INTRODUCTION TO RELIABILITY ANALYSIS

Introduction

- Goal is to analyze the reliability of complex systems
- Approach:
  - Breakdown the complex system into small elements
  - Identify the interconnecting patterns and compose the overall reliability in terms of elemental reliabilities
  - Obtain elemental reliabilities by experimental means – analyzing failure data

Why Analyze Reliability?

- Reliability gives the probability of failure for a particular design
- Obviously, cost is another factor in the design
- Reliability analysis helps the engineer make quantitative decisions in trading-off risk versus cost

Combinatorial Reliability

- We consider combinatorial reliability where given a decomposition of the system, the overall reliability is determined by analyzing the failure scenarios
- For simplicity, we don’t consider repairs in the reliability analysis – i.e., we consider only up to the first failure
- Considering repairs makes the quantitative analysis lot harder

Series Configuration

- Functional operation depends on the correct operation of **all** the modules

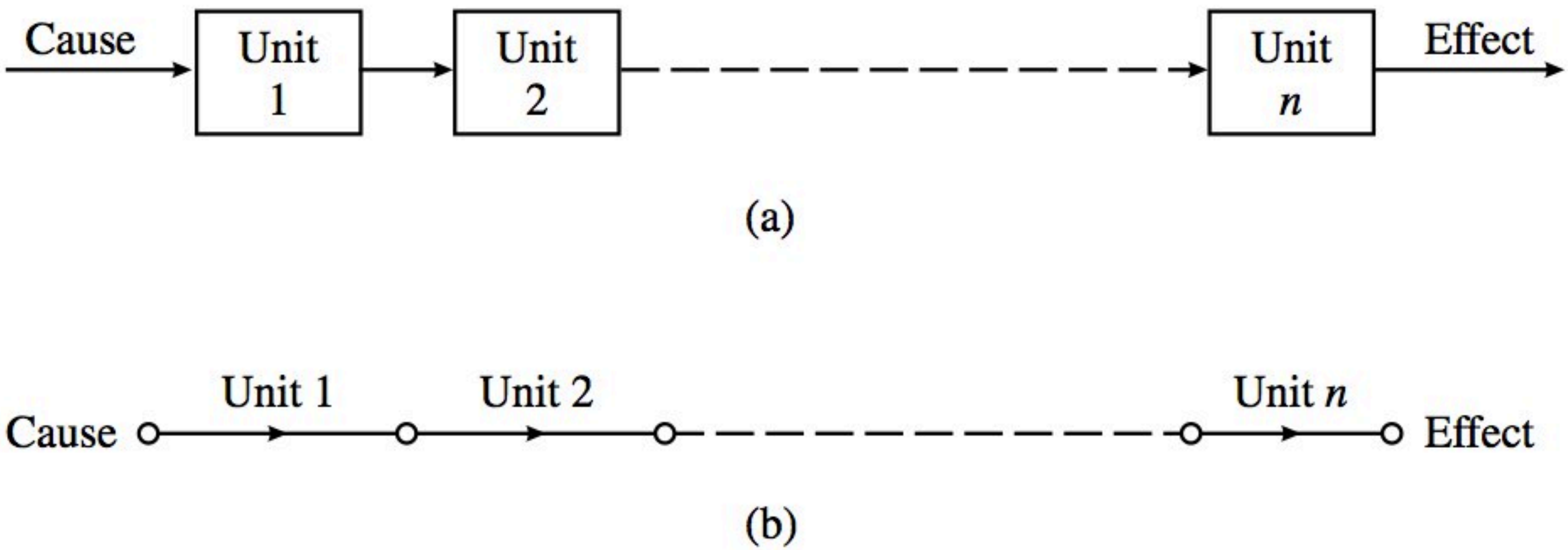


Figure B1 Series reliability configuration: (a) reliability block diagram (RBD); (b) reliability graph.

Series Configuration…

- Probability a module n is successful is $P(x_n)$
- Probability module n fails is $P(y_n)$, where $y = \overline{x}$
- Probability of system success P_s
- System reliability $R = P_s$

$$R = P_s = P(x_1 x_2 x_3 \cdots x_n) \tag{B1}$$

Expansion of Eq. (B1) yields

$$P_s = P(x_1)P(x_2|x_1)P(x_3|x_1x_2)\cdots P(x_n|x_1x_2\cdots x_{n-1}) \tag{B2}$$

Series Configuration…

- $P(x_3|x_1x_2)$ is the conditional probability of module 3 working given modules 1 and 2 are working
- Common mode factors such as power dissipation can impact the failure of modules in the same way therefore we use conditional probabilities.
- If the failure or success are independent, we can simplify the above expression for reliability to

$$P_s = P(x_1)P(x_2)P(x_3)\cdots P(x_n) \tag{B3}$$

Series Configuration…

- Reliability of a series configuration is smaller than the reliability of the weakest among the components
- Alternative is to compute probability of failure

$$P_f = P(\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \cdots + \bar{x}_n) \tag{B4}$$

Expansion of Eq. (B4) yields

$$\begin{aligned} P_f = {} & [P(\bar{x}_1) + P(\bar{x}_2) + P(\bar{x}_3) + \cdots + P(\bar{x}_n)] \\ & - [P(\bar{x}_1\bar{x}_2) + P(\bar{x}_1\bar{x}_3) + \cdots + \underset{i \neq j}{P(\bar{x}_i\bar{x}_j)}] \\ & + \cdots + (-1)^{n-1}[P(\bar{x}_1\bar{x}_2\cdots\bar{x}_n)] \end{aligned} \tag{B5}$$

Series Configuration…

Since

$$P_s = 1 - P_f \tag{B6}$$

the probability of system success becomes

$$\begin{aligned} P_s = 1 &- P(\bar{x}_1) - P(\bar{x}_2) - P(\bar{x}_3) - \cdots - P(\bar{x}_n) + P(\bar{x}_1)P(\bar{x}_2|\bar{x}_1) \\ &+ P(\bar{x}_1)P(\bar{x}_3|\bar{x}_1) + \cdots + P(\bar{x}_i)P\underset{i \neq j}{(\bar{x}_i|\bar{x}_j)} \\ &- \cdots + (-1)^n P(\bar{x}_1)P(\bar{x}_2|\bar{x}_1)\cdots P(\bar{x}_n|\bar{x}_1\cdots\bar{x}_{n-1}) \end{aligned} \tag{B7}$$

- With independent failures

$$\begin{aligned} P_s = 1 &- P(\bar{x}_1) - P(\bar{x}_2) - P(\bar{x}_3) - \cdots - P(\bar{x}_n) \\ &+ P(\bar{x}_1)P(\bar{x}_2) + P(\bar{x}_1)P(\bar{x}_3) + \cdots + P\underset{i \neq j}{(\bar{x}_i)}P(\bar{x}_j) \\ &- \cdots + (-1)^n P(\bar{x}_1)P(\bar{x}_2)\cdots P(\bar{x}_n) \end{aligned} \tag{B8}$$

Parallel Configuration

- Unlike series systems, in parallel configurations the system remains operational if at least one module is operational

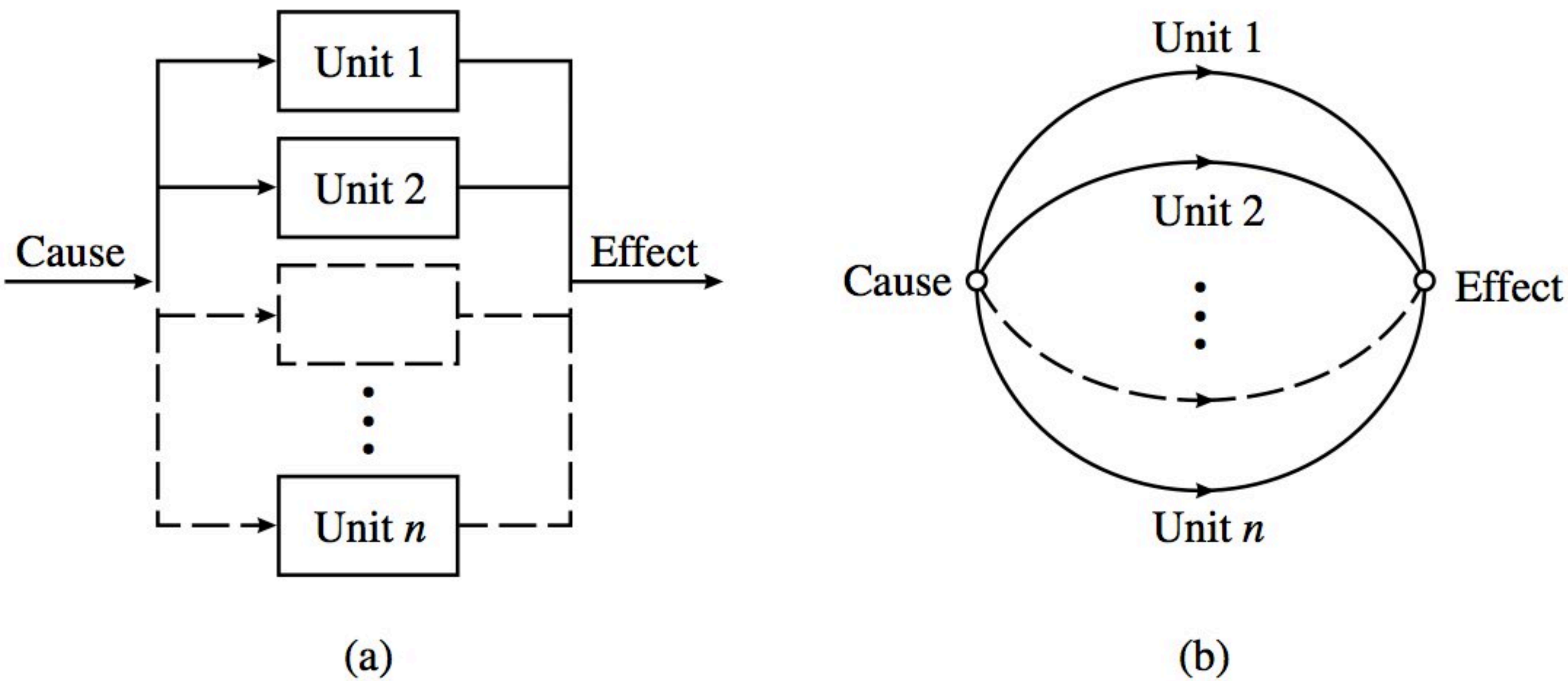


Figure B2 Parallel reliability configuration: (a) reliability block diagram; (b) reliability graph.

Parallel Configuration…

- Reliability of the parallel configuration is:

$$P_s = P(x_1 + x_2 + x_3 + \cdots + x_n) \tag{B9}$$

Expansion of Eq. (B9) yields

$$\begin{aligned} P_s = {} & [P(x_1) + P(x_2) + P(x_3) + \cdots + P(x_n)] \\ & - [P(x_1x_2) + P(x_1x_3) + \cdots + \underset{i \neq j}{P(x_ix_j)}] \\ & + \cdots + (-1)^{n-1} P(x_1x_2 \cdots x_n) \end{aligned} \tag{B10}$$

- Simpler formula can be obtained by considering failure probabilities:

$$P_f = P(\bar{x}_1\bar{x}_2\bar{x}_3 \cdots \bar{x}_n) \tag{B11}$$

Parallel Configuration…

- If the failures are independent, above expression can be simplified as follows:

$$P_s = 1 - P(\bar{x}_1)P(\bar{x}_2)\cdots P(\bar{x}_n) \qquad \text{(B14)}$$

R-out-of-N Configuration

- Many systems function if *r* systems out of *n* function
- If the modules are identical, probability of *r* successes out of *n* modules is given by:

$$B(r;n,p) = \binom{n}{r} p^r (1-p)^{n-r} \qquad \text{for } r = 0, 1, 2 \cdots n \qquad \text{(B15)}$$

Problem: 4 out of 5 Configuration

- Show that the reliability of the configuration below is given by $5p^4 - 4p^5$, where p is prob. of success.

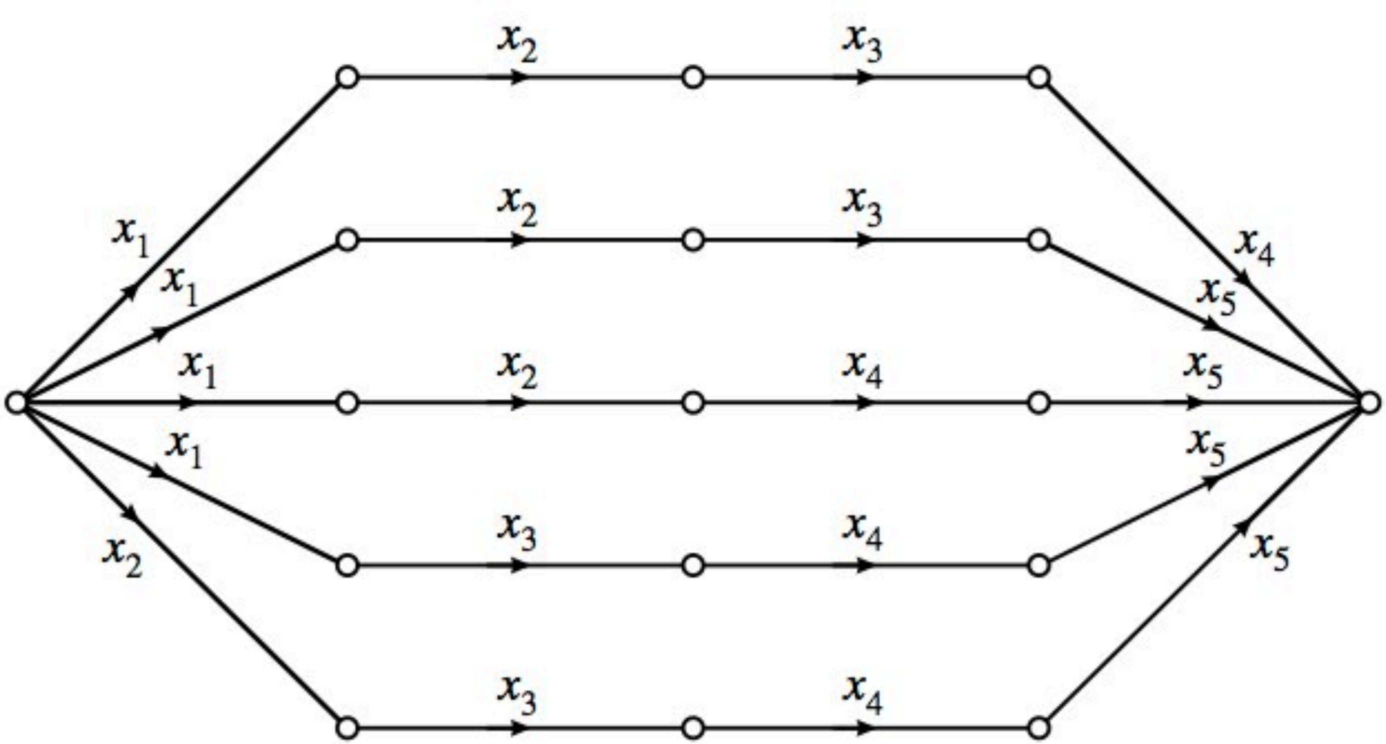


Figure B3 Reliability graph for a 4-out-of-5 system.

Cut-Set and Tie-Set Methods

- Tie set: A group of branches that form a connection between input and output in a reliability graph
- Minimal tie set: tie set containing minimum number of elements
- If T_1, T_2, T_3 are the tie sets, system reliability is given by $R = P(T_1 + T_2 + T_3)$
- Cut set: set of branches that interrupt connection from input to output
- If C_1, C_2, C_3 are the cut sets, system reliability is given by $R = 1 - P(\bar{C}_1 + \bar{C}_2 + \bar{C}_3)$

Cut-Set and Tie-Set Methods

- Minimal tie sets of the following reliability graph are given below:

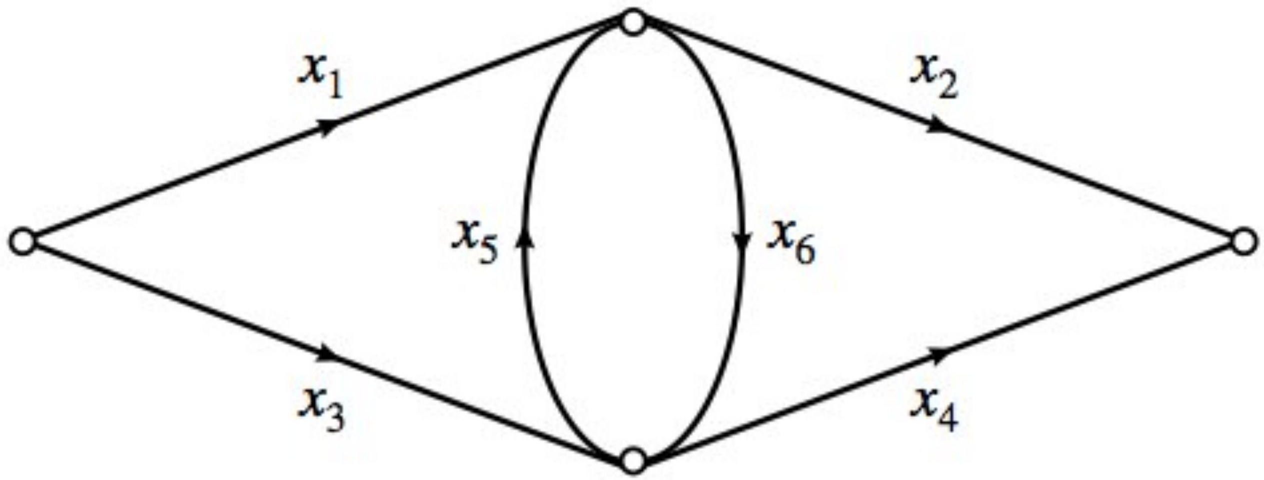


Figure B4 Reliability graph for a six-element system.

- Minimal tie sets are:

$$T_1 = x_1 x_2 \qquad T_2 = x_3 x_4 \qquad T_3 = x_1 x_6 x_4 \qquad T_4 = x_3 x_5 x_2$$

Failure Rate Models

- System reliability analysis needs elemental reliability
- To construct elemental failure models
  - Need test data or plan a test on parts that are same as those to be used
  - Failure rate should be computed or graphed
  - Failure rate model should be chosen based on it
  - Parameters of the model are estimated from the graph or computed using statistical techniques
  - Emphasis should be placed on simple models – easy to work with models

Treatment of Failure Data

- Failure data generally obtained from two sources:
  - Failure data from a population placed on life test
  - Repair reports listing operating hours of replaced parts from field technicians
- Use this data to compute
  - Failure density function
  - Hazard rate function
- Assume N items placed in operation at time $t = 0$
- As time progresses, items fail. Suppose $n(t)$ survive at time t

Treatment of Failure Data…

- Assume *N* items placed in operation at time *t = 0*
- As time progresses, items fail. Suppose *n(t)* survive at time *t*
- Density function is given by the ratio of the failures occurring in an interval to the size of the original population, divided by the length of the interval

$$f_d(t) = \frac{[n(t_i) - n(t_i + \Delta t_i)]/N}{\Delta t_i} \qquad \text{for } t_i < t \le t_i + \Delta t_i \qquad \text{(B23)}$$

- Hazard rate is defined as the ratio of failures to survivors at the beginning of the interval

$$z_d(t) = \frac{[n(t_i) - n(t_i + \Delta t_i)]/n(t_i)}{\Delta t_i} \qquad \text{for } t_i < t \le t_i + \Delta t_i \qquad \text{(B24)}$$

Treatment of Failure Data…

TABLE B1 Failure Data for 10 Hypothetical Electronic Components

Failure Number	Operating Time, h
1	8
2	20
3	34
4	46
5	63
6	86
7	111
8	141
9	186
10	266

TABLE B2 Computation of Data Failure Density and Data Hazard Rate

Time Interval, h	Failure Density per Hour, $f_d(t)(\times 10^{-2})$	Hazard Rate per Hour, $z_d(t)(\times 10^{-2})$
0–8	$\frac{1}{10\times 8} = 1.25$	$\frac{1}{10\times 8} = 1.25$
8–20	$\frac{1}{10\times 12} = 0.84$	$\frac{1}{9\times 12} = 0.93$
20–34	$\frac{1}{10\times 14} = 0.72$	$\frac{1}{8\times 14} = 0.96$
34–46	$\frac{1}{10\times 12} = 0.84$	$\frac{1}{7\times 12} = 1.19$
46–63	$\frac{1}{10\times 17} = 0.59$	$\frac{1}{6\times 17} = 0.98$
63–86	$\frac{1}{10\times 23} = 0.44$	$\frac{1}{5\times 23} = 0.87$
86–111	$\frac{1}{10\times 25} = 0.40$	$\frac{1}{4\times 25} = 1.00$
111–141	$\frac{1}{10\times 30} = 0.33$	$\frac{1}{3\times 30} = 1.11$
141–186	$\frac{1}{10\times 45} = 0.22$	$\frac{1}{2\times 45} = 1.11$
186–266	$\frac{1}{10\times 80} = 0.13$	$\frac{1}{1\times 80} = 1.25$

Treatment of Failure Data…

- Since $f_d(t)$ is a density function, we can define a failure distribution function

$$F_d(t) = \int_0^t f_d(\xi)\, d\xi$$

- And success distribution function

$$R_d(t) = 1 - F_d(t) = 1 - \int_0^t f_d(\xi)\, d\xi$$

Treatment of Failure Data...

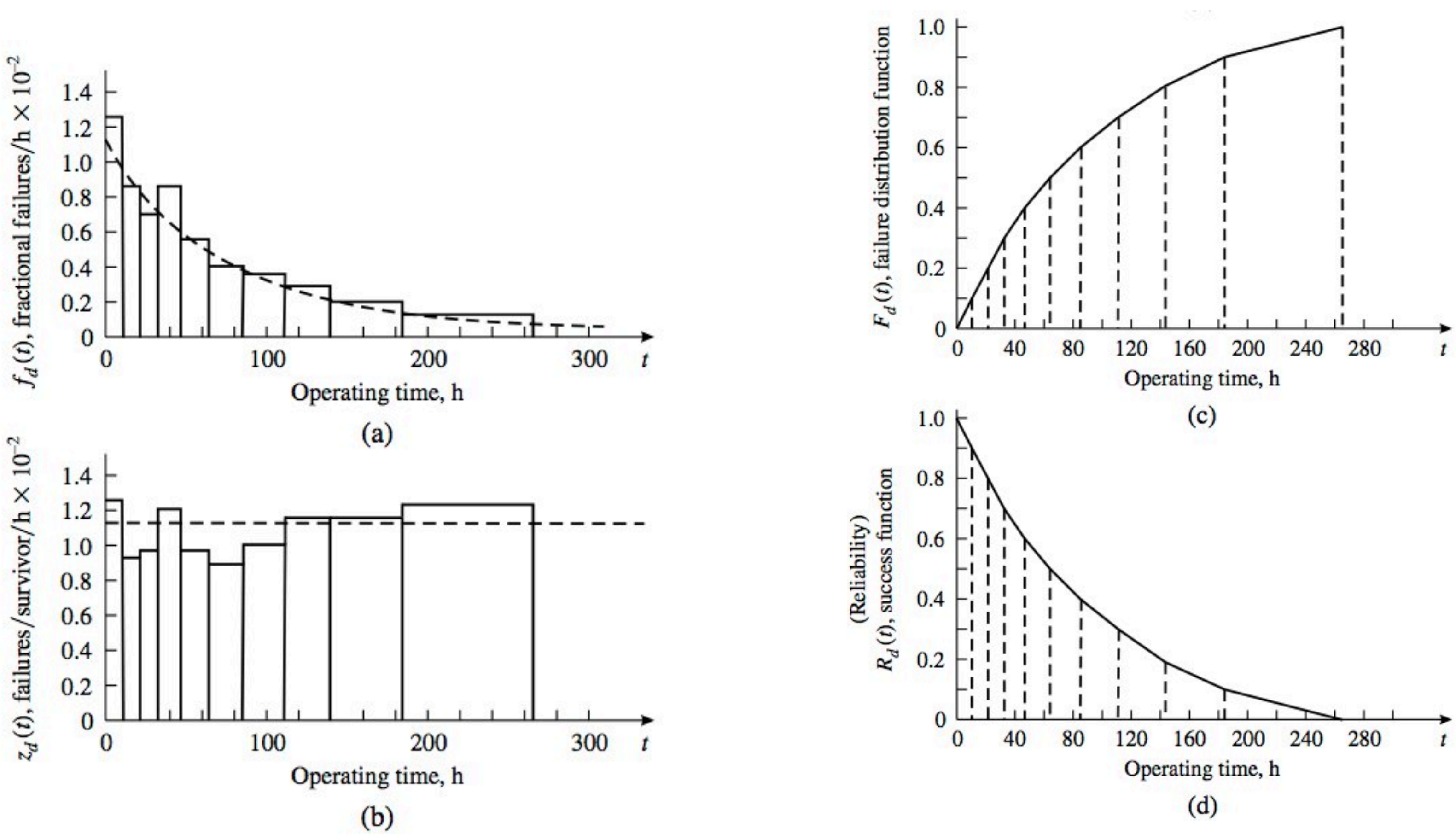
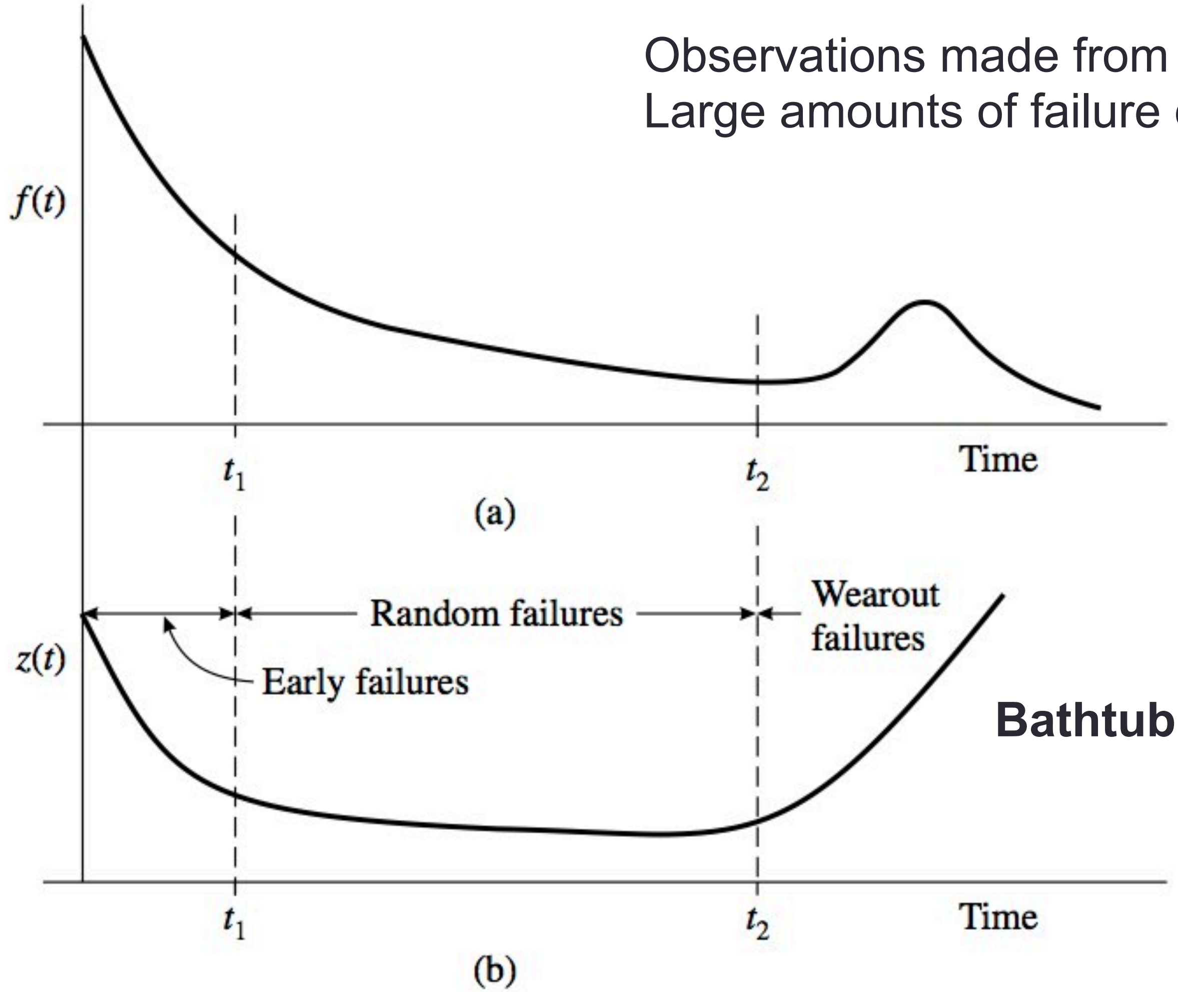


Figure B5 Density and hazard functions for the data of Table B1. (a) Data failure density functions; (b) data hazard rate; (c) data failure distribution function; (d) data success function.

Failure Data Patterns

Observations made from
Large amounts of failure data



Bathtub curve

Figure B6 General form of failure curves. (a) Failure density; (b) hazard rate.

Reliability, Hazard Rate, and Failure Density

The random variable **t** is defined as the failure time of the item in question.[6] Thus, the probability of failure as a function of time is given as

$$P(\mathbf{t} \leq t) = F(t) \tag{B28}$$

which is simply the definition of the failure distribution function. We can define the reliability, which is a probability of success in terms of $F(t)$, as

$$R(t) = P_s(t) = 1 - F(t) = P(\mathbf{t} \geq t) \tag{B29}$$

The failure density function is of course given by

$$\frac{dF(t)}{dt} = f(t) \tag{B30}$$

Reliability, Hazard Rate, and Failure Density

We now consider a population of N items with the same failure-time distribution. The items fail independently with probability of failure given by $F(t) = 1 - R(t)$ and probability of success given by $R(t)$. If the random variable $\mathbf{N}(t)$ represents the number of units surviving at time t, then $\mathbf{N}(t)$ has a binomial distribution with $p = R(t)$. Therefore,

$$P[\mathbf{N}(t) = n] = B[n; \mathbf{N}, R(t)] = \frac{N!}{n!(N-n)!}[R(t)]^n[1 - R(t)]^{N-n}$$

$$n = 0, 1, \ldots, N \qquad \text{(B31)}$$

The number of units operating at any time t is a random variable and is not fixed; however, we can compute the expected value $\mathbf{N}(t)$. From Table A1 we see that the expected value of a random variable with a binomial distribution is given by $NR(t)$ and leads to

$$n(t) \equiv E[N(t)] = NR(t) \qquad \text{(B32)}$$

Solving for the reliability yields

$$R(t) = \frac{n(t)}{N} \qquad \text{(B33)}$$

Reliability, Hazard Rate, and Failure Density

$$F(t) = 1 - \frac{n(t)}{N} = \frac{N - n(t)}{N} \qquad \text{(B34)}$$

$$f(t) = \frac{dF(t)}{dt} = -\frac{1}{N}\frac{dn(t)}{dt}$$

$$f(t) \equiv \lim_{\Delta t \to 0} \frac{n(t) - n(t + \Delta t)}{N\Delta t}$$

$$z(t) \equiv -\lim_{\Delta t \to 0} \frac{n(t) - n(t + \Delta t)}{n(t)\Delta t} \qquad \text{(B36)}$$

The definition of $z(t)$ in Eq. (B36) of course agrees with the definition of $z_d(t)$ in Eq. (B24). We can relate $z(t)$ and $f(t)$ using Eqs. (B35) and (B36):

$$z(t) = -\lim_{\Delta t \to 0} \frac{n(t) - n(t + \Delta t)}{\Delta t} \frac{1}{n(t)} = Nf(t)\frac{1}{n(t)}$$

$$z(t) = -\frac{1}{N}\,\frac{dn(t)}{dt}\,\frac{N}{n(t)} = -\frac{d}{dt}\ln n(t)$$

Solving the differential equation yields:

$$\ln n(t) = -\int_0^t z(\xi)\,d\xi + c$$

where ξ is a dummy variable and c is the constant of integration. Taking the antilog of both sides of the equation gives:

$$n(t) = e^c \exp\left[-\int_0^t z(\xi)\,d\xi\right]$$

Inserting initial conditions

$$n(0) = N = e^c$$

gives

$$n(t) = N \exp\left[-\int_0^t z(\xi)\,d\xi\right]$$

Substitution of Eq. (B33) completes the derivation

$$R(t) = \exp\left[-\int_0^t z(\xi)\,d\xi\right] \tag{B39}$$