3

Point loading of an elastic half-space

3.1 Potential functions of Boussinesq and Cerruti

In this chapter we consider the stresses and deformations produced in an elastic half-space, bounded by the plane surface $z = 0$, under the action of normal and tangential tractions applied to a closed area S of the surface in the neighbourhood of the origin. Outside the loaded area both normal and tangential tractions are zero. Thus the problem in elasticity is one in which the tractions are specified throughout the whole surface $z = 0$. In view of the restricted area to which the loads are applied, it follows that all the components of stress fall to zero at a long distance from the origin. The loading is two-dimensional: the normal pressure $p(x, y)$ and the tangential tractions $q_x(x, y)$ and $q_y(x, y)$, in general, vary in both x and y directions. The stress system is three-dimensional therefore; in general all six components of stress, σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx}, will appear.

A special case arises when the loading is axi-symmetric about the z-axis. In cylindrical polar coordinates (r, θ, z) the pressure $p(r)$ and the tangential traction $q(r)$ are independent of θ and $q(r)$, if it is present, acts in a radial direction. The stress components $\tau_{r\theta}$ and $\tau_{\theta z}$ vanish and the other stress components are independent of θ.

The classical approach to finding the stresses and displacements in an elastic half-space due to surface tractions is due to Boussinesq (1885) and Cerruti (1882) who made use of the theory of potential. This approach is presented by Love (1952): only selected results will be quoted here.

The half-space is shown in Fig. 3.1. We take $C(\xi, \eta)$ to be a general surface point within the loaded area S, whilst $A(x, y, z)$ is a general point within the body of the solid. The distance

$$CA \equiv \rho = \{(\xi - x)^2 + (\eta - y)^2 + z^2\}^{1/2} \tag{3.1}$$

Distributions of traction $p(\xi, \eta)$, $q_x(\xi, \eta)$ and $q_y(\xi, \eta)$ act on the area S. The

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following potential functions, each satisfying Laplace's equation, are defined:

$$F_1 = \int_S \int q_x(\xi, \eta)\Omega \, d\xi \, d\eta$$

$$G_1 = \int_S \int q_y(\xi, \eta)\Omega \, d\xi \, d\eta \qquad (3.2)$$

$$H_1 = \int_S \int p(\xi, \eta)\Omega \, d\xi \, d\eta$$

where

$$\Omega = z \ln(\rho + z) - \rho \qquad (3.3)$$

In addition we define the potential functions

$$F = \frac{\partial F_1}{\partial z} = \int_S \int q_x(\xi, \eta) \ln(\rho + z) \, d\xi \, d\eta$$

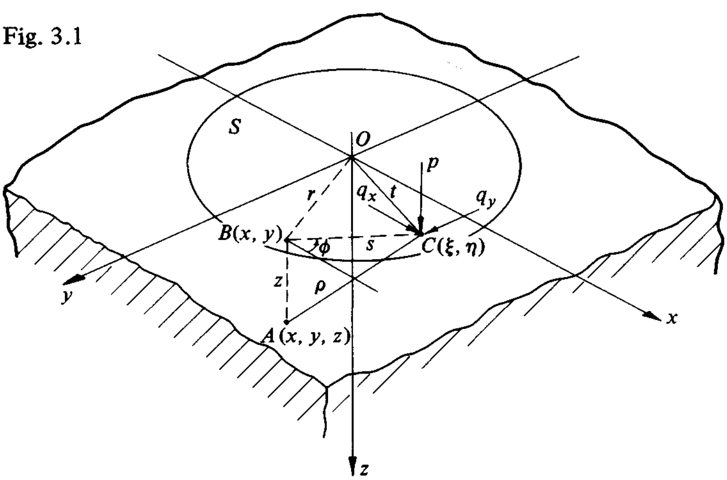
$$G = \frac{\partial G_1}{\partial z} = \int_S \int q_y(\xi, \eta) \ln(\rho + z) \, d\xi \, d\eta \qquad (3.4)$$

$$H = \frac{\partial H_1}{\partial z} = \int_S \int p(\xi, \eta) \ln(\rho + z) \, d\xi \, d\eta$$

We now write

$$\psi_1 = \frac{\partial F_1}{\partial x} + \frac{\partial G_1}{\partial y} + \frac{\partial H_1}{\partial z} \qquad (3.5)$$

Fig. 3.1



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and

$$\psi = \frac{\partial \psi_1}{\partial z} = \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} \tag{3.6}$$

Love (1952) shows that the components of elastic displacement u_x, u_y and u_z at any point $A(x, y, z)$ in the solid can be expressed in terms of the above functions as follows:

$$u_x = \frac{1}{4\pi G}\left\{2\frac{\partial F}{\partial z} - \frac{\partial H}{\partial x} + 2\nu\frac{\partial \psi_1}{\partial x} - z\frac{\partial \psi}{\partial x}\right\} \tag{3.7a}$$

$$u_y = \frac{1}{4\pi G}\left\{2\frac{\partial G}{\partial z} - \frac{\partial H}{\partial y} + 2\nu\frac{\partial \psi_1}{\partial y} - z\frac{\partial \psi}{\partial y}\right\} \tag{3.7b}$$

$$u_z = \frac{1}{4\pi G}\left\{\frac{\partial H}{\partial z} + (1-2\nu)\psi - z\frac{\partial \psi}{\partial z}\right\} \tag{3.7c}$$

These expressions decrease as $(1/\rho)$ at large distances from the loaded region. They represent, therefore, the elastic displacements of points close to the loaded region relative to the points in the solid at a large distance from the loaded region ($\rho \to \infty$) where the half-space may be looked upon as fixed. This behaviour in two-dimensional loading, where a datum for displacements can be taken at infinity, compares favourably with one-dimensional loading, considered in the previous chapter, where a variation of displacements as $\ln \rho$ precludes taking a datum at infinity and imposes an arbitrary choice of datum.

The displacements having been found, the stresses are calculated from the corresponding strains by Hooke's law:

$$\sigma_x = \frac{2\nu G}{1-2\nu}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2G\frac{\partial u_x}{\partial x} \tag{3.8a}$$

$$\sigma_y = \frac{2\nu G}{1-2\nu}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2G\frac{\partial u_y}{\partial y} \tag{3.8b}$$

$$\sigma_z = \frac{2\nu G}{1-2\nu}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2G\frac{\partial u_z}{\partial z} \tag{3.8c}$$

$$\tau_{xy} = G\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \tag{3.8d}$$

$$\tau_{yz} = G\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \tag{3.8e}$$

$$\tau_{zx} = G\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) \tag{3.8f}$$

Under the action of a purely normal pressure $p(\xi, \eta)$, which would occur in a frictionless contact, the above equations may be simplified. Here

$$F = F_1 = G = G_1 = 0$$

whence

$$\psi_1 = \frac{\partial H_1}{\partial z} = H = \int_S\int p(\xi,\eta) \ln(\rho + z)\, d\xi\, d\eta \tag{3.9}$$

$$\psi = \frac{\partial H}{\partial z} = \frac{\partial \psi_1}{\partial z} = \int_S\int p(\xi,\eta)\ \frac{1}{\rho}\ d\xi\, d\eta \tag{3.10}$$

$$u_x = -\frac{1}{4\pi G}\left\{(1-2\nu)\frac{\partial\psi_1}{\partial x} + z\frac{\partial\psi}{\partial x}\right\} \tag{3.11a}$$

$$u_y = -\frac{1}{4\pi G}\left\{(1-2\nu)\frac{\partial\psi_1}{\partial y} + z\frac{\partial\psi}{\partial y}\right\} \tag{3.11b}$$

$$u_z = \frac{1}{4\pi G}\left\{2(1-\nu)\psi - z\frac{\partial\psi}{\partial z}\right\} \tag{3.11c}$$

Remembering that ψ and ψ_1 are harmonic functions of x, y and z, i.e. they both satisfy Laplace's equation,

$$\nabla^2\psi = 0, \quad \nabla^2\psi_1 = 0$$

the dilatation Δ is given by

$$\Delta \equiv \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \frac{1-2\nu}{2\pi G}\frac{\partial\psi}{\partial z} \tag{3.12}$$

Substitution of equations (3.11) and (3.12) into equations (3.8) gives expressions for the components of stress at any point in the solid. These are:

$$\sigma_x = \frac{1}{2\pi}\left\{2\nu\frac{\partial\psi}{\partial z} - z\frac{\partial^2\psi}{\partial x^2} - (1-2\nu)\frac{\partial^2\psi_1}{\partial x^2}\right\} \tag{3.13a}$$

$$\sigma_y = \frac{1}{2\pi}\left\{2\nu\frac{\partial\psi}{\partial z} - z\frac{\partial^2\psi}{\partial y^2} - (1-2\nu)\frac{\partial^2\psi_1}{\partial y^2}\right\} \tag{3.13b}$$

$$\sigma_z = \frac{1}{2\pi}\left\{\frac{\partial\psi}{\partial z} - z\frac{\partial^2\psi}{\partial z^2}\right\} \tag{3.13c}$$

$$\tau_{xy} = -\frac{1}{2\pi}\left\{(1-2\nu)\frac{\partial^2\psi_1}{\partial x\partial y} + 2\frac{\partial^2\psi}{\partial x\partial y}\right\} \tag{3.13d}$$

$$\tau_{yz} = -\frac{1}{2\pi}z\frac{\partial^2\psi}{\partial y\partial z} \tag{3.13e}$$

$$\tau_{zx} = -\frac{1}{2\pi}z\frac{\partial^2\psi}{\partial x\partial z} \tag{3.13f}$$

We note that the stress components σ_z, τ_{yz} and τ_{zx} depend upon the function ψ only. The stress components σ_x and σ_y depend upon the function ψ_1 but their sum does not, thus

$$\sigma_x + \sigma_y = \frac{1}{2\pi}\left\{(1+2\nu)\frac{\partial \psi}{\partial z} + z\frac{\partial^2 \psi}{\partial z^2}\right\} \tag{3.14}$$

At the surface of the solid the normal stress

$$\bar{\sigma}_z = \frac{1}{2\pi}\left(\frac{\partial \psi}{\partial z}\right)_{z=0} = \left\{\begin{matrix} -p(\xi,\eta) & \text{inside } S \\ 0 & \text{outside } S \end{matrix}\right\} \tag{3.15}$$

and the surface displacements are

$$\bar{u}_x = -\frac{1-2\nu}{4\pi G}\left(\frac{\partial \psi_1}{\partial x}\right)_{z=0} \tag{3.16a}$$

$$\bar{u}_y = -\frac{1-2\nu}{4\pi G}\left(\frac{\partial \psi_1}{\partial y}\right)_{z=0} \tag{3.16b}$$

$$\bar{u}_z = \frac{1-\nu}{2\pi G}\left(\frac{\partial \psi_1}{\partial z}\right)_{z=0} = \frac{1-\nu}{2\pi G}(\psi)_{z=0} \tag{3.16c}$$

Equations (3.15) and (3.16*c*) show that the normal pressure and normal displacement within the loaded area depend only on the potential function ψ.

The equations quoted above provide a formal solution to the problem of stresses and deformations in an elastic half-space with prescribed tractions acting on the surface. If the distributions of traction within the area S are known explicitly then, in principle, the stresses and displacements at any point in the solid can be found. In practice, obtaining expressions in closed form for the stresses in any but the simplest problems presents difficulties. In particular circumstances more sophisticated analytical techniques have been developed to overcome some of the difficulties of the classical approach. A change from rectangular to ellipsoidal coordinates enables problems in which the loaded area is bounded by an ellipse to be handled more conveniently (Lur'e, 1964; Galin, 1953; de Pater, 1964). For circular contact areas, the use of a special complex stress function suggested by Rostovtzev (1953) (see also Green & Zerna, 1954) enables the stresses to be found when the displacements are specified within the loaded area.

For the case of axial symmetry Sneddon (1951) has put forward integral transform methods which have been developed by Noble & Spence (1971) (see Gladwell, 1980, for a full discussion of this approach).

An alternative approach, which is the one generally followed in this book, is to start from the stresses and displacements produced by concentrated normal and tangential forces. The stress distribution and deformation resulting from any distributed loading can then be found by superposition. This approach has the merit that it lends itself to numerical analysis and makes possible the solution of problems in which the geometry makes analytical methods impossible.

3.2 Concentrated normal force

The stresses and displacements produced by a concentrated point force P acting normally to the surface at the origin (Fig. 3.2) can be found* directly in several ways. Using the results of the previous section, the area S over which the normal traction acts is made to approach zero, thus

$$\rho = (x^2 + y^2 + z^2)^{1/2}$$

and

$$\int_S\int p(\xi, \eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = P \tag{3.17}$$

The Boussinesq potential functions ψ_1 and ψ, defined in equations (3.9) and (3.10), in this case reduce to

$$\psi_1 = \frac{\partial H_1}{\partial z} = H = P \ln(\rho + z)$$

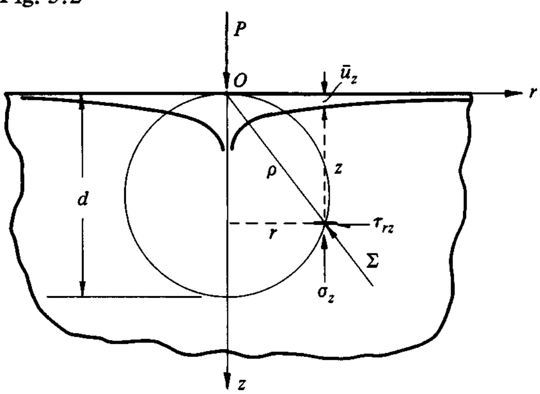
$$\psi = \frac{\partial H}{\partial z} = P/\rho$$

Substituting (3.11) for the elastic displacements at any point in the solid gives

$$u_x = \frac{P}{4\pi G}\left\{\frac{xz}{\rho^3} - (1-2\nu)\frac{x}{\rho(\rho+z)}\right\} \tag{3.18a}$$

$$u_y = \frac{P}{4\pi G}\left\{\frac{yz}{\rho^3} - (1-2\nu)\frac{y}{\rho(\rho+z)}\right\} \tag{3.18b}$$

Fig. 3.2



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$$u_z = \frac{P}{4\pi G}\left\{\frac{z^2}{\rho^3} + \frac{2(1-\nu)}{\rho}\right\} \tag{3.18c}$$

where ρ is given by (3.17).

The stress components are then given by equations (3.13) with the results:

$$\sigma_x = \frac{P}{2\pi}\left[\frac{(1-2\nu)}{r^2}\left\{\left(1-\frac{z}{\rho}\right)\frac{x^2-y^2}{r^2} + \frac{zy^2}{\rho^3}\right\} - \frac{3zx^2}{\rho^5}\right] \tag{3.19a}$$

$$\sigma_y = \frac{P}{2\pi}\left[\frac{(1-2\nu)}{r^2}\left\{\left(1-\frac{z}{\rho}\right)\frac{y^2-x^2}{r^2} + \frac{zx^2}{\rho^3}\right\} - \frac{3zy^2}{\rho^5}\right] \tag{3.19b}$$

$$\sigma_z = -\frac{3P}{2\pi}\frac{z^3}{\rho^5} \tag{3.19c}$$

$$\tau_{xy} = \frac{P}{2\pi}\left[\frac{(1-2\nu)}{r^2}\left\{\left(1-\frac{z}{\rho}\right)\frac{xy}{r^2} - \frac{xyz}{\rho^3}\right\} - \frac{3xyz}{\rho^5}\right] \tag{3.19d}$$

$$\tau_{xz} = -\frac{3P}{2\pi}\frac{xz^2}{\rho^5} \tag{3.19e}$$

$$\tau_{yz} = -\frac{3P}{2\pi}\frac{yz^2}{\rho^5} \tag{3.19f}$$

where $r^2 = x^2 + y^2$. The equations in this form are useful for finding, by direct superposition, the stresses due to a distributed normal load.

Alternatively, we may recognise from the outset that the system is axi-symmetric and use polar coordinates. Timoshenko & Goodier (1951) start by introducing a suitable stress function for axi-symmetric problems and use it to deduce the stresses produced by a concentrated normal force acting on the surface of an elastic half-space with the results:

$$\sigma_r = \frac{P}{2\pi}\left\{(1-2\nu)\left(\frac{1}{r^2} - \frac{z}{\rho r^2}\right) - \frac{3zr^2}{\rho^5}\right\} \tag{3.20a}$$

$$\sigma_\theta = -\frac{P}{2\pi}(1-2\nu)\left(\frac{1}{r^2} - \frac{z}{\rho r^2} - \frac{z}{\rho^3}\right) \tag{3.20b}$$

$$\sigma_z = -\frac{3P}{2\pi}\frac{z^3}{\rho^5} \tag{3.20c}$$

$$\tau_{rz} = -\frac{3P}{2\pi}\frac{rz^2}{\rho^5} \tag{3.20d}$$

It is easy to see that equations (3.19) and (3.20) are identical by putting $x = r$, $y = 0$, $\sigma_x = \sigma_r$ and $\sigma_y = \sigma_\theta$ in equations (3.19). Note also that

$$\sigma_r + \sigma_\theta + \sigma_z = -\frac{P}{\pi}\frac{(1+\nu)z}{\rho^3} \tag{3.20e}$$

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The direct and shear stresses σ_z and τ_{rz} which act on planes within the solid parallel to the free surface are independent of Poisson's ratio. Consider the resultant stress Σ acting on elements of such parallel planes. Where those planes intersect the spherical surface of diameter d which is tangential to the surface of the half-space at O, as shown in Fig. 3.2, Σ is given by

$$\Sigma = (\sigma_z^2 + \tau_{rz}^2)^{1/2} = \frac{3P}{2\pi}\frac{z^2}{\rho^4} = \frac{3P}{2\pi d^2} = \text{constant}$$

and the direction of the resultant stress acts towards O. There is some analogy in this result with that for a concentrated line load discussed in the last chapter (§2.2 and Fig. 2.2). In the three-dimensional case, however, it should be noted that Σ is not a principal stress and that the principal stress does not act in a radial direction, nor are the surfaces of constant principal shear stress spherical.

Timoshenko & Goodier (1951) derive the strains from the stresses and integrate to obtain the displacements as was done in §2.2 with the results:

$$u_r = \frac{P}{4\pi G}\left\{\frac{rz}{\rho^3} - (1-2\nu)\frac{\rho - z}{\rho r}\right\} \tag{3.21a}$$

$$u_z = \frac{P}{4\pi G}\left\{\frac{z^2}{\rho^3} + \frac{2(1-\nu)}{\rho}\right\} \tag{3.21b}$$

The results are consistent with equations (3.18). On the surface of the solid $(z = 0)$

$$\bar{u}_r = -\frac{(1-2\nu)}{4\pi G}\frac{P}{r} \tag{3.22a}$$

$$\bar{u}_z = \frac{(1-\nu)}{2\pi G}\frac{P}{r} \tag{3.22b}$$

From equation (3.22*b*) we note that the profile of the deformed surface is a rectangular hyperboloid, which is asymptotic to the undeformed surface at a large distance from O and exhibits a theoretically infinite deflexion at O, as shown in Fig. 3.2.

The stresses and deflexions produced by a normal pressure distributed over an area S of the surface can now be found by superposition using results of the last section for a concentrated force. Referring to Fig. 3.1, we require the surface depression $\bar{u}_z$ at a general surface point $B(x, y)$ and the stress components at an interior point $A(x, y, z)$ due to a distributed pressure $p(\xi, \eta)$ acting on the surface area S. We change to polar coordinates (s, ϕ) with origin at B such that the pressure $p(s, \phi)$ acting on a surface element at C is equivalent to a force of magnitude $ps\,\mathrm{d}s\,\mathrm{d}\phi$. The displacement of the surface at B due to this force can be written down from equation (3.22*b*) in which $r = BC = s$. The displacement at B due to the pressure distributed over the whole of S is thus:

$$\bar{u}_z = \frac{1-\nu^2}{\pi E} \int_S \int p(s, \phi) \, ds \, d\phi \tag{3.23}$$

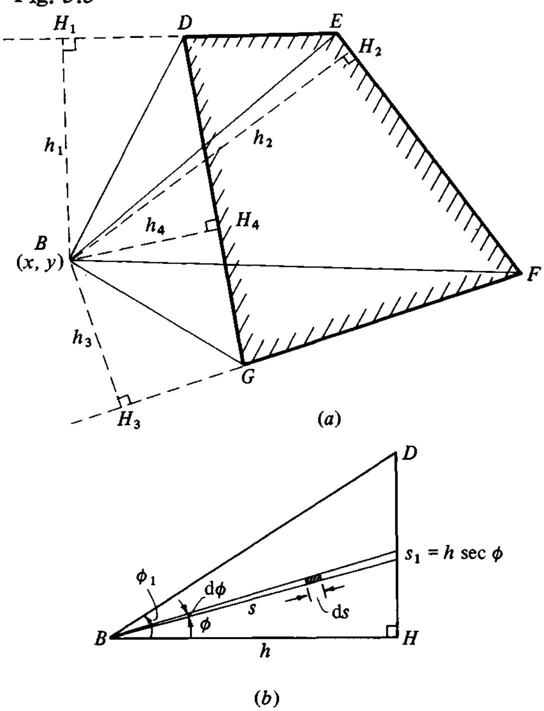
The stress components at A can be found by integrating the stress components for a concentrated force given by equation (3.19).

3.3 Pressure applied to a polygonal region

(a) Uniform pressure

We shall consider in this section a uniform pressure p applied to a region of the surface consisting of a straight-sided polygon, as shown in Fig. 3.3(a). It is required to find the depression $\bar{u}_z$ at a general point $B(x, y)$ on the surface and the stress components at a subsurface point $A(x, y, z)$. BH_1, BH_2, etc. are perpendiculars of length h_1, h_2, etc. from B onto the sides of the polygon DE, EF respectively. The loaded polygon is then made up of the algebraic addition of

Fig. 3.3



eight right-angle triangles:

$$DEFG = [BEH_1 + BEH_2 + BFH_2 + BFH_3] - [BDH_1 + BDH_4 + BGH_3 + BGH_4]$$

A similar breakdown into rectangular triangles would have been possible if B had lain inside the polygon. A typical triangular area is shown in Fig. 3.3(*b*).

If the pressure is uniformly distributed, equation (3.23) becomes

$$\begin{aligned}(\bar{u}_z)_B &= \frac{1-\nu^2}{\pi E}\, p \int_0^{\phi_1} \mathrm{d}\phi \int_0^{s_1} \mathrm{d}s \\ &= \frac{1-\nu^2}{\pi E}\, p \int_0^{\phi_1} h \sec\phi \,\mathrm{d}\phi \\ &= \frac{1-\nu^2}{\pi E}\, p \frac{h}{2} \ln\left\{\frac{1+\sin\phi_1}{1-\sin\phi_1}\right\} \end{aligned} \tag{3.24}$$

The total displacement at B due to a uniform pressure on the polygonal region $DEFG$ can then be found by combining the results of equation (3.24) for the eight constitutive triangles. The stress components at an interior point $A(x, y, z)$ below B can be found by integration of the stress components due to a point force given by equations (3.19), but the procedure is tedious.

The effect of a uniform pressure acting on a rectangular area $2a \times 2b$ has been analysed in detail by Love (1929). The deflexion of a general point (x, y) on the surface is given by

$$\begin{aligned}\frac{\pi E}{1-\nu^2}\frac{\bar{u}_z}{p} &= (x+a)\ln\left[\frac{(y+b)+\{(y+b)^2+(x+a)^2\}^{1/2}}{(y-b)+\{(y-b)^2+(x+a)^2\}^{1/2}}\right] \\ &\quad + (y+b)\ln\left[\frac{(x+a)+\{(y+b)^2+(x+a)^2\}^{1/2}}{(x-a)+\{(y+b)^2+(x-a)^2\}^{1/2}}\right] \\ &\quad + (x-a)\ln\left[\frac{(y-b)+\{(y-b)^2+(x-a)^2\}^{1/2}}{(y+b)+\{(y+b)^2+(x-a)^2\}^{1/2}}\right] \\ &\quad + (y-b)\ln\left[\frac{(x-a)+\{(y-b)^2+(x-a)^2\}^{1/2}}{(x+a)+\{(y-b)^2+(x+a)^2\}^{1/2}}\right]\end{aligned} \tag{3.25}$$

Expressions have been found by Love (1929) from which the stress components at a general point in the solid can be found. Love comments on the fact that the component of shear stress τ_{xy} has a theoretically infinite value at the corner of the rectangle. Elsewhere all the stress components are finite. On the surface, at the centre of the rectangle:

$$[\sigma_x]_0 = -p\{2\nu + (2/\pi)(1-2\nu)\tan^{-1}(b/a)\} \tag{3.26a}$$

$$[\sigma_y]_0 = -p\{2\nu + (2/\pi)(1 - 2\nu)\tan^{-1}(a/b)\} \tag{3.26b}$$

$$[\sigma_z]_0 = -p \tag{3.26c}$$

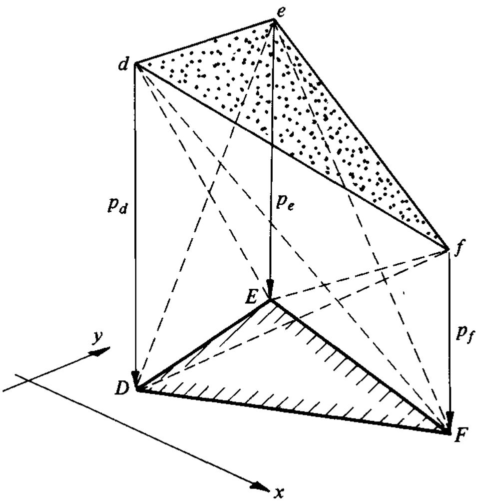
These results are useful when a uniformly loaded rectangle is used as a 'boundary element' in the numerical solution of more general contact problems (see §5.9).

(b) Non-uniform pressure

Any general variation in pressure over a polygonal region can only be analysed numerically. In this connection it is useful to consider a triangular area *DEF* on which the pressure varies linearly from p_d at D to p_e at E to p_f at F, as shown in Fig. 3.4, i.e. the pressure distribution forms a plane facet *def*. Any general polygonal region can be divided into triangular elements such as *DEF*. In this way a continuous distribution of pressure over the whole polygonal surface may be approximated by linear facets acting on triangular elements such as the one shown in Fig. 3.4. This representation of a continuous pressure distribution may be regarded as an improved approximation upon a series of boundary elements in each of which the pressure is taken to be uniform, since discontinuities in pressure along the sides of the elements have been eliminated and replaced by discontinuities in pressure gradient.

The element shown in Fig. 3.4 can be further simplified by splitting it into three tetragonal pressure elements: the first having pressure p_d at D which falls linearly to zero along the side of the triangle EF; the second having pressure p_e

Fig. 3.4



at E which falls linearly to zero along FD; and the third having pressure p_f at F which falls to zero along DE.

Influence coefficients for the normal deflexion of the surface $\bar{u}_z(x,y)$ due to the pressure distribution shown in Fig. 3.4 have been calculated by Kalker & van Randen (1972) in connection with the numerical solution of contact problems. Similarly Johnson & Bentall (1977) considered the deflexion of a surface under the action of a pyramidal distribution of pressure acting on a uniform hexagonal area of an elastic half-space. The maximum pressure p_0 acts at the centre O of the hexagon and falls to zero along the edges. The deflexion at the centre $(\bar{u}_z)_0$ was found to be $3\sqrt{3}\ln 3(1-\nu^2)p_0c/2\pi E$ and that at an apex to be $(\bar{u}_z)_0/3$, where c is the length of the side of the hexagon. Explicit results for a polynomial distribution of pressure acting on a triangular area have been found by Svec & Gladwell (1971).

The stresses and deformations produced by a pressure distribution of the form $p_0(1-x^2/a^2)^{1/2}$ acting on the rectangle $x=\pm a, y=\pm b$ have been calculated by Kunert (1961).

3.4 Pressure applied to a circular region

A circular region of radius a is shown in Fig. 3.5. It is required to find the displacement at a surface point B and the stresses at an internal point A due to pressure distributed over the circular region. Solutions in closed form can be found for axi-symmetrical pressure distributions of the form:

$$p = p_0(1-r^2/a^2)^n \tag{3.27}$$

We will consider in detail some particular values of n.

(a) Uniform pressure ($n = 0$)

Regarding the pressure p at C, acting on a surface element of area $s\,\mathrm{d}s\,\mathrm{d}\phi$, as a concentrated force, the normal displacement is given by equation (3.23), i.e.

$$\bar{u}_z = \frac{1-\nu^2}{\pi E}\,p\int_S\int \mathrm{d}\phi\,\mathrm{d}s$$

We will consider first the case where B lies inside the circle (Fig. 3.5(a)). The limits on s are

$$s_{1,2} = -r\cos\phi \pm \{r^2\cos^2\phi + (a^2-r^2)\}^{1/2} \tag{3.28}$$

Thus

$$\bar{u}_z = \frac{1-\nu^2}{\pi E}\,p\int_0^{\pi} 2\{r^2\cos^2\phi + (a^2-r^2)\}^{1/2}\,\mathrm{d}\phi$$

$$= \frac{4(1-\nu^2)pa}{\pi E}\int_0^{\pi/2}\{1-(r^2/a^2)\sin^2\phi\}^{1/2}\,\mathrm{d}\phi$$

i.e.

$$\bar{u}_z = \frac{4(1-\nu^2)pa}{\pi E}\, \mathbf{E}(r/a), \quad (r < a) \tag{3.29a}$$

where $\mathbf{E}(r/a)$ is the complete elliptic integral of the second kind with modulus (r/a). At the centre of the circle: $r = 0$, $\mathbf{E}(0) = \pi/2$, thus

$$(\bar{u}_z)_0 = 2(1-\nu^2)pa/E.$$

At the edge of the circle: $r/a = 1$, $\mathbf{E}(1) = 1$, then

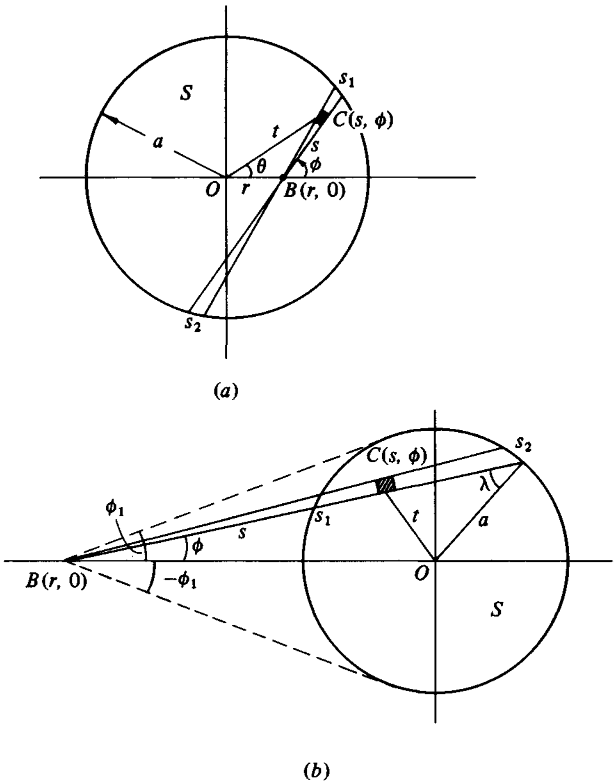
$$(\bar{u}_z)_a = 4(1-\nu^2)pa/\pi E$$

The mean displacement of the loaded circle is $16(1-\nu^2)pa/3\pi E$.

In view of the axial symmetry, the displacement tangential to the surface must be radial. By equation (3.22*a*) the tangential displacement at B due to an element of load at C is

$$\frac{(1-2\nu)(1+\nu)}{2\pi E}\,\frac{ps\,\mathrm{d}s\,\mathrm{d}\phi}{s}$$

Fig. 3.5. Pressure applied to a circular region. Displacement: (*a*) at an internal point B; (*b*) at an external point B.



in the direction from B towards C. Thus the radial component of this displacement is:

$$\bar{u}_r = \frac{(1-2\nu)(1+\nu)}{2\pi E} p \cos\phi \, ds \, d\phi$$

The total displacement due to the whole load is then

$$\bar{u}_r = \frac{(1-2\nu)(1+\nu)}{2\pi E} p \int_0^{2\pi} \{-r\cos\phi + (r^2\cos^2\phi + a^2 - r^2)^{1/2}\} \times \cos\phi \, d\phi$$

The second term in the integral vanishes when integrated over the limits 0 to 2π, hence

$$\bar{u}_r = -(1-2\nu)(1+\nu)pr/2E, \quad r \leqslant a \tag{3.29b}$$

We turn now to a point B on the surface outside the circle (Fig. 3.5(*b*)). In this case the limits on ϕ are $\pm\phi_1$, so that

$$\bar{u}_z = \frac{2(1-\nu^2)p}{\pi E} \int_0^{\phi_1} (a^2 - r^2\sin^2\phi)^{1/2} \, d\phi$$

We change the variable to the angle λ, shown in Fig. 3.5(*b*), which is related to ϕ by

$$a\sin\lambda = r\sin\phi$$

whereupon the expression for $\bar{u}_z$ becomes:

$$\bar{u}_z = \frac{4(1-\nu^2)}{\pi E} p \int_0^{\pi/2} \frac{a^2\cos^2\lambda \, d\lambda}{r\{1-(a^2/r^2)\sin^2\lambda\}}$$

$$= \frac{4(1-\nu^2)}{\pi E} pr\{\mathbf{E}(a/r) - (1-a^2/r^2)\mathbf{K}(a/r)\}, \quad r > a \tag{3.30a}$$

where $\mathbf{K}(a/r)$ is the complete elliptic integral of the first kind with modulus (a/r).

The tangential displacement at B is radial and is given by

$$\bar{u}_r = -\frac{2(1-2\nu)(1+\nu)}{\pi E} p \int_0^{\phi_1} \cos\phi \, (a^2 - r^2\sin^2\phi)^{1/2} \, d\phi$$

Changing the variable to λ, as before, gives

$$\bar{u}_r = -\frac{2(1-2\nu)(1+\nu)}{\pi E} p\frac{a^2}{r} \int_0^{\pi/2} \cos^2\lambda \, d\lambda$$

$$= -\frac{(1-2\nu)(1+\nu)}{2E} p\frac{a^2}{r}, \quad r > a \tag{3.30b}$$

Since the $p\pi a^2$ is equal to the total load P acting on the whole area, we note that the tangential displacement outside the loaded region, given by (3.30*b*), is the same as though the whole load were concentrated at the centre of the circle (see equation (3.22*a*)). It follows by superposition that this conclusion is true for any axially symmetrical distribution of pressure acting in the circle.

The stresses at the surface within the circle may now be found from equations (3.29). Thus:

$$\bar{\epsilon}_r = \frac{\partial \bar{u}_r}{\partial r} = \bar{\epsilon}_\theta = \frac{\bar{u}_r}{r} = -\frac{(1-2\nu)(1+\nu)}{2E}p \tag{3.31}$$

from which, by Hooke's law, we get

$$\bar{\sigma}_r = \bar{\sigma}_\theta = -\tfrac{1}{2}(1+2\nu)p, \quad \bar{\sigma}_z = -p \tag{3.32}$$

To find the stresses within the half-space along the z-axis we make use of equations (3.20) for the stresses due to a concentrated force. Consider an annular element of area $2\pi r\,\mathrm{d}r$ at radius r. The load on the annulus is $2\pi rp\,\mathrm{d}r$, so substituting in (3.20*c*) and integrating over the circle gives

$$\begin{aligned}\sigma_z &= -3p\int_0^a \frac{rz^3}{(r^2+z^2)^{5/2}}\,\mathrm{d}r \\ &= -p\{1 - z^3/(a^2+z^2)^{3/2}\}\end{aligned} \tag{3.33a}$$

Along Oz, $\sigma_r = \sigma_\theta$, hence applying equation (3.20*e*) to an annulus of pressure

$$\begin{aligned}\sigma_r + \sigma_\theta + \sigma_z &= -\frac{(1+\nu)}{\pi}p\int_0^a \frac{2\pi rz\,\mathrm{d}r}{(r^2+z^2)^{3/2}} \\ &= 2(1+\nu)p\{z(a^2+z^2)^{-1/2} - 1\}\end{aligned}$$

so that

$$\sigma_r = \sigma_\theta = -p\left\{\frac{1+2\nu}{2} - \frac{(1+\nu)z}{(a^2+z^2)^{1/2}} + \frac{z^3}{2(a^2+z^2)^{3/2}}\right\} \tag{3.33b}$$

The stress components at other points throughout the half-space have been investigated by Love (1929).

(b) Uniform normal displacement $(n = -\frac{1}{2})$

We shall proceed to show that a pressure distribution of the form

$$p = p_0(1 - r^2/a^2)^{-1/2} \tag{3.34}$$

gives rise to a uniform normal displacement of the loaded circle. This is the pressure, therefore, which would arise on the face of a flat-ended, frictionless cylindrical punch pressed squarely against an elastic half-space. It is the axi-symmetrical analogue of the two-dimensional problem discussed in §2.8.

Referring to Fig. 3.5(*a*):

$$t^2 = r^2 + s^2 + 2rs\cos\phi$$

so that

$$p(s, \phi) = p_0 a(\alpha^2 - 2\beta s - s^2)^{-1/2} \tag{3.35}$$

where $\alpha^2 = a^2 - r^2$ and $\beta = r \cos\phi$. The displacement within the loaded circle, using equation (3.23), is

$$\bar{u}_z(r) = \frac{1-\nu^2}{\pi E} p_0 a \int_0^{2\pi} \mathrm{d}\phi \int_0^{s_1} (\alpha^2 - 2\beta s - s^2)^{-1/2}\, \mathrm{d}s$$

where the limit s_1 is the positive root of

$$\alpha^2 - 2\beta s - s^2 = 0$$

Now

$$\int_0^{s_1} (\alpha^2 - 2\beta s - s^2)^{-1/2}\, \mathrm{d}s = \frac{\pi}{2} - \tan^{-1}(\beta/\alpha)$$

and

$$\tan^{-1}\{\beta(\phi)/\alpha\} = -\tan^{-1}\{\beta(\phi + \pi)/\alpha\}$$

so that the integral of $\tan^{-1}(\beta/\alpha)$ vanishes as ϕ varies from 0 to 2π, whereupon

$$\bar{u}_z = \frac{1-\nu^2}{\pi E} p_0 a \int_0^{2\pi} \left\{\frac{\pi}{2} - \tan^{-1}(\beta/\alpha)\right\} \mathrm{d}\phi = \pi(1-\nu^2)p_0 a/E \tag{3.36}$$

which is constant and independent of r. The total force

$$P = \int_0^a 2\pi r p_0 (1 - r^2/a^2)^{-1/2}\, \mathrm{d}r = 2\pi a^2 p_0 \tag{3.37}$$

When B lies outside the loaded circle (Fig. 3.5(*b*))

$$p(s, \phi) = p_0 a(\alpha^2 + 2\beta s - s^2)^{-1/2}$$

and the limits $s_{1,2}$ are the root of $\alpha^2 + 2\beta s - s^2 = 0$, whereupon

$$\int_{s_1}^{s_2} (\alpha^2 + 2\beta s - s^2)^{-1/2}\, \mathrm{d}s = \pi$$

The limits on ϕ are $\phi_{1,2} = \sin^{-1}(a/r)$, so that

$$\bar{u}_z(r) = \frac{2(1-\nu^2)}{E} p_0 a \sin^{-1}(a/r) \tag{3.38}$$

Like the two-dimensional punch, the pressure is theoretically infinite at the edge of the punch and the surface has an infinite gradient just outside the edge. Stresses within the half-space have been found by Sneddon (1946).

(c) Hertz pressure ($n = \frac{1}{2}$)

The pressure given by the Hertz theory (see Chapter 4), which is exerted between two frictionless elastic solids of revolution in contact, is given by

$$p(r) = p_0(a^2 - r^2)^{1/2}/a \tag{3.39}$$

from which the total load $P = 2\pi p_0 a^2/3$. The method of finding the deflexions is identical to that in the previous problem (§3.4*b*) and uses the same notation Thus, within the loaded circle the normal displacement is given by

$$\bar{u}_z(r) = \frac{1-\nu^2}{\pi E}\,\frac{p_0}{a}\int_0^{2\pi} d\phi \int_0^{s_1} (\alpha^2 - 2\beta s - s^2)^{1/2}\, ds \tag{3.40}$$

$$\int_0^{s_1} (\alpha^2 - 2\beta s - s^2)^{1/2}\, ds = -\tfrac{1}{2}\alpha\beta + \tfrac{1}{2}(\alpha^2 + \beta^2)\,\{(\pi/2) - \tan^{-1}(\beta/\alpha)\}$$

The terms $\beta\alpha$ and $\tan^{-1}(\beta/\alpha)$ vanish when integrated with respect to ϕ between the limits 0 and 2π, whereupon

$$\bar{u}_z(r) = \frac{1-\nu^2}{\pi E}\,\frac{p_0}{a}\int_0^{2\pi} \frac{\pi}{4}(a^2 - r^2 + r^2\cos^2\phi)\, d\phi$$

$$= \frac{1-\nu^2}{E}\,\frac{\pi p_0}{4a}(2a^2 - r^2), \quad r \leqslant a \tag{3.41a}$$

To find the tangential displacement at B, which by symmetry must be radial, we make use of equation (3.22*a*). The element of pressure at C causes a tangential displacement at B:

$$\frac{(1-2\nu)(1+\nu)}{2\pi E}\, p\, ds\, d\phi$$

directed from B towards C. The radial component of this displacement is

$$\frac{(1-2\nu)(1+\nu)}{2\pi E}\cos\phi\, p\, ds\, d\phi$$

so that the resultant tangential displacement at B due to the whole pressure distribution is

$$\bar{u}_r(r) = \frac{(1-2\nu)(1+\nu)}{2\pi E}\,\frac{p_0}{a}\int_0^{2\pi}\cos\phi\, d\phi \int_0^{s_1}(\alpha^2 - 2\beta s - s^2)^{1/2}\, ds$$

The integration with respect to s is the same as before; integrating with respect to ϕ gives

$$\bar{u}_r(r) = -\frac{(1-2\nu)(1+\nu)}{3E}\,\frac{a^2}{r}\, p_0\{1 - (1 - r^2/a^2)^{3/2}\}, \quad r \leqslant a \tag{3.41b}$$

When the point B lies outside the loaded circle, proceeding in the same way as in the previous case, we find

$$\bar{u}_z = \frac{(1-\nu^2)}{E}\,\frac{p_0}{2a}$$

$$\times\{(2a^2 - r^2)\sin^{-1}(a/r) + r^2(a/r)(1 - a^2/r^2)^{1/2}\}, \quad r > a \tag{3.42a}$$

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The tangential displacement outside the loaded circle is the same as if the load were concentrated at the centre, so that, by equation (3.22*a*)

$$\bar{u}_r = -\frac{(1-2\nu)(1+\nu)}{3E} p_0 \frac{a^2}{r}, \quad r > a \tag{3.42b}$$

The surface strain components: $\bar{\varepsilon}_r = \partial \bar{u}_r/\partial r$ and $\bar{\varepsilon}_\theta = \bar{u}_r/r$ can be found from equations (3.41) and (3.42) which, together with the pressure, determine the stresses in the surface $z = 0$ with the result that, inside the loaded circle,

$$\bar{\sigma}_r/p_0 = \frac{1-2\nu}{3}(a^2/r^2)\{1-(1-r^2/a^2)^{3/2}\} - (1-r^2/a^2)^{1/2} \tag{3.43a}$$

$$\bar{\sigma}_\theta/p_0 = -\frac{1-2\nu}{3}(a^2/r^2)\{1-(1-r^2/a^2)^{3/2}\} - 2\nu(1-r^2/a^2)^{1/2} \tag{3.43b}$$

$$\bar{\sigma}_z/p_0 = -(1-r^2/a^2)^{1/2} \tag{3.43c}$$

and outside the circle

$$\bar{\sigma}_r/p_0 = -\bar{\sigma}_\theta/p_0 = (1-2\nu)a^2/3r^2 \tag{3.44}$$

The radial stress is therefore tensile outside the loaded circle. It reaches its maximum value at the edge of the circle at $r = a$. This is the maximum tensile stress occurring anywhere. The stresses along the z-axis may be calculated without difficulty by considering a ring of concentrated force at radius r. They are:

$$\frac{\sigma_r}{p_0} = \frac{\sigma_\theta}{p_0} = -(1+\nu)\{1-(z/a)\tan^{-1}(a/z)\} + \tfrac{1}{2}(1+z^2/a^2)^{-1} \tag{3.45a}$$

$$\frac{\sigma_z}{p_0} = -(1+z^2/a^2)^{-1} \tag{3.45b}$$

The stresses at other points throughout the solid have been calculated by Huber (1904) and Morton & Close (1922). This stress distribution is illustrated in Fig. 4.3 where it is compared with the stresses produced by a uniform pressure acting on a circular area given by equation (3.33). Along the z-axis σ_r, σ_θ and σ_z are principal stresses. The principal shear stress, $\tau_1 = \frac{1}{2}|\sigma_z - \sigma_\theta|$, is also plotted in Fig. 4.3. It has a maximum value which lies below the surface. For the Hertz pressure distribution

$$(\tau_1)_{\max} = 0.31 p_0 = 0.47P/\pi a^2 \tag{3.46}$$

at $z = 0.57a$.

For the uniform pressure distribution

$$(\tau_1)_{\max} = 0.33p = 0.33P/\pi a^2 \tag{3.47}$$

at $z = 0.64a$. Both the values above are computed for $\nu = 0.30$.

(d) General pressure ($n = m - \frac{1}{2}$)

A pressure distribution of the form

$$p_m = p_0 a^{1-2m}(a^2 - r^2)^{m-1/2} \tag{3.48}$$

where m is an integer, will produce a normal displacement within the loaded region given in the notation of equation (3.35) by:

$$\bar{u}_z = \frac{1-\nu^2}{\pi E}\frac{p_0}{a^{2m-1}}\int_0^{\pi} \mathrm{d}\phi \int_0^{s_1} (\alpha^2 - 2\beta s - s^2)^{m-1/2}\, \mathrm{d}s \tag{3.49}$$

Reduction formulae for the integral with respect to s enable solutions to be found in closed form (see Lur'e, 1964). The resulting expressions for $\bar{u}_z$ are polynomials in r of order $2m$. Thus, in the examples considered above: $m = 0$ results in a constant displacement; $m = 1$ results in a displacement quadratic in r.

Alternative methods, which link polynomial variation of displacement with pressure distribution of the form $p_m = p_0 a^{1-2m} r^{2m}(a^2 - r^2)^{-1/2}$, have been developed by Popov (1962) using Legendre polynomials and by Steuermann (1939). In this way, if the displacement within the loaded circle can be represented by an even polynomial in r, then the corresponding pressure distributions can be expressed in the appropriate sum of distributions having the form p_m defined above (see §5.3).

3.5 Pressure applied to an elliptical region

It is shown in Chapter 4 that two non-conforming bodies loaded together make contact over an area which is elliptical in shape, so that the stresses and deformation due to pressure and traction on an elliptical region are of practical importance. A circle is a particular case of an ellipse, so that we might expect results for an elliptical region to be qualitatively similar to those derived for a circular region in the previous section. This is indeed the case, so that we are led to consider pressure distributions of the form

$$p(x, y) = p_0\{1 - (x/a)^2 - (y/b)^2\}^n \tag{3.50}$$

which act over the region bounded by the ellipse

$$(x/a)^2 + (y/b)^2 - 1 = 0$$

The classical approach, using the potential functions of Boussinesq, is usually followed. Thus, by equation (3.10),

$$\psi(x, y, z) = \int_S\int \{1 - (\xi/a)^2 - (\eta/b)^2\}^n \rho^{-1}\, \mathrm{d}\xi\, \mathrm{d}\eta \tag{3.51}$$

where $\rho^2 = (\xi - x)^2 + (\eta - y)^2 + z^2$. The normal displacement of the surface is then given by equation (3.16c), viz.:

$$\bar{u}_z(x, y) = \frac{1-\nu}{2\pi G}(\psi)_{z=0}$$

It then follows from potential theory (see, for example, Routh, 1908, p. 129) that for a general point in the solid

$$\psi(x,y,z) = \frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+3/2)} p_0 ab \int_{\lambda_1}^{\infty} \left(1 - \frac{x^2}{a^2+w} - \frac{y^2}{b^2+w} - \frac{z^2}{w}\right)^{n+1/2} \times \frac{\mathrm{d}w}{\{(a^2+w)(b^2+w)w\}^{1/2}} \tag{3.52}$$

where Γ denotes a gamma function and where λ_1 is the positive root of the equation

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{\lambda} = 1 \tag{3.53}$$

λ_1 may be interpreted geometrically as the parameter of an ellipsoid, confocal with the given elliptical pressure region, whose surface passes through the point in question (x, y, z). To find $\psi(x, y, 0)$ at a surface point *within* the loaded region the lower limit of the integral in (3.52) is taken to be zero.

A few cases in which n takes different values will be discussed.

(a) Uniform displacement $(n = -\frac{1}{2})$

Putting $n = -\frac{1}{2}$ in equation (3.52) gives:

$$\psi(x,y,z) = \pi p_0 ab \int_{\lambda_1}^{\infty} \frac{\mathrm{d}w}{\{(a^2+w)(b^2+w)w\}^{1/2}} \tag{3.54}$$

and on the surface $z = 0$, within the loaded region,

$$\psi(x,y,0) = \pi p_0 ab \int_{0}^{\infty} \frac{\mathrm{d}w}{\{(a^2+w)(b^2+w)w\}^{1/2}} \tag{3.55}$$

This integral is a constant independent of x and y; it is an elliptic integral which, when put into standard form and substituted into equation (3.16*c*), quoted above, gives

$$\bar{u}_z = \frac{1-\nu^2}{E} 2p_0 b\mathbf{K}(e) \tag{3.56}$$

where $e = (1 - b^2/a^2)^{1/2}$ is the eccentricity of the ellipse, and $a \geqslant b$. This is the uniform displacement of a half-space under that action of a rigid frictionless punch of elliptical plan-form. The pressure on the face of the punch is

$$p(x,y) = p_0\{1 - (x/a)^2 - (y/b)^2\}^{-1/2} \tag{3.57}$$

where the total load $P = 2\pi abp_0$. The displacement under a cylindrical punch given by equation (3.36) could be obtained by putting $a = b$ or $e = 1$ in equation (3.56).

(b) Hertz pressure ($n = \frac{1}{2}$)

In this important case the pressure is

$$p = p_0\{1 - (x/a)^2 - (y/b)^2\}^{1/2} \tag{3.58}$$

Thus

$$\psi(x, y, z) = \tfrac{1}{2}\pi ab p_0 \int_{\lambda_1}^{\infty} \left(1 - \frac{x^2}{a^2 + w} - \frac{y^2}{b^2 + w} - \frac{z^2}{w}\right) \times \frac{\mathrm{d}w}{\{(a^2 + w)(b^2 + w)w\}^{1/2}} \tag{3.59}$$

and on the surface, within the loaded region,

$$\psi(x, y, 0) = \tfrac{1}{2}\pi ab p_0 \int_0^{\infty} \left(1 - \frac{x^2}{a^2 + w} - \frac{y^2}{b^2 + w}\right) \times \frac{\mathrm{d}w}{\{(a^2 + w)(b^2 + w)w\}^{1/2}} \tag{3.60}$$

The surface displacement within the loaded region may then be written:

$$\bar{u}_z = \frac{1 - \nu^2}{\pi E}(L - Mx^2 - Ny^2) \tag{3.61}$$

where

$$M = \frac{\pi p_0 ab}{2} \int_0^{\infty} \frac{\mathrm{d}w}{\{(a^2 + w)^3(b^2 + w)w\}^{1/2}} = \frac{\pi p_0 b}{e^2 a^2}\{\mathbf{K}(e) - \mathbf{E}(e)\} \tag{3.62a}$$

$$N = \frac{\pi p_0 ab}{2} \int_0^{\infty} \frac{\mathrm{d}w}{\{(a^2 + w)(b^2 + w)^3 w\}^{1/2}} = \frac{\pi p_0 b}{e^2 a^2}\left\{\frac{a^2}{b^2}\mathbf{E}(e) - \mathbf{K}(\mathrm{e})\right\} \tag{3.62b}$$

and

$$L = \frac{\pi p_0 ab}{2} \int_0^{\infty} \frac{\mathrm{d}w}{\{(a^2 + w)(b^2 + w)w\}^{1/2}} = \pi p_0 b \mathbf{K}(e) \tag{3.62c}$$

The total load acting on the ellipse is given by

$$P = 2\pi ab p_0/3 \tag{3.63}$$

Finding the components of displacement and stress at a general point in the solid from equation (3.59) is not straightforward, firstly because the limit λ_1 is the root of a cubic equation (3.53) and secondly because, for certain stress components, it is necessary to determine the auxiliary function $\psi_1 = \int_z^{\infty} \psi \, \mathrm{d}z$.

The difficulties are least when finding the stresses along the z-axis. In this case $\lambda_1 = z^2$, and the integration with respect to z to determine the derivatives of ψ_1 is straightforward. These calculations have been performed by Thomas & Hoersch (1930), Belajev (1917) and Lundberg & Sjövall (1958) with the results that along the z-axis

$$\frac{\sigma_x}{p_0} = \frac{2b}{e^2 a}(\Omega_x + \nu\Omega_x') \tag{3.64a}$$

$$\frac{\sigma_y}{p_0} = \frac{2b}{e^2 a}(\Omega_y + \nu\Omega_y') \tag{3.64b}$$

$$\frac{\sigma_z}{p_0} = -\frac{b}{e^2 a}\left(\frac{1-T^2}{T}\right) \tag{3.64c}$$

where

$$\Omega_x = -\tfrac{1}{2}(1-T) + \zeta\{\mathbf{F}(\phi, e) - \mathbf{E}(\phi, e)\}$$

$$\Omega_x' = 1 - (a^2 T/b^2) + \zeta\{(a^2/b^2)\mathbf{E}(\phi, e) - \mathbf{F}(\phi, e)\}$$

$$\Omega_y = \tfrac{1}{2} + (1/2T) - (Ta^2/b^2) + \zeta\{(a^2/b^2)\mathbf{E}(\phi, e) - \mathbf{F}(\phi, e)\}$$

$$\Omega_y' = -1 + T + \zeta\{\mathbf{F}(\phi, e) - \mathbf{E}(\phi, e)\}$$

$$T = \left(\frac{b^2 + z^2}{a^2 + z^2}\right)^{1/2}, \quad \zeta = \frac{z}{a} = \cot\phi$$

The elliptic integrals $\mathbf{F}(\phi, e)$ and $\mathbf{E}(\phi, e)$ are tabulated.† Within the surface of contact, along the x-axis

$$\frac{\sigma_x}{p_0} = -2\nu\gamma - (1-2\nu)\frac{b}{ae^2}\left\{(1 - b\gamma/a) - \frac{x}{ae}\tanh^{-1}\left(\frac{ex}{a + b\gamma}\right)\right\} \tag{3.65a}$$

$$\frac{\sigma_y}{p_0} = -2\nu\gamma - (1-2\nu)\frac{b}{ae^2}\left\{\left(\frac{a\gamma}{b} - 1\right) + \frac{x}{ae}\tanh^{-1}\left(\frac{ex}{a + b\gamma}\right)\right\} \tag{3.65b}$$

and along the y-axis

$$\frac{\sigma_x}{p_0} = -2\nu\gamma - (1-2\nu)\frac{b}{ae^2}\left\{(1 - b\gamma/a) - \frac{y}{ae}\tan^{-1}\left(\frac{aey}{b(a\gamma + b)}\right)\right\} \tag{3.66a}$$

$$\frac{\sigma_y}{p_0} = -2\nu\gamma - (1-2\nu)\frac{b}{ae^2}\left\{\left(\frac{a\gamma}{b} - 1\right) + \frac{y}{ae}\tan^{-1}\left(\frac{aey}{b(a\gamma + b)}\right)\right\} \tag{3.66b}$$

where $\gamma = \{1 - (x/a)^2 - (y/b)^2\}^{1/2}$.

At the centre ($x = y = 0$)

$$\frac{\sigma_x}{p_0} = -2\nu - (1-2\nu)\frac{b}{a+b} \tag{3.67a}$$

† Abramowitz, M. & Stegun, I. A., *Handbook of Math. Functions*, Dover, 1965.

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$$\frac{\sigma_y}{p_0} = -2\nu - (1-2\nu)\frac{a}{a+b} \tag{3.67b}$$

Outside the loaded ellipse the surface stresses are equal and opposite, i.e. there is a state of pure shear:

$$\frac{\sigma_x}{p_0} = -\frac{\sigma_y}{p_0} = -(1-2\nu)\frac{b}{ae^2}\left\{1 - \frac{x}{ae}\tanh^{-1}\left(\frac{ex}{a}\right) - \frac{y}{ae}\tan^{-1}\left(\frac{aey}{b^2}\right)\right\} \tag{3.68a}$$

and

$$\tau_{xy} = -(1-2\nu)\frac{b}{ae^2}\left\{\frac{y}{ae}\tanh^{-1}\left(\frac{ex}{a}\right) - \frac{x}{ae}\tan^{-1}\left(\frac{aey}{b^2}\right)\right\} \tag{3.68b}$$

Fessler & Ollerton (1957) determined the shear stresses τ_{zx} and τ_{yz} in the planes of symmetry $y = 0$, and $x = 0$ respectively, with the result

$$\frac{\tau_{zx}}{p_0} = -\frac{b}{a}\frac{x}{a}\left(\frac{z}{a}\right)^2 \frac{\left\{\left(1+\frac{\lambda_1}{a^2}\right)\frac{\lambda_1}{a^2}\right\}^{-3/2}\left(\frac{b^2}{a^2}+\frac{\lambda_1}{a^2}\right)^{-1/2}}{\left(\frac{ax}{a^2+\lambda_1}\right)^2 + \left(\frac{az}{\lambda_1}\right)^2} \tag{3.69a}$$

and

$$\frac{\tau_{yz}}{p_0} = -\frac{a}{b}\frac{y}{b}\left(\frac{z}{b}\right)^2 \frac{\left\{\left(1-\frac{\lambda_1}{b^2}\right)\frac{\lambda_1}{b^2}\right\}^{-3/2}\left(\frac{a^2}{b^2}+\frac{\lambda_1}{b^2}\right)^{-1/2}}{\left(\frac{by}{b^2+\lambda_1}\right)^2 + \left(\frac{bz}{\lambda_1}\right)^2} \tag{3.69b}$$

where λ_1 is the positive root of equation (3.53). Expressions from which the stress components at a general point can be computed have been obtained by Sackfield & Hills (1983*a*).

(c) General pressure ($n = m - \frac{1}{2}$)

In this case it follows from equation (3.52) that the surface displacements within the ellipse are given by

$$\bar{u}_z(x,y) = \frac{1-\nu^2}{\pi E}[\psi]_{z=0} = \frac{1-\nu^2}{E}\frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})}p_0ab \times \int_0^\infty \left(1 - \frac{x^2}{a^2+w} - \frac{y^2}{b^2+w}\right)^m \frac{dw}{\{(a^2+w)(b^2+w)w\}^{1/2}} \tag{3.70}$$

By expanding the bracket under the integral sign it is apparent that the expres-

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sions for the displacement will take the form

$$\bar{u}_z = C_0 + \sum_{l=1}^{l=m} C_l x^{2l} y^{2(m-l)} \tag{3.71}$$

Alternatively if the displacements within the ellipse are specified by a polynomial of the form (3.71), which would be the case if the half-space were being indented by a rigid frictionless punch, then the pressure distribution on the elliptical face of the punch would take the form

$$p(x, y) = p_0(ab)^{-m} \sum_{l=1}^{l=m} C_l' x^{2l} y^{2(m-l)} \{1 - (x/a)^2 - (y/b)^2\}^{-1/2} \tag{3.72}$$

General expressions for finding the relation between the coefficient C_l and C_l' are given by Shail (1978) and Gladwell (1980).

3.6 Concentrated tangential force

In this and the next section we shall investigate the displacements and stresses due to a tangential traction $q_x(\xi, \eta)$ acting over the loaded area S. The tangential traction parallel to the y-axis q_y and the normal pressure p are both taken to be zero. Thus in equations (3.2) to (3.7) we put

$$G_1 = H_1 = G = H = 0$$

thus

$$\psi_1 = \frac{\partial F_1}{\partial x}, \quad \psi = \frac{\partial^2 F_1}{\partial x \partial z}$$

whence

$$u_x = \frac{1}{4\pi G} \left\{ 2 \frac{\partial^2 F_1}{\partial z^2} + 2\nu \frac{\partial^2 F_1}{\partial x^2} - z \frac{\partial^3 F_1}{\partial x^2 \partial z} \right\} \tag{3.73a}$$

$$u_y = \frac{1}{4\pi G} \left\{ 2\nu \frac{\partial^2 F_1}{\partial x \partial y} - z \frac{\partial^3 F_1}{\partial x \partial y \partial z} \right\} \tag{3.73b}$$

$$u_z = \frac{1}{4\pi G} \left\{ (1 - 2\nu) \frac{\partial^2 F_1}{\partial x \partial z} - z \frac{\partial^3 F_1}{\partial x \partial z^2} \right\} \tag{3.73c}$$

where

$$F_1 = \int_S \int q_x(\xi, \eta) \{ z \ln(\rho + z) - \rho \} \, d\xi \, d\eta$$

and

$$\rho^2 = (\xi - x)^2 + (\eta - y)^2 - z^2$$

When the appropriate derivatives are substituted in equations (3.73), we get

$$u_x = \frac{1}{4\pi G}\int_S\int q_x(\xi,\eta)$$

$$\times\left\{\frac{1}{\rho}+\frac{1-2\nu}{\rho+z}+\frac{(\xi-x)^2}{\rho^3}-\frac{(1-2\nu)(\xi-x)^2}{\rho(\rho+z)^2}\right\}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{3.74a}$$

$$u_y = \frac{1}{4\pi G}\int_S\int q_x(\xi,\eta)$$

$$\times\left\{\frac{(\xi-x)(\eta-y)}{\rho^3}-(1-2\nu)\frac{(\xi-x)(\eta-y)}{\rho(\rho+z)^2}\right\}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{3.74b}$$

$$u_z = -\frac{1}{4\pi G}\int_S\int q_x(\xi,\eta)\left\{\frac{(\xi-x)z}{\rho^3}+(1-2\nu)\frac{(\xi-x)}{\rho(\rho+z)}\right\}\mathrm{d}\xi\,\mathrm{d}\eta \tag{3.74c}$$

The tangential traction is now taken to be concentrated on a vanishingly small area at the origin, so that $\int_S\int q_x(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$ reduces to a concentrated force Q_x acting at the origin ($\xi=\eta=0$) in a direction parallel to the x-axis. Equations (3.74) for the displacements throughout the solid reduce to

$$u_x = \frac{Q_x}{4\pi G}\left[\frac{1}{\rho}+\frac{x^2}{\rho^3}+(1-2\nu)\left\{\frac{1}{\rho+z}-\frac{x^2}{\rho(\rho+z)^2}\right\}\right] \tag{3.75a}$$

$$u_y = \frac{Q_x}{4\pi G}\left[\frac{xy}{\rho^3}-(1-2\nu)\frac{xy}{\rho(\rho+z)^2}\right] \tag{3.75b}$$

$$u_z = \frac{Q_x}{4\pi G}\left[\frac{xz}{\rho^3}+(1-2\nu)\frac{x}{\rho(\rho+z)}\right] \tag{3.75c}$$

where now $\rho^2 = x^2+y^2+z^2$.

By differentiating equations (3.75) the strain components and hence the stress components are found, with the results

$$\frac{2\pi\sigma_x}{Q_x} = -\frac{3x^3}{\rho^5}+(1-2\nu)$$

$$\times\left\{\frac{x}{\rho^3}-\frac{3x}{\rho(\rho+z)^2}+\frac{x^3}{\rho^3(\rho+z)^2}+\frac{2x^3}{\rho^2(\rho+z)^3}\right\} \tag{3.76a}$$

$$\frac{2\pi\sigma_y}{Q_x} = -\frac{3xy^2}{\rho^5}+(1-2\nu)$$

$$\times\left\{\frac{x}{\rho^3}-\frac{x}{\rho(\rho+z)^2}+\frac{xy^2}{\rho^3(\rho+z)^2}+\frac{2xy^2}{\rho^2(\rho+z)^3}\right\} \tag{3.76b}$$

$$\frac{2\pi\sigma_z}{Q_x} = -\frac{3xz^2}{\rho^5} \tag{3.76c}$$

$$\frac{2\pi\tau_{xy}}{Q_x} = -\frac{3x^2y}{\rho^5} + (1-2\nu) \times \left\{ -\frac{y}{\rho(\rho+z)^2} + \frac{x^2y}{\rho^3(\rho+z)^2} + \frac{2x^2y}{\rho^2(\rho+z)^3} \right\} \tag{3.76d}$$

$$\frac{2\pi\tau_{yz}}{Q_x} = -\frac{3xyz}{\rho^5} \tag{3.76e}$$

$$\frac{2\pi\tau_{zx}}{Q_x} = -\frac{3x^2z}{\rho^5} \tag{3.76f}$$

$$\frac{2\pi}{Q_x}(\sigma_x + \sigma_y + \sigma_z) = -2(1+\nu)x/\rho^3 \tag{3.76g}$$

The stresses and displacement on the surface, excluding the origin, are obtained by putting $z = 0$, and $\rho = r$. These expressions can be used to build up the stress components within the solid due to any known distribution of tangential traction by superposition.

3.7 Uni-directional tangential tractions on elliptical and circular regions

The influence of tangential traction has not been studied so extensively as that of normal pressure, but tractions of the form

$$q_x(x, y) = q_0\{1 - (x/a)^2 - (y/b)^2\}^n \tag{3.77}$$

which act parallel to the x-axis on an elliptical area bounded by the curve

$$(x/a)^2 + (y/b)^2 - 1 = 0 \tag{3.78}$$

are important in the theory of contact stresses. These tractions are comparable with the pressure distributions considered in §5 (see equation (3.50)). To examine the extent to which the problems are analogous we should compare equations (3.73) for the displacements due to tangential traction with equations (3.11) for the displacements due to normal pressure. Remembering that $\psi_1 = \partial H_1/\partial z$ and $\psi = \partial^2 H_1/\partial z^2$, it is immediately apparent that there is no complete analogy between the two sets of equations: the displacements due to a tangential traction cannot be written down directly from the known displacements due to a similar distribution of normal pressure. However a very restricted analogy does exist. In the case where Poisson's ratio is zero, the surface displacements due to a tangential traction may be written:

$$\bar{u}_x = \frac{1}{2\pi G}\frac{\partial^2 F_1}{\partial z^2}, \quad \bar{u}_y = 0$$
$$\bar{u}_z = \frac{1}{4\pi G}\frac{\partial^2 F_1}{\partial x \partial z} \tag{3.79}$$

whilst the corresponding displacements due to normal pressure are

$$\bar{u}_x = \bar{u}_y = -\frac{1}{4\pi G}\frac{\partial^2 H_1}{\partial x \partial z}$$
$$\bar{u}_z = \frac{1}{2\pi G}\frac{\partial^2 H_1}{\partial z^2} \tag{3.80}$$

We recall from the definitions given in (3.2) that F_1 and H_1 are analogous so that, under the action of identical distributions of tangential and normal traction q_x and p,

$$(\bar{u}_x)_q = (\bar{u}_z)_p$$

and (3.81)

$$(\bar{u}_z)_q = -(\bar{u}_x)_p$$

provided that $\nu = 0$. It will be remembered that this analogy exists for two-dimensional loading of a half-space whatever the value of Poisson's ratio (cf. equation (2.30)).

For non-zero values of Poisson's ratio it is not possible to use equation (3.52) from potential theory to find the function $\partial^2 F_1/\partial z^2$ corresponding to ψ, and we must proceed instead from equations (3.74).

(a) Circular region, $n = -\frac{1}{2}$

We will consider first the distribution of traction:

$$q_x(x, y) = q_0(1 - r^2/a^2)^{-1/2} \tag{3.82}$$

acting parallel to Ox on the circular region of radius a shown in Fig. 3.6. A pressure distribution of this form (equation (3.34)) produces a constant normal displacement of the surface within the circle. By the analogy we have just been discussing, for $\nu = 0$, the tangential traction given by (3.82) would produce a uniform tangential displacement of the surface $\bar{u}_x$ in the direction of the traction. We shall now proceed to show that the given traction still results in a uniform tangential displacement for non-zero values of ν.

Restricting the discussion to surface displacements within the loaded circle ($r \leqslant a$) equations (3.74) reduce to

$$\bar{u}_x = \frac{1}{2\pi G}\int_S\int q_x(\xi, \eta)\left\{\frac{1-\nu}{s} + \nu\frac{(\xi - x)^2}{s^3}\right\} \mathrm{d}\xi\, \mathrm{d}\eta \tag{3.83a}$$

$$\bar{u}_y = \frac{\nu}{2\pi G} \int_S \int q_x(\xi,\eta) \frac{(\xi - x)(\eta - y)}{s^3} \, d\xi \, d\eta \tag{3.83b}$$

$$\bar{u}_z = -\frac{1-2\nu}{4\pi G} \int_S \int q_x(\xi,\eta) \frac{\xi - x}{s^2} \, d\xi \, d\eta \tag{3.83c}$$

where $s^2 = (\xi - x)^2 + (\eta - y)^2$.

These expressions for the surface displacements could also have been derived by superposition, using equations (3.75) for the displacements at a general point $B(x, y)$ due to a concentrated tangential force $Q_x = q_x \, d\xi \, d\eta$ acting at $C(\xi, \eta)$.

In order to perform the surface integration we change the coordinates from (ξ, η) to (s, ϕ) as shown, where

$$\xi^2 + \eta^2 = (x + s \cos \phi)^2 + (y + s \sin \phi)^2$$

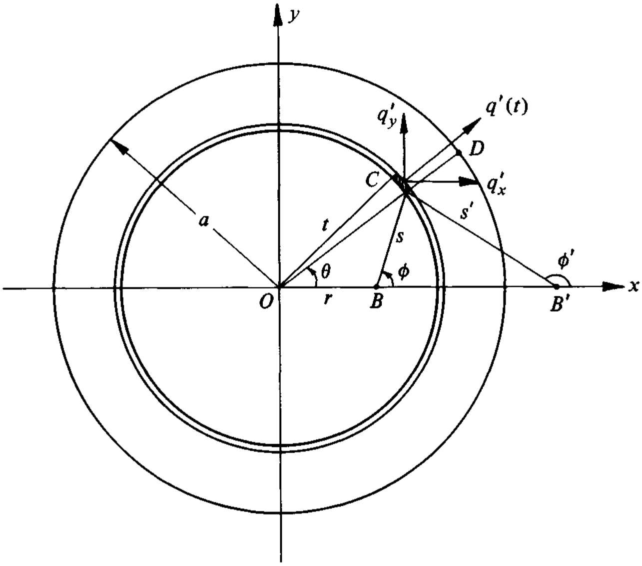
Putting $\alpha^2 = a^2 - x^2 - y^2$ and $\beta = x \cos \phi + y \sin \phi$

$$q_x(s, \phi) = q_0 a(\alpha^2 - 2\beta s - s^2)^{-1/2} \tag{3.84}$$

Equations (3.83) then become

$$\bar{u}_x = \frac{1}{2\pi G} \int_0^{2\pi} \int_0^{s_1} q_x(s, \phi)\{(1-\nu) + \nu \cos^2 \phi\} \, d\phi \, ds \tag{3.85a}$$

Fig. 3.6



$$\bar{u}_y = \frac{\nu}{2\pi G} \int_0^{2\pi} \int_0^a q_x(s, \phi) \sin\phi \cos\phi \, d\phi \, ds \tag{3.85b}$$

$$\bar{u}_z = -\frac{1-2\nu}{4\pi G} \int_0^{2\pi} \int_0^{s_1} q_x(s, \phi) \cos\phi \, d\phi \, ds \tag{3.85c}$$

The limit s_1 is given by point D lying on the boundary of the circle, for which

$$s_1 = -\beta + (\alpha^2 + \beta^2)^{1/2}$$

Integrating first with respect to s, we have

$$\int_0^{s_1} (\alpha^2 - 2\beta s - s^2)^{-1/2} \, ds = \pi/2 - \tan^{-1}(\beta/\alpha)$$

When performing the integration with respect to ϕ between the limits 0 and 2π, we note that $\beta(\phi) = -\beta(\phi + \pi)$, so that for $(r \leqslant a)$

$$\bar{u}_x = \frac{q_0 a}{4G} \int_0^{2\pi} \{(1-\nu) + \nu \cos^2\phi\} \, d\phi$$

$$= \frac{\pi(2-\nu)}{4G} q_0 a = \text{constant} \tag{3.86a}$$

$$\bar{u}_y = 0 \tag{3.86b}$$

$$\bar{u}_z = \frac{(1-2\nu)q_0 a}{4\pi G} \int_0^{2\pi} \cos\phi \tan^{-1}(\beta/\alpha) \, d\phi$$

$$= -\frac{(1-2\nu)q_0 a}{2G} \left\{ \frac{a}{r} - \frac{(a^2 - r^2)^{1/2}}{r} \right\} \tag{3.86c}$$

The normal traction given by (3.34) which produces a constant normal displacement of the surface within the circle $(r \leqslant a)$ was interpreted physically as the pressure which would be exerted on the flat face of a rigid frictionless cylindrical punch pressed into contact with the surface of an elastic half-space. By analogy, therefore, we are tempted to ask whether the tangential traction we have just been considering (3.82) represents the shear stress in the adhesive when a rigid cylindrical punch, whose flat face adheres to the surface of an elastic half-space, is given a tangential displacement parallel to the x-axis. However difficulty arises due to the non-zero normal displacements given by (3.86c), so that the punch face would not fit flush with the surface of the half-space without introducing additional tractions, both normal and tangential, at the interface.

The tractions acting on the surface of a rigid cylindrical punch which adheres to the surface of a half-space and is given a displacement in the *normal* direction have been found by Mossakovski (1954) and Spence (1968). The pressure distribution on the face of the punch with adhesion is not very different from that when the punch is frictionless. This was also shown to be the case for a two-dimensional punch (§2.8). The problem of finding the tractions for an adhesive cylindrical punch which is given a tangential displacement has been solved by Ufliand (1967). By analogy with the case of normal displacements, the shear traction on the punch face was found to be close to that given by (3.82). This approximation amounts to neglecting the mismatch of normal displacements with the flat face of the punch.

Obtaining expressions in closed form for the stress components within the solid is extremely involved. The variation of τ_{xz} along the z-axis due to the traction (3.82) is given by

$$\tau_{xz} = -p_0(1 - z^2/a^2)^{-2} \tag{3.87}$$

(b) Elliptical region, $n = -\frac{1}{2}$

When a tangential traction

$$q_x = q_0 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-1/2} \tag{3.88}$$

acts parallel to Ox on an elliptical region of semi-axes a and b Mindlin (1949) has shown that the tangential displacement of the surface is again constant and in the Ox direction. Within the elliptical region

$$\bar{u}_x = \left\{\begin{array}{ll} \dfrac{q_0 b}{G}\left[\mathbf{K}(e) - \dfrac{\nu}{e^2}\{(1-e^2)\mathbf{K}(e) - \mathbf{E}(e)\}\right], & a > b \\ \dfrac{q_0 a}{G}\left[\mathbf{K}(e) - \dfrac{\nu}{e^2}\{\mathbf{K}(e) - \mathbf{E}(e)\}\right], & a < b \end{array}\right\} \tag{3.89a}$$

$$\bar{u}_y = 0 \tag{3.89b}$$

(c) Circular region, $n = \frac{1}{2}$

The distribution of traction

$$q_x = q_0(1 - r^2/a^2)^{1/2} \tag{3.90}$$

acting on a circular region of radius a may be treated in the same way, by substituting (3.90) into equations (3.85) to find the surface displacements within the circle ($r \leqslant a$). The integration with respect to s is the same as in (3.40). Using that result and omitting the terms which do not contribute to the integration with respect to ϕ, we find

$$\bar{u}_x = \frac{\pi q_0}{32Ga}\{4(2-\nu)a^2 - (4-3\nu)x^2 - (4-\nu)y^2\} \tag{3.91a}$$

Similarly

$$\bar{u}_y = \frac{\pi q_0}{32Ga} 2\nu xy \tag{3.91b}$$

In this case $\bar{u}_x$ is not constant throughout the loaded circle and $\bar{u}_y$ does not vanish. Again we note that the normal displacements are not zero although we shall not evaluate them explicitly.

The tangential surface displacements outside the loaded circle ($r > a$) due to the traction given by (3.90) have been found by Illingworth (see Johnson, 1955) with the results:

$$\bar{u}_x = \frac{q_0}{8Ga} [(2-\nu)\{(2a^2 - r^2)\sin^{-1}(a/r) + ar(1 - a^2/r^2)^{1/2}\}$$
$$+ \tfrac{1}{2}\nu\{r^2 \sin^{-1}(a/r) + (2a^2 - r^2)(1 - a^2/r^2)^{1/2}(a/r)\}(x^2 - y^2)] \tag{3.92a}$$

$$\bar{u}_y = \frac{q_0 \nu}{8Ga}\{r^2 \sin^{-1}(a/r) + (2a^2 - r^2)(1 - a^2/r^2)^{1/2}(a/r)\}xy \tag{3.92b}$$

A comprehensive investigation of the stresses within the solid due to the traction (3.90) has been carried out by Hamilton & Goodman (1966) and Hamilton (1983). Their results are discussed in §7.1(*b*) in relation to sliding contact.

(d) Elliptical region, $n = \frac{1}{2}$

The corresponding traction

$$q_x = q_0 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2} \tag{3.93}$$

acting on an elliptical region of semi-axes a and b gives rise to tangential diplacements within the ellipse given by

$$\bar{u}_x = \frac{q_0 a}{2G}(C - Ax^2 - By^2) \tag{3.94a}$$

$$\bar{u}_y = \frac{q_0 a}{2G} Dxy \tag{3.94b}$$

where A, B, C and D are functions of shape and size of the ellipse. They have been expressed in terms of tabulated elliptic integrals by Vermeulen & Johnson (1964). Stresses within the half-space have been found by Bryant & Keer (1982) and Sackfield & Hills (1983*b*).

(e) Elliptical region, $n = m - \frac{1}{2}$

Finally we consider a general distribution of traction of the form

$$q_x(x,y) = q_0\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{m-1/2} \tag{3.95}$$

acting on an elliptical region of the surface. We saw in §5 that a pressure distribution of this form gave rise to normal displacements which varied throughout the elliptical region as a polynomial in x and y of order $2m$. The two examples of tangential traction which we have investigated, i.e. $m = 0$ and $m = 1$, have resulted in surface displacements which, in the first case are constant - a polynomial of zero order - and in the second case (equation (3.94)) vary through the elliptical region as a polynomial of second order in x and y. Kalker (1967*a*) has proved that the general tangential traction (3.95) does give rise to tangential surface displacements of order $2m$ in x and y, and has shown how the coefficients of the polynomial can be computed. Thus if the displacements within the ellipse are specified and can be approximated by a finite number of terms in a polynomial series, then the resultant tangential traction which would give rise to those specified displacements can be found as the sum of an equal number of terms having the form of (3.95).

3.8 Axi-symmetrical tractions

An important special case of 'point' loading arises when the half-space is loaded over a circular region by surface tractions, both normal and tangential, which are rotationally symmetrical about the z-axis. The magnitudes of the tractions are thereby independent of θ and the direction of the tangential traction is radial at all points. The system of stresses induced in solids of revolution by an axially symmetrical distribution of surface tractions is discussed by Timoshenko & Goodier (1951, Chapter 13). It follows from the symmetry that the components of shearing stress $\tau_{r\theta}$ and $\tau_{\theta z}$ vanish, whilst the remaining stress components are independent of θ.

A number of examples of axi-symmetric distributions of normal traction were considered in §3.4. In this section we shall approach the problem somewhat differently and include radial tangential tractions.

Referring to Fig. 3.6, we start by considering a normal line load of intensity p' per unit length acting on a ring of radius t. The normal and tangential displacements at a surface point $B(r, 0)$ are found from equations (3.18*a* and *c*) for a concentrated normal force, with the result:

$$\bar{u}_z'(r) = \frac{1-\nu^2}{\pi E}\, 2\int_0^{\pi} \frac{p't\,\mathrm{d}\theta}{s}$$

$$= \frac{1-\nu^2}{\pi E} 4p't \int_0^\pi \{(t+r)^2 - 4tr\sin^2(\theta/2)\}^{-1/2} \, d(\theta/2)$$

$$= \frac{1-\nu^2}{\pi E} \frac{4p't}{t+r} \mathbf{K}(k) \tag{3.96a}$$

where $k^2 = 4tr/(t+r)^2$.

$$\bar{u}_r'(r) = \frac{(1-2\nu)(1+\nu)}{2\pi E} 2p' \int_0^\pi \frac{t\cos\phi \, d\theta}{s}$$

$$= \begin{cases} -\dfrac{(1-2\nu)(1+\nu)}{E} p't/r, & r > t \quad (3.96b) \\ 0, & r < t \quad (3.96c) \end{cases}$$

Now consider a tangential line load of intensity q' per unit length acting radially at radius t. At any point such as C, q' is resolved into $q_x' = q' \cos\theta$ and $q_y' = q' \sin\theta$. The displacements at a surface point B due to concentrated forces $q_x' t \, d\theta$ and $q_y' t \, d\theta$ acting at C are found to be from equations (3.75). Integrating for a complete ring gives

$$\bar{u}_r'(r) = \frac{2(1-\nu^2)}{\pi E} q'(t) t \int_0^\pi \frac{\cos\theta}{s} \, d\theta$$

$$= \frac{4(1-\nu^2)}{\pi E} q'(t) \frac{t}{t+r} \left\{ \left(\frac{2}{k^2} - 1 \right) \mathbf{K}(k) - \frac{2}{k^2} \mathbf{E}(k) \right\} \tag{3.97a}$$

and

$$\bar{u}_z'(r) = \begin{cases} -\dfrac{(1-2\nu)(1+\nu)}{E} q'(t), & r \leqslant t \quad (3.97b) \\ 0, & r > t \quad (3.97c) \end{cases}$$

The surface displacement due to a distributed pressure $p(t)$ and traction $q(t)$ can be built up from equations (3.96) and (3.97) with the result

$$\bar{u}_z = \frac{4(1-\nu^2)}{\pi E} \int_0^a \frac{t}{t+r} p(t) \mathbf{K}(k) \, dt - \frac{(1-2\nu)(1+\nu)}{\pi E} \int_r^a q(t) \, dt \tag{3.98a}$$

$$\bar{u}_r = \frac{4(1-\nu^2)}{\pi E} \int_0^a \frac{t}{t+r} q(t) \left\{ \left(\frac{2}{k^2} - 1 \right) \mathbf{K}(k) - \frac{2}{k^2} \mathbf{E}(k) \right\} dt$$

$$- \frac{(1-2\nu)(1+\nu)}{Er} \int_0^r tp(t) \, dt \tag{3.98b}$$

When $r > a$ the second term in equation (3.98a) should be ignored and the

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upper limit in the second term in (3.98*b*) becomes *a*. These expressions enable the surface displacements to be calculated, numerically at least, for any axisymmetric distributions of traction. They are not convenient however when the surface displacements are specified and the surface tractions are unknown. Integral transform methods have been developed for this purpose. This mathematical technique is beyond the scope of this book and the interested reader is referred to the books by Sneddon (1951) and Gladwell (1980) and the work of Spence (1968). However we shall quote the following useful results.

Noble & Spence (1971) introduce the functions

$$L(\lambda) = \frac{1}{2G}\int_{\lambda}^{1} \frac{\rho p(\rho)\,\mathrm{d}\rho}{(\rho^2-\lambda^2)^{1/2}}, \quad M(\lambda) = \frac{\lambda}{2G}\int_{\lambda}^{1} \frac{q(\rho)\,\mathrm{d}\rho}{(\rho^2-\lambda^2)^{1/2}} \tag{3.99}$$

which can be evaluated if the pressure $p(\rho)$ and traction $q(\rho)$ are known within the loaded circle $\rho(=r/a) \leqslant 1$. Alternatively, if the normal surface displacement $\bar{u}_z(\rho)$ is known within the circle due to $p(\rho)$ *acting alone*, then

$$L(\lambda) = \frac{1}{2(1-\nu)a}\frac{\mathrm{d}}{\mathrm{d}\lambda}\int_0^{\lambda} \frac{\rho\bar{u}_z(\rho)\,\mathrm{d}\rho}{(\lambda^2-\rho^2)^{1/2}} \tag{3.100}$$

or if the tangential displacement $\bar{u}_r(\rho)$ due to $q(\rho)$ *acting alone* is known, then

$$M(\lambda) = \frac{1}{2(1-\nu)a}\frac{\mathrm{d}}{\mathrm{d}\lambda}\int_0^{\lambda} \frac{\bar{u}_r(\rho)\,\mathrm{d}\rho}{(\lambda^2-\rho^2)^{1/2}} \tag{3.101}$$

The displacements and stresses throughout the surface of the half-space may now be expressed in terms of $L(\lambda)$ and $M(\lambda)$, thus

$$\frac{\bar{u}_r(\rho)}{a} = \begin{cases} \dfrac{2(1-2\nu)}{\pi\rho}\left\{\displaystyle\int_{\rho}^{1} \frac{\lambda L(\lambda)\,\mathrm{d}\lambda}{(\lambda^2-\rho^2)^{1/2}} - \int_{\rho}^{1} L(\lambda)\,\mathrm{d}\lambda\right\} \\ \quad + \dfrac{4(1-\nu)}{\pi\rho}\displaystyle\int_0^{\rho} \frac{\lambda M(\lambda)\,\mathrm{d}\lambda}{(\rho^2-\lambda^2)^{1/2}}, \quad \rho \leqslant 1 \\ \dfrac{2(1-2\nu)}{\pi\rho}\displaystyle\int_0^{1} L(\lambda)\,\mathrm{d}\lambda + \dfrac{4(1-\nu)}{\pi\rho}\int_0^{1} \frac{\lambda M(\lambda)\,\mathrm{d}\lambda}{(\rho^2-\lambda^2)^{1/2}}, \\ \qquad \rho > 1 \end{cases} \tag{3.102a}$$

$$\frac{\bar{u}_z(\rho)}{a} = \begin{cases} \dfrac{4(1-\nu)}{\pi}\displaystyle\int_0^{\rho} \frac{L(\lambda)\,\mathrm{d}\lambda}{(\rho^2-\lambda^2)^{1/2}} - \dfrac{2(1-2\nu)}{\pi}\int_{\rho}^{1} \frac{M(\lambda)\,\mathrm{d}\lambda}{(\lambda^2-\rho^2)^{1/2}}, \\ \qquad \rho \leqslant 1 \\ \dfrac{4(1-\nu)}{\pi}\displaystyle\int_0^{\rho} \frac{L(\lambda)\,\mathrm{d}\lambda}{(\rho^2-\lambda^2)^{1/2}}, \quad \rho > 1 \end{cases} \tag{3.102b}$$

$$\frac{\tau_{rz}(\rho)}{2G}=\begin{cases}-\dfrac{q(\rho)}{2G}=\dfrac{2}{\pi\rho}\dfrac{\mathrm{d}}{\mathrm{d}\rho}\displaystyle\int_{\rho}^{1}\frac{M(\lambda)\,\mathrm{d}\lambda}{(\lambda^2-\rho^2)^{1/2}}, & \rho\leqslant 1\\ 0, & \rho>1\end{cases}\tag{3.103a}$$

$$\frac{\sigma_z(\rho)}{2G}=\begin{cases}-\dfrac{p(\rho)}{2G}=\dfrac{2}{\pi\rho}\dfrac{\mathrm{d}}{\mathrm{d}\rho}\displaystyle\int_{\rho}^{1}\frac{\lambda L(\lambda)\,\mathrm{d}\lambda}{(\lambda^2-\rho^2)^{1/2}}, & \rho\leqslant 1\\ 0, & \rho>1\end{cases}\tag{3.103b}$$

$$\frac{\sigma_r(\rho)}{2G}=\begin{cases}-\dfrac{p(\rho)}{2G}-\dfrac{2(1-2\nu)}{\pi\rho^2}\left\{\displaystyle\int_{\rho}^{1}\frac{\lambda L(\lambda)\,\mathrm{d}\lambda}{(\lambda^2-\rho^2)^{1/2}}-\int_0^1 L(\lambda)\,\mathrm{d}\lambda\right\}\\ +\dfrac{4}{\pi\rho}\left(\dfrac{\mathrm{d}}{\mathrm{d}\rho}-\dfrac{1-\nu}{\rho}\right)\displaystyle\int_0^{\rho}\frac{\lambda M(\lambda)\,\mathrm{d}\lambda}{(\rho^2-\lambda^2)^{1/2}}, & \rho\leqslant 1\\ \dfrac{2(1-2\nu)}{\pi\rho^2}\displaystyle\int_0^1 L(\lambda)\,\mathrm{d}\lambda+\dfrac{4}{\pi\rho}\left(\dfrac{\mathrm{d}}{\mathrm{d}\rho}-\dfrac{1-\nu}{\rho}\right)\displaystyle\int_0^1\frac{\lambda M(\lambda)\,\mathrm{d}\lambda}{(\rho^2-\lambda^2)^{1/2}},\\ & \rho>1\end{cases}\tag{3.103c}$$

$$\frac{\sigma_\theta(\rho)}{2G}=\begin{cases}-\dfrac{2\nu}{2G}p(\rho)+\dfrac{2(1-2\nu)}{\pi\rho^2}\left\{\displaystyle\int_{\rho}^{1}\frac{\lambda L(\lambda)\,\mathrm{d}\lambda}{(\lambda^2-\rho^2)^{1/2}}-\int_0^1 L(\lambda)\,\mathrm{d}\lambda\right\}\\ +\dfrac{4}{\pi\rho}\left(\nu\dfrac{\mathrm{d}}{\mathrm{d}\rho}+\dfrac{1-\nu}{\rho}\right)\displaystyle\int_0^{\rho}\frac{\lambda L(\lambda)\,\mathrm{d}\lambda}{(\rho^2-\lambda^2)^{1/2}}, & \rho\leqslant 1\\ -\dfrac{2(1-2\nu)}{\pi\rho^2}\displaystyle\int_0^1 L(\lambda)\,\mathrm{d}\lambda+\dfrac{4}{\pi\rho}\left(\nu\dfrac{\mathrm{d}}{\mathrm{d}\rho}+\dfrac{1-\nu}{\rho}\right)\\ \times\displaystyle\int_0^1\frac{\lambda M(\lambda)\,\mathrm{d}\lambda}{(\rho^2-\lambda^2)^{1/2}}, & \rho>1\end{cases}\tag{3.103d}$$

In the case where both $\bar{u}_r$ and $\bar{u}_z$ are specified within the loaded circle, equations (3.102*a* and *b*) are coupled integral equations for $L(\lambda)$ and $M(\lambda)$. They have been reduced to a single integral equation by Abramian *et al.* (1966) and Spence (1968), from which $L(\lambda), M(\lambda)$ and hence $p(\rho)$ and $q(\rho)$ can be found.

A problem of this type arises when a rigid flat-ended cylindrical punch of radius a is pressed normally in contact with an elastic half-space under conditions in which the flat face of the punch adheres to the surface. This is the axially symmetrical analogue of the two-dimensional rigid punch discussed in §2.8(*b*). The punch indents the surface with a uniform displacement δ. Thus the boundary conditions within the circle of contact, $r \leqslant a$, are

$$\bar{u}_z = \delta, \quad \bar{u}_r = 0 \tag{3.104}$$

Mossakovski (1954) and Spence (1968) solve this problem and show that the load on the punch P is related to the indentation δ by

$$P = 4Ga\delta \ln\{(3-4\nu)/(1-2\nu)\} \tag{3.105}$$

The load on the face of a *frictionless* punch is given by equations (3.36) and (3.37) which, for comparison with (3.105), may be written

$$P = 4Ga\delta/(1-\nu)$$

The 'adhesive' load is greater than the 'frictionless' load by an amount which varies from 10% when $\nu = 0$, to zero when $\nu = 0.5$.

Spence (1975) has also examined the case of partial slip. During monotonic loading the contact circle is divided into a central region of radius c which does not slip and an annulus $c \leqslant r \leqslant a$ where the surface of the half-space slips radially inwards under the face of the punch. Turner (1979) has examined the behaviour on unloading. As the load is reduced the inner boundary of the slip zone $r = c$ shrinks in size with the slip there maintaining its *inward* direction. At the same time a thin annulus at the periphery adheres to the punch without slip until, when the load has decreased to about half its maximum value, *outward* slip begins at $r = a$ and rapidly spreads across the contact surface.

The surface displacements produced by an axi-symmetric distribution of pressure, calculated in §3 by the classical method, could equally well have been found by substituting $p(\rho)$ into equation (3.99) and then (3.102). The surface stress could also be found directly from equations (3.103).

3.9 Torsional loading

In this section we examine tangential tractions which act in a circumferential direction, that is perpendicular to the radius drawn from the origin. Such tractions induce a state of torsion in the half-space.

(a) Circular region

For the circular region shown in Fig. 3.7 we shall assume that the magnitude of the traction $q(r)$ is a function of r only. Thus

$$q_x = -q(r)\sin\theta = -q(t)\eta/t \tag{3.106a}$$

$$q_y = q(r)\cos\theta = q(t)\xi/t \tag{3.106b}$$

The expressions for the displacements u_x, u_y and u_z can be written in the form of equations (3.7), where $H = 0$ and F and G are given by

$$F = -\int_S\int \frac{q(t)}{t}\eta \ln(\rho + z)\,d\xi\,d\eta \tag{3.107a}$$

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and

$$G = \int_S \int \frac{q(t)}{t} \xi \ln(\rho + z) \, d\xi \, d\eta \tag{3.107b}$$

In this case, from the reciprocal nature of F and G with respect to the coordinates, it follows that $\partial G/\partial y = -\partial F/\partial x$, so that the expressions for the displacements on the surface reduce to

$$\bar{u}_x = \frac{1}{2\pi G} \frac{\partial F}{\partial z} = -\frac{1}{2\pi G} \int_0^{2\pi} \int_0^{s_1} \frac{q(t)}{t} \eta \, ds \, d\phi \tag{3.108a}$$

$$\bar{u}_y = \frac{1}{2\pi G} \frac{\partial G}{\partial z} = \frac{1}{2\pi G} \int_0^{2\pi} \int_0^{s_1} \frac{q(t)}{t} \xi \, ds \, d\phi \tag{3.108b}$$

$$\bar{u}_z = 0 \tag{3.108c}$$

If we consider the displacement of the point $B(x, 0)$, as shown in Fig. 3.7, then $\eta/t = \sin\theta$ and it is apparent that the surface integral in (3.108a) vanishes. So we are left with the circumferential component $\bar{u}_y$ as the only non-zero displacement, which was to be expected in a purely torsional deformation.

Now consider the traction

$$q(r) = q_0 r(a^2 - r^2)^{-1/2}, \quad r \leqslant a \tag{3.109}$$

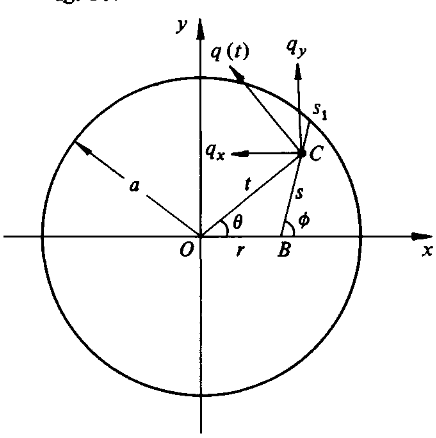
Substituting in (3.108b) for the surface displacement

$$\bar{u}_y = \frac{q_0}{2\pi G} \int_0^{2\pi} \int_0^{s_1} (a^2 - x^2 - 2xs\cos\phi - s^2)^{-1/2}(x + s\cos\phi) \, ds \, d\phi$$

The integral is of the form met previously and gives

$$\bar{u}_y = \pi q_0 x/4G$$

Fig. 3.7



In view of the circular symmetry we can write

$$\bar{u}_\theta = \pi q_0 r/4G$$
$$\bar{u}_r = \bar{u}_z = 0 \tag{3.110}$$

Thus the traction (3.109) produces a rigid rotation of the loaded circle through an angle $\beta = \pi q_0/4G$. The traction gives rise to a resultant twisting moment

$$M_z = \int_0^a q(r) 2\pi r \, dr$$
$$= 4\pi a^3 q_0/3 \tag{3.111}$$

Hence equation (3.109) gives the traction acting on the surface of a flat-ended cylindrical punch which adheres to the surface of a half-space when given a twist about its axis. Since the normal displacements $\bar{u}_z$ due to the twist are zero, the pressure distribution on the face of the punch is not influenced by the twist. This is in contrast to the behaviour of a punch which is given a uniaxial tangential displacement, where we saw (in §7) that the normal pressure and tangential tractions are not independent.

Hetenyi & McDonald (1958) have considered the distribution of traction

$$q(r) = \bar{\tau}_{z\theta} = q_0(1 - r^2/a^2)^{1/2}, \quad r \leqslant a \tag{3.112}$$

Expressions have been found for u_θ, $\tau_{r\theta}$ and $\tau_{z\theta}$ and values of the stress components $\tau_{r\theta}$ (r, z) have been tabulated. The maximum value is $0.73q_0$ on the surface at $r = a$.

(b) Elliptical region

We turn now to a loaded region of elliptical shape in order to find the distribution of traction which will again result in a rigid rotation of the loaded ellipse. In this case there is no rotational symmetry and we tentatively put

$$q_x = -q_0' y \{1 - (x/a)^2 - (y/b)^2\}^{-1/2} \tag{3.113a}$$

and

$$q_y = q_0'' x \{1 - (x/a)^2 - (y/b)^2\}^{-1/2} \tag{3.113b}$$

These expressions are substituted in equations (3.2) to obtain the potential functions F_1 and G_1, which in turn are substituted in (3.7) to obtain the tangential displacements of a general surface point (x, y). Performing the integrations in the usual way Mindlin (1949) showed that the displacements correspond to a rigid rotation of the elliptical region through a small angle β i.e. $\bar{u}_x = -\beta y$ and $\bar{u}_y = \beta x$, provided that

$$q_0' = \frac{G\beta}{2a} \frac{\mathbf{B} - 2\nu(1 - e^2)\mathbf{C}}{\mathbf{BD} - \nu\mathbf{CE}} \tag{3.114a}$$

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and

$$q_0'' = \frac{G\beta}{2a} \frac{\mathbf{D} - 2\nu\mathbf{C}}{\mathbf{BD} - \nu\mathbf{CE}} \tag{3.114b}$$

where $\mathbf{D}(e)$, $\mathbf{B}(e)$ and $\mathbf{C}(e)$ can be expressed in terms of the standard elliptic integrals $\mathbf{E}(e)$ and $\mathbf{K}(e)$ and $e = (1 - a^2/b^2)^{1/2}$ is the eccentricity of the ellipse, viz.:

$$\mathbf{D} = (\mathbf{K} - \mathbf{E})/e^2$$

$$\mathbf{B} = \{\mathbf{E} - (1 - e^2)\mathbf{K}\}/e^2$$

$$\mathbf{C} = \{(2 - e^2)\mathbf{K} - 2\mathbf{E}\}/e^4$$

The twisting moment M_z is given by

$$M_z = \tfrac{2}{3}\pi b^3 G\beta \frac{\mathbf{E} - 4\nu(1 - e^2)\mathbf{C}}{\mathbf{BD} - \nu\mathbf{CE}} \tag{3.115}$$

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