2

Line loading of an elastic half-space

2.1 The elastic half-space

Non-conforming elastic bodies in contact whose deformation is sufficiently small for the linear small strain theory of elasticity to be applicable inevitably make contact over an area whose dimensions are small compared with the radii of curvature of the undeformed surfaces. The contact stresses are highly concentrated close to the contact region and decrease rapidly in intensity with distance from the point of contact, so that the region of practical interest lies close to the contact interface. Thus, provided the dimensions of the bodies themselves are large compared with the dimensions of the contact area, the stresses in this region are not critically dependent upon the shape of the bodies distant from the contact area, nor upon the precise way in which they are supported. The stresses may be calculated to good approximation by considering each body as a semi-infinite elastic solid bounded by a plane surface: i.e. an elastic half-space. This idealisation, in which bodies of arbitrary surface profile are regarded as semi-infinite in extent and having a plane surface, is made almost universally in elastic contact stress theory. It simplifies the boundary conditions and makes available the large body of elasticity theory which has been developed for the elastic half-space.

In this chapter, therefore, we shall study the stresses and deformations in an elastic half-space loaded one-dimensionally over a narrow strip ('line loading'). In our frame of reference the boundary surface is the x–y plane and the z-axis is directed into the solid. The loaded strip lies parallel to the y-axis and has a width $(a + b)$ in the x-direction; it carries normal and tangential tractions which are a function of x only. We shall assume that a state of plane strain ($\epsilon_y = 0$) is produced in the half-space by the line loading.

For the assumption of plane strain to be justified the thickness of the solid should be large compared with the width of the loaded region, which is usually

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the case. The other extreme of plane stress ($\sigma_y = 0$) would only be realised by the edge loading of a plate whose thickness is small compared with the width of the loaded region, which is a very impractical situation.

The elastic half-space is shown in cross-section in Fig. 2.1. Surface tractions $p(x)$ and $q(x)$ act on the surface over the region from $x = -b$ to $x = a$ while the remainder of the surface is free from traction. It is required to find the stress components σ_x, σ_z and τ_{xz} at all points throughout the solid and the components u_x and u_z of the elastic displacement of any point from its undeformed position. In particular we are interested in the deformed shape of the surface $\bar{u}_z(x)$ (the over bar is used throughout to denote values of the variable at the surface $z = 0$).

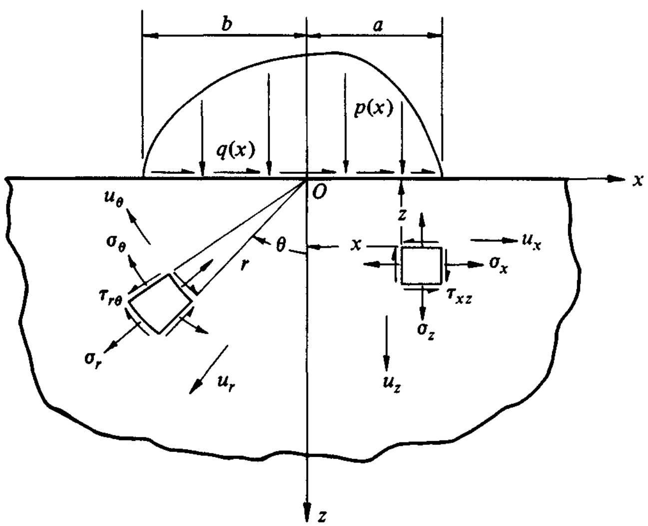
The reader is referred to Timoshenko & Goodier: *Theory of Elasticity*, McGraw-Hill, 1951, for a derivation of the elastic equations. For convenience they are summarised below. The stress components must satisfy the equilibrium equations throughout the solid:

$$\left.\begin{aligned}\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} &= 0\\ \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} &= 0\end{aligned}\right\} \tag{2.1}$$

The corresponding strains ϵ_x, ϵ_z and γ_{xz} must satisfy the compatibility condition:

$$\frac{\partial^2 \epsilon_x}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial x^2} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial z} \tag{2.2}$$

Fig. 2.1



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where the strains are related to the displacements by

$$\epsilon_x = \frac{\partial u_x}{\partial x}, \quad \epsilon_z = \frac{\partial u_z}{\partial z}, \quad \gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \tag{2.3}$$

Under conditions of plane strain,

$$\begin{aligned} &\epsilon_y = 0 \\ &\sigma_y = \nu(\sigma_x + \sigma_z) \end{aligned} \tag{2.4}$$

whereupon Hooke's law, relating the stresses to the strains, may be written:

$$\left.\begin{aligned} \epsilon_x &= \frac{1}{E}\{(1-\nu^2)\sigma_x - \nu(1+\nu)\sigma_z\} \\ \epsilon_z &= \frac{1}{E}\{(1-\nu^2)\sigma_z - \nu(1+\nu)\sigma_x\} \\ \gamma_{xz} &= \frac{1}{G}\tau_{xz} = \frac{2(1+\nu)}{E}\tau_{xz} \end{aligned}\right\} \tag{2.5}$$

If a stress function $\phi(x, z)$ is defined by:

$$\sigma_x = \frac{\partial^2 \phi}{\partial z^2}, \quad \sigma_z = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xz} = -\frac{\partial^2 \phi}{\partial x \partial z} \tag{2.6}$$

then the equations of equilibrium (2.1), compatibility (2.2) and Hooke's law (2.5) are satisfied, provided that $\phi(x, z)$ satisfies the biharmonic equation:

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right\}\left\{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2}\right\} = 0 \tag{2.7}$$

In addition the boundary conditions must be satisfied. For the half-space shown in Fig. 2.1 these are as follows. On the boundary $z = 0$, outside the loaded region, the surface is free of stress, i.e.

$$\bar{\sigma}_z = \bar{\tau}_{xz} = 0, \quad x < -b, \quad x > +a \tag{2.8}$$

Within the loaded region

$$\left.\begin{aligned} \bar{\sigma}_z &= -p(x) \\ \bar{\tau}_{xz} &= -q(x) \end{aligned}\right\}, \quad -b \leqslant x \leqslant a \tag{2.9}$$

and the tangential and normal displacements are $\bar{u}_x(x)$ and $\bar{u}_z(x)$. Finally, at a large distance from the loaded region ($x, z \to \infty$) the stresses must become vanishingly small ($\sigma_x, \sigma_z, \tau_{xz} \to 0$).

To specify a particular problem for solution two of the four quantities $p(x)$, $q(x)$, $\bar{u}_x(x)$ and $\bar{u}_z(x)$ must be prescribed within the loaded region. Various combinations arise in different contact problems. For example, if a rigid punch is pressed into contact with an elastic half-space the normal displacement $\bar{u}_z(x)$ is prescribed by the known profile of the punch. If the interface is frictionless

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the second boundary condition is that the shear traction $q(x)$ is zero. Alternatively, if the surface adheres to the punch without slip at the interface, the tangential displacement $\bar{u}_x(x)$ is specified whilst $q(x)$ remains to be found. Special boundary conditions arise if the punch is sliding on the surface of the half-space with a coefficient of friction μ. Only $\bar{u}_z(x)$ is specified but a second boundary condition is provided by the relationship:

$$q(x) = \pm\mu p(x)$$

In some circumstances it is convenient to use cylindrical polar coordinates (r, θ, y). The corresponding equations for the stress components σ_r, σ_θ and $\tau_{r\theta}$, strain components ϵ_r, ϵ_θ and $\gamma_{r\theta}$ and radial and circumferential displacements u_r and u_θ will now be summarised.

The stress function $\phi(r, \theta)$ must satisfy the biharmonic equation:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = 0 \tag{2.10}$$

where

$$\left.\begin{aligned} \sigma_r &= \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \\ \sigma_\theta &= \frac{\partial^2 \phi}{\partial r^2} \\ \tau_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) \end{aligned}\right\} \tag{2.11}$$

The strains are related to the displacements by

$$\left.\begin{aligned} \epsilon_r &= \frac{\partial u_r}{\partial r} \\ \epsilon_\theta &= \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \\ \gamma_{r\theta} &= \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \end{aligned}\right\} \tag{2.12}$$

Equations (2.4) and (2.5) for the stress–strain relationships remain the same with x and z replaced by r and θ.

We shall now proceed to discuss the solutions to particular problems relevant to elastic contact stress theory.

2.2 Concentrated normal force

In this first problem we investigate the stresses produced by a concentrated force of intensity P per unit length distributed along the y-axis and acting

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in a direction normal to the surface. This loading may be visualised as that produced by a knife-edge pressed into contact with the half-space along the y-axis (see Fig. 2.2).

This problem was first solved by Flamant (1892). It is convenient to use polar coordinates in the first instance. The solution is given by the stress function

$$\phi(r, \theta) = Ar\theta \sin\theta \tag{2.13}$$

where A is an arbitrary constant.

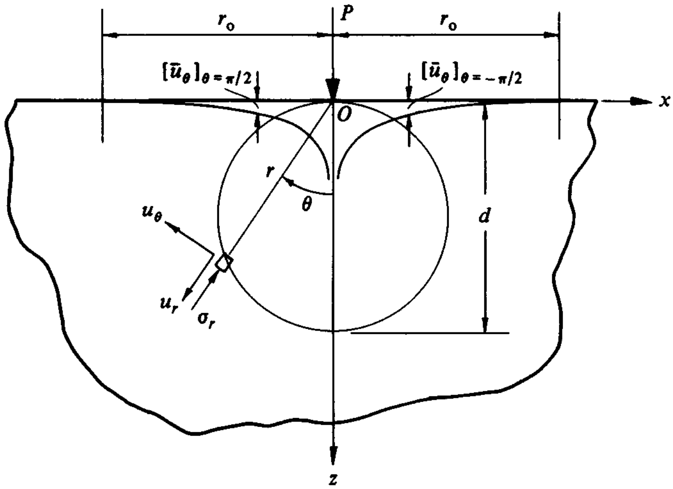
Using equations (2.11), the stress components are

$$\left.\begin{aligned} \sigma_r &= 2A \frac{\cos\theta}{r} \\ \sigma_\theta &= \tau_{r\theta} = 0 \end{aligned}\right\} \tag{2.14}$$

This system of stresses is referred to as a simple radial distribution directed towards the point of application of the force at O. At the surface $\theta = \pm\pi/2$, so that normal stress $\bar{\sigma}_\theta = 0$ except at the origin itself, and the shear stress $\bar{\tau}_{r\theta} = 0$. At a large distance from the point of application of the force ($r \to \infty$) the stresses approach zero, so that all the boundary conditions are satisfied. We note that the stresses decrease in intensity as $1/r$. The theoretically infinite stress at O is obviously a consequence of assuming that the load is concentrated along a line. The constant A is found by equating the vertical components of stress acting on a semi-circle of radius r to the applied force P. Thus

$$-P = \int_{-\pi/2}^{\pi/2} \sigma_r \cos\theta \, r \, d\theta = \int_0^{\pi/2} 4A\cos^2\theta \, d\theta = A\pi$$

Fig. 2.2



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Hence

$$\sigma_r = -\frac{2P}{\pi}\frac{\cos\theta}{r} \tag{2.15}$$

We note that σ_r has a constant magnitude $-2P/\pi d$ on a circle of diameter d which passes through O. Since $\tau_{r\theta} = 0$, σ_r and σ_θ are principal stresses. The principal shear stress τ_1 at (r, θ) has the value $(\sigma_r/2)$ and acts on planes at 45° to the radial direction. Hence contours of τ_1 are also a family of circles passing through O. This pattern is clearly demonstrated by the isochromatic fringes in a photo-elastic experiment, as shown in Fig. 4.6(a).

Changing the radial stress distribution of (2.15) into rectangular coordinates we obtain the equivalent stress components

$$\sigma_x = \sigma_r \sin^2\theta = -\frac{2P}{\pi}\frac{x^2 z}{(x^2+z^2)^2} \tag{2.16a}$$

$$\sigma_z = \sigma_r \cos^2\theta = -\frac{2P}{\pi}\frac{z^3}{(x^2+z^2)^2} \tag{2.16b}$$

$$\tau_{zx} = \sigma_r \sin\theta\cos\theta = -\frac{2P}{\pi}\frac{xz^2}{(x^2+z^2)^2} \tag{2.16c}$$

To find the distortion of the solid under the action of the load, we substitute the stresses given by (2.14) and (2.15) into Hooke's law (2.5); this yields the strains, from which we may find the displacements by using equations (2.12) with the result

$$\frac{\partial u_r}{\partial r} = \epsilon_r = -\frac{(1-\nu^2)}{E}\frac{2P}{\pi}\frac{\cos\theta}{r} \tag{2.17a}$$

$$\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = \epsilon_\theta = \frac{\nu(1+\nu)}{E}\frac{2P}{\pi}\frac{\cos\theta}{r} \tag{2.17b}$$

$$\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = \gamma_{r\theta} = \frac{\tau_{r\theta}}{G} = 0 \tag{2.17c}$$

From these three equations, in the manner demonstrated for plane stress by Timoshenko & Goodier (1951), p. 90, we obtain

$$u_r = -\frac{(1-\nu^2)}{\pi E} 2P\cos\theta\ln r - \frac{(1-2\nu)(1+\nu)}{\pi E} P\theta\sin\theta + C_1\sin\theta + C_2\cos\theta \tag{2.18a}$$

and

$$u_\theta = \frac{(1-\nu^2)}{\pi E} 2P\sin\theta\ln r + \frac{\nu(1+\nu)}{\pi E} 2P\sin\theta$$

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$$- \frac{(1-2\nu)(1+\nu)}{\pi E} P\theta \cos\theta + \frac{(1-2\nu)(1+\nu)}{\pi E} P \sin\theta$$

$$+ C_1 \cos\theta - C_2 \sin\theta + C_3 r \tag{2.18b}$$

If the solid does not tilt, so that points on the z-axis displace only along Oz, then $C_1 = C_3 = 0$. At the surface, where $\theta = \pm\pi/2$,

$$[\bar{u}_r]_{\theta=\frac{\pi}{2}} = [\bar{u}_r]_{\theta=-\frac{\pi}{2}} = -\frac{(1-2\nu)(1+\nu)P}{2E} \tag{2.19a}$$

$$[\bar{u}_\theta]_{\theta=\frac{\pi}{2}} = -[\bar{u}_\theta]_{\theta=-\frac{\pi}{2}} = \frac{(1-\nu^2)}{\pi E} 2P \ln r + C \tag{2.19b}$$

where the constant C is determined by choosing a point on the surface at a distance r_0, say (or alternatively on the z-axis below the surface), as a datum for normal displacements. Then

$$[\bar{u}_\theta]_{\theta=\frac{\pi}{2}} = -[\bar{u}_\theta]_{\theta=-\frac{\pi}{2}} = -\frac{(1-\nu^2)}{\pi E} 2P \ln(r_0/r)$$

The deformed shape of the surface is shown in Fig. 2.2. The infinite displacement at O is to be expected in view of the singularity in stress at that point. Choice of an appropriate value of r_0 presents some difficulty in view of the logarithmic variation of $\bar{u}_\theta$ with r. This is an inevitable feature of two-dimensional deformation of an elastic half-space. To surmount the difficulty it is necessary to consider the actual shape and size of the body and its means of support. This question is discussed further in §5.6.

2.3 Concentrated tangential force

A concentrated force Q per unit length of the y-axis, which acts tangentially to the surface at O as shown in Fig. 2.3, produces a radial stress field similar to that due to a normal force but rotated through 90°. If we measure θ from the line of action of the force, in this case the Ox direction, the expressions for the stresses are the same as for a normal force, viz.:

$$\left.\begin{aligned} \sigma_r &= -\frac{2Q}{\pi}\frac{\cos\theta}{r} \\ \sigma_\theta &= \tau_{r\theta} = 0 \end{aligned}\right\} \tag{2.20}$$

Contours of constant stress are now semi-circular through O, as shown in Fig. 2.3. Ahead of the force, in the quadrant of positive x, σ_r is compressive, whilst behind the force σ_r is tensile, as we might expect. The expressions for the stress

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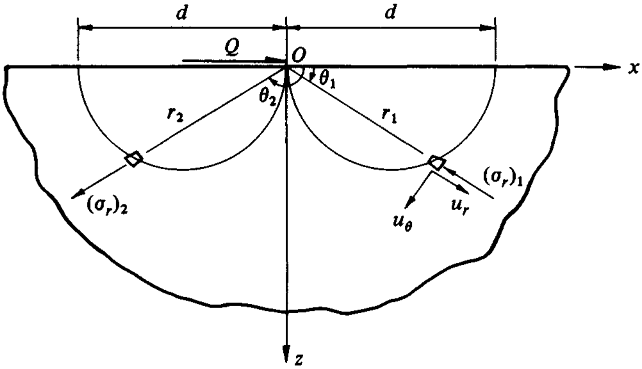
Fig. 2.3



in x–z coordinates may be obtained as before:

$$\sigma_x = -\frac{2Q}{\pi}\frac{x^3}{(x^2+z^2)^2} \tag{2.21a}$$

$$\sigma_z = -\frac{2Q}{\pi}\frac{xz^2}{(x^2+z^2)^2} \tag{2.21b}$$

$$\tau_{xz} = -\frac{2Q}{\pi}\frac{x^2 z}{(x^2+z^2)^2} \tag{2.21c}$$

With the appropriate change in the definition of θ, equations (2.18) for the displacement still apply. If there is no rigid body rotation of the solid, nor vertical displacement of points on the z-axis, the surface displacements turn out to be:

$$-[\bar{u}_r]_{\theta=\pi} = [\bar{u}_r]_{\theta=0} = -\frac{(1-\nu^2)}{\pi E}\,2Q\ln r + C \tag{2.22a}$$

$$[\bar{u}_\theta]_{\theta=\pi} = [\bar{u}_\theta]_{\theta=0} = \frac{(1-2\nu)(1+\nu)}{2E}\,Q \tag{2.22b}$$

which compare with (2.19) due to normal force. Equation (2.22b) snows that the whole surface ahead of the force ($x > 0$) is depressed by an amount proportional to Q whilst the surface behind Q ($x < 0$) rises by an equal amount. Once again the tangential displacement of the surface varies logarithmically with the distance from O and the datum chosen for this displacement determines the value of the constant C.

2.4 Distributed normal and tangential tractions

In general, a contact surface transmits tangential tractions due to friction in addition to normal pressure. An elastic half-space loaded over the

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strip ($-b < x < a$) by a normal pressure $p(x)$ and tangential traction $q(x)$ distributed in any arbitrary manner is shown in Fig. 2.4. We wish to find the stress components due to $p(x)$ and $q(x)$ at any point A in the body of the solid and the displacement of any point C on the surface of the solid.

The tractions acting on the surface at B, distance s from O, on an elemental area of width $\mathrm{d}s$ can be regarded as concentrated forces of magnitude $p\,\mathrm{d}s$ acting normal to the surface and $q\,\mathrm{d}s$ tangential to the surface. The stresses at A due to these forces are given by equations (2.16) and (2.21) in which x is replaced by $(x - s)$. Integrating over the loaded region gives the stress components at A due to the complete distribution of $p(x)$ and $q(x)$. Thus:

$$\sigma_x = -\frac{2z}{\pi}\int_{-b}^{a} \frac{p(s)(x-s)^2\,\mathrm{d}s}{\{(x-s)^2+z^2\}^2} - \frac{2}{\pi}\int_{-b}^{a} \frac{q(s)(x-s)^3\,\mathrm{d}s}{\{(x-s)^2+z^2\}^2} \qquad (2.23a)$$

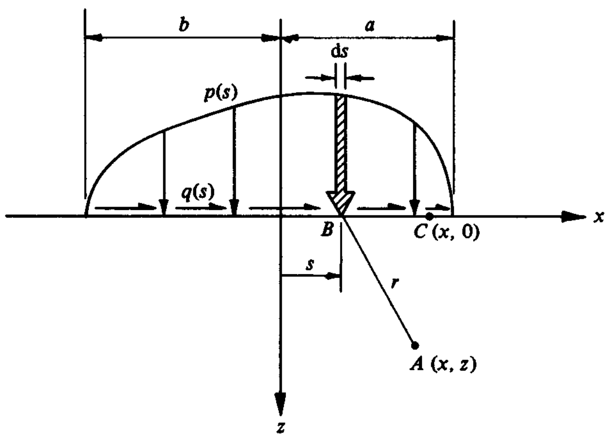
$$\sigma_z = -\frac{2z^3}{\pi}\int_{-b}^{a} \frac{p(s)\,\mathrm{d}s}{\{(x-s)^2+z^2\}^2} - \frac{2z^2}{\pi}\int_{-b}^{a} \frac{q(s)(x-s)\,\mathrm{d}s}{\{(x-s)^2+z^2\}^2} \qquad (2.23b)$$

$$\tau_{xz} = -\frac{2z^2}{\pi}\int_{-b}^{a} \frac{p(s)(x-s)\,\mathrm{d}s}{\{(x-s)^2+z^2\}^2} - \frac{2z}{\pi}\int_{-b}^{a} \frac{q(s)(x-s)^2\,\mathrm{d}s}{\{(x-s)^2+z^2\}^2} \qquad (2.23c)$$

If the distributions of $p(x)$ and $q(x)$ are known then the stresses can be evaluated although the integration in closed form may be difficult.

The elastic displacements on the surface are deduced in the same way by summation of the displacements due to concentrated forces given in equations (2.19) and (2.22). Denoting the tangential and normal displacement of point C due to the combined action of $p(x)$ and $q(x)$ by $\bar{u}_x$ and $\bar{u}_z$ respectively,

Fig. 2.4



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we find

$$\bar{u}_x = -\frac{(1-2\nu)(1+\nu)}{2E}\left\{\int_{-b}^{x} p(s)\,\mathrm{d}s - \int_{x}^{a} p(s)\,\mathrm{d}s\right\}$$

$$-\frac{2(1-\nu^2)}{\pi E}\int_{-b}^{a} q(s)\ln|x-s|\,\mathrm{d}s + C_1 \qquad (2.24a)$$

$$\bar{u}_z = -\frac{2(1-\nu^2)}{\pi E}\int_{-b}^{a} p(s)\ln|x-s|\,\mathrm{d}s$$

$$+\frac{(1-2\nu)(1+\nu)}{2E}\left\{\int_{-b}^{x} q(s)\,\mathrm{d}s - \int_{x}^{a} q(s)\,\mathrm{d}s\right\} + C_2 \qquad (2.24b)$$

The step changes in displacement at the origin which occur in equations (2.19*a*) and (2.22*b*) lead to the necessity of splitting the range of integration in the terms in curly brackets in equations (2.24). These equations take on a much neater form, and also a form which is more useful for calculation if we choose to specify the *displacement gradients* at the surface $\partial\bar{u}_x/\partial x$ and $\partial\bar{u}_z/\partial x$ rather than the absolute values of $\bar{u}_x$ and $\bar{u}_z$. The artifice also removes the ambiguity about a datum for displacements inherent in the constants C_1 and C_2. The terms in curly brackets can be differentiated with respect to the limit x, and the other integrals can be differentiated within the integral signs to give

$$\frac{\partial\bar{u}_x}{\partial x} = -\frac{(1-2\nu)(1+\nu)}{E}p(x) - \frac{2(1-\nu^2)}{\pi E}\int_{-b}^{a}\frac{q(s)}{x-s}\,\mathrm{d}s \qquad (2.25a)$$

$$\frac{\partial\bar{u}_z}{\partial x} = -\frac{2(1-\nu^2)}{\pi E}\int_{-b}^{a}\frac{p(s)}{x-s}\,\mathrm{d}s + \frac{(1-2\nu)(1+\nu)}{E}q(x) \qquad (2.25b)$$

The gradient $\partial\bar{u}_x/\partial x$ will be recognised as the tangential component of strain $\bar{\epsilon}_x$ at the surface and the gradient $\partial\bar{u}_z/\partial x$ is the actual slope of the deformed surface.

An important result follows directly from (2.25). Due to the normal pressure $p(x)$ alone ($q(x) = 0$)

$$\bar{\epsilon}_x = \frac{\partial\bar{u}_x}{\partial x} = -\frac{(1-2\nu)(1+\nu)}{E}p(x)$$

But from Hooke's law in plane strain (the first of (2.5)), at the boundary

$$\bar{\epsilon}_x = \frac{1}{E}\{(1-\nu^2)\bar{\sigma}_x - \nu(1+\nu)\bar{\sigma}_z\}$$

Equating the two expressions for $\bar{\epsilon}_x$ and remembering that $\bar{\sigma}_z = -p(x)$ gives

$$\bar{\sigma}_x = \bar{\sigma}_z = -p(x) \qquad (2.26)$$

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Thus under any distribution of surface pressure the tangential and normal direct stresses at the surface are compressive and equal. This state of affairs restricts the tendency of the surface layer to yield plastically under a normal contact pressure.

2.5 Uniform distributions of traction

(a) Normal pressure

The simplest example of a distributed traction arises when the pressure is uniform over the strip ($-a \leqslant x \leqslant a$) and the shear traction is absent. In equations (2.23) the constant pressure p can be taken outside the integral sign and $q(s)$ is everywhere zero. Performing the integrations and using the notation of Fig. 2.5, we find

$$\sigma_x = -\frac{p}{2\pi}\{2(\theta_1 - \theta_2) + (\sin 2\theta_1 - \sin 2\theta_2)\} \tag{2.27a}$$

$$\sigma_z = -\frac{p}{2\pi}\{2(\theta_1 - \theta_2) - (\sin 2\theta_1 - \sin 2\theta_2)\} \tag{2.27b}$$

$$\tau_{xz} = \frac{p}{2\pi}(\cos 2\theta_1 - \cos 2\theta_2) \tag{2.27c}$$

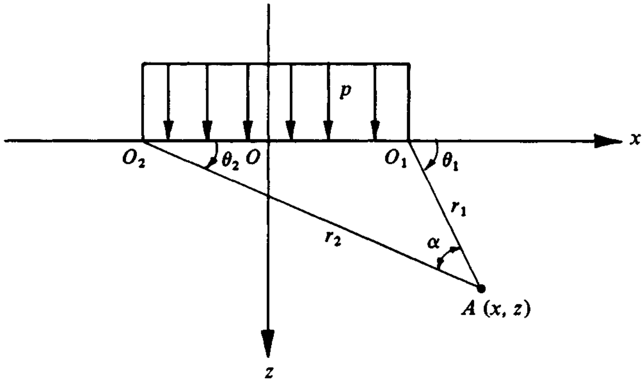
where

$$\tan \theta_{1,2} = z/(x \mp a)$$

If the angle $(\theta_1 - \theta_2)$ is denoted by α, the principal stresses shown by Mohr's circle in Fig. 2.6 are given by:

$$\sigma_{1,2} = -\frac{p}{\pi}(\alpha \mp \sin \alpha) \tag{2.28}$$

Fig. 2.5



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at an angle $(\theta_1 + \theta_2)/2$ to the surface. The principal shear stress has the value

$$\tau_1 = \frac{p}{\pi} \sin \alpha \tag{2.29}$$

Expressed in this form it is apparent that contours of constant principal stress and constant τ_1 are a family of circles passing through the points O_1 and O_2 as shown in Fig. 2.7(*a*) and by the photoelastic fringes in Fig. 4.6(*b*). The principal shear stress reaches a uniform maximum value p/π along the semi-circle $\alpha = \pi/2$. The trajectories of principal stress are a family of confocal ellipses and hyperbolae with foci O_1 and O_2 as shown in Fig. 2.7(*b*). Finally we note that the stress system we have just been discussing approaches that due to a concentrated normal force at O (§2.2) when r_1 and r_2 become large compared with a.

To find the displacements on the surface we use equation (2.25). For a point lying inside the loaded region $(-a \leqslant x \leqslant a)$

$$\frac{\partial \bar{u}_x}{\partial x} = -\frac{(1-2\nu)(1+\nu)}{E} p$$

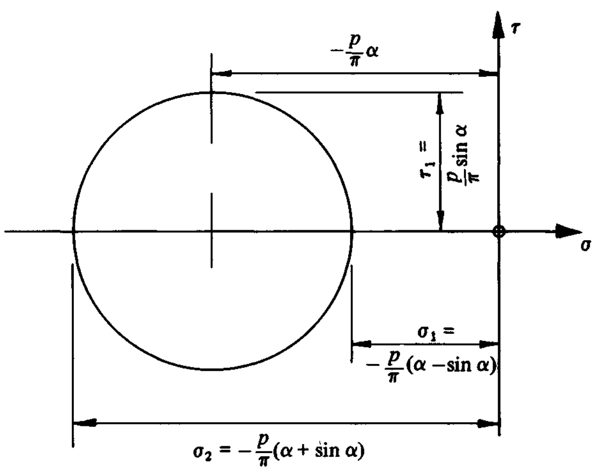
Then, assuming that the origin does not displace laterally,

$$\bar{u}_x = -\frac{(1-2\nu)(1+\nu)}{E} px \tag{2.30a}$$

Now

$$\frac{\partial \bar{u}_z}{\partial x} = -\frac{2(1-\nu^2)}{\pi E} \int_{-a}^{a} \frac{\mathrm{d}s}{x-s}$$

Fig. 2.6. Mohr's circle for stress due to loading of Fig. 2.5.



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This integral calls for comment: the integrand has a singularity at $s = x$ and changes sign. The integration must be carried out in two parts, from $s = -a$ to $x - \epsilon$ and from $s = x + \epsilon$ to a, where ϵ can be made vanishingly small. The result is then known as the Cauchy Principal Value of the integral, i.e.

$$\int_{-a}^{a} \frac{\mathrm{d}s}{x-s} = \int_{-a}^{x-\epsilon} \frac{\mathrm{d}s}{x-s} - \int_{x+\epsilon}^{a} \frac{\mathrm{d}s}{s-x}$$
$$= [\ln(x-s)]_{-a}^{x-\epsilon} - [\ln(s-x)]_{x+\epsilon}^{a}$$
$$= \ln(a+x) - \ln(a-x)$$

Fig. 2.7. Stresses due to loading of Fig. 2.5: (*a*) Contours of principal stresses σ_1, σ_2 and τ_1; (*b*) Trajectories of principal stress directions.

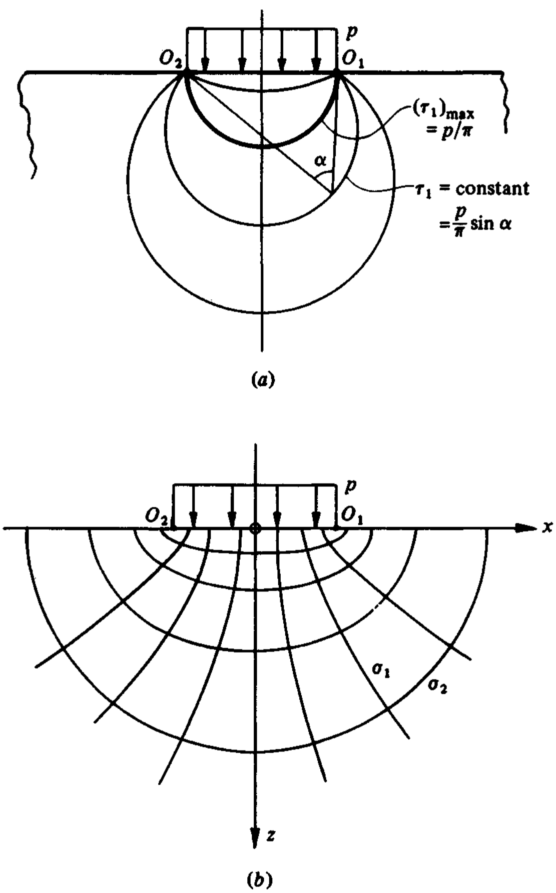

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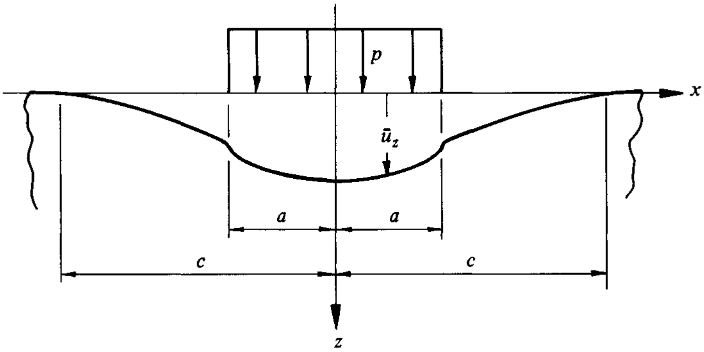
Fig. 2.8



Hence

$$\frac{\partial \bar{u}_z}{\partial x} = -\frac{2(1-\nu^2)}{\pi E} p\{\ln(a+x) - \ln(a-x)\}$$

$$\bar{u}_z = -\frac{(1-\nu^2)}{\pi E} p \left\{(a+x)\ln\left(\frac{a+x}{a}\right)^2 + (a-x)\ln\left(\frac{a-x}{a}\right)^2\right\} + C$$

$$(2.30b)$$

For a point outside the loaded region ($|x| > a$)

$$\bar{u}_x = \left\{\begin{array}{ll} +\dfrac{(1-2\nu)(1+\nu)}{E} pa, & x < -a \\ -\dfrac{(1-2\nu)(1+\nu)}{E} pa, & x > a \end{array}\right\} \qquad (2.30c)$$

In this case the integrand in (2.25*b*) is continuous so that we find

$$\bar{u}_z = -\frac{(1-\nu^2)p}{\pi E} \left\{(x+a)\ln\left(\frac{x+a}{a}\right)^2 - (x-a)\ln\left(\frac{x-a}{a}\right)^2\right\} + C \qquad (2.30d)$$

which is identical with equation (2.30*b*). The constant C in equations (2.30*b* and *d*) is the same and is fixed by the datum chosen for normal displacements. In Fig. 2.8 the normal displacement is illustrated on the assumption that $\bar{u}_z = 0$ when $x = \pm c$.

(b) Tangential traction

The stresses and surface displacements due to a uniform distribution of tangential traction acting on the strip ($-a \leqslant x \leqslant a$) can be found in the same

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way. From equations (2.23) putting $p(x) = 0$, we obtain

$$\sigma_x = \frac{q}{2\pi}\{4\ln(r_1/r_2) - (\cos 2\theta_1 - \cos 2\theta_2)\} \tag{2.31a}$$

$$\sigma_z = \frac{q}{2\pi}(\cos 2\theta_1 - \cos 2\theta_2) \tag{2.31b}$$

$$\tau_{xz} = -\frac{q}{2\pi}\{2(\theta_1 - \theta_2) + (\sin 2\theta_1 - \sin 2\theta_2)\} \tag{2.31c}$$

where $r_{1,2} = \{(x \mp a)^2 + z^2\}^{1/2}$.

Examination of the equations (2.24) for general surface displacements reveals that the surface displacements in the present problem may be obtained directly from those given in equations (2.30) due to uniform normal pressure. Using suffixes p and q to denote displacements due to similar distributions of normal and tangential tractions respectively, we see that

$$(\bar{u}_x)_q = (\bar{u}_z)_p \tag{2.32a}$$

and

$$(\bar{u}_z)_q = -(\bar{u}_x)_p \tag{2.32b}$$

provided that the same point is taken as a datum in each case.

The stress distributions in an elastic half-space due to uniformly distributed normal and tangential tractions p and q, given in equations (2.27) and (2.31), have been found by summing the stress components due to concentrated normal or tangential forces (equations (2.16) and (2.21)). An alternative approach is by superposition of appropriate Airy stress functions and subsequent derivation of the stresses by equations (2.6) or (2.11). This method has been applied to the problem of uniform loading of a half-space by Timoshenko & Goodier (1951). Although calculating the stresses by this method is simpler, there is no direct way of arriving at the appropriate stress functions other than by experience and intuition.

It is instructive at this juncture to examine the influence of the discontinuities in p and q at the edges of a uniformly loaded region upon the stresses and displacements at those points. Taking the case of a normal load first, we see from equations (2.27) that the stresses are everywhere finite, but at O_1 and O_2 there is a jump in σ_x from zero outside the region to $-p$ inside it. There is also a jump in τ_{xz} from zero at the surface to p/π just beneath. The surface displacements given by (2.30*b*) are also finite everywhere (taking a finite value for C) but the *slope* of the surface becomes theoretically infinite at O_1 and O_2. The discontinuity in q at the edge of a region which is loaded tangentially has a strikingly different effect. In equation (2.31*a*) the logarithmic term leads to an infinite value of σ_x, compressive at O_1 and tensile at O_2, as shown in Fig. 2.9. The

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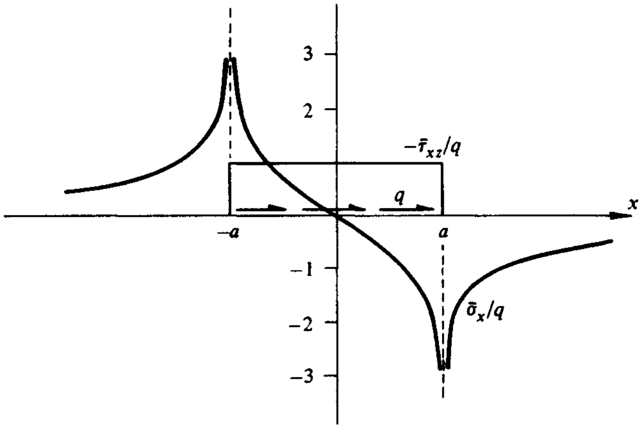
Fig. 2.9

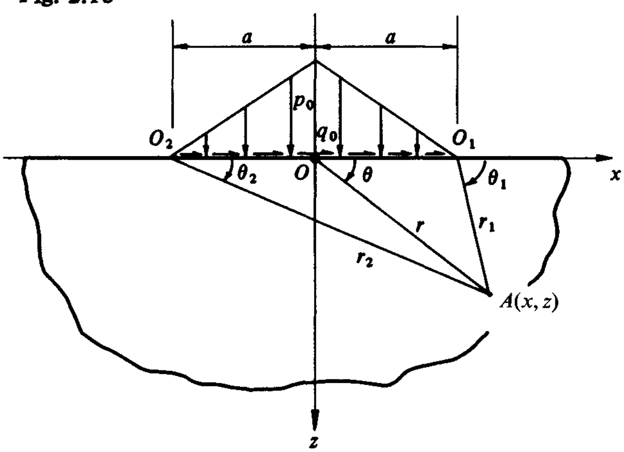


normal displacements of the surface given by equations (2.32) together with (2.30*a* and *c*) are continuous but there is a discontinuity in *slope* at O_1 and O_2. The concentrations of stress implied by the singularities at O_1 and O_2 undoubtedly play a part in the fatigue failure of surfaces subjected to oscillating friction forces - the phenomenon known as fretting fatigue.

2.6 Triangular distributions of traction

Another simple example of distributed loading will be considered. The tractions, normal and tangential, increase uniformly from zero at the surface points O_1 and O_2, situated at $x = \pm a$, to maximum values p_0 and q_0 at O ($x = 0$),

Fig. 2.10



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as shown in Fig. 2.10, i.e.

$$p(x) = \frac{p_0}{a}(a - |x|), \quad |x| \leqslant a \tag{2.33}$$

and

$$q(x) = \frac{q_0}{a}(a - |x|), \quad |x| \leqslant a \tag{2.34}$$

These triangular distributions of traction provide the basis for the numerical procedure for two-dimensional contact stress analysis described in §5.9.

When these expressions are substituted into equations (2.23) the integrations are straightforward so that the stresses at any point $A(x, z)$ in the solid may be found. Due to the normal pressure:

$$\sigma_x = \frac{p_0}{\pi a}\{(x-a)\theta_1 + (x+a)\theta_2 - 2x\theta + 2z \ln(r_1 r_2/r^2)\} \tag{2.35a}$$

$$\sigma_z = \frac{p_0}{\pi a}\{(x-a)\theta_1 + (x+a)\theta_2 - 2x\theta\} \tag{2.35b}$$

$$\tau_{xz} = -\frac{p_0 z}{\pi a}(\theta_1 + \theta_2 - 2\theta) \tag{2.35c}$$

and due to the tangential traction:

$$\sigma_x = \frac{q_0}{\pi a}\{2x \ln(r_1 r_2/r^2) + 2a \ln(r_2/r_1) - 3z(\theta_1 + \theta_2 - 2\theta)\} \tag{2.36a}$$

$$\sigma_z = -\frac{q_0 z}{\pi a}(\theta_1 + \theta_2 - 2\theta) \tag{2.36b}$$

$$\tau_{xz} = \frac{q_0}{\pi a}\{(x-a)\theta_1 + (x+a)\theta_2 - 2x\theta + 2z \ln(r_1 r_2/r^2)\} \tag{2.36c}$$

where $r_1^2 = (x-a)^2 + z^2, r_2^2 = (x+a)^2 + z^2, r^2 = x^2 + z^2$ and $\tan\theta_1 = z/(x-a)$, $\tan\theta_2 = z/(x+a)$, $\tan\theta = z/x$.

The surface displacements are found from equations (2.25). Due to the normal pressure $p(x)$ acting alone, at a point within the loaded region:

$$\frac{\partial \bar{u}_x}{\partial x} = -\frac{(1-2\nu)(1+\nu)}{E}\frac{p_0}{a}(a - |x|) \tag{2.37a}$$

i.e.

$$\bar{u}_x = -\frac{(1-2\nu)(1+\nu)}{E}\frac{p_0}{a}x(a - \tfrac{1}{2}|x|), \quad |x| \leqslant a \tag{2.37b}$$

relative to a datum at the origin. At a point outside the loaded region:

$$\bar{u}_x = \mp\frac{(1-2\nu)(1+\nu)}{E}\frac{p_0 a}{2} \quad \text{for } x \gtrless 0$$

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The normal displacement throughout the surface is given by

$$\frac{\partial \bar{u}_z}{\partial x} = -\frac{(1-\nu^2)}{\pi E}\frac{p_0}{a}\left\{(x+a)\ln\left(\frac{x+a}{x}\right)^2 + (x-a)\ln\left(\frac{x-a}{x}\right)^2\right\}$$

i.e.

$$\bar{u}_z = -\frac{(1-\nu^2)}{2\pi E}\frac{p_0}{a}\left\{(x+a)^2\ln\left(\frac{x+a}{a}\right)^2 + (x-a)^2\ln\left(\frac{x-a}{a}\right)^2 - 2x^2\ln(x/a)^2\right\} + C \tag{2.37c}$$

The surface displacements due to a triangular distribution of shear stress are similar and follow from the analogy expressed in equations (2.32).

Examining the stress distributions in equations (2.35) and (2.36) we see that the stress components are all finite and continuous. Equations (2.37) show that the slope of the deformed surface is also finite everywhere. This state of affairs contrasts with that discussed in the last section where there was a discontinuity in traction at the edge of the loaded region.

2.7 Displacements specified in the loaded region

So far we have discussed the stresses and deformations of an elastic half-space to which specified distributions of surface tractions are applied in the loaded region. Since the surface tractions are zero outside the loaded region, the boundary conditions in these cases amount to specifying the distribution of traction over the complete boundary of the half-space. In most contact problems, however, it is the displacements, or a combination of displacements and surface tractions, which are specified within the contact region, whilst outside the contact the surface tractions are specifically zero. It is to these 'mixed boundary-value problems' that we shall turn our attention in this section.

It will be useful to classify the different combinations of boundary conditions with which we have to deal. In all cases the surface of the half-space is considered to be free from traction outside the loaded region and, within the solid, the stresses should decrease as $(1/r)$ at a large distance r from the centre of the loaded region. There are four classes of boundary conditions within the contact region:

Class I: Both tractions, $p(x)$ and $q(x)$, specified. These are the conditions we have discussed in the previous sections. The stresses and surface displacements may be calculated by equations (2.23) and (2.24) respectively.

Class II: Normal displacements $\bar{u}_z(x)$ and tangential traction $q(x)$ specified *or* tangential displacements $\bar{u}_x(x)$ and normal pressure $p(x)$ specified.

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The first alternative in this class arises most commonly in the contact of frictionless surfaces, where $q(x)$ is zero everywhere, and the displacements $\bar{u}_z(x)$ are specified by the profile of the two contacting surfaces before deformation. The second alternative arises where the frictional traction $q(x)$ is sought between surfaces which do not slip over all or part of the contact interface, and where the normal traction $p(x)$ is known.

Class III: Normal and tangential displacements $\bar{u}_z(x)$ and $\bar{u}_x(x)$ specified. These boundary conditions arise when surfaces of known profile make contact without interfacial slip. The distributions of both normal and tangential traction are sought.

Class IV: The normal displacement $\bar{u}_z(x)$ is specified, while the tractions are related by $q(x) = \pm\mu p(x)$, where μ is a constant coefficient of friction. This class of boundary conditions clearly arises with solids in sliding contact; $\bar{u}_z(x)$ is specified by their known profiles.

It should be noted that the boundary conditions on different sectors of the loaded region may fall into different classes. For example, two bodies in contact may slip over some portions of the interface, to which the boundary conditions of class IV apply, while not slipping over the remaining portion of the interface where the boundary conditions are of class III.

To formulate two-dimensional problems of an elastic half-space in which displacements are specified over the interval $(-b \leqslant x \leqslant a)$ we use equations (2.25). Using a prime to denote $\partial/\partial x$, we may write these equations:

$$\int_{-b}^{a} \frac{q(s)}{x-s}\,\mathrm{d}s = -\frac{\pi(1-2\nu)}{2(1-\nu)}\,p(x) - \frac{\pi E}{2(1-\nu^2)}\,\bar{u}_x'(x) \tag{2.38a}$$

$$\int_{-b}^{a} \frac{p(s)}{x-s}\,\mathrm{d}s = \frac{\pi(1-2\nu)}{2(1-\nu)}\,q(x) - \frac{\pi E}{2(1-\nu^2)}\,\bar{u}_z'(x) \tag{2.38b}$$

With known displacements, (2.38) are coupled integral equations for the unknown tractions $p(x)$ and $q(x)$. Within the limits of integration there is a point of singularity when $s = x$, which has led to their being known as 'singular integral equations'. Their application to the theory of elasticity has been advanced notably by Muskhelishvili (1946, 1949) and the Soviet school: Mikhlin (1948) and Galin (1953). The development of this branch of the subject is beyond the scope of this book and only the immediately relevant results will be quoted.

When the boundary conditions are in the form of class II, e.g. $\bar{u}_z'(x)$ and $q(x)$ prescribed, then equations (2.38*a*) and (2.38*b*) become uncoupled. Each equation takes the form

$$\int_{-b}^{a} \frac{F(s)}{x-s}\,\mathrm{d}s = g(x) \tag{2.39}$$

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where $g(x)$ is a known function, made up from a combination of the known component of traction and the known component of displacement gradient, and $F(x)$ is the unknown component of traction. This is a singular integral equation of the first kind; it provides the basis for the solution of most of the two-dimensional elastic contact problems discussed in this book. It has a general solution of the form (see Söhngen, 1954; or Mikhlin, 1948)

$$F(x) = \frac{1}{\pi^2\{(x+b)(a-x)\}^{1/2}} \int_{-b}^{a} \frac{\{(s+b)(a-s)\}^{1/2} g(s)\,\mathrm{d}s}{x-s} + \frac{C}{\pi^2\{(x+b)(a-x)\}^{1/2}} \tag{2.40}$$

If the origin is taken at the centre of the loaded region the solution simplifies to

$$F(x) = \frac{1}{\pi^2(a^2-x^2)^{1/2}} \int_{-a}^{a} \frac{(a^2-s^2)^{1/2} g(s)\,\mathrm{d}s}{x-s} + \frac{C}{\pi^2(a^2-x^2)^{1/2}} \tag{2.41}$$

The constant C is determined by the total load, normal or tangential, from the relationship

$$C = \pi \int_{-a}^{a} F(x)\,\mathrm{d}x \tag{2.42}$$

The integrals in equations (2.40) and (2.41) have a singularity at $s = x$. The *principal value* of these integrals is required, as defined by:

$$\text{P.V.} \int_{-b}^{a} \frac{f(s)\,\mathrm{d}s}{x-s} \equiv \underset{\epsilon \to 0}{\text{Limit}} \left[\int_{-b}^{x-\epsilon} \frac{f(s)\,\mathrm{d}s}{x-s} + \int_{x+\epsilon}^{a} \frac{f(s)\,\mathrm{d}s}{x-s} \right] \tag{2.43}$$

The principal values of a number of integrals which arise in contact problems are listed in Appendix 1.

The integral equation in which $g(x)$ is of polynomial form:

$$g(x) = Ax^n \tag{2.44}$$

is of technical importance. An obvious example arises when a rigid frictionless punch or stamp is pressed into contact with an elastic half-space as shown in Fig. 2.11. If the profile of the stamp is of polynomial form

$$z = Bx^{n+1}$$

the normal displacements of the surface are given by

$$\bar{u}_z(x) = \bar{u}_z(0) - Bx^{n+1}$$

thus

$$\bar{u}_z'(x) = -(n+1)Bx^n$$

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If the punch is frictionless $q(x) = 0$, so that substituting in equation (2.38*b*) gives

$$\int_{-b}^{a} \frac{p(s)}{x-s}\,ds = \frac{\pi E}{2(1-\nu^2)}(n+1)Bx^n \tag{2.45}$$

This is an integral equation of the type (2.39) for the pressure $p(x)$, where $g(x)$ is of the form Ax^n. If the contact region is symmetrical about the origin $b = a$, equation (2.45) has a solution of the form expressed in (2.41). The principal value of the following integral is required:

$$I_n \equiv \text{P.V.} \int_{-1}^{+1} \frac{S^n(1-S^2)^{1/2}\,dS}{X-S} \tag{2.46}$$

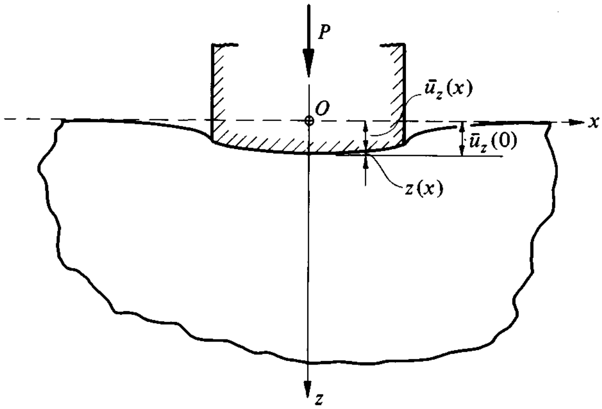
where $X = x/a$, $S = s/a$. From the table in Appendix 1

$$I_0 = \text{P.V.} \int_{-1}^{+1} \frac{(1-S^2)^{1/2}}{X-S}\,dS = \pi X$$

A series for I_n may be developed by writing

$$\begin{aligned} I_n &= X \int_{-1}^{+1} \frac{S^{n-1}(1-S^2)^{1/2}\,dS}{X-S} - \int_{-1}^{+1} S^{n-1}(1-S^2)^{1/2}\,dS \\ &= XI_{n-1} - J_{n-1} \\ &= X^n I_0 - X^{n-1}J_0 - X^{n-2}J_1 - \cdots - XJ_{n-2} - J_{n-1} \end{aligned}$$

Fig. 2.11



where

$$J_m = \int_{-1}^{+1} S^m(1-S^2)^{1/2}\,\mathrm{d}S$$

$$= \begin{cases} \pi/2 & \text{for } m = 0 \\ \dfrac{1\cdot 3\cdot 5\ldots(m-1)}{2\cdot 4\ldots m(m+2)}\,\pi & \text{for } m \text{ even} \\ 0 & \text{for } m \text{ odd} \end{cases}$$

Hence

$$I_n = \begin{cases} \pi\left\{X^{n+1} - \tfrac{1}{2}X^{n-1} - \tfrac{1}{8}X^{n-3} - \ldots - \dfrac{1\cdot 3\cdot 5\ldots(n-3)}{2\cdot 4\ldots n}\,X\right\} & \\ \quad\text{for } n \text{ even} & (2.47a) \\ \pi\left\{X^{n+1} - \tfrac{1}{2}X^{n-1} - \tfrac{1}{8}X^{n-3} - \ldots - \dfrac{1\cdot 3\ldots(n-2)}{2\cdot 4\ldots(n+1)}\right\} & \\ \quad\text{for } n \text{ odd} & (2.47b) \end{cases}$$

If P is the total load on the punch, then by equation (2.42)

$$C = \pi P$$

The pressure distribution under the face of the punch is then given by equation (2.41), i.e.

$$p(x) = -\frac{E(n+1)Ba^{n+1}}{2(1-\nu^2)\pi}\,\frac{I_n}{(a^2-x^2)^{1/2}} + \frac{P}{\pi(a^2-x^2)^{1/2}} \qquad (2.48)$$

In this example it is assumed that the load on the punch is sufficient to maintain contact through a positive value of pressure over the whole face of the punch. If n is odd the profile of the punch and the pressure distribution given by equation (2.48) are symmetrical about the centre-line. On the other hand, if n is even, the punch profile is anti-symmetrical and the line of action of the compressive load will be eccentric giving rise to a moment

$$M = \int_{-a}^{a} xp(x)\,\mathrm{d}x \qquad (2.49)$$

Finally it is apparent from the expression for the pressure given in (2.48) that, in general, the pressure at the edges of the punch rises to a theoretically infinite value.

We turn now to boundary conditions in classes III and IV. When both components of boundary displacement are specified (class III) the integral equations (2.38) can be combined by expressing the required surface tractions as a single

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complex function:

$$F(x) = p(x) + iq(x) \tag{2.50}$$

Then by adding (2.38*a*) and (2.38*b*), we get

$$F(x) - i\frac{2(1-\nu)}{\pi(1-2\nu)}\int_{-b}^{a}\frac{F(s)\,ds}{x-s} = -\frac{E}{(1-2\nu)(1+\nu)} \times \{\bar{u}_x'(x) - i\bar{u}_z'(x)\} \tag{2.51}$$

In the case of sliding motion, where $\bar{u}_z(x)$ is given together with $q(x) = \mu p(x)$ (boundary conditions of class IV), equation (2.38*b*) becomes

$$p(x) - \frac{2(1-\nu)}{\pi\mu(1-2\nu)}\int_{-b}^{a}\frac{p(s)}{x-s}\,ds = \frac{E}{\mu(1-2\nu)(1+\nu)}\bar{u}_z'(x) \tag{2.52}$$

To simplify equations (2.51) and (2.52) we shift the origin to the mid-point of the contact region (i.e. put $b = a$), and put $X = x/a, S = s/a$. Equations (2.51) and (2.52) are both integral equations of the second kind having the form

$$F(X) + \frac{\lambda}{\pi}\int_{-1}^{+1}\frac{F(S)\,dS}{X-S} = G(X) \tag{2.53}$$

where $G(X)$, $F(X)$ and λ can be real or complex. The function $G(X)$ is known and it is required to find the function $F(X)$. λ is a parameter whose value depends upon the particular problem. The solution to (2.53) is given by Söhngen (1954) in the form

$$F(X) = F_1(X) + F_0(X) \tag{2.54}$$

where $F_0(X)$ is the solution of the homogeneous equation, i.e. equation (2.53) with the right-hand side put equal to zero. He gives

$$F_1(X) = \frac{1}{1+\lambda^2}G(X) - \frac{\lambda}{1+\lambda^2}\frac{1}{\pi(1-X^2)^{1/2}}\left(\frac{1+X}{1-X}\right)^{\gamma} \times \int_{-1}^{1}(1-S^2)^{1/2}\left(\frac{1-S}{1+S}\right)^{\gamma}\frac{G(S)}{X-S}\,dS \tag{2.55}$$

where γ is a complex constant related to λ by $\cot(\pi\gamma) = \lambda$, i.e. $e^{2\pi i\gamma} = (i\lambda - 1)/(i\lambda + 1)$, restricted so that its real part $\text{Re}(\gamma)$ lies within the interval $-\frac{1}{2}$ to $+\frac{1}{2}$, and

$$F_0(X) = -\frac{\lambda}{(1+\lambda^2)^{1/2}}\frac{1}{\pi(1-X^2)^{1/2}}\left(\frac{1+X}{1-X}\right)^{\gamma}C \tag{2.56}$$

where the constant

$$C = \int_{-1}^{+1}F(X)\,dX = \frac{1}{a}(P + iQ)$$

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If λ is imaginary, so that $\lambda = i\lambda_1$, then λ_1 must lie outside the interval -1 to $+1$. This condition is met in the problems considered here.

We will take first the case of both boundary displacements specified (class III), where we require the solution of equation (2.51). Comparing the general solution given in (2.55) and (2.56) with equation (2.51) we see that λ is imaginary ($\lambda = i\lambda_1$) where

$$\lambda_1 = -\frac{2-2\nu}{1-2\nu} \tag{2.57}$$

Since ν lies between 0 and $\frac{1}{2}$, $\lambda_1 < -2$ which makes the solution given by (2.56) valid. Thus γ is also imaginary, so that, putting $\gamma = i\eta$, we have

$$e^{-2\pi\eta} = \frac{-\lambda_1 - 1}{-\lambda_1 + 1} = \frac{1}{3-4\nu}$$

giving

$$\eta = \frac{1}{2\pi} \ln(3-4\nu) \tag{2.58}$$

Substituting for λ and γ from equations (2.57) and (2.58) in (2.56) the required solution is

$$p(X) + iq(X) = F(X) = F_1(X) + F_0(X)$$

where

$$F_1(X) = \frac{(1-2\nu)E}{(3-4\nu)(1+\nu)} \{\bar{u}_x'(X) - i\bar{u}_z'(X)\} + i\,\frac{2(1-\nu)E}{(3-4\nu)(1+\nu)}$$
$$\times \frac{1}{\pi(1-X^2)^{1/2}} \left(\frac{1+X}{1-X}\right)^{i\eta} \int_{-1}^{+1} (1-S^2)^{1/2} \left(\frac{1-S}{1+S}\right)^{i\eta}$$
$$\times \left\{\frac{\bar{u}_x'(S) - i\bar{u}_z'(S)}{X-S}\right\} dS \tag{2.59a}$$

and

$$F_0(X) = \frac{2(1-\nu)}{(3-4\nu)^{1/2}} \frac{P+iQ}{\pi a(1-X^2)^{1/2}} \left(\frac{1+X}{1-X}\right)^{i\eta} \tag{2.59b}$$

To obtain expressions for the surface tractions $p(X)$ and $q(X)$ requires the evaluation of the integral in (2.59*a*). So far only a few problems, in which the distributions of displacement $\bar{u}_x'(X)$ and $\bar{u}_z'(X)$ are particularly simple, have been solved in closed form. For an incompressible material ($\nu = 0.5$) the basic integral equations (2.38) become uncoupled. In this case we see from (2.58) that $\eta = 0$, whereupon the general solution to the coupled equations given by

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equation (2.59), when real and imaginary parts are separated, reduces to the solution of two uncoupled equations of the form of (2.41).

When the boundary conditions are of class IV, by comparing equations (2.52) with the general form (2.53), we see that λ is real: i.e.

$$\lambda = -\frac{2(1-\nu)}{\mu(1-2\nu)} = \cot \pi\gamma \tag{2.60}$$

Substituting for λ in the general solution (2.55) and (2.56) gives

$$\begin{aligned} p(X) = &-\frac{E \sin \pi\gamma \cos \pi\gamma}{2(1-\nu^2)} \bar{u}_z'(X) + \frac{E \cos^2 \pi\gamma}{2(1-\nu^2)} \frac{1}{\pi(1-X^2)^{1/2}} \left(\frac{1+X}{1-X}\right)^{\gamma} \\ &\times \int_{-1}^{+1} (1-S^2)^{1/2} \left(\frac{1-S}{1+S}\right)^{\gamma} \frac{\bar{u}_z'(S)}{X-S} \, \mathrm{d}S + \frac{P \cos \pi\gamma}{\pi a (1-X^2)^{1/2}} \\ &\times \left(\frac{1+X}{1-X}\right)^{\gamma} \end{aligned} \tag{2.61}$$

and

$$q(X) = \mu p(X).$$

Once again, for an incompressible material, or when the coefficient of friction approaches zero, γ approaches zero and the integral equations become uncoupled. Equation (2.61) then degenerates into the uncoupled solution (2.41).

Having found the surface tractions $p(X)$ and $q(X)$ to satisfy the displacement boundary conditions, we may find the internal stresses in the solid, in principle at least, by the expressions for stress given in equations (2.23).

An example in the application of the results presented in this section is provided by the indentation of an elastic half-space by a rigid two-dimensional punch which has a flat base. This example will be discussed in the next section.

2.8 Indentation by a rigid flat punch

In this section we consider the stresses produced in an elastic half-space by the action of a rigid punch pressed into the surface as shown in Fig. 2.12. The punch has a flat base of width $2a$ and has sharp square corners; it is long in the y-direction so that plane-strain conditions can be assumed. Since the punch is rigid the surface of the elastic solid must remain flat where it is in contact with the punch. We shall restrict our discussion to indentations in which the punch does not tilt, so that the interface, as well as being flat, remains parallel to the undeformed surface of the solid. Thus our first boundary condition within the contact region is one of specified normal displacement:

$$\bar{u}_z(x) = \text{constant} = \delta_z \tag{2.62}$$

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The second boundary condition in the loaded region depends upon the frictional conditions at the interface. We shall consider four cases:

(*a*) that the surface of the punch is frictionless, so that $q(x) = 0$;
(*b*) that friction at the interface is sufficient to prevent any slip between the punch and the surface of the solid so that $\bar{u}_x(x) = \text{constant} = \delta_x$;
(*c*) that partial slip occurs to limit the tangential traction $|q(x)| \leqslant \mu p(x)$; and
(*d*) that the punch is sliding along the surface of the half-space from right to left, so that $q(x) = \mu p(x)$ at all points on the interface, where μ is a constant coefficient of sliding friction.

No real punch, of course, can be perfectly rigid, although this condition will be approached closely when a solid of low elastic modulus such as a polymer or rubber is indented by a metal punch. Difficulties arise in allowing for the elasticity of the punch, since the deformation of a square-cornered punch cannot be calculated by the methods appropriate to a half-space. However the results of this section are of importance in circumstances other than that of a punch indentation. We shall use the stresses arising from constant displacements δ_x and δ_z in the solution of other problems (see §§5.5 & 7.2).

(a) Frictionless punch

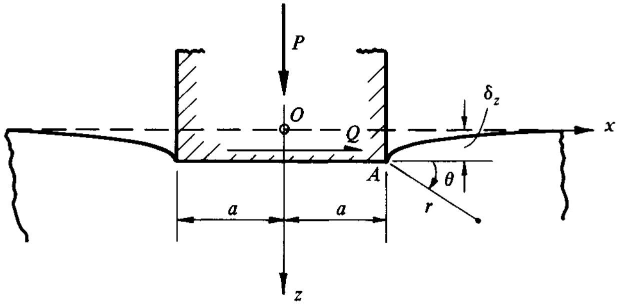
The boundary conditions:

$$\bar{u}_z(x) = \text{constant}, \qquad q(x) = 0 \tag{2.63}$$

are of class II as defined in the last section so that the pressure distribution is given by the integral equation (2.38*b*) which has the general solution (2.41) in which

$$g(s) = -\frac{\pi E}{2(1-\nu^2)}\,\bar{u}_z'(x) = 0$$

Fig. 2.12



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In this case the result reduces to the homogeneous solution while $C = \pi P$:

$$p(x) = \frac{P}{\pi(a^2 - x^2)^{1/2}} \tag{2.64}$$

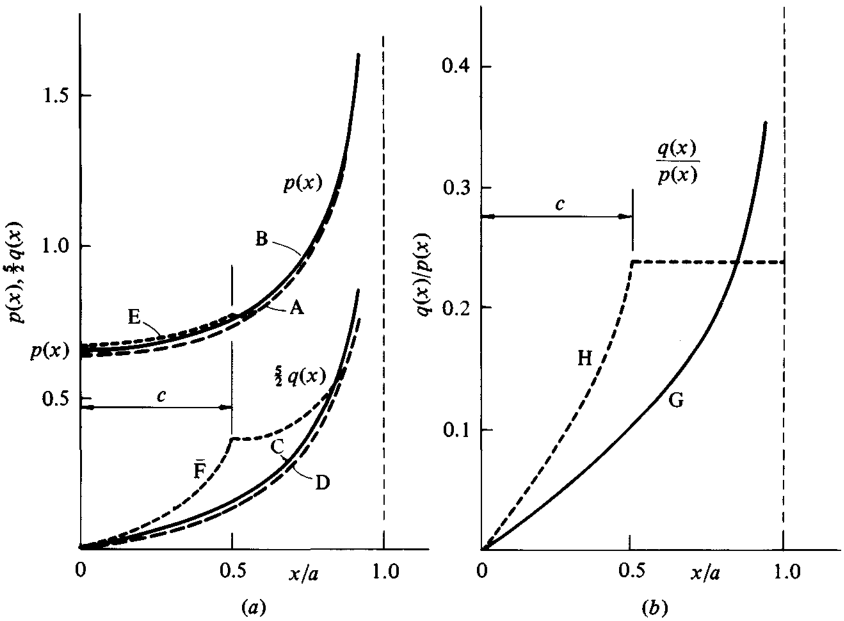
This pressure distribution is plotted in Fig. 2.13(*a*) (curve A). The pressure reaches a theoretically infinite value at the edges of the punch ($x = \pm a$). The stresses within the solid in the vicinity of the corners of the punch have been found by Nadai (1963). The sum of the principal stresses is given by

$$\sigma_1 + \sigma_2 \approx -\frac{2P}{\pi(2ar)^{1/2}} \sin(\theta/2) \tag{2.65a}$$

and the principal shear stress

$$\tau_1 \approx -\frac{P}{2\pi(2ar)^{1/2}} \sin\theta \tag{2.65b}$$

Fig. 2.13. (*a*) Tractions on the face of flat punch shown in Fig. 2.12: curve A – Frictionless, eq. (2.64) for $p(x)$; curve B – No slip, exact eq. (2.69) for $p(x)$; curve C – No slip, exact eq. (2.69) for $q(x)$; curve D – No slip, approx. eq. (2.72) for $q(x)$; curve E – Partial slip, $p(x)$; curve F – Partial slip, $q(x)$ (curves E & F from Spence, 1973). (*b*) Ratio of tangential traction $q(x)$ to normal traction $p(x)$: curve G – No slip, approx. eqs (2.72) and (2.64); curve H – Partial slip, from Spence (1973) ($\nu = 0.3, \mu = 0.237$ giving $c = 0.5$).



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where r and θ are polar coordinates from an origin at $x = \pm a$ and $r \ll a$ (see the photoelastic fringe pattern in Fig. 4.6(c)). At the surface of the solid $\sigma_1 = \sigma_2 = \sigma_x = \sigma_z$. From (2.65$b$) the principal shear stress is seen to reach a theoretically infinite value as $r \to 0$ so that we would expect a real material to yield plastically close to the corners of the punch even at the lightest load. The displacement of the surface outside the punch can be calculated from (2.24b) with the result

$$\bar{u}_z(x) = \delta_z - \frac{2(1-\nu^2)P}{\pi E} \ln\left\{\frac{x}{a} + \left(\frac{x^2}{a^2} - 1\right)^{1/2}\right\} \tag{2.66}$$

where, as usual, δ_z can only be determined relative to an arbitrarily chosen datum. We note that the surface gradient $\bar{u}'_z$ is infinite at $x = \pm a$. From (2.24a) we find the tangential displacements under the punch to be

$$\bar{u}_x(x) = -\frac{(1-2\nu)(1+\nu)P}{\pi E} \sin^{-1}(x/a) \tag{2.67}$$

For a compressible material ($\nu < 0.5$), this expression shows that points on the surface move towards the centre of the punch. In practice this motion would be opposed by friction, and, if the coefficient of friction were sufficiently high, it might be prevented altogether. We shall now examine this possibility.

(b) No slip

If the surface of the solid adheres completely to the punch during indentation then the boundary conditions are

$$\bar{u}_x(x) = \delta_x \quad \text{and} \quad \bar{u}_z(x) = \delta_z \tag{2.68}$$

where δ_x and δ_z are the (constant) displacements of the punch. These boundary conditions, in which both displacements are specified, are of class III. The integral equations (2.38) for the tractions at the surface of the punch are now coupled and their general solution is given by equation (2.59). Since the displacements are constant, $\bar{u}'_x(x) = \bar{u}'_z(x) = 0$, so that only the solution to the homogeneous equation (2.59b) remains, viz.:

$$\begin{aligned} p(x) + iq(x) &= \frac{2(1-\nu)}{(3-4\nu)^{1/2}} \frac{P + iQ}{\pi(a^2 - x^2)^{1/2}} \left(\frac{a+x}{a-x}\right)^{i\eta} \\ &= \frac{2(1-\nu)}{(3-4\nu)^{1/2}} \frac{P + iQ}{\pi(a^2 - x^2)^{1/2}} \\ &\quad \times \left[\cos\left\{\eta \ln\left(\frac{a+x}{a-x}\right)\right\} + i \sin\left\{\eta \ln\left(\frac{a+x}{a-x}\right)\right\}\right] \end{aligned} \tag{2.69}$$

where $\eta = (1/2\pi) \ln(3-4\nu)$.

The tractions $p(x)$ and $q(x)$ under the action of a purely normal load ($Q = 0$) have been computed for $\nu = 0.3$ and are plotted in Fig. 2.13(a) (curves B and C).

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The nature of the singularities at $x = \pm a$ is startling. From the expression (2.69) it appears that the tractions fluctuate in sign an infinite number of times as $x \to a$! However the maximum value of η is (ln 3)/2π which results in the pressure first becoming negative when $x = \pm a \tanh(\pi^2/2 \ln 3)$ i.e. when $x = \pm 0.9997a$, which is very close to the edge of the punch. We conclude that this anomalous result arises from the inadequacy of the linear theory of elasticity to handle the high strain gradients in the region of the singularity. Away from those points, we might expect equation (2.69) to provide an accurate measure of the stresses.

If a tangential force Q acts on the punch in addition to a normal force, additional shear and normal tractions arise at the interface. With complete adhesion, these are also given by equation (2.69) such that, due to unit loads

$$\left.\begin{aligned} [q(x)]_Q &= [p(x)]_P \\ \text{and} \qquad & \\ [p(x)]_Q &= -[q(x)]_P \end{aligned}\right\} \tag{2.70}$$

Since $[q(x)]_P$ is an odd function of x, the influence of the tangential force is to reduce the pressure on the face of the punch where x is positive and to increase it where x is negative. A moment is then required to keep the punch face square. Close to $x = +a$ the pressure would tend to become negative unless the punch were permitted to tilt to maintain contact over the whole face. Problems of a tilted punch have been solved by Muskhelishvili (1949) and are discussed by Gladwell (1980).

At this juncture it is instructive to compare the pressure distribution in the presence of friction computed from equation (2.69) with that in the absence of friction from equation (2.64) (see Fig. 2.13(*a*)). The difference is not large, showing that the influence of the tangential traction on the normal pressure is small for $\nu = 0.3$. Larger values of ν will make the difference even smaller. Therefore, in more difficult problems than the present one, the integral equations can be uncoupled by assuming that the pressure distribution in the presence of friction is the same as that without friction. Thus we put $q(x) = 0$ in equation (2.38*b*) and solve it to find $p(x)$ without friction. This solution for $p(x)$ is then substituted in equation (2.38*a*) to find an approximate solution for $q(x)$. Each integral equation is then of the first kind having a solution of the form (2.41).

If this expedient is used in the present example, the pressure given by (2.64) is substituted in equation (2.38*a*) to give

$$\int_{-a}^{a} \frac{q(s)}{x-s}\,\mathrm{d}s = -\frac{(1-2\nu)}{2(1-\nu)} \frac{P}{(a^2-x^2)^{1/2}} \tag{2.71}$$

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Using the general solution to this equation given by (2.41), we get

$$q(x) = -\frac{(1-2\nu)}{2\pi^2(1-\nu)}\frac{P}{(a^2-x^2)^{1/2}}\int_{-a}^{a}\frac{ds}{x-s}+\frac{Q}{\pi(a^2-x^2)^{1/2}}$$

$$= -\frac{(1-2\nu)}{2\pi^2(1-\nu)}\frac{P}{(a^2-x^2)^{1/2}}\ln\left(\frac{a+x}{a-x}\right)+\frac{Q}{\pi(a^2-x^2)^{1/2}} \qquad (2.72)$$

This approximate distribution of tangential traction is also plotted in Fig. 2.13(a) for $Q = 0$ (curve D). It is almost indistinguishable from the exact solution given by (2.69).

(c) Partial slip

In case (b) above it was assumed that friction was capable of preventing slip entirely between the punch and the half-space. The physical possibility of this state of affairs under the action of a purely normal load P can be examined by considering the ratio of tangential to normal traction $q(x)/p(x)$. This ratio is plotted in Fig. 2.13(b) (curve G) using the approximate expressions for $q(x)$ and $p(x)$, i.e. equations (2.72) and (2.64) respectively, from which it is apparent that theoretically infinite values are approached at the edges of the contact. (The same conclusion would be reached if the exact expressions for $q(x)$ and $p(x)$ were used.) This means that, in practice, some slip must take place at the edges on the contact.

The problem of partial slip was studied first by Galin (1945) and more completely by Spence (1973). Under a purely normal load the contact is symmetrical about the centre-line so that the no-slip region will be centrally placed from $x = -c$ to $x = +c$, say. The boundary condition $\bar{u}_z(x) = \delta_z$ = constant still applies for $-a < x < a$, but the condition $\bar{u}_x(x) = \delta_x = 0$ is restricted to the no-slip zone $-c < x < c$. In the slip zones $c \leqslant |x| \leqslant a$

$$q(x) = \pm\mu p(x)$$

The extent of the no-slip zone is governed by the values of Poisson's ratio ν and the coefficient of friction μ. The problem is considerably simplified if the integral equations are uncoupled by neglecting the influence of tangential traction on normal pressure. With this approximation Spence has shown that c is given by the relationship

$$\mathbf{K}'(c/a)/\mathbf{K}(c/a) = (1-2\nu)/2(1-\nu)\mu \qquad (2.73)$$

where $\mathbf{K}(c/a)$ is the complete elliptic integral of the second kind and $\mathbf{K}'(c/a) = \mathbf{K}(1-c^2/a^2)^{1/2}$. The values of $p(x)$ and $q(x)$ and the ratio $q(x)/p(x)$ with partial slip have been calculated for $\nu = 0.3$, $\mu = 0.237$ and are plotted in curves E, F and H in Fig. 2.13(a and b).

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(d) Sliding punch

A punch which is sliding over the surface of a half-space at a speed much less than the velocity of elastic waves, so that inertia forces can be neglected, has the boundary conditions:

$$\bar{u}_z(x) = \text{constant} = \delta_z, \quad q(x) = \mu p(x) \tag{2.74}$$

These are boundary conditions in class IV, so that the coupled integral equations for the surface tractions combine to give equation (2.52) having as its general solution equation (2.61). In the case of the flat punch, $\bar{u}_z'(x) = 0$, so that once again we only require the solution to the homogeneous equation, viz.:

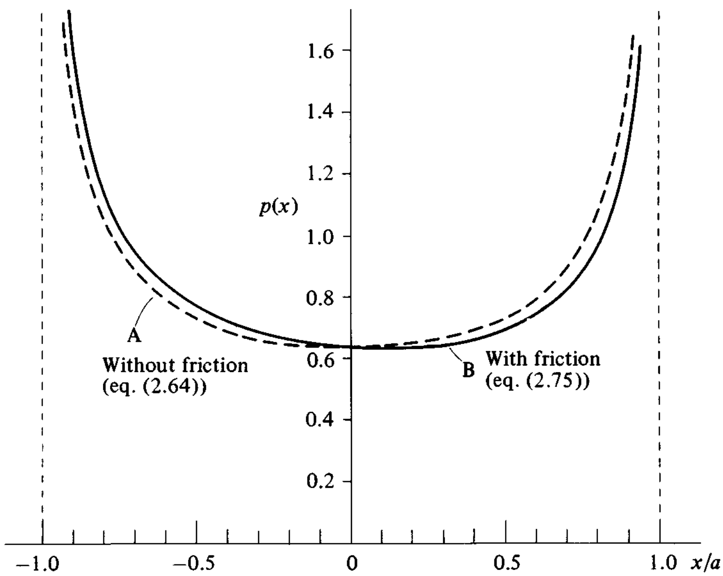
$$p(x) = \frac{P \cos \pi\gamma}{\pi(a^2 - x^2)^{1/2}} \left(\frac{a+x}{a-x}\right)^{\gamma} \tag{2.75}$$

where

$$\cot \pi\gamma = -\frac{2(1-\nu)}{\mu(1-2\nu)}$$

This pressure distribution is plotted for $\nu = 0.3$ and $\mu = 0.5$ in Fig. 2.14, where it is compared with the pressure distribution in the absence of friction.

Fig. 2.14. Pressure on the face of a sliding punch: curve A frictionless; curve B with friction ($\nu = 0.3$, $\mu = 0.5$).



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The punch is moving from right to left so that the effect of friction is to reduce the pressure in the front half of the punch and to increase it on the rear. In this case also it is apparent that the influence of frictional traction upon the normal pressure is relatively small.

2.9 Traction parallel to y-axis

A different form of two-dimensional deformation occurs when tangential tractions, whose magnitude and distribution are independent of the y-coordinate, act on the surface of the half-space in the y-direction. Clearly x–z cross-sections of the solid will not remain plane but will be warped by the action of the surface traction. Since all cross-sections will deform alike, however, the stress field and resulting deformations will be independent of y.

We will consider first a concentrated tangential force of magnitude Q_y per unit length acting on the surface along the y-axis, as shown in Fig. 2.15. This force produces a simple shear-stress distribution which can be derived easily as follows.

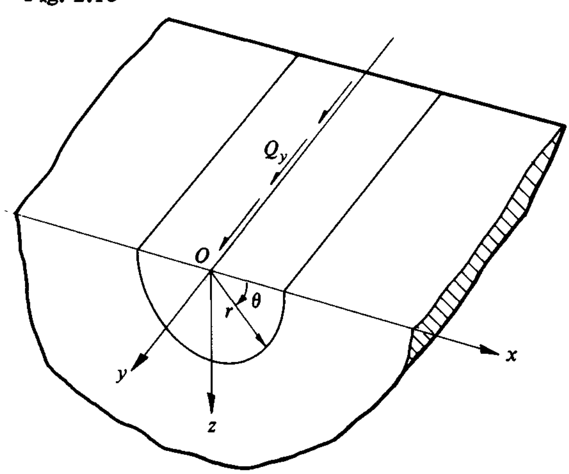
We think first of two half-spaces with their surfaces glued together to make a complete infinite solid. A force $2Q_y$ per unit length acts along the y-axis. In cylindrical coordinates r, θ, y the stress system in the complete solid must be axially symmetrical and independent of θ and y. By considering the equilibrium of a cylinder of radius r we find

$$2\pi r\tau_{ry} = -2Q_y$$

i.e.

$$\tau_{ry} = -\frac{Q_y}{\pi r} \tag{2.76a}$$

Fig. 2.15



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The other stress components must vanish, i.e.

$$\sigma_r = \sigma_\theta = \sigma_y = 0 \tag{2.76b}$$

The strains associated with these stresses are compatible and result in an identical warping of each cross-section of the solid. Now no stresses act across the interface where the two half-spaces are glued together, so that they can be separated without changing the stress distribution. Thus equations (2.76) describe the required stress distribution. In Cartesian coordinates

$$\tau_{xy} = \tau_{ry}\cos\theta = -\frac{Q_y x}{\pi r^2} \tag{2.77a}$$

$$\tau_{yz} = \tau_{ry}\sin\theta = -\frac{Q_y z}{\pi r^2} \tag{2.77b}$$

The deformations are found from the only strain component

$$\frac{\partial u_y}{\partial r} + \frac{\partial u_r}{\partial y} = \gamma_{ry} = \tau_{ry}/G$$

Since the deformation is independent of y, $\partial u_r/\partial y = 0$, so

$$\frac{\partial u_y}{\partial r} = \tau_{ry}/G = -\frac{Q_y}{\pi G r} \tag{2.78}$$

At the surface $z = 0$, this becomes

$$\frac{\partial \bar{u}_y}{\partial x} = -\frac{Q_y}{\pi G x} \tag{2.79}$$

or

$$\bar{u}_y = -\frac{Q_y}{\pi G}\ln|x| + C \tag{2.80}$$

and, neglecting rigid body motions, $\bar{u}_x = \bar{u}_z = 0$. Comparing this result with equations (2.19) and (2.22), we note that the displacement in the direction of the force is similar in form to that produced by a concentrated normal force, or by a tangential force in the x-direction. However in this case the displacement perpendicular to the force is absent.

The stresses and displacements produced by a tangential traction, distributed over the strip $-a \leqslant x \leqslant a$, are found as before by summing the effects of a concentrated load $Q_y = q_y(s)\,ds$ acting on an elemental strip of width ds. As an example we shall consider the tangential traction

$$q_y(x) = q_0(1 - x^2/a^2)^{1/2} \tag{2.81}$$

acting on the strip $-a \leqslant x \leqslant a$. From equations (2.77)

$$\tau_{xy} = -\frac{q_0}{\pi a}\int_{-a}^{a}\frac{(a^2 - s^2)^{1/2}(x - s)\,ds}{(x - s)^2 + z^2} \tag{2.82a}$$

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and

$$\tau_{yz} = -\frac{q_0 z}{\pi a} \int_{-a}^{a} \frac{(a^2 - s^2)^{1/2}\, ds}{(x-s)^2 + z^2} \tag{2.82b}$$

Evaluation of the integrals is straightforward. On the surface ($z = 0$) it is found that

$$\bar{\tau}_{xy} = \left\{ \begin{array}{ll} -q_0 x/a, & -a \leqslant x \leqslant a \\ -q_0 [(x/a) \mp \{(x/a)^2 - 1\}^{1/2}], & |x| > a \end{array} \right\} \tag{2.83a}$$

and on the axis of symmetry ($x = 0$)

$$\tau_{yz} = -q_0 [\{1 + (z/a)^2\}^{1/2} - z/a] \tag{2.83b}$$

The surface displacements follow from equation (2.79)

$$\frac{\partial \bar{u}_y}{\partial x} = -\frac{q_0}{\pi G} \int_{-a}^{a} \frac{(a^2 - s^2)^{1/2}}{(x-s)}\, ds$$

From the list of principal values of integrals of this type given in Appendix 1 it follows that

$$\frac{\partial \bar{u}_y}{\partial x} = -\frac{q_0}{G}\frac{x}{a}$$

Thus

$$\bar{u}_y = -\frac{q_0 x^2}{2Ga} + C \tag{2.84}$$

The results obtained in this example are used later (§8.3) when studying the contact stresses in rolling cylinders which transmit a tangential force parallel to the axes of the cylinders.

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