1

Motion and forces at a point of contact

1.1 Frame of reference

This book is concerned with the stresses and deformation which arise when the surfaces of two solid bodies are brought into contact. We distinguish between *conforming* and *non-conforming* contacts. A contact is said to be conforming if the surfaces of the two bodies 'fit' exactly or even closely together without deformation. Flat slider bearings and journal bearings are examples of conforming contact. Bodies which have dissimilar profiles are said to be non-conforming. When brought into contact without deformation they will touch first at a point – 'point contact' – or along a line – 'line contact'. For example, in a ball-bearing the ball makes point contact with the races, whereas in a roller bearing the roller makes line contact. Line contact arises when the profiles of the bodies are conforming in one direction and non-conforming in the perpendicular direction. The contact area between non-conforming bodies is generally small compared with the dimensions of the bodies themselves; the stresses are highly concentrated in the region close to the contact zone and are not greatly influenced by the shape of the bodies at a distance from the contact area. These are the circumstances with which we shall be mainly concerned in this book.

The points of surface contact which are found in engineering practice frequently execute complex motions and are called upon to transmit both forces and moments. For example, the point of contact between a pair of gear teeth itself moves in space, while at that point the two surfaces move relative to each other with a motion which combines both rolling and sliding. In this preliminary chapter we begin by defining a frame of reference in which the motions and forces which arise in any particular circumstances can be generalised. This approach enables the problems of contact mechanics to be formulated and studied independently of technological particularities and,

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further, it facilitates the application of the results of such studies to the widest variety of engineering problems.

Non-conforming surfaces brought into contact by a negligibly small force touch at a single point. We take this point O as origin of rectangular coordinate axes $Oxyz$. The two bodies, lower and upper as shown in Fig. 1.1, are denoted by suffixes 1 and 2 respectively. The Oz axis is chosen to coincide with the common normal to the two surfaces at O. Thus the x–y plane is the tangent plane to the two surfaces, sometimes called the osculating plane. The directions of the axes Ox and Oy are chosen for convenience to coincide, where possible, with axes of symmetry of the surface profiles.

Line contact, which arises when two cylindrical bodies are brought into contact with their axes parallel, appears to constitute a special case. Their profiles are non-conforming in the plane of cross-section, but they do conform along a line of contact in the plane containing the axes of the cylinders. Nevertheless this important case is covered by the general treatment as follows: we choose the x-axis to lie in the plane of cross-section and the y-axis parallel to the axes of the cylinders.

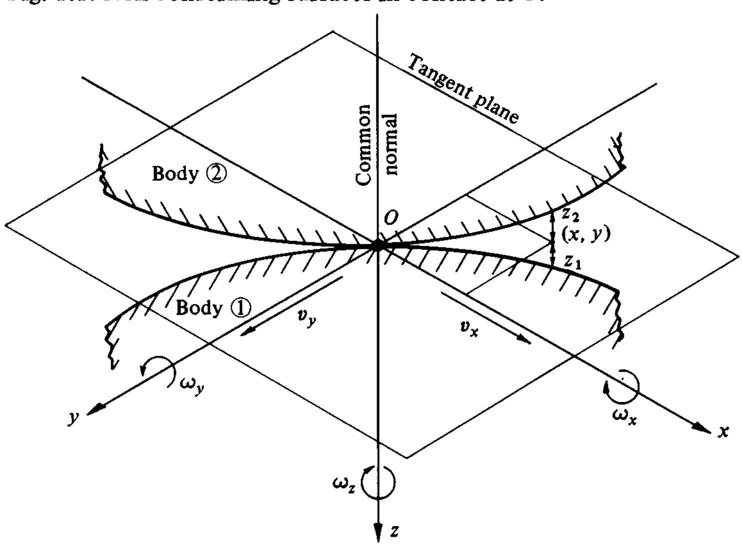
The undeformed shapes of two surfaces are specified in this frame by the functions:

$$z_1 = f_1(x, y)$$
$$z_2 = f_2(x, y)$$

Thus the separation between them before loading is given by

$$h = z_1 + z_2 = f(x, y) \tag{1.1}$$

Fig. 1.1. Non-conforming surfaces in contact at O.



1.2 Relative motion of the surfaces – sliding, rolling and spin

The motion of a body at any instant of time may be defined by the linear velocity vector of an arbitrarily chosen point of reference in the body together with the angular velocity vector of the body. If we now take reference points in each body coincident with the point of contact O at the given instant, body (1) has linear velocity $\mathbf{V}_1$ and angular velocity $\mathbf{\Omega}_1$, and body (2) has linear velocity $\mathbf{V}_2$ and angular velocity $\mathbf{\Omega}_2$. The frame of reference defined above moves with the linear velocity of the contact point $\mathbf{V}_O$ and rotates with angular velocity $\mathbf{\Omega}_O$ in order to maintain its orientation relative to the common normal and tangent plane at the contact point.

Within the frame of reference the two bodies have linear velocities at O:

$$\left.\begin{aligned} \mathbf{v}_1 &= \mathbf{V}_1 - \mathbf{V}_O \\ \mathbf{v}_2 &= \mathbf{V}_2 - \mathbf{V}_O \end{aligned}\right\} \tag{1.2}$$

and angular velocities:

$$\left.\begin{aligned} \boldsymbol{\omega}_1 &= \mathbf{\Omega}_1 - \mathbf{\Omega}_O \\ \boldsymbol{\omega}_2 &= \mathbf{\Omega}_2 - \mathbf{\Omega}_O \end{aligned}\right\} \tag{1.3}$$

We now consider the cartesian components of $\mathbf{v}_1$, $\mathbf{v}_2$, $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. If contact is continuous, so the surfaces are neither separating nor overlapping, their velocity components along the common normal must be equal, viz:

$$\left.\begin{aligned} &V_{z1} = V_{z2} = V_{zO} \\ \text{i.e.} \quad & \\ &v_{z1} = v_{z2} = 0 \end{aligned}\right\} \tag{1.4}$$

We now define *sliding* as the relative linear velocity between the two surfaces at O and denote it by $\Delta\mathbf{v}$.

$$\Delta\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 = \mathbf{V}_1 - \mathbf{V}_2$$

The sliding velocity has components:

$$\left.\begin{aligned} &\Delta v_x = v_{x1} - v_{x2} \\ \text{and} \quad & \\ &\Delta v_y = v_{y1} - v_{y2} \end{aligned}\right\} \tag{1.5}$$

Rolling is defined as a relative angular velocity between the two bodies about an axis lying in the tangent plane. The angular velocity of roll has components:

$$\left.\begin{aligned} &\Delta\omega_x = \omega_{x1} - \omega_{x2} = \Omega_{x1} - \Omega_{x2} \\ \text{and} \quad & \\ &\Delta\omega_y = \omega_{y1} - \omega_{y2} = \Omega_{y1} - \Omega_{y2} \end{aligned}\right\} \tag{1.6}$$

Finally *spin* motion is defined as a relative angular velocity about the common normal, viz.:

$$\Delta\omega_z = \omega_{z1} - \omega_{z2} = \Omega_{z1} - \Omega_{z2} \tag{1.7}$$

Any motion of contacting surfaces must satisfy the condition of continuous contact (1.4) and can be regarded as the combination of sliding, rolling and spin. For example, the wheels of a vehicle normally roll without slide or spin. When it turns a corner spin is introduced; if it skids with the wheels locked, it slides without rolling.

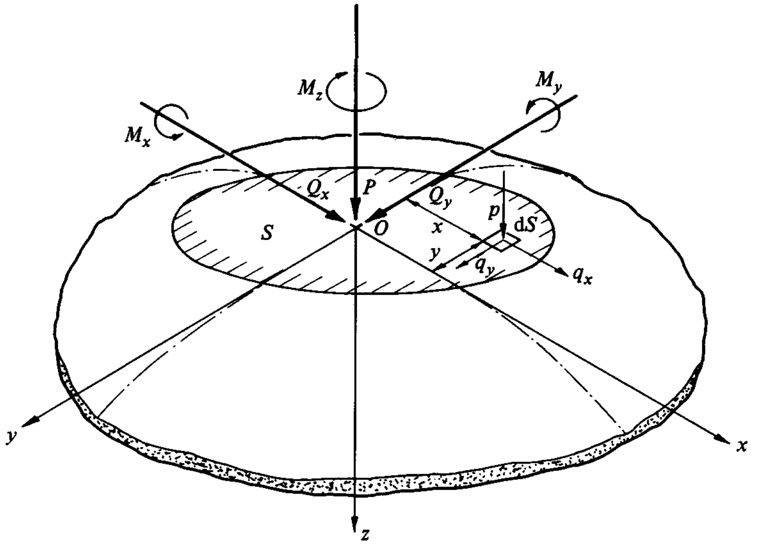
1.3 Forces transmitted at a point of contact

The resultant force transmitted from one surface to another through a point of contact is resolved into a *normal force P* acting along the common normal, which generally must be compressive, and a *tangential force* **Q** in the tangent plane sustained by friction. The magnitude of **Q** must be less than or, in the limit, equal to the force of limiting friction, i.e.

$$Q \leqslant \mu P \tag{1.8}$$

where μ is the coefficient of limiting friction. **Q** is resolved into components Q_x and Q_y parallel to axes Ox and Oy. In a purely sliding contact the tangential force reaches its limiting value in a direction opposed to the sliding velocity,

Fig. 1.2. Forces and moments acting on contact area S.



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from which:

$$\left.\begin{aligned} Q_x &= -\frac{\Delta v_x}{|\Delta v|}\mu P \\ Q_y &= -\frac{\Delta v_y}{|\Delta v|}\mu P \end{aligned}\right\} \tag{1.9}$$

The force transmitted at a nominal point of contact has the effect of compressing deformable solids so that they make contact over an area of finite size. As a result it becomes possible for the contact to transmit a resultant moment in addition to a force (Fig. 1.2). The components of this moment M_x and M_y are defined as *rolling moments.* They provide the resistance to a rolling motion commonly called 'rolling friction' and in most practical problems are small enough to be ignored.

The third component M_z, acting about the common normal, arises from friction within the contact area and is referred to as the *spin moment.* When spin accompanies rolling the energy dissipated by the spin moment is combined with that dissipated by the rolling moments to make up the overall rolling resistance.

At this point it is appropriate to define *free rolling* ('inertia rolling' in the Russian literature). We shall use this term to describe a rolling motion in which spin is absent and where the *tangential force* **Q** *at the contact point is zero.* This is the condition of the unpowered and unbraked wheels of a vehicle if rolling resistance and bearing friction are neglected; it is in contrast with the driving wheels or braked wheels which transmit sizable tangential forces at their points of contact with the road or rail.

1.4 Surface tractions

The forces and moments which we have just been discussing are transmitted across the contact interface by surface tractions at the interface. The normal traction (pressure) is denoted by p and the tangential traction (due to friction) by q, shown acting positively on the lower surface in Fig. 1.2. While nothing can be said at this stage about the distribution of p and q over the area of contact S, for overall equilibrium:

$$P = \int_S p \, \mathrm{d}S \tag{1.10}$$

$$Q_x = \int_S q_x \, \mathrm{d}S, \quad Q_y = \int_S q_y \, \mathrm{d}S \tag{1.11}$$

With non-conforming contacts (including cylinders having parallel axes) the contact area lies approximately in the x–y plane and slight warping is neglected, whence

$$M_x = \int_S py \, \mathrm{d}S, \quad M_y = -\int_S px \, \mathrm{d}S \tag{1.12}$$

and

$$M_z = \int_S (q_y x - q_x y) \, \mathrm{d}S \tag{1.13}$$

When the bodies have closely conforming curved surfaces, as for example in a deep-groove ball-bearing, the contact area is warped appreciably out of the tangent plane and the expressions for M_x and M_y (1.12) have to be modified to include terms involving the shear tractions q_x and q_y. Examples of the treatment of such problems are given later in §8.5.

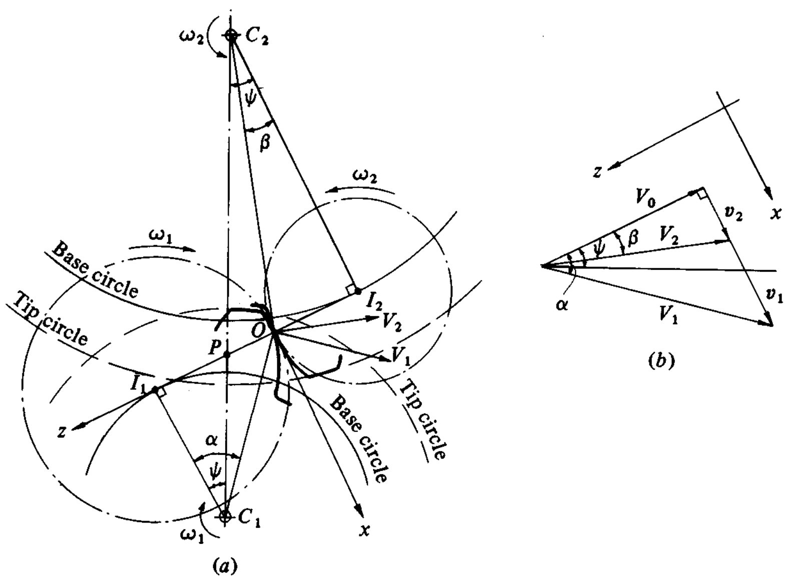
To illustrate the approach to contact kinematics and statics presented in this chapter, two examples from engineering practice will be considered briefly.

1.5 Examples

Example (1): involute spur gears

The meshing of a pair of involute spur gear teeth is shown in Fig. 1.3(a). The wheels rotate about centres C_1 and C_2. The line $I_1 I_2$ is the common tangent

Fig. 1.3. Contact of involute spur gear teeth.



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to the two base circles from which the involute profiles are generated. P is the pitch point. The teeth are shown in contact at O, which is taken as origin of our coordinate frame of reference. The common normal to the two teeth through O coincides with I_1I_2 and is taken as the z-axis. The x-axis lies in the tangent plane and is taken to be in the plane of rotation as shown.

The point of contact moves along the path I_1I_2 with a velocity V_O: points on the two teeth coincident with O have velocities V_1 and V_2 perpendicular to the radial lines C_1O and C_2O. Since the path of contact is straight, the frame of reference does not rotate ($\Omega_O = 0$); the wheels rotate with angular velocities $-\omega_1$ and ω_2. (Since the motion lies entirely in the x–z plane, we can omit the suffix y from the angular velocities and the suffix x from the linear velocities.)

Velocities within the frame of reference are shown in Fig. 1.3(b). Applying equation (1.4) for continuity of contact:

$$V_1 \cos\alpha = V_2 \cos\beta = V_O$$

i.e.

$$\omega_1(C_1I_1) = \omega_2(C_2I_2)$$

therefore

$$\frac{\omega_2}{\omega_1} = \frac{C_1I_1}{C_2I_2} = \frac{C_1P}{C_2P} \tag{1.14}$$

The angular velocity of rolling about the y-axis is

$$\Delta\omega = -(\omega_1 + \omega_2) \tag{1.15}$$

The velocity of sliding is

$$\begin{aligned}\Delta v &= v_1 - v_2\\ &= V_1 \sin\alpha - V_2 \sin\beta\\ &= \omega_1(OI_1) - \omega_2(OI_2)\\ &= \omega_1(PI_1 + OP) - \omega_2(PI_2 - OP)\end{aligned}$$

i.e.

$$\Delta v = (\omega_1 + \omega_2)OP \tag{1.16}$$

since triangles C_1PI_1 and C_2PI_2 are similar.

Thus the velocity of sliding is equal to the angular velocity of rolling multiplied by the distance of the point of contact from the pitch point. The direction of sliding changes from the arc of approach to the arc of recess and at the pitch point there is pure rolling.

We note that the motion of rolling and sliding at a given instant in the meshing cycle can be reproduced by two circular discs of radii I_1O and I_2O rotating with angular velocities $-\omega_1$ and $+\omega_2$ about fixed centres at I_1 and I_2. This is the basis of the *disc machine*, originally developed by Merritt (1935), to simulate the conditions of gear tooth contact in the simple laboratory test. Since the radii of curvature of the involute teeth at O are the same as those of the discs, I_1O and I_2O, the contact stresses under a given contact load are also simulated by the disc machine. The obvious departure from similarity arises from replacing the cyclic behaviour of tooth meshing by a steady motion which reproduces the conditions at only one instant in the meshing cycle.

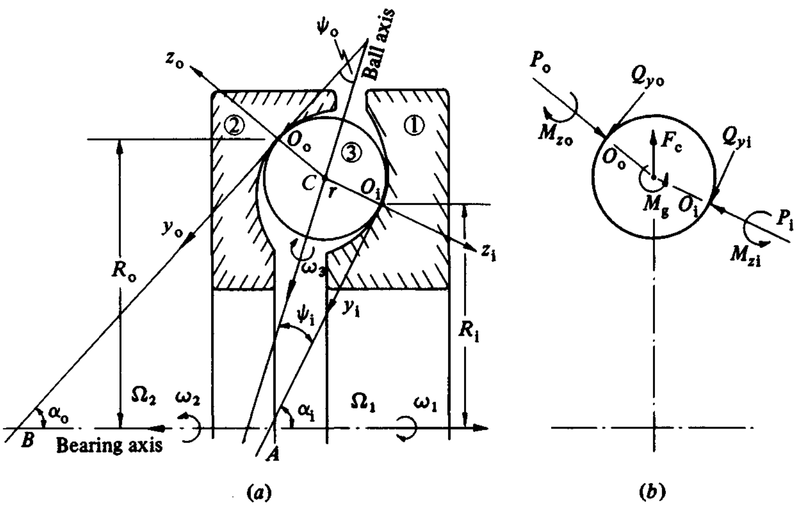
Example (2): angular contact ball-bearings

An axial cross-section through an angular contact ball-bearing is given in Fig. 1.4, showing a typical ball. The inner and outer races, and the cage (i.e. ball centre C) rotate about the bearing axis with angular velocities Ω_1, Ω_2 and Ω_c respectively. To bring to rest our standard frames of reference, which move with the points of contact between the races and the ball O_i and O_o, we subtract the cage speed from the race speeds, thus

$$\omega_1 = \Omega_1 - \Omega_c, \quad \omega_2 = \Omega_2 - \Omega_c$$

Although the two contact points O_i and O_o are frequently assumed to lie at opposite ends of a ball diameter, they will not do so in general and are deliberately displaced from a diameter in Fig. 1.4. Thus the two sets of axes $O_ix_iy_iz_i$ and $O_ox_oy_oz_o$ will not be in line. If the ball rolls without sliding at the two points of contact, the axis of rotation of the ball (in our prescribed frames of

Fig. 1.4. Angular contact ball bearing, showing contact of the ball (3) with the inner race (1) at O_i and with the outer race (2) at O_o.



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reference) must lie in the y-z plane. Its direction in that plane, however, remains to be determined. It is drawn in an arbitrary direction in Fig. 1.4(a), inclined at angle ψ_i, to O_iy_i and ψ_o to O_oy_o. The axes O_iy_i and O_oy_o intersect the bearing axis at points A and B, and at angles α_i and α_o respectively.

For no sliding at O_i:

$$v_{x3} = v_{x1}$$

i.e.

$$\omega_3 r \cos\psi_i = \omega_1 R_i$$

Similarly at O_o

$$\omega_3 r \cos\psi_o = \omega_2 R_o$$

Thus, eliminating ω_3,

$$\frac{\omega_2}{\omega_1} = \frac{R_i \cos\psi_o}{R_o \cos\psi_i} \tag{1.17}$$

If the points of contact are diametrically opposed, the contact angles α_i and α_o are equal so that $\psi_i = \psi_o$. Only then is the ratio of the race speeds, (1.17), independent of the direction of the axis of rotation of the ball.

We now examine the spin motion at O_i. The angular velocity of spin

$$\begin{aligned}(\Delta\omega_z)_i &= \omega_{z1} - \omega_{z3}\\ &= \omega_1 \sin\alpha_i - \omega_3 \sin\psi_i\end{aligned}$$

i.e.

$$(\Delta\omega_z)_i = \omega_1 \frac{R}{r}\left(\frac{r}{AO_i} - \tan\psi_i\right) \tag{1.18}$$

From this expression we see that the spin motion at O_i will vanish if the axis of rotation of the ball passes through point A on the axis of the bearing (whereupon $\tan\psi_i = r/(AO_i)$). Similarly, for spin to be absent at O_o, the axis of rotation of the ball must intersect the bearing axis at B. For spin to be absent at both points of contact, either the two tangents O_iy_i and O_oy_o are parallel to the bearing axis, as in a simple radial bearing, or O_i and O_o are so disposed that O_iy_i and O_oy_o intersect the bearing axis at a common point. This latter circumstance is achieved in a taper-roller bearing where the conical races have a common apex on the bearing axis, but never occurs in an angular contact ball-bearing.

We turn now to the forces acting on the ball shown in Fig. 1.4(b). The bearing is assumed to carry a purely axial load so that each ball is identically loaded. Each contact point transmits a normal force $P_{i,o}$ and a tangential force $(Q_y)_{i,o}$. Pressure and friction between the ball and cage pockets introduce small tangential forces in the x-direction at O_i and O_o which are neglected in this example. The rolling friction moments $(M_y)_{i,o}$ will be neglected also, but the spin moments $(M_z)_{i,o}$ play an important role in governing the direction of the axis of rotation of the ball. At high rotational speeds the ball is subjected to an appreciable centrifugal force F_c and a gyroscopic moment M_g.

Consider the equilibrium of the ball; taking moments about the line $O_i O_o$, it follows that

$$(M_z)_i = (M_z)_o \tag{1.19}$$

But the positions of the contact points O_i and O_o and the direction of the ball axis ψ_i are not determined by statics alone. In order to proceed further with the analysis it is necessary to know how the tangential forces $(Q_y)_{i,o}$ and the spin moments $M_{zi,o}$ are related to the motions of rolling and spin at O_i and O_o. This question will be considered in Chapter 8, §4.

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