can admit an exact potential. □

Our next step is to prove that any weighted multi-commodity network congestion game with linear resource delays admits at least one PNE, which can be computed in pseudo-polynomial time. Recall that Rosenthal's (exact) potential function [7] for unweighted congestion games is

$$\forall \mathbf{r} \in \times_{i\in N}\Pi_i, \qquad \Phi_{ros}(\mathbf{r}) \equiv \sum_{e\in\cup_{i\in N} r_i} \sum_{k=1}^{\theta_e(\mathbf{r})} d_e(k).$$

We already know that even the case of weighted ℓ-layered network congestion games with delays equal to the loads cannot have any exact potential. [5] We next show that a non-trivial generalization of Rosenthal's potential function is a **b**-potential for weighted multi-commodity network congestion games with linear resource delays.

Theorem 1. *For any weighted multi-commodity network congestion game with linear resource delays, at least one PNE exists and can be computed in pseudo-polynomial time.*

Proof. Fix an arbitrary network $G = (V, E)$ with linear resource/edge delays $d_e(x) = a_e x + b_e$, $e \in E$, $a_e, b_e \geqslant 0$. Let $\mathbf{r} \in \Pi$ be an arbitrary configuration for the corresponding weighted multi-commodity congestion game on G. For the configuration r we consider the potential $\Phi(\mathbf{r}) = C(\mathbf{r}) + W(\mathbf{r})$, where [6]

$$C(\mathbf{r}) = \sum_{e\in E} d_e(\theta_e(\mathbf{r}))\theta_e(\mathbf{r}) = \sum_{e\in E} \left[a_e\theta_e^2(\mathbf{r}) + b_e\theta_e(\mathbf{r})\right],$$

and

$$W(\mathbf{r}) = \sum_{i=1}^{n}\sum_{e\in r_i} d_e(w_i)w_i = \sum_{e\in E}\sum_{i\in\Lambda_e(\mathbf{r})} d_e(w_i)w_i = \sum_{e\in E}\sum_{i\in\Lambda_e(\mathbf{r})} \left(a_e w_i^2 + b_e w_i\right).$$

[4] For any logical expression $\mathcal{E}$, $\mathbb{I}_{[\mathcal{E}]}$ is the indicator variable of this expression being true.

[5] The example at the end of the Proof of Lemma 3 involves the 3-layered network of Fig. 1.

[6] For linear resource delays, Rosenthal's potential function is essentially identical to $C(\mathbf{r})$.

Let i be a user of demand w_i, let $\pi_i \in \Pi_i$ be a $s_i - t_i$ path different from r_i, and let $\mathbf{r}' \equiv \mathbf{r}^{-\mathbf{i}} \oplus \pi_i$. First of all, we observe that for each edge $e \in E \setminus ((\pi_i \setminus r_i) \cup (r_i \setminus \pi_i))$, $\Lambda_e(\mathbf{r}) = \Lambda_e(\mathbf{r}')$ and $\theta_e(\mathbf{r}) = \theta_e(\mathbf{r}')$. On the other hand, for each edge $e \in \pi_i \setminus r_i$, $\Lambda_e(\mathbf{r}') = \Lambda_e(\mathbf{r}) \cup \{i\}$ and $\theta_e(\mathbf{r}') = \theta_e(\mathbf{r}) + w_i$, while for each edge $e \in r_i \setminus \pi_i$, $\Lambda_e(\mathbf{r}') = \Lambda_e(\mathbf{r}) \setminus \{i\}$ and $\theta_e(\mathbf{r}') = \theta_e(\mathbf{r}) - w_i$. Therefore,

$$
\begin{aligned}
C(\mathbf{r}') - C(\mathbf{r}) &= \sum_{e\in E} d_e(\theta_e(\mathbf{r}'))\theta_e(\mathbf{r}') - \sum_{e\in E} d_e(\theta_e(\mathbf{r}))\theta_e(\mathbf{r}) \\
&= \sum_{e\in\pi_i\setminus r_i} \left[d_e(\theta_e(\mathbf{r}) + w_i)(\theta_e(\mathbf{r}) + w_i) - d_e(\theta_e(\mathbf{r}))\theta_e(\mathbf{r})\right] \\
&\quad + \sum_{e\in r_i\setminus \pi_i} \left[d_e(\theta_e(\mathbf{r}) - w_i)(\theta_e(\mathbf{r}) - w_i) - d_e(\theta_e(\mathbf{r}))\theta_e(\mathbf{r})\right] \\
&= \sum_{e\in\pi_i\setminus r_i} \left[a_e(\theta_e(\mathbf{r}) + w_i)^2 + b_e(\theta_e(\mathbf{r}) + w_i) - a_e\theta_e^2(\mathbf{r}) - b_e\theta_e(\mathbf{r})\right] \\
&\quad + \sum_{e\in r_i\setminus \pi_i} \left[a_e(\theta_e(\mathbf{r}) - w_i)^2 + b_e(\theta_e(\mathbf{r}) - w_i) - a_e\theta_e^2(\mathbf{r}) - b_e\theta_e(\mathbf{r})\right] \\
&= 2w_i \sum_{e\in\pi_i\setminus r_i} \left[a_e(\theta_e(\mathbf{r}) + w_i) + b_e\right] - \sum_{e\in\pi_i\setminus r_i} \left[a_e w_i^2 + b_e w_i\right] \\
&\quad -2w_i \sum_{e\in r_i\setminus \pi_i} \left[a_e\theta_e(\mathbf{r}) + b_e\right] + \sum_{e\in r_i\setminus \pi_i} \left[a_e w_i^2 + b_e w_i\right] \\
&= 2w_i \left[\sum_{e\in\pi_i\setminus r_i} d_e\left(\theta_e(\mathbf{r}')\right) - \sum_{e\in r_i\setminus \pi_i} d_e\left(\theta_e(\mathbf{r})\right)\right] \\
&\quad - \left[\sum_{e\in\pi_i\setminus r_i} \left(a_e w_i^2 + b_e w_i\right) - \sum_{e\in r_i\setminus \pi_i} \left(a_e w_i^2 + b_e w_i\right)\right].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
W(\mathbf{r}') - W(\mathbf{r}) &= \sum_{e\in E} \sum_{j\in\Lambda_e(\mathbf{r}')} \left(a_e w_j^2 + b_e w_j\right) - \sum_{e\in E} \sum_{j\in\Lambda_e(\mathbf{r})} \left(a_e w_j^2 + b_e w_j\right) \\
&= \sum_{e\in E} \left[\sum_{j\in\Lambda_e(\mathbf{r}')\setminus\Lambda_e(\mathbf{r})} \left(a_e w_j^2 + b_e w_j\right) - \sum_{j\in\Lambda_e(\mathbf{r})\setminus\Lambda_e(\mathbf{r}')} \left(a_e w_j^2 + b_e w_j\right)\right] \\
&= \sum_{e\in\pi_i\setminus r_i} \left(a_e w_i^2 + b_e w_i\right) - \sum_{e\in r_i\setminus \pi_i} \left(a_e w_i^2 + b_e w_i\right)
\end{aligned}
$$

since for each $e \in \pi_i \setminus r_i$, $\Lambda_e(\mathbf{r}') = \Lambda_e(\mathbf{r}) \cup \{i\}$, for each $e \in r_i \setminus \pi_i$, $\Lambda_e(\mathbf{r}') = \Lambda_e(\mathbf{r}) \setminus \{i\}$, and for each $e \notin (\pi_i \setminus r_i) \cup (r_i \setminus \pi_i)$, $\Lambda_e(\mathbf{r}') = \Lambda_e(\mathbf{r})$.

Combining the equalities above, we obtain that

$$
\Phi(\mathbf{r}') - \Phi(\mathbf{r}) = 2w_i \left[\sum_{e\in\pi_i\setminus r_i} d_e(\theta_e(\mathbf{r}')) - \sum_{e\in r_i\setminus \pi_i} d_e(\theta_e(\mathbf{r}))\right] = 2w_i \left[\sum_{e\in\pi_i} d_e(\theta_e(\mathbf{r}')) - \sum_{e\in r_i} d_e(\theta_e(\mathbf{r}))\right].
$$

The second equality follows from the fact that for each edge $e \in \pi_i \cap r_i$, $\theta_e(\mathbf{r}') = \theta_e(\mathbf{r})$.

Since $\lambda^i(\mathbf{r}') = \lambda_{\pi_i}(\mathbf{r}') = \sum_{e\in\pi_i} d_e(\theta_e(\mathbf{r}'))$ and $\lambda^i(\mathbf{r}) = \lambda_{r_i}(\mathbf{r}) = \sum_{e\in r_i} d_e(\theta_e(\mathbf{r}))$, we conclude that

$$
\Phi(\mathbf{r}') - \Phi(\mathbf{r}) = 2w_i \left[\lambda^i(\mathbf{r}') - \lambda^i(\mathbf{r})\right],
$$

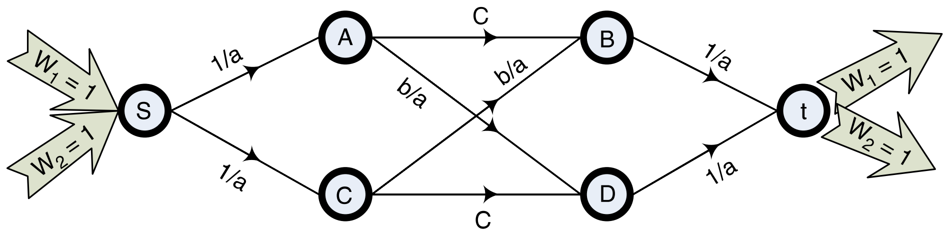

Fig. 2. Example of an ℓ-layered network with linear resource delays and unbounded anarchy.

and Φ is a $\mathbf{b}$-potential for $b_i = 1/2w_i$, $i \in N$. Therefore, any weighted multi-commodity network congestion game with linear resource delays admits a PNE. [7]

Wlog we can assume that all w_i's, a_e's and b_e's are integers. Then each user performing any improving defection must reduce its cost by at least 1 and thus the potential function decreases by at least $2w_{\min} \geqslant 2$ along each arc of the Dynamics Graph of the game. Consequently, the naive algorithm that, starting from an arbitrary initial configuration $\mathbf{r} \in \Pi$, follows any improvement path that leads to a sink (i.e., a PNE) of the Dynamics Graph, cannot move more than $W_{\text{tot}} \sum_{e \in E} d_e(W_{\text{tot}})$ times, since $\forall \mathbf{r} \in \Pi$, $\Phi(\mathbf{r}) \leqslant 2W_{\text{tot}} \sum_{e \in E} d_e(W_{\text{tot}})$. $\square$

7. The price of anarchy in ℓ-layered networks

In this section we focus our interest on weighted ℓ-layered network congestion games where the resource delays are identical to their loads. This case comprises a non-trivial generalization of routing through identical parallel channels.

The main reason why we focus on this specific category of resource delays is that selfish unsplittable flows can have unbounded price of anarchy even for linear resource delays. In [23, p. 256] an example is given where the price of anarchy is unbounded. This example is trivially converted in an ℓ-layered network. The resource delay functions used are either constant or M/M/1-like delay functions. Thus, one might claim that is actually the main reason for the unboundedness of the price of anarchy in this setting. Nevertheless, we can be equally bad even with linear resource delay functions: observe the following example of Fig. 2. Two users, each of unit demand, want to move selfishly from s to t. The edge delays are shown above them. We assume that $a \gg b \gg 1 \geqslant c$. It is easy to see that the configuration (sCBt,sADt) is a PNE of social cost $2 + b$ while the optimum configuration is (sABt,sCDt) whose social optimum is $2 + c$. Thus, $\mathcal{R} = (b + 2)/(c + 2)$.

In the following, we restrict our attention to ℓ-layered networks whose resource delays are equal to their loads. Our main tool is to interpret a strategies profile as a flow in the underlying network.

Flows and mixed strategies profiles: Fix an arbitrary ℓ-layered network $G = (V, E)$ and n distinct users willing to satisfy their own traffic demands from the unique source $s \in V$ to the unique destination $t \in V$. Again, $\mathbf{w} = (w_i)_{i \in [n]}$ denotes the varying demands of the users. W log we assume that w_i's are non-negative integers. Fix an arbitrary mixed strategies profile $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. A *feasible flow* for the n users is a function $\rho : \mathcal{P} \mapsto \mathbb{R}_{\geqslant 0}$, s.t. $\sum_{\pi \in \mathcal{P}} \rho(\pi) = W_{\text{tot}} \equiv \sum_{i \in [n]} w_i$, i.e., all users' demands are actually met. We distinguish between *unsplittable* and *splittable* (feasible) flows. A flow is unsplittable if each user's traffic demand is satisfied by a unique path of $\mathcal{P}$. A flow is splittable if the traffic demand of each user can be routed over several paths of $\mathcal{P}$.

We map the mixed strategies profile $\mathbf{p}$ to a flow $\rho_{\mathbf{p}}$ as follows: for each $s - t$ path $\pi \in \mathcal{P}$, $\rho_{\mathbf{p}}(\pi) \equiv \sum_{i \in [n]} w_i \cdot p_i(\pi)$. That is, we handle the *expected load traveling along* π *according to* $\mathbf{p}$ as a splittable flow, where user i routes a fraction of $p_i(\pi)$ of its total demand w_i along π. Observe that, if $\mathbf{p}$ is actually a pure strategies profile, the corresponding flow is then unsplittable. Recall now that for each edge $e \in E$,

$$\theta_e(\mathbf{p}) \equiv \sum_{i=1}^{n} \sum_{\pi: e \in \pi} w_i p_i(\pi) = \sum_{\pi: e \in \pi} \rho_{\mathbf{p}}(\pi) \equiv \theta_e(\rho_{\mathbf{p}})$$

[7] Actually, even if for a finite game there is a function s.t. the sign of its differences for adjacent nodes in the Dynamics Graph agrees with the sign of the differences of the corresponding costs whenever these are non-zero (this is called a *generalized ordinal potential function*), the Finite Improvement Property holds and thus we have a PNE [19].

denotes the expected load (and in our case, also the expected delay) of e wrt $\mathbf{p}$. As for the expected delay along a path $\pi \in \mathcal{P}$ according to $\mathbf{p}$, this is

$$\theta_\pi(\mathbf{p}) \equiv \sum_{e\in\pi} \theta_e(\mathbf{p}) = \sum_{e\in\pi} \sum_{\pi':e\in\pi'} \rho_{\mathbf{p}}(\pi') = \sum_{\pi'\in\mathcal{P}} |\pi \cap \pi'| \rho_{\mathbf{p}}(\pi') \equiv \theta_\pi(\rho_{\mathbf{p}}).$$

Let $\theta^{\min}(\rho) = \min_{\pi\in\mathcal{P}}\{\theta_\pi(\rho)\}$ be the minimum expected delay among all $s-t$ paths. From now on for simplicity we drop the subscript of $\mathbf{p}$ from its corresponding flow $\rho_{\mathbf{p}}$, when this is clear by the context. We evaluate flow ρ using the objective of *maximum latency*, which is defined as

$$L(\rho) \equiv \max_{\pi:\rho(\pi)>0} \{\theta_\pi(\rho)\} = \max_{\pi:\exists i,\, p_i(\pi)>0} \{\theta_\pi(\mathbf{p})\} \equiv L(\mathbf{p}). \tag{6}$$

$L(\rho)$ is nothing but the *maximum expected delay paid by the users*, wrt $\mathbf{p}$. From now on, we use ρ^* and ρ^*_f to denote the optimal unsplittable and splittable flows, respectively.

In addition, we sometimes evaluate flow ρ using the objective of *total latency*, which is defined as

$$C(\rho) \equiv \sum_{\pi\in\mathcal{P}} \rho(\pi)\theta_\pi(\rho) = \sum_{e\in E} \theta_e^2(\rho) = \sum_{e\in E} \theta_e^2(\mathbf{p}) \equiv C(\mathbf{p}). \tag{7}$$

The second equality is obtained by summing over the edges of π and reversing the order of the summation.

Flows at Nash equilibrium: Let $\mathbf{p}$ be a mixed strategies profile and let ρ be the corresponding flow. For a ℓ-layered network with resource delays equal to the loads, the cost of user i on path π is $\lambda^i_\pi(\mathbf{p}) = \ell w_i + \theta^{-i}_\pi(\mathbf{p})$, where $\theta^{-i}_\pi(\mathbf{p})$ is the expected delay along path π if the demand of user i was removed from the system:

$$\theta^{-i}_\pi(\mathbf{p}) = \sum_{\pi'\in\mathcal{P}} |\pi\cap\pi'| \sum_{j\neq i} w_j p_j(\pi') = \theta_\pi(\mathbf{p}) - w_i \sum_{\pi'\in\mathcal{P}} |\pi\cap\pi'| p_i(\pi'). \tag{8}$$

Thus, $\lambda^i_\pi(\mathbf{p}) = \theta_\pi(\mathbf{p}) + \left[\ell - \sum_{\pi'\in\mathcal{P}} |\pi\cap\pi'| p_i(\pi')\right] w_i$. Observe now that, if $\mathbf{p}$ is a NE, then $L(\mathbf{p}) = L(\rho) \leqslant \theta^{\min}(\rho) + \ell w_{\max}$. Otherwise, the users routing their traffic on a path of expected delay greater than $\theta^{\min}(\rho)+\ell w_{\max}$ could improve their delay by defecting to a path of expected delay $\theta^{\min}(\rho)$. We sometimes say that a flow ρ corresponding to a mixed strategies profile $\mathbf{p}$ is a NE with the understanding that it is actually $\mathbf{p}$ which is a NE.

Maximum latency versus total latency: We show that if the resource delays are equal to their loads, a splittable flow is optimal wrt the objective of maximum latency if and only if it is optimal wrt the objective of total latency. As a corollary, we obtain that the optimal splittable flow defines a NE where all users adopt the same mixed strategy.

Lemma 3. *There is a unique feasible splittable flow ρ which minimizes both $L(\rho)$ and $C(\rho)$.*

Proof. For every feasible flow ρ, the average latency of ρ cannot exceed its maximum latency:

$$C(\rho) = \sum_{\pi\in\mathcal{P}} \rho(\pi)\theta_\pi(\rho) = \sum_{\pi:\rho(\pi)>0} \rho(\pi)\theta_\pi(\rho) \leqslant L(\rho)\, W_{\text{tot}}. \tag{9}$$

A splittable flow ρ minimizes $C(\rho)$ if and only if for every $\pi', \pi \in \mathcal{P}$ with $\rho(\pi') > 0$, $\theta_{\pi'}(\rho) \leqslant \theta_\pi(\rho)$ (e.g., [2], [21, Section 7.2], [23, Corollary 4.2]). In addition, for every flow ρ, there exists a $\pi' \in \mathcal{P}$ with $\rho(\pi') > 0$ such that $L(\rho) = \theta_{\pi'}(\rho)$. Hence, if ρ is optimal wrt the objective of total latency, for all paths $\pi \in \mathcal{P}$, $\theta_{\pi'}(\rho) = L(\rho) \leqslant \theta_\pi(\rho)$ because $\rho(\pi') > 0$. Moreover, by definition of L, for all paths $\pi \in \mathcal{P}$ with $\rho(\pi) > 0$, $L(\rho) \geqslant \theta_\pi(\rho)$. Thus, $L(\rho) = \theta_\pi(\rho)$ for every path π with $\rho(\pi) > 0$. Therefore, if ρ minimizes $C(\rho)$, the average latency is equal to the maximum latency:

$$C(\rho) = \sum_{\pi\in\mathcal{P}:\rho(\pi)>0} \rho(\pi)\theta_\pi(\rho) = L(\rho) W_{\text{tot}}. \tag{10}$$

Let ρ be the feasible splittable flow that minimizes the total latency and let ρ' be the feasible splittable flow that minimizes the maximum latency. We prove the lemma by establishing that the two flows are identical.

Observe that $L(\rho') \geqslant C(\rho')/W_{\text{tot}} \geqslant C(\rho)/W_{\text{tot}} = L(\rho)$. The first inequality follows from Ineq. (9), the second from the assumption that ρ minimizes the total latency and the last equality from Eq. (10). On the other hand, it must be $L(\rho') \leqslant L(\rho)$ because of the assumption that the flow ρ' minimizes the maximum latency. Hence, it must be $L(\rho') = L(\rho)$ and $C(\rho') = C(\rho)$. In addition, since the function $C(\rho)$ is strictly convex and the set of feasible splittable flows forms a convex polytope, there is a unique flow which minimizes the total latency. Thus, ρ and ρ' must be identical. □

The following corollary states that the optimal splittable flow defines a mixed NE where all users adopt exactly the same strategy.

Corollary 1. *Let ρ^*_f be the optimal splittable flow and let* $\mathbf{p}$ *be the mixed strategies profile where every user routes its traffic on each path π with probability $\rho^*_f(\pi)/W_{\text{tot}}$. Then,* $\mathbf{p}$ *is a NE.*

Proof. By construction, the expected path loads corresponding to $\mathbf{p}$ are equal to the values of ρ^*_f on these paths. Since all users follow exactly the same strategy and route their demand on each path π with probability ρ^*_f/W_{tot}, for each user i,

$$\theta^{-i}_\pi(\mathbf{p}) = \theta_\pi(\mathbf{p}) - w_i \sum_{\pi' \in \mathcal{P}} |\pi \cap \pi'| \frac{\rho^*_f(\pi')}{W_{\text{tot}}} = \left(1 - \frac{w_i}{W_{\text{tot}}}\right) \theta_\pi(\mathbf{p}).$$

Since the flow ρ^*_f also minimizes the total latency, for every $\pi_1, \pi_2 \in \mathcal{P}$ with $\rho^*_f(\pi_1) > 0$, $\theta_{\pi_1}(\mathbf{p}) \leqslant \theta_{\pi_2}(\mathbf{p})$ (e.g., [2], [21, Section 7.2], [23, Corollary 4.2]), which also implies that $\theta^{-i}_{\pi_1}(\mathbf{p}) \leqslant \theta^{-i}_{\pi_2}(\mathbf{p})$. Therefore, for every user i and every $\pi_1, \pi_2 \in \mathcal{P}$ such that the user i routes its demand on π_1 with positive probability, $\lambda^i_{\pi_1}(\mathbf{p}) = \ell w_i + \theta^{-i}_{\pi_1}(\mathbf{p}) \leqslant \ell w_i + \theta^{-i}_{\pi_2}(\mathbf{p}) = \lambda^i_{\pi_2}(\mathbf{p})$. Consequently, $\mathbf{p}$ is a NE. □

An upper bound on the social cost: Next we derive an upper bound on the social cost of every strategy profile whose maximum expected delay (i.e., the maximum latency of its associated flow) is within a constant factor from the maximum latency of the optimal unsplittable flow.

Lemma 4. *Let ρ^* be the optimal unsplittable flow, and let* $\mathbf{p}$ *be a mixed strategies profile and ρ its corresponding flow. If $L(\mathbf{p}) = L(\rho) \leqslant \alpha L(\rho^*)$, for some $\alpha \geqslant 1$, then*

$$\mathrm{SC}(\mathbf{p}) \leqslant (\alpha + 1) \mathrm{O}\left(\frac{\log m}{\log \log m}\right) L(\rho^*),$$

where $m = |E|$ denotes the number of edges in the network.

Proof. For each edge $e \in E$ and each user i, let $X_{e,i}$ be the random variable describing the actual load routed through e by i. The random variable $X_{e,i}$ is equal to w_i if i routes its demand on a path π including e and 0 otherwise. Consequently, the expectation of $X_{e,i}$ is equal to $\mathbb{E}[X_{e,i}] = \sum_{\pi: e \in \pi} w_i p_i(\pi)$. Since each user selects its path independently, for every fixed edge e, the random variables $\{X_{e,i}, i \in [n]\}$ are independent from each other.

For each edge $e \in E$, let $X_e = \sum_{i=1}^n X_{e,i}$ be the random variable that describes the actual load routed through e, and thus, also the actual delay paid by any user traversing e. X_e is the sum of n independent random variables with values in $[0, w_{\max}]$. By linearity of expectation,

$$\mathbb{E}[X_e] = \sum_{i=1}^n \mathbb{E}[X_{e,i}] = \sum_{i=1}^n w_i \sum_{\pi: e \in \pi} p_i(\pi) = \theta_e(\rho).$$

By applying the standard Hoeffding bound [8] with $w = w_{\max}$ and $t = ek \max\{\theta_e(\rho), w_{\max}\}$, we obtain that for every $k \geqslant 1$,

$$\mathbb{P}[X_e \geqslant ek \max\{\theta_e(\rho), w_{\max}\}] \leqslant k^{-ek}.$$

For $m \equiv |E|$, we apply the union bound and conclude that for every $k \geqslant 1$,

$$\mathbb{P}[\exists e \in E : X_e \geqslant ek \max\{\theta_e(\rho), w_{\max}\}] \leqslant mk^{-ek}. \tag{11}$$

[8] We use the standard version of Hoeffding bound [12]: let $X_1, X_2, \ldots, X_n$ be independent random variables with values in the interval $[0, w]$. Let $X = \sum_{i=1}^n X_i$ and let $\mathbb{E}[X]$ denote its expectation. Then, $\forall t > 0$, $\mathbb{P}[X \geqslant t] \leqslant (e\mathbb{E}[X]/t)^{t/w}$.

For each path $\pi \in \mathcal{P}$ with $\rho(\pi) > 0$, we define the random variable $X_\pi = \sum_{e\in\pi} X_e$ describing the actual delay along π. The social cost of **p**, which is equal to the expected maximum delay experienced by some user cannot exceed the expected maximum delay among paths π with $\rho(\pi) > 0$. Formally,

$$\mathrm{SC}(\mathbf{p}) \leqslant \mathbb{E}\left[\max_{\pi:\rho(\pi)>0}\{X_\pi\}\right] \leqslant \sum_{i=0}^{\infty}\mathbb{P}\left[\max_{\pi:\rho(\pi)>0}\{X_\pi\} \geqslant i\right]. \tag{12}$$

The last inequality holds because $\max_{\pi:\rho(\pi)>0}\{X_\pi\}$ takes only non-negative integer values since w_i's are non-negative integers. It is well-known that for all non-negative integer-valued random variables X, $\mathbb{E}[X] = \sum_{i=0}^{\infty}\mathbb{P}[X > i]$ (e.g. [20, Proposition C.7]).

If for all $e \in E$, $X_e < ek\max\{\theta_e(\rho), w_{\max}\}$, then for every path $\pi \in \mathcal{P}$ with $\rho(\pi) > 0$,

$$\begin{aligned} X_\pi = \sum_{e\in\pi} X_e &< ek\sum_{e\in\pi}\max\{\theta_e(\rho), w_{\max}\} \leqslant ek\sum_{e\in\pi}(\theta_e(\rho) + w_{\max}) \\ &= ek(\theta_\pi(\rho) + \ell w_{\max}) \leqslant ek(L(\rho) + \ell w_{\max}) \leqslant e(\alpha+1)kL(\rho^*). \end{aligned}$$

The third equality follows from $\theta_\pi(\rho) = \sum_{e\in\pi}\theta_e(\rho)$, the fourth inequality from $\theta_\pi(\rho) \leqslant L(\rho)$ since $\rho(\pi) > 0$, and the last inequality from the hypothesis that $L(\rho) \leqslant \alpha L(\rho^*)$ and the fact that $\ell w_{\max} \leqslant L(\rho^*)$ because ρ^* is an unsplittable flow. Therefore, using Ineq. (11), we conclude that

$$\mathbb{P}\left[\max_{\pi:\rho(\pi)>0}\{X_\pi\} \geqslant e(\alpha+1)kL(\rho^*)\right] \leqslant mk^{-ek}. \tag{13}$$

In other words, the probability that the actual maximum delay caused by **p** exceeds the optimal maximum delay by a factor $e(\alpha+1)k$ or greater is at most mk^{-ek}.

Using Ineq. (12), we obtain that for every $\kappa_0 \geqslant 2$,

$$\begin{aligned} \mathrm{SC}(\mathbf{p}) &\leqslant e(\alpha+1)\kappa_0 L(\rho^*) + \sum_{i=e(\alpha+1)\kappa_0 L(\rho^*)}^{\infty}\mathbb{P}\left[\max_{\pi:\rho(\pi)>0}\{X_\pi\} \geqslant i\right] \\ &\leqslant e(\alpha+1)\kappa_0 L(\rho^*) + \sum_{k=\kappa_0}^{\infty} e(\alpha+1)L(\rho^*)\mathbb{P}\left[\max_{\pi:\rho(\pi)>0}\{X_\pi\} \geqslant e(\alpha+1)kL(\rho^*)\right] \\ &\leqslant e(\alpha+1)\kappa_0 L(\rho^*) + \sum_{k=\kappa_0}^{\infty} e(\alpha+1)L(\rho^*)\, mk^{-ek} \\ &= e(\alpha+1)L(\rho^*)\left(\kappa_0 + \sum_{k=\kappa_0}^{\infty} mk^{-ek}\right) \leqslant e(\alpha+1)L(\rho^*)(\kappa_0 + 2m\kappa_0^{-e\kappa_0}). \end{aligned}$$

For the first inequality, we use Ineq. (12) and $\mathbb{P}[\max_{\pi:\rho(\pi)>0}\{X_\pi\} \geqslant i] \leqslant 1$ for $i = 0, \ldots, e(\alpha+1)\kappa_0 L(\rho^*) - 1$. In the second inequality, we change the variable of the summation. The third inequality follows from Ineq. (13). For the last inequality, we first observe that $k^{-ek} \leqslant \kappa_0^{-ek}$ for all $k \geqslant \kappa_0$. Hence,

$$\sum_{k=\kappa_0}^{\infty} mk^{-ek} \leqslant m\sum_{k=\kappa_0}^{\infty}(k_0^{-e})^k \leqslant 2m\kappa_0^{-e\kappa_0}$$

because $k_0^{-e} \leqslant 1/2$. For $\kappa_0 = 2\log m/\log\log m$, we obtain that $\kappa_0^{-e\kappa_0} \leqslant m^{-1}$, for every $m \geqslant 4$. Therefore, $\mathrm{SC}(\mathbf{p}) \leqslant 2e(\alpha+1)(\log m/\log\log m + 1)L(\rho^*)$. $\square$

Bounding the coordination ratio: Our final step is to show that the maximum expected delay of every NE is a good approximation to the optimal maximum latency. Then, we can apply Lemma 7 to bound the coordination ratio for our selfish routing game.

Lemma 5. *Let ρ^* be the optimal unsplittable flow. For every flow ρ corresponding to a mixed strategies profile* **p** *at NE, $L(\rho) \leqslant 3L(\rho^*)$.*

Proof. The proof is based on Dorn's Theorem [5] which establishes strong duality in quadratic programming. [9] Let ρ_f^* be the optimal splittable flow. We use quadratic programming duality to prove that for any flow ρ at Nash equilibrium, the minimum expected delay $\theta^{\min}(\rho)$ cannot exceed $L(\rho_f^*) + \ell w_{\max}$. This implies the lemma because $L(\rho) \leqslant \theta^{\min}(\rho) + \ell w_{\max}$, since ρ is at Nash equilibrium, and $L(\rho^*) \geqslant \max\{L(\rho_f^*), \ell w_{\max}\}$, since ρ^* is an unsplittable flow.

Let Q be the square matrix describing the number of edges shared by each pair of paths. Formally, Q is a $|\mathcal{P}| \times |\mathcal{P}|$ matrix and for every $\pi, \pi' \in \mathcal{P}$, $Q[\pi, \pi'] = |\pi \cap \pi'|$. By definition, Q is symmetric. Next we prove that Q is positive semi-definite[10]

$$x^{\mathrm{T}} Q x = \sum_{\pi \in \mathcal{P}} x(\pi) \sum_{\pi' \in \mathcal{P}} Q[\pi, \pi'] x(\pi') = \sum_{\pi \in \mathcal{P}} x(\pi) \sum_{\pi' \in \mathcal{P}} |\pi \cap \pi'| x(\pi') = \sum_{\pi \in \mathcal{P}} x(\pi) \sum_{e \in \pi} \sum_{\pi' : e \in \pi'} x(\pi')$$

$$= \sum_{\pi \in \mathcal{P}} x(\pi) \sum_{e \in \pi} \theta_e(x) = \sum_{e \in E} \theta_e(x) \sum_{\pi : e \in \pi} x(\pi) = \sum_{e \in E} \theta_e^2(x) \geqslant 0.$$

First recall that for each edge e, $\theta_e(x) \equiv \sum_{\pi : e \in \pi} x(\pi)$. The third and the fifth equalities follow by reversing the order of summation. In particular, in the third equality, instead of considering the edges shared by π and π', for all $\pi' \in \mathcal{P}$, we consider all the paths π' using each edge $e \in \pi$. On both sides of the fifth inequality, for every edge $e \in E$, $\theta_e(x)$ is multiplied by the sum of $x(\pi)$ over all the paths π using e.

Let ρ also denote the $|\mathcal{P}|$-dimensional vector corresponding to the flow ρ. Then, the πth coordinate of $Q\rho$ is equal to the expected delay $\theta_\pi(\rho)$ on the path π, and the total latency of ρ is $C(\rho) = \rho^{\mathrm{T}} Q \rho$.

Therefore, the problem of computing a feasible splittable flow of minimum total latency is equivalent to computing the optimal solution to the following quadratic program: $\min \left\{ \rho^{\mathrm{T}} Q \rho : 1^{\mathrm{T}} \rho \geqslant W_{\text{tot}}, \rho \geqslant 0 \right\}$, where 1/0 denotes the $|\mathcal{P}|$-dimensional vector having 1/0 in each coordinate. Also notice that no flow of value strictly greater than W_{tot} can be optimal for this program. This quadratic program is clearly feasible and its optimal solution is ρ_f^* (Lemma 7).

The Dorn's dual of this quadratic program is: $\max \left\{ z W_{\text{tot}} - \rho^{\mathrm{T}} Q \rho : 2Q\rho \geqslant \mathbf{1} z, z \geqslant 0 \right\}$ (e.g., [5], [1, Chapter 6]). We observe that any flow ρ can be regarded as a feasible solution to the dual program by setting $z = 2\theta^{\min}(\rho)$. Hence, both the primal and the dual programs are feasible. By Dorn's Theorem [5], the objective value of the optimal dual solution is exactly $C(\rho_f^*)$. [11]

Let ρ be any feasible flow at Nash equilibrium. Setting $z = 2\theta^{\min}(\rho)$, we obtain a dual feasible solution. By the discussion above, the objective value of the feasible dual solution $(\rho, 2\theta^{\min}(\rho))$ cannot exceed $C(\rho_f^*)$. In other words,

$$2\theta^{\min}(\rho) W_{\text{tot}} - C(\rho) \leqslant C(\rho_f^*). \tag{14}$$

Since ρ is at Nash equilibrium, $L(\rho) \leqslant \theta^{\min}(\rho) + \ell w_{\max}$. In addition, by Ineq. (9), the average latency of ρ cannot exceed its maximum latency. Thus,

$$C(\rho) \leqslant L(\rho) W_{\text{tot}} \leqslant \theta^{\min}(\rho) W_{\text{tot}} + \ell w_{\max} W_{\text{tot}}.$$

Combining the inequality above with Ineq. (14), we obtain that $\theta^{\min}(\rho) W_{\text{tot}} \leqslant C(\rho_f^*) + \ell w_{\max} W_{\text{tot}}$. Using $C(\rho_f^*) = L(\rho_f^*) W_{\text{tot}}$, we conclude that $\theta^{\min}(\rho) \leqslant L(\rho_f^*) + \ell w_{\max}$. □

The following theorem is an immediate consequence of Lemmas 7 and 7.

Theorem 2. *The price of anarchy of any ℓ-layered network congestion game with resource delays equal to their loads is at most* $8e(\log m / \log\log m + 1)$.

[9] Let $\min \left\{ x^{\mathrm{T}} Q x + c^{\mathrm{T}} x : Ax \geqslant b, x \geqslant \mathbf{0} \right\}$ be the primal quadratic program. The Dorn's dual of this program is $\max \left\{ -y^{\mathrm{T}} Q y + b^{\mathrm{T}} u : A^{\mathrm{T}} u - 2Qy \leqslant c, u \geqslant \mathbf{0} \right\}$. Dorn [5] proved strong duality when the matrix Q is symmetric and positive semi-definite. Thus, if Q is symmetric and positive semi-definite and both the primal and the dual programs are feasible, their optimal solutions have the same objective value.

[10] A $n \times n$ matrix Q is positive semi-definite if for every vector $x \in \mathbb{R}^n$, $x^{\mathrm{T}} Q x \geqslant 0$.

[11] More specifically, the optimal dual solution is obtained from ρ_f^* by setting $z = 2\theta^{\min}(\rho_f^*)$. Since $L(\rho_f^*) = \theta^{\min}(\rho_f^*)$ and $C(\rho_f^*) = L(\rho_f^*) W_{\text{tot}}$, the objective value of this solution is $2\theta^{\min}(\rho_f^*) W_{\text{tot}} - C(\rho_f^*) = C(\rho_f^*)$.

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