Hypothesis testing: randomisation, privacy and minimax optimality

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Hypothesis testing

Testing hypotheses is at the heart of statistics. Basic inferential questions: making statements of *confidence* and *quantifying uncertainty*.


```
> attach(mtcars)
> summary(lm(mpg~disp+hp+drat+wt))

Call:
lm(formula = mpg ~ disp + hp + drat + wt)

Residuals:
    Min      1Q  Median      3Q     Max
-3.5077 -1.9052 -0.5057  0.9821  5.6883

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) 29.148738   6.293588   4.631 8.2e-05 ***
disp         0.003815   0.010805   0.353 0.72675
hp          -0.034784   0.011597  -2.999 0.00576 **
drat         1.768049   1.319779   1.340 0.19153
wt          -3.479668   1.078371  -3.227 0.00327 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 2.602 on 27 degrees of freedom
Multiple R-squared: 0.8376,    Adjusted R-squared: 0.8136
F-statistic: 34.82 on 4 and 27 DF,  p-value: 2.704e-10

>
```
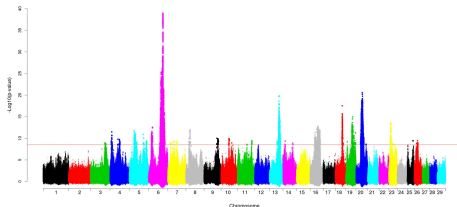
In scientific application we often want to answer a qualitative question: is this variable important? is the new treatment better than the old one? Does my model explain what is happening?

Property testing

As well as in statistics, modern research is done in computer science and information theory (e.g. Paninski, 2008; Diakonikolas & Kane, 2016; Canonne, 2020).

In *distribution testing* we see that properties of complex distributions can be tested with much less data than would be needed to estimate them.
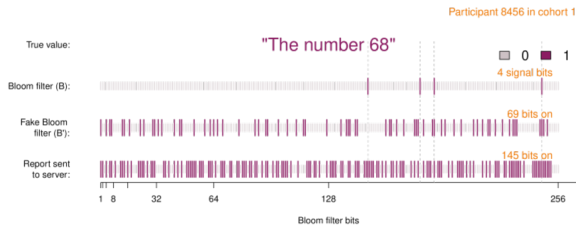
Figure: RAPPOR mechanism (Erlingsson, Pihur & Korolova, 2014)

When distributions are complex, traditional asymptotic approaches are typically not appropriate. Instead we take a finite-sample, minimax view.

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Minimax optimality theory

Hypothesis testing is the foundation for much modern statistical theory.

$$\mathcal{R}_n := \inf_{\hat{\theta}} \sup_{\theta \in \Theta} \mathbb{E}_\theta L(\hat{\theta}, \theta).$$

Most minimax lower bounds are based on reductions from estimation to testing problems for minimax lower bounds.

If we can prove that $\mathcal{R}_n \asymp \psi_n$ we can

- Certify that our estimators are *rate optimal*
- Quantify the inherent difficulty of a statistical problem, study the effect on statistical utility of
  - computational/communication resources
  - contaminated/missing data
  - privacy/fairness constraints
  - ...

Overview

1 Introduction

2 Simple null hypotheses
- The likelihood ratio test and Neyman–Pearson
- Composite alternatives: Goodness-of-fit testing

3 Permutation testing for composite nulls
- Independence testing – minimax optimality
- Conditional independence testing – non-uniform permutations

4 Local differential privacy
- Two-point testing
- Goodness-of-fit testing

Simple hypothesis testing

One of the simplest and most studied statistical problems there is. Test

$$H_0 : P = P_0 \quad \text{vs.} \quad H_1 : P = P_1$$

given independent and indentically distributed data $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} P$.

For us, a *test* is a function $\phi : \mathcal{X}^n \to [0, 1]$ with the interpretation that we reject H_0 with probability $\phi(X_1, \ldots, X_n)$.

Write $L(x) \equiv L(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{p_1(x_i)}{p_0(x_i)}$ where p_0, p_1 are densities of P_0, P_1 w.r.t. some dominating measure. *Likelihood Ratio Tests* are of the form

$$\phi_k(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } L(x) > k \\ c & \text{if } L(x) = k \\ 0 & \text{if } L(x) < k \end{cases},$$

where $k > 0$ and $c \in [0, 1]$ are chosen such that $\mathbb{E}_{P_0} \phi_k(X) = \alpha$, and $\alpha \in (0, 1)$ is the desired significance level.

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Neyman–Pearson Lemma

We are interested in LRTs because they are exactly *most powerful*.

> ## Theorem (Neyman–Pearson Lemma, Lehmann & Romano 2005)
>
> *Fix $\alpha \in (0,1)$ and let ϕ_k be a LRT such that $\mathbb{E}_{P_0}\phi_k(X) = \alpha$. Let ϕ be any test satisfying $\mathbb{E}_{P_0}\phi(X) \leq \alpha$. Then*
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> $$\mathbb{E}_{P_1}\phi(X) \leq \mathbb{E}_{P_1}\phi_k(X).$$

For fixed significance level α, no *valid* test correctly rejects the null more often than LRTs.

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Optimal error rate

If we care about Type I and Type II errors equally, then ϕ_1 is optimal and we can characterise its error rate through the Total Variation distance: for any test $\phi(X_1, \ldots, X_n)$ we have

$$\mathbb{E}_{P_0}\phi + \mathbb{E}_{P_1}(1 - \phi) \geq 1 - \sup_{f:\mathcal{X}^n \to [0,1]} (\mathbb{E}_{P_1}f - \mathbb{E}_{P_0}f)$$
$$= 1 - \{\mathbb{P}_{P_1}(L \geq 1) - \mathbb{P}_{P_0}(L \geq 1)\}$$
$$= 1 - \mathrm{TV}(P_0^{\otimes n}, P_1^{\otimes n})$$
$$= \mathbb{E}_{P_0}\phi_1 + \mathbb{E}_{P_1}(1 - \phi_1).$$

There exist good tests of H_0 vs. H_1 if and only if $\mathrm{TV}(P_0^{\otimes n}, P_1^{\otimes n})$ is large.

When n is large, $\mathrm{TV}(P_0^{\otimes n}, P_1^{\otimes n})$ is not easy to interpret.

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Hellinger distance

Recall the Hellinger distance $H^2(Q_1, Q_2) = 1 - \int \sqrt{q_1 q_2}$ and its relationship to TV distance:

$$H^2(Q_1, Q_2) \leq \mathrm{TV}(Q_1, Q_2) \leq \sqrt{2} H(Q_1, Q_2) \left\{ 1 - \frac{1}{2} H^2(Q_1, Q_2) \right\}^{1/2}.$$

Hellinger is useful for its tensorisation property

$$H_n^2 := H^2(P_0^{\otimes n}, P_1^{\otimes n}) = 1 - \left\{ 1 - H^2(P_0, P_1) \right\}^n =: 1 - (1 - H^2)^n.$$

These properties are used to show that

$$\frac{1}{2}(1 - H^2)^{2n} \leq 1 - \sqrt{1 - (1 - H^2)^{2n}} = 1 - \sqrt{2 H_n^2 (1 - H_n^2/2)}$$

$$\leq 1 - \mathrm{TV}(P_0^{\otimes n}, P_1^{\otimes n}) \leq 1 - H_n^2 = (1 - H^2)^n.$$

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Behaviour for large n

We have

$$\frac{1}{2}(1 - H^2)^{2n} \leq 1 - \text{TV}(P_0^{\otimes n}, P_1^{\otimes n}) \leq (1 - H^2)^n.$$

The optimal error decays exponentially quickly in n, and the rate of decay is like $H^2 = H^2(P_0, P_1)$, up to constants.

When studying test performance it is convenient to think about separation conditions or sample complexity. We can check that

- $H^2 \geq \frac{\log(1/\delta)}{n} \implies \exists$ test with Type I + Type II error $\leq \delta$
- $H^2 < \frac{\log \frac{1}{4\delta}}{4n} \leq \frac{1}{2} \implies$ all tests have Type I + Type II error $> \delta$.

Roughly, there are good tests if and only if P_0, P_1 are separated in Hellinger distance[1].

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Two–point testing – conclusion

- The Neyman–Pearson Lemma gives us the exactly optimal test for the classical model.
- We can give bounds on the power of this test, thus characterising the fundamental difficulty of the problem through the Hellinger distance.

Later, we will see that Hellinger is no longer the right distance for robust/private problems.

Beyond this simple problem (e.g. when alternatives are *composite*) it is rare to find exactly optimal tests. However, if we fix a separation metric of interest, we can often find good tests and establish their rate optimality.

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Goodness-of-fit testing

Given $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} P$, a basic problem with composite alternative is

$$H_0 : P = P_0 \quad \text{vs.} \quad H_1 : P \neq P_0,$$

It is not possible to have uniformly powerful tests here, so we study power by introducing separation.
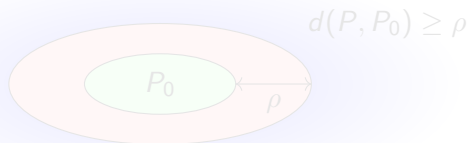
The minimal separation needed for acceptable error, e.g.

$$\rho^*(\phi) = \inf\left\{\rho \in (0,1) : \mathbb{E}_{P_0}(\phi) + \sup_{P : d(P, P_0) \geq \rho} \mathbb{E}(1 - \phi) \leq 1/4\right\}$$

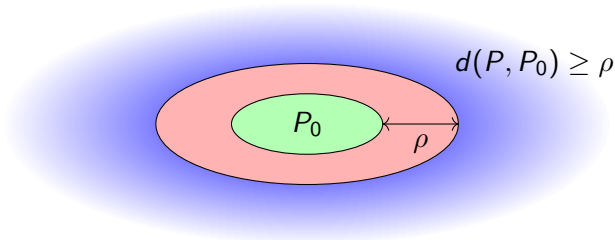measures the power of ϕ. The *minimax separation* is $\rho^* = \inf_\phi \rho^*(\phi)$.

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Discrete uniformity testing

The minimax separation $\rho^* = \inf_\phi \rho^*(\phi)$ quantifies the difficulty of the testing problem.

Suppose $X_1, \ldots, X_n \sim P$ take values in $[d] = \{1, \ldots, d\}$ and we test

$$H_0 : P = \mathrm{Unif}([d]) \quad \text{vs.} \quad H_1(\rho) : \mathrm{TV}(P, \mathrm{Unif}([d])) \geq \rho.$$

Theorem (e.g. Balakrishnan & Wasserman 2019)

In this setting, when $d^{1/2} \leq n$ there exist universal constants $c, C > 0$ such that

$$\frac{c\, d^{1/4}}{n^{1/2}} \leq \rho^* \leq \frac{C\, d^{1/4}}{n^{1/2}},$$

i.e. the minimax optimal separation rate is $d^{1/4}/n^{1/2}$.

Note that the optimal rate for estimation of P is $\mathrm{TV}(\hat{P}, P) \asymp (d/n)^{1/2}$. In this sense, testing is easier than estimation.

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Optimal tests – Minimax upper bound

We can prove an upper bound on ρ^* by choosing a test ϕ and bounding its error, since $\rho^* \leq \rho^*(\phi)$.

Define kernel $h(x, y) = \mathbb{1}_{\{x=y\}} - 1/d$ and test statistic

$$T = T(X_1, \ldots, X_n) = \frac{1}{\binom{n}{2}} \sum_{1 \leq i_1 < i_2 \leq n} h(X_{i_1}, X_{i_2}) = \frac{1}{n(n-1)} \sum_{j=1}^{d} N_j^2 + \text{const.}$$

For suitable critical value $C_{d,n}$ we will consider the test $\phi = \mathbb{1}_{\{T \geq C_{d,n}\}}$.

To analyse this test it suffices to bound the mean and variance of T. First,

$$\mathbb{E}_P(T) = \mathbb{E}\{h(X_1, X_2)\} = \sum_{j=1}^{d} p(j)^2 - 1/d = \sum_{j=1}^{d} \{p(j) - 1/d\}^2$$

$$= \|p - p_0\|_2^2 \stackrel{H_0}{=} 0.$$

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Since T is a second order U-statistic, we know (e.g. Serfling, 2009) that

$$\mathrm{Var}_P(T) = \frac{2(n-2)}{\binom{n}{2}}\mathrm{Var}\Big(\mathbb{E}\{h(X_1,X_2)|X_1\}\Big) + \frac{1}{\binom{n}{2}}\mathrm{Var}\big(h(X_1,X_2)\big)$$

$$= \frac{4(n-2)}{n(n-1)}\mathrm{Var}\big(p(X_1)\big) + \frac{2}{n(n-1)}\mathrm{Var}(\mathbb{1}_{\{X_1=X_2\}}).$$

Under H_0 the first term vanishes and we get $\mathrm{Var}_P(T) \leq \frac{2}{dn(n-1)}$.

By Chebyshev's inequality we see that taking $C_{d,n} = 4/\{dn(n-1)\}^{1/2}$ gives a test with Type I error $\leq 1/8$.

Similarly, we have small Type II error when $\mathbb{E}_P(T) = \|p - p_0\|_2^2 \geq 2C_{d,n}$ and $\mathrm{Var}_P(T) \ll (\mathbb{E}_P T)^2$.

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More generally, we can see that

$$\mathrm{Var}_P(T) \lesssim \frac{\mathbb{E}_P(T)}{n}(1/d + 1/n) + \frac{\{\mathbb{E}_P(T)\}^{3/2}}{n} + \frac{1}{dn^2}.$$

Thus, when $\mathbb{E}_P(T) = \|p - p_0\|_2^2 \geq 2C_{d,n} \asymp 1/(n\sqrt{d})$ we have

$$\frac{\mathrm{Var}_P(T)}{(\mathbb{E}_P T)^2} \lesssim \frac{\sqrt{d}}{n} + \frac{1}{\sqrt{d}} + \frac{d^{1/4}}{\sqrt{n}} + 1 \lesssim 1.$$

By Chebyshev's inequality, there exists universal constant $B > 0$ such that Type II error $\leq 1/8$ when $\|p - p_0\|_2^2 \geq B/(n\sqrt{d})$.

Since $\|p - p_0\|_2^2 \geq (4/d)\mathrm{TV}(P, P_0)^2$, we have

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If π is any (prior) distribution over $\mathcal{P}_1(\rho) = \{P : \mathrm{TV}(P, P_0) \geq \rho\}$, we have
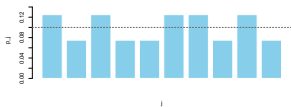
$$\inf_\phi \left\{ \mathbb{E}_{P_0}(\phi) + \sup_{P \in \mathcal{P}_1(\rho)} \mathbb{E}_P(1 - \phi) \right\} \geq \inf_\phi \left\{ \mathbb{E}_{P_0}(\phi) + \mathbb{E}_\pi \mathbb{E}_P(1 - \phi) \right\}$$

$$= 1 - \mathrm{TV}(P_0^{\otimes n}, \mathbb{E}_\pi(P^{\otimes n})),$$

where $\mathbb{E}_\pi(P^{\otimes n})$ is the mixture distribution on $[d]^n$ with mass function

$$(x_1, \ldots, x_n) \mapsto \mathbb{E}_\pi \left[\prod_{i=1}^n p(x_i) \right].$$

Construct π (Paninski, 2008) by generating $\xi_1, \ldots, \xi_{d/2} \sim$ Rademacher, and setting

$$p(j) = \begin{cases} (1/d)(1 + 2\rho\xi_{j/2}) & j \text{ even} \\ (1/d)(1 - 2\rho\xi_{(j+1)/2}) & j \text{ odd} \end{cases}$$

By construction, $\mathrm{TV}(P, P_0) = (1/2)\|p - p_0\|_1 = \rho$ with prob. 1

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Let p' be an independent copy of p. Taking $\rho = \{\frac{d \log(1+\delta^2/4)}{16n^2}\}^{1/4}$ we have

$$1 + 4\mathrm{TV}(P_0^{\otimes n}, \mathbb{E}_\pi(P^{\otimes n}))^2 \leq 1 + \chi^2(P_0^{\otimes n}, \mathbb{E}_\pi(P^{\otimes n}))$$

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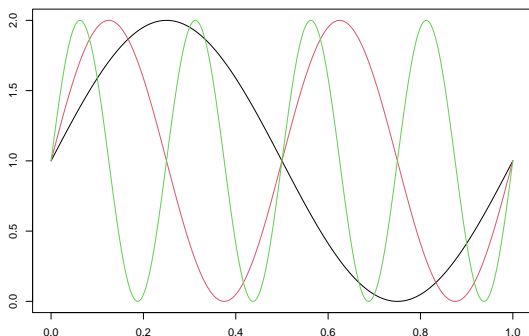$$1 + 4\mathrm{TV}(P_0^{\otimes n}, \mathbb{E}_\pi(P^{\otimes n}))^2 \leq 1 + \chi^2(P_0^{\otimes n}, \mathbb{E}_\pi(P^{\otimes n}))$$

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Continuous case

Similar ideas can be used to find optimal tests for continuous settings, though there is an important conceptual difference.
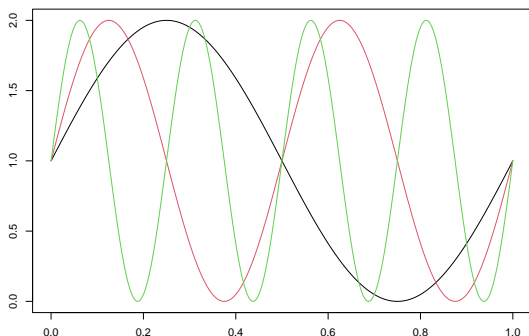

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Continuous case – smoothness

To give tests with uniform power guarantees we must impose structure on the alternative. It is common to make smoothness assumptions such as

$$|p(x) - p(y)| \leq L\|x - y\|_2^\beta \quad \text{for all } x, y \in [0, 1]^d$$

for some $\beta \in (0, 1], L > 0$. Write $p \in \mathcal{H}(\beta, L)$.

Our problem becomes: given $X_1, \ldots, X_n \sim P$ on $[0, 1]^d$, test

$$H_0 : P = \text{Unif}([0, 1]^d) \quad \text{vs.} \quad H_1(\rho) : \|p - p_0\|_2 \geq \rho, p \in \mathcal{H}(\beta, L)$$

Theorem (Arias-Castro et al. 2018)

In this setting, there exist constants $c, C > 0$ depending only on d, β, L such that

$$c\, n^{-\frac{2\beta}{4\beta+d}} \leq \rho^* \leq C\, n^{-\frac{2\beta}{4\beta+d}}.$$

Again, this rate is faster than the optimal estimation rate of $n^{-\frac{\beta}{2\beta+d}}$.

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Instance optimality

So far we have considered GoF testing with uniform nulls. Recent work (Diakonikolas & Kane, 2016; Valiant & Valiant, 2017; Balakrishnan & Wasserman, 2019) has shown that the difficulty of the problem depends intricately on P_0.

Theorem (Chhor & Carpentier 2022)

Assuming that $p_0(1) \geq p_0(2) \geq \ldots \geq p_0(d)$, we have

$$\rho^* \asymp \sqrt{\frac{\|(p_0^{-\max})_{\leq I}\|_{2/3}}{n}} + \|p_{>I}\|_1 + \frac{1}{n}$$

where $I = \min\{J : \sum_{i>J} p_0(i)^2 \leq cn^{-2}\}$ and $p_0^{-\max} = (p_0(2), \ldots, p_0(d))$.

Roughly speaking, the difficulty is determined by the '2/3-norm' of p_0:

- $P_0 = \mathrm{Unif}([d])$ is the hardest problem, with $\frac{d^{1/4}}{n^{1/2}}$ a global upper bound
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Goodness-of-fit testing – conclusion

- Separation rates give a convenient way to study the power of tests – how strong does a signal need to be before we can detect it?
- U-statistics are often a convenient basis for powerful tests.
- Lower bounds can be proved by constructing alternatives that are difficult to detect.

However, the picture is more complex than for point hypotheses:

- We typically do not know the exactly optimal test.
- In general (e.g. for continuous data) uniform power is impossible, and we must make (smoothness) assumptions.
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Composite null hypotheses

So far we have seen how to construct powerful tests in settings where the null hypothesis is *simple*. Given T we can find C_α such that

$$\mathbb{P}_{H_0}(T \geq C_\alpha) \leq \alpha.$$

When $H_0 : P \in \mathcal{P}_0$ is *composite*, it is harder to guarantee validity. We will say that a test ϕ is *uniformly valid* if

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Two canonical problems are:

Two-sample testing

Given $X_1, \ldots, X_n \sim f$ and $Y_1, \ldots, Y_m \sim g$, we test $H_0 : f = g$.

Independence testing

Given $(X_1, Y_1), \ldots, (X_n, Y_n) \sim f$ we test $H_0 : f(x, y) = f_X(x)f_Y(y) \ \forall x, y$.

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Independence testing

Independence testing is one of the most studied problems in statistics. With discrete data we usually see a contingency table, e.g.

	Middle school or lower	High school	Bachelor's	Master's	PhD or higher
Never married	18	36	21	9	6
Married	12	36	45	36	21
Divorced	6	9	9	3	3
Widowed	3	9	9	6	3

Source: https://www.spss-tutorials.com/chi-square-independence-test/.

Writing $p_{jk} = \mathbb{P}(X = j, Y = k)$, we test

$$H_0 : p_{jk} = r_j q_k \quad \text{for some } (r_j), (q_k).$$

The χ^2 test

It is standard practice to use Pearson's χ^2 test, where we use the fact that

$$T = \sum_{j=1}^{J} \sum_{k=1}^{K} \frac{(o_{jk} - e_{jk})^2}{e_{jk}} \xrightarrow{\mathrm{d}} \chi^2_{(J-1)(K-1)}$$

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This convergence is not uniform and validity at finite sample sizes is not
guaranteed.

Asymptotic null distributions

For fixed $\lambda > 0$ set $p = \sqrt{\lambda/n}$ and consider the null distribution given by:

j/k	1	2
1	p^2	$p(1-p)$
2	$p(1-p)$	$(1-p)^2$

Comparing T to the 99th quantile of the χ_1^2 distribution, for large n we get
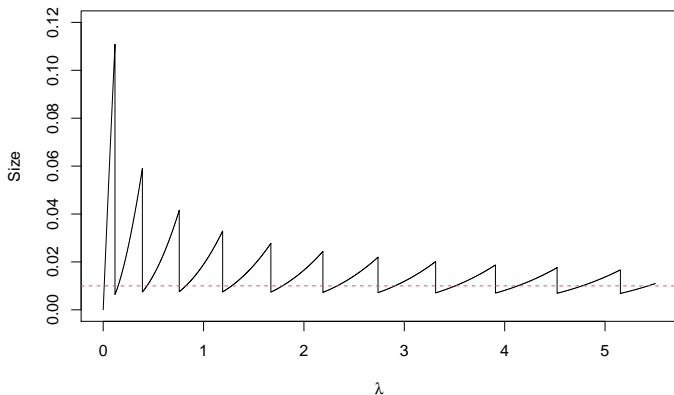
Permutation tests

The issue is exacerbated in more complex problems, where

- asymptotic approximations may not be known; or

- rates of convergence depend on unverifiable smoothness assumptions.

Fortunately, for some problems permutation tests exist and give exact
Type I error guarantees.

Exact tests

Permutation testing is a classical idea (Fisher, 1935; Pitman, 1938). The simplest example is the *Lady tasting tea* experiment, using an *exact test*.

Figure: Source: `e10v.me/tea-tasting-rdd/`

Can the taster tell whether milk or tea was first? We can calculate the exact significance of any outcome.

Exact tests

Thought of as an independence test: data $(X_1, Y_1), \ldots, (X_n, Y_n) \in \{0, 1\}^2$

X/Y	Guess tea first	Guess milk first	Total
Tea first	n_{11}	n_{10}	n_{1+}
Milk first	n_{01}	n_{00}	n_{0+}
Total	n_{+1}	n_{+0}	n

Permutations $(X_{\pi(1)}, Y_1), \ldots, (X_{\pi(n)}, Y_n)$ preserve the row/column totals and lead to n_{11}^π with a Hypergeometric distribution.

Compare n_{11} to $(n_{11}^\pi : \pi \in \mathcal{S}_n)$, rejecting $H_0 : X \perp\!\!\!\perp Y$ if it is sufficiently extreme.
$$p = (n!)^{-1} \sum_{\pi \in \mathcal{S}_n} \mathbb{1}_{\{n_{11}^\pi \geq n_{11}\}}.$$

Strong Type I error guarantees: $\mathbb{P}_{H_0}(p \leq \alpha) \leq \alpha$ for all $\alpha \in [0, 1]$.

Beyond discrete data – Two-sample testing

Two-sample testing

Given $X_1, \ldots, X_n \sim f$ and $Y_1, \ldots, Y_m \sim g$, we test $H_0 : f = g$.

$$\mathbf{Z} = (X_1, \ldots, X_n, Y_1, \ldots, Y_m), \quad \mathbf{Z}_\pi = (Z_{\pi(1)}, \ldots, Z_{\pi(m+n)}) \text{ for } \pi \in \mathcal{S}_{n+m}.$$
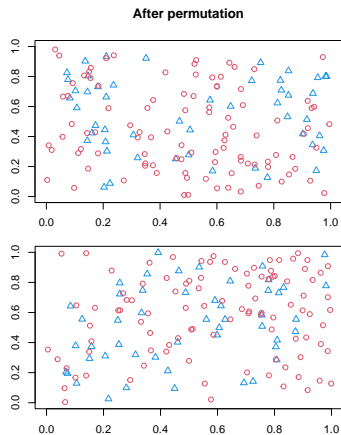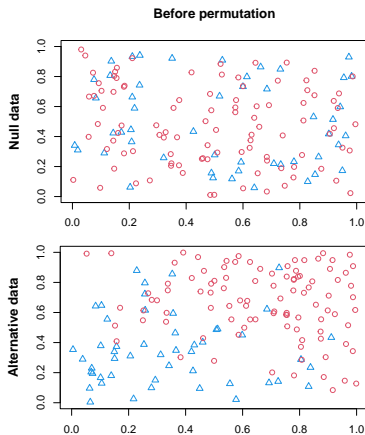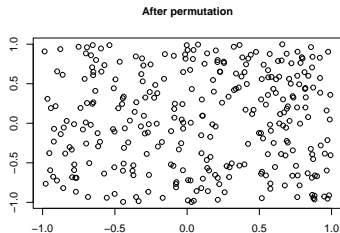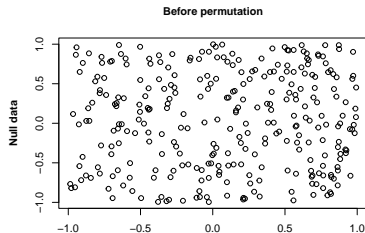
Independence testing

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Given $(X_1, Y_1), \ldots, (X_n, Y_n) \sim f$ we test $H_0 : f(x, y) = f_X(x)f_Y(y) \; \forall x, y$.

$$\mathbf{Z} = (X_1, Y_1, \ldots, X_n, Y_n), \quad \mathbf{Z}_\pi = (X_{\pi(1)}, Y_1, \ldots, X_{\pi(n)}, Y_n) \text{ for } \pi \in \mathcal{S}_n.$$

Type I error control

Typically infeasible to use all permutations, and permutation distributions (e.g. Hypergeometric) are not known, so we sample uniformly.

For any test statistic $T : \mathcal{X}^n \times \mathcal{Y}^n \to \mathbb{R}$ and $\Pi_1, \ldots, \Pi_B \overset{\text{i.i.d.}}{\sim} \mathrm{Unif}(\mathcal{S}_n)$ we calculate the p-value

$$p = \frac{1 + \sum_{b=1}^{B} \mathbb{1}\{T(\mathbf{Z}) \leq T(\mathbf{Z}_{\Pi_b})\}}{1 + B} = \frac{\mathrm{rank}(T(\mathbf{Z}))}{1 + B}.$$

Theorem

Suppose that H_0 holds. Then

$$\mathbb{P}(p \leq \alpha) \leq \frac{\lfloor \alpha(B + 1) \rfloor}{B + 1} \leq \alpha$$

for all $\alpha \in [0, 1]$.

If we change p to break ties at random, we can replace the first $\leq$ by $=$.

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Type I error control – proof

Under H_0, if we condition on the multi-set $\mathcal{X}_n := \{X_1, \ldots, X_n\}$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ then $\mathbf{Z} = (X_1, Y_1, \ldots, X_n, Y_n)$ is a uniform assignment.

So, given $\mathcal{X}_n$ and $\mathbf{Y}$, the datasets $\mathbf{Z}, \mathbf{Z}_{\Pi_1}, \ldots, \mathbf{Z}_{\Pi_B}$ are conditionally i.i.d.

So $\mathbf{Z}, \mathbf{Z}_{\Pi_1}, \ldots, \mathbf{Z}_{\Pi_B}$ are exchangeable.

So

$$
\begin{aligned}
&(B+1)\mathbb{P}(p \leq \alpha) \\
&= \mathbb{E}\left[\mathbb{1}\{\mathrm{rank}(T(\mathbf{Z}) \leq \alpha(B+1)\} + \sum_{b=1}^{B} \mathbb{1}\{\mathrm{rank}(T(\mathbf{Z}_{\Pi_b}) \leq \alpha(B+1)\} \right] \\
&\leq \lfloor \alpha(B+1) \rfloor.
\end{aligned}
$$

Asymptotic power

While Type I error control is simple, power results are difficult to prove.

Theorem (Hoeffding 1952)

Say $(X_1, Y_1) \sim P_n$ for some sequence (P_n). Suppose that

$$\mathbb{P}(T(\mathbf{Z}_{\Pi_1}) \leq u, T(\mathbf{Z}_{\Pi_2}) \leq v) \to R(u)R(v)$$

and $\mathbb{P}(T(\mathbf{Z}) \leq u) \to H(u)$ for distribution functions H, R. If H and R are continuous at $R^{-1}(1 - \alpha)$ and R is strictly increasing at $R^{-1}(1 - \alpha)$, then, if $B \to \infty$, we have

$$\mathbb{P}(p \leq \alpha) \to 1 - H(R^{-1}(1 - \alpha))$$

as $n \to \infty$.

Used for results on power against sequences (P_n) of local alternatives.

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> *as $n \to \infty$.*

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Permutation tests

Permutation tests:

- require no assumptions for non-asymptotic Type I error control;

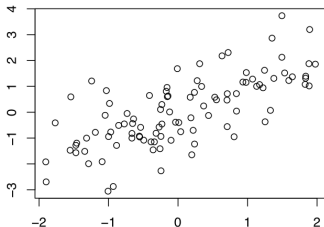- can be used with any test statistic.

In the rest of the lecture we will see that they:

- can achieve minimax rate optimal power

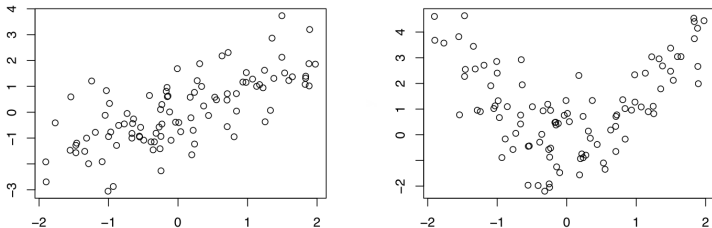- can be adapted for problems with non-exchangeable null hypotheses.

Independence testing

There is a vast literature on testing independence of continuous variables, with classical measure such as Pearson's correlation (e.g. Pearson, 1920), Kendall's tau (Kendall, 1938), Hoeffding's D (Hoeffding, 1948). These are limited to linear or monotonic dependence, or bivariate settings.

Modern datasets exhibit complex dependence not captured by classical measures. More flexible measures and tests include HSIC (Gretton et al., 2005), Distance covariance (Székely, Rizzo & Bakirov, 2007), mutual information (B. & Samworth, 2019), multivariate rank-based tests (Deb & Sen, 2023; Chatterjee, 2021), etc...

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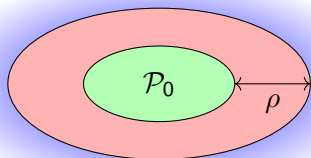
Independence testing – problem statement

> **Independence testing**
>
> Given $(X_1, Y_1), \ldots, (X_n, Y_n) \sim f$ we test $H_0 : f(x, y) = f_X(x)f_Y(y) \;\forall x, y$.

Given a suitable (e.g. smooth) class $\mathcal{P}_1$, define the alternatives

$$\mathcal{P}_1(\rho) = \left\{ P \in \mathcal{P}_1 : D(P) \geq \rho^2 \right\}$$

$$D(f) = \int (f - f_X f_Y)^2$$

For a test ϕ, we define the worst-case risk

$$\mathcal{R}_n(\mathcal{P}_1(\rho), \phi) = \sup_{P \in \mathcal{P}_0} \mathbb{E}_P(\phi) + \sup_{P \in \mathcal{P}_1(\rho)} \mathbb{E}_P(1 - \phi),$$

and the associated separation radius $\rho_n^*(\mathcal{P}_1, \phi)$.

Continuous case – basis expansion

Suppose $\mathcal{X} \times \mathcal{Y} = [0,1]^{d_X + d_Y}$. As we mentioned before, we will need smoothness assumptions for uniform power.

Let $(p_{jk})_{j \in \mathcal{J}, k \in \mathcal{K}} = (p_j^X p_k^Y)_{j \in \mathbb{N}^{d_X}, k \in \mathbb{N}^{d_Y}}$ be the Fourier basis and for density f define

$$a_{jk} = \int p_{jk} f \, d\mu, \quad a_{j\bullet} = \int p_j^X f_X \, d\mu_X, \quad a_{\bullet k} = \int p_k^Y f_Y \, d\mu_Y$$

Define the Sobolev smoothness of $f - f_X f_Y$ by

$$S_{s_X, s_Y}(f) = \sum_{j \in \mathbb{N}^{d_X}, k \in \mathbb{N}^{d_Y}} (\|j\|_1^{2s_X} \vee \|k\|_1^{2s_Y}) \{a_{jk}(f) - a_{j\bullet}(f) a_{\bullet k}(f)\}^2.$$

We will consider classes of the form

$$\mathcal{P}_1 = \{P : S_{s_X, s_Y}(f) \leq r^2, \max(\|f\|_\infty, \|f_X\|_\infty, \|f_Y\|_\infty) \leq A\}.$$

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Continuous case – minimax rates

Suppose $\mathcal{X} \times \mathcal{Y} = [0,1]^{d_X + d_Y}$ with (s_X, s_Y)-Sobolev smoothness.

Theorem (B., Kontoyiannis, & Samworth 2021)

Fix $\gamma \in (0,1)$. Writing $d = d_X + d_Y$ and $s = d/(d_X/s_X + d_Y/s_Y)$ we have

$$\inf_\phi \rho_n^*(\mathcal{P}_1, \phi) \asymp n^{-2s/(4s+d)}.$$

Moreover, this rate is attained by a permutation test.

- Similar rates for univariate problems known since Ingster (1989).
- In the multivariate setting, Albert et al. (2022) finds minimax rates with a test using an oracle critical value.
- In independent work, Kim, Balakrishnan & Wasserman (2022) proved the minimax rate optimality of HSIC-based permutation tests. Also consider two-sample testing.

Upper bound

As for goodness-of-fit testing, U-statistics are useful.

For a subset $\mathcal{M} \subseteq \mathbb{N}^{d_X + d_Y}$ we write

$$h((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4))$$
$$= \sum_{(j,k) \in \mathcal{M}} \{ p_{jk}(x_1, y_1) p_{jk}(x_2, y_2) - 2 p_{jk}(x_1, y_1) p_{jk}(x_2, y_3)$$
$$+ p_{jk}(x_1, y_2) p_{jk}(x_3, y_4) \}$$

and take test statistic

$$\hat{D}_n = \frac{1}{4! \binom{n}{4}} \sum_{\substack{i_1, i_2, i_3, i_4 \\ \text{distinct}}} h((X_{i_1}, Y_{i_1}), (X_{i_2}, Y_{i_2}), (X_{i_3}, Y_{i_3}), (X_{i_4}, Y_{i_4})).$$

This is an unbiased estimator of a truncated version of $D(f)$:

$$\mathbb{E}(\hat{D}_n) = \sum_{(j,k) \in \mathcal{M}} (a_{jk}^2 - 2 a_{jk} a_{j\bullet} a_{\bullet k} + a_{j\bullet}^2 a_{\bullet k}^2) = \sum_{(j,k) \in \mathcal{M}} (a_{jk} - a_{j\bullet} a_{\bullet k})^2.$$

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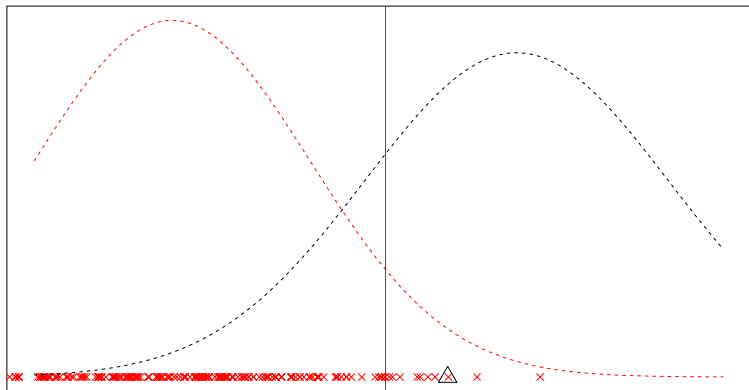\end{aligned}
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Upper bound

To analyse the power of the permutation test, it suffices to control the first and second moments of $\hat{D}_n$ and its permuted versions $\hat{D}_n^{(1)}, \ldots, \hat{D}_n^{(B)}$.

Can then use Chebyshev's inequality to show that $\hat{D}_n > \hat{D}_n^{(1)}$ with high probability.

Upper bound – original statistic

By construction, we have $\mathbb{E}_f(\hat{D}_n) = \sum_{(j,k) \in \mathcal{M}} (a_{jk} - a_{j\bullet} a_{\bullet k})^2$. Bias is due to truncation to $\mathcal{M}$. For suitable choices we have

$$|\mathbb{E}_f(\hat{D}) - D(f)| = \sum_{(j,k) \notin \mathcal{M}} (a_{jk} - a_{j\bullet} a_{\bullet k})^2$$

$$\leq \frac{r^2}{\inf\{\|j\|_1^{2s_X} \vee \|k\|_1^{2s_Y} : (j,k) \notin \mathcal{M}\}} \lesssim \frac{r^2}{|\mathcal{M}|^{2s/d}}.$$

Variance can be controlled using U-statistic results (Serfling, 2009), and

$$\mathrm{Var}_f(\hat{D}_n) \lesssim \frac{D(f)}{n} + \frac{|\mathcal{M}|}{n^2}.$$

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Upper bound – permuted statistic

Under permutation it is harder to analyse the statistic, but it can be done. If $\Pi \sim \mathrm{Unif}(\mathcal{S}_n)$ we have, for example, $(\Pi(1), \Pi(2)) \stackrel{\mathrm{d}}{=} (\Pi(1), \Pi(3))$, and we can check

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We can also prove that $\mathrm{Var}_f(\hat{D}_n^{(1)}) \lesssim \frac{|\mathcal{M}|}{n^2}$.

Up to constants, the moments of $\hat{D}_n^{(1)}$ behave like those of $\hat{D}_n$ under H_0.

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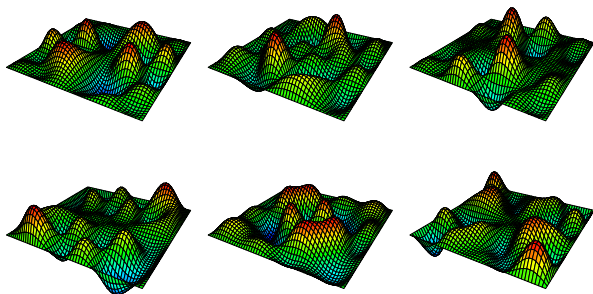We can also prove that $\mathrm{Var}_f(\hat{D}_n^{(1)}) \lesssim \frac{|\mathcal{M}|}{n^2}$.

Up to constants, the moments of $\hat{D}_n^{(1)}$ behave like those of $\hat{D}_n$ under H_0.

Lower bound

As for goodness-of-fit, we define a prior over $\mathcal{P}_1(\rho)$. Here we take

$$f(x, y) = 1 + \sum_{(j,k) \in \mathbb{N}^{d_X + d_Y}} w_{jk} \xi_{jk} p_{jk}(x, y).$$

for (ξ_{jk}) independent Rademacher and (w_{jk}) appropriate weights.

The marginal distribution of the data with this prior can be shown to be indistinguishable from the uniform distribution.

Summary so far

- With composite nulls, Type I error control can be difficult.
- For some problems, random permutation can take any test statistic and give us a uniformly valid test.
- Theoretical analysis is more complex, but for independence testing we can prove that appropriate permutation tests are rate optimal.

Not discussed here:

- Choice of optimal truncation set depends on unknown smoothness, but we can be adaptive.
- We can prove stronger distributional results about $\hat{D}_n^{(1)}$ to get approximate power functions.
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This works for independence and two-sample testing, but these are 'nice' problems.

Summary so far

- With composite nulls, Type I error control can be difficult.
- For some problems, random permutation can take any test statistic and give us a uniformly valid test.
- Theoretical analysis is more complex, but for independence testing we can prove that appropriate permutation tests are rate optimal.

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Non-exchangeable nulls

Validity of traditional permutation tests relies on certain exchangeabilities under H_0, but many interesting problems do not have these.

Conditional independence testing

Given an i.i.d. sample $(X_1, Y_1, W_1), \ldots, (X_n, Y_n, W_n)$ we aim to test

$$H_0 : X \perp\!\!\!\perp Y | W.$$

Conditional two-sample testing

Given $(X_1, W_1), \ldots, (X_n, W_n) \sim f$ and $(Y_1, W_1'), \ldots, (Y_m, W_m') \sim g$, we want to test

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Conditional independence testing

Conditional independence is central to statistical theory and modelling (Dawid, 1979).

We say that $X \perp\!\!\!\perp Y | W$ if the conditional density of $(X, Y) | W$ factorises as $f_{XY|W} = f_{X|W} f_{Y|W}$. Equivalently, if $f_{Y|X,W} = f_{Y|W}$.
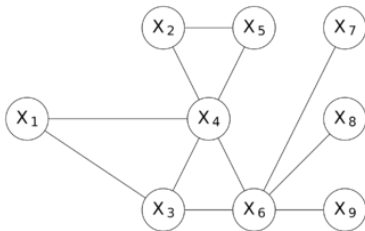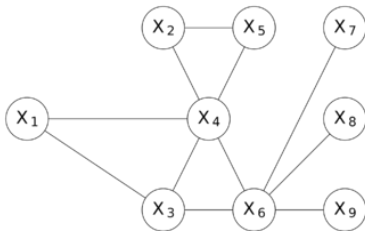

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Hardness of conditional independence testing

Such conditional problems are significantly more complex. E.g. for conditional independence testing:

Hardness result (Shah & Peters, 2020)

For $n \in \mathbb{N}$ and $\alpha \in (0, 1)$, let ψ_n be a test with $\sup_{P \in \mathcal{P}} \mathbb{P}_P(\psi_n = 1) \leq \alpha$. Then

$$\sup_{Q \in \mathcal{Q}} \mathbb{P}_Q(\psi_n = 1) \leq \alpha,$$

where $\mathcal{P}$ is the class of continuous null ($X \perp\!\!\!\perp Y | W$) distributions of (X, Y, W) and $\mathcal{Q}$ is the class of continuous alternatives ($X \not\perp\!\!\!\perp Y | W$).

The null hypothesis is so complex that any test with uniform Type I error control must be trivial.
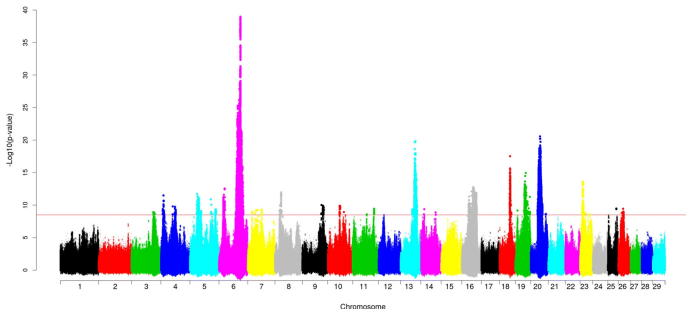
We will need to make assumptions to make any progress.

Model-X framework

In many settings we have more information that can help.

In the 'Model-X' framework (Candès et al., 2018) we assume that we know (an approximation to) the conditional distribution $Q(\cdot|w)$ of $X|W = w$.

In some cases (X, W) data is abundant while labeled data (X, Y, W) is relatively scarce.

The conditional permutation test

This additional information allows the construction of a permutation test.

Generate[2] $\Pi_1, \ldots, \Pi_B \in \mathcal{S}_n$ independently from the distribution

$$\mathbb{P}(\Pi = \pi | \mathbf{X}, \mathbf{Y}, \mathbf{W}) \propto \prod_{i=1}^{n} q(X_{\pi(i)} | W_i),$$

where $q(\cdot | w)$ is the density of $X | \{W = w\}$, and set $X_i^{(b)} = X_{\Pi_b(i)}$.

Generates $\mathbf{X}^{(b)}$ from $Q^n(\cdot | \mathbf{W})$ conditional on the event that the marginal distribution is preserved.[3]

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[3]Without this condition, we have the CRT of Candès et al. (2018)

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48

Validity of the CPT

$$\mathbb{P}(\Pi = \pi | \mathbf{X}, \mathbf{Y}, \mathbf{W}) \propto \prod_{i=1}^{n} q(X_{\pi(i)} | W_i),$$

This distribution ensures that $(\mathbf{X}, \mathbf{Y}, \mathbf{W}), (\mathbf{X}^{(1)}, \mathbf{Y}, \mathbf{W}), \ldots (\mathbf{X}^{(B)}, \mathbf{Y}, \mathbf{W})$ are exchangeable under H_0.

CPT (B. et al., 2020)

Reject if
$$p = \frac{1 + \sum_{b=1}^{B} \mathbb{1}\{T(\mathbf{X}^{(b)}, \mathbf{Y}, \mathbf{W}) \geq T(\mathbf{X}, \mathbf{Y}, \mathbf{W})\}}{1 + B} \leq \alpha.$$

Non-uniform permutation distributions also in testing exchangeability (Ramdas et al., 2022) and weighted conformal prediction (Tibshirani et al., 2019).

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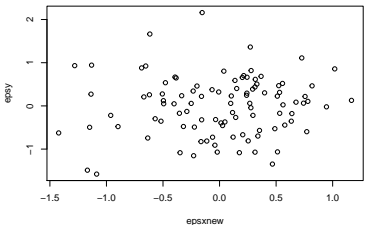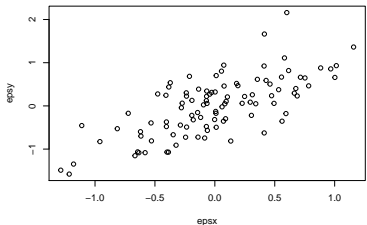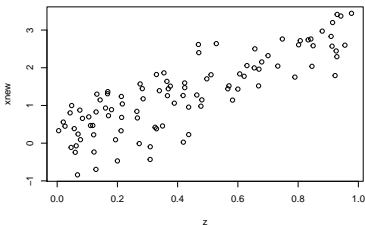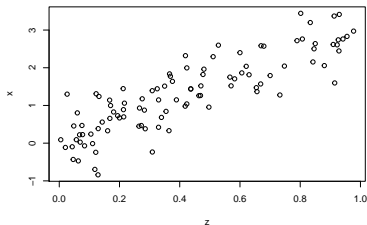
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The conditional permutation test

Consider $X|W \sim \mathcal{N}(\beta^T W, \sigma^2)$ and $Y|X, W \sim \mathcal{N}(\beta^T W + \gamma X, \sigma^2)$. We can compare the residuals $\hat{\epsilon}_X$ and $\hat{\epsilon}_Y$ after regressing X and Y on W.

Robustness

The input $Q(\cdot|z)$ will generally be different to the true conditional distribution $Q_*(\cdot|z)$, and the Type I error control may not hold exactly.

Theorem

Under H_0, for any test statistic T and significance level $\alpha \in [0, 1]$ we have that

$$\mathbb{P}(p \leq \alpha | \mathbf{Y}, \mathbf{W}) \leq \alpha + \mathrm{TV}\big(Q_*^n(\cdot|\mathbf{W}), Q^n(\cdot|\mathbf{W})\big).$$
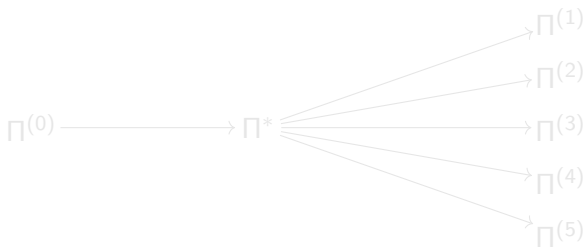
If we can choose Q with $\mathrm{TV}\big(Q_*^n(\cdot|\mathbf{W}), Q^n(\cdot|\mathbf{W})\big) = o(1)$ then we have an approximately valid test.

Sampling

We construct Markov chains whose stationary distribution is the desired distribution, allowing for (approximate) sampling.
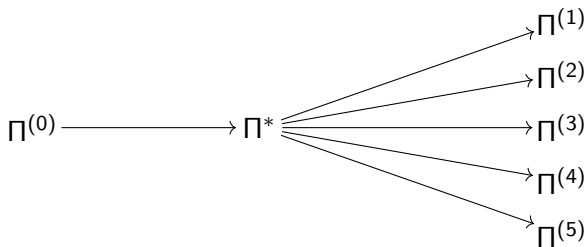
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We overcome (weak) dependence between permutations using a star-shaped sampler (Besag & Clifford, 1989) to retain uniform validity.

$$\Pi^{(0)} \longrightarrow \Pi^* \begin{cases} \nearrow \Pi^{(1)} \\ \nearrow \Pi^{(2)} \\ \rightarrow \Pi^{(3)} \\ \searrow \Pi^{(4)} \\ \searrow \Pi^{(5)} \end{cases}$$

Parallelised algorithm

Algorithm Parallelized pairwise sampler for the CPT

Input: Initial permutation $\Pi^{[0]}$, integer $S \geq 1$.

for $s = 1, 2, \ldots, S$ **do**

    Sample uniformly without replacement from $\{1, \ldots, n\}$ to obtain disjoint pairs

$$(i_{s,1}, j_{s,1}), \ldots, (i_{s,\lfloor n/2 \rfloor}, j_{s,\lfloor n/2 \rfloor}).$$

    Draw independent Bernoulli variables $B_{s,1}, \ldots, B_{s,\lfloor n/2 \rfloor}$ with odds ratios

$$\frac{\mathbb{P}(B_{s,k} = 1)}{\mathbb{P}(B_{s,k} = 0)} = \frac{q(X_{(\Pi^{[s-1]}(j_{s,k}))}|Z_{i_{s,k}}) \cdot q(X_{(\Pi^{[s-1]}(i_{s,k}))}|Z_{j_{s,k}})}{q(X_{(\Pi^{[s-1]}(i_{s,k}))}|Z_{i_{s,k}}) \cdot q(X_{(\Pi^{[s-1]}(j_{s,k}))}|Z_{j_{s,k}})}.$$

    Define $\Pi^{[s]}$ by swapping $\Pi^{[s-1]}(i_{s,k})$ and $\Pi^{[s-1]}(j_{s,k})$ for each k with $B_{s,k} = 1$.

end for

Capital bikeshare data set

We implemented the CPT on the Capital bikeshare data set.

This records for each ride the start and end times and locations and a user type that can be 'Member' or 'Casual'.

Capital bikeshare data set

Problem

Test $X \perp\!\!\!\perp Y | W$, where

- X is the duration of the ride
- W the start and end locations and time of day
- Y the user type, date **or** day of the week.

For our conditional distribution we use

$$Q(\cdot | w) = \mathcal{N}(\hat{\mu}(w), \hat{\sigma}^2(w)),$$

where $\hat{\mu}(w)$ and $\hat{\sigma}^2(w)$ are calculated using the training data for each combination of start and end location with a kernel weighting times of day.

- Test data: all rides taken on weekdays in Oct 2011. Sample size n=7,346 rides (after screening).
- Training data: all rides taken on weekdays in Sep and Nov 2011. Sample size N=149,912 rides.

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Writing $R_i = X_i - \hat{\mu}(W_i)$ we take T to be the correlation between $\mathbf{R}$ and $\mathbf{Y}$ when Y is the user type. When Y is the day of the week we take T to be

$$\max_{y \in \{\text{Mon},\ldots,\text{Fri}\}} \left| \mathrm{Corr}\big(\mathbf{R}, (\mathbb{1}\{Y_1 = y\}, \ldots, \mathbb{1}\{Y_n = y\})\big) \right|.$$

With $M = 100$ we obtain the following average p-values over ten trials of the experiment:

Variable Y	CPT
User type	0.0010 (0.0000)
Date	0.1146 (0.0032)
Day of week	0.1980 (0.0037)

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Towards power results...

The complexity of the permutation distribution poses a challenge to power results. We can consider a simpler problem.

Problem

Given data $X_1, \ldots, X_n \sim f$ and $Y_1, \ldots, Y_m \sim g$ on domain $\mathcal{X}$, we aim to test the hypothesis

$$H_0 : g \propto r \cdot f,$$

where $r : \mathcal{X} \to (0, \infty)$ is a known function.

- Goodness-of-fit test for the ratio g/f.
- Conditional two-sample testing $H_0 : f_{X|W} = g_{Y|W}$ when f_W/g_W is known.
- Two-sample testing with *biased data* (e.g. Cox, 1969; Qin, 1993; Efromovich, 2004).

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Conclusion

Permutation testing is a robust and powerful approach that can be widely used.

- Minimax rate optimal power guarantees for independence and two-sample testing.

- With additional knowledge and non-uniform permutations we can introduce conditional independence tests with Type I error guarantees.

Beyond classical models

So far we have been considering the classical model $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} P$ and answering questions about P. We have:

- found (rate) optimal tests of various hypotheses;
- given tight characterisations of the power of optimal tests, or the fundamental difficulties of the problems.

In modern data science, we may have other concerns than simply minimising statistical error. We may also have to deal with

- limited computational/communication resources
- contaminated/missing data
- privacy/fairness constraints
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These considerations may need to balanced with statistical accuracy. With minimax optimality results we can quantify trade-offs.

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Personal data

In this lecture we will consider *differential privacy*.

The collection and use of personal data is increasingly common in modern society.
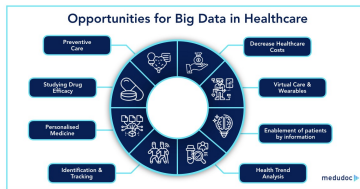
Figure: Source: `medium.com`

Sensitive information is routinely collected in modern technology, the census, medicine, finance...

Data breaches

At the same time, data breaches have become a feature of everyday life.

Biggest fines for data breaches
Fines over £250,000

- ● Fines under the **Data Protection Act 1998** (Max. fine £500,000)
- ◉ Fine under the **new GDPR** (Max. fine €20 million, or 4% annual global turnover)

Sussex Hospitals NHS Trust £350k
Scottish Borders Council £250k

2012

Sony £250k

2013

2014

2015

2016 — TalkTalk £400k

Crown Prosecution Service £325k

2017 — Carphone Warehouse £400k — **Equifax £500k** — **Facebook £500k**

2018 — Yahoo £250k — Uber £400k

Proposed fine for British Airways £183.39m

2019

Source: ICO - Information Commisioner's Office
BBC

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Traditional anonymisation is not enough

Removing names/addresses is insufficient to prevent re-identification.
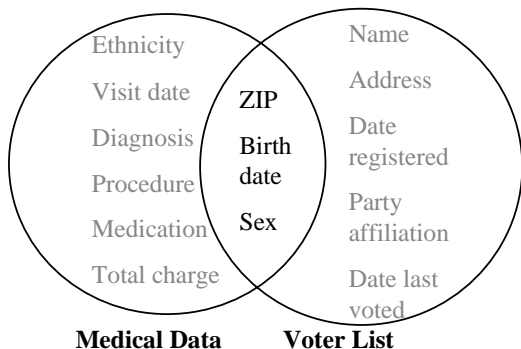
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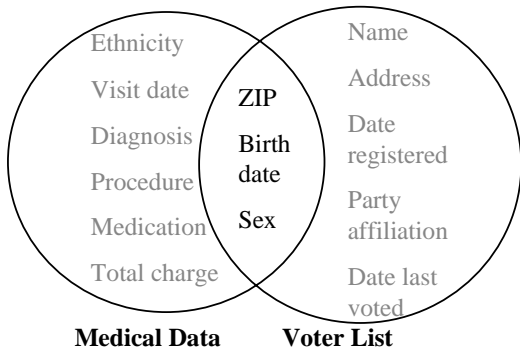
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Summary statistics may also be insecure

Suppose $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} N(0, I_d)$ is used to calculate $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$, which is published.

Let $X_1', \ldots, X_n' \overset{\text{i.i.d.}}{\sim} N(0, I_d)$ be data from individuals not in the sample.
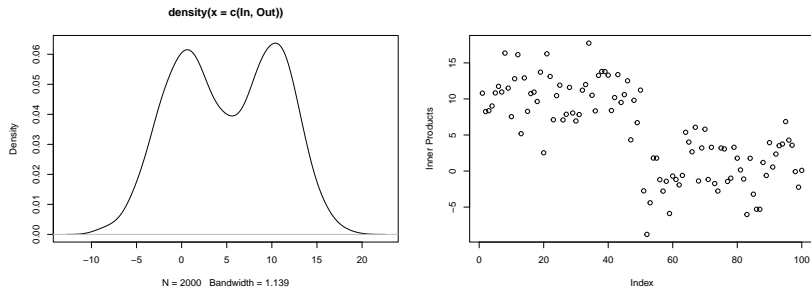
Figure: $n = 1000$, $d = 10000$

Distributions of $\bar{X}_n^T X_i$ and $\bar{X}_n^T X_i'$ can be very different (e.g. Homer et al., 2008).

Privacy mechanisms

Privacy mechanisms are algorithms taking input $\mathbf{X} = (X_1, \ldots, X_n)$ and producing randomised publishable $\mathbf{Z}$.

Formally, they are sets of conditional distributions $Q = \{Q(\cdot|\mathbf{x}) : \mathbf{x} \in \mathcal{X}\}$ such that

$$\mathbf{Z}|\{\mathbf{X} = \mathbf{x}\} \sim Q(\cdot|\mathbf{x}).$$
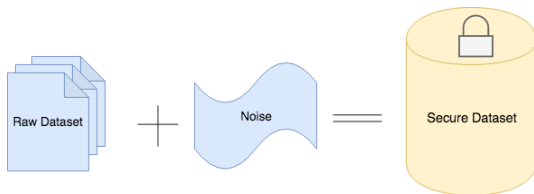
Figure: Source: medium.com

We choose Q to preserve the most pertinent information. *How much* noise should we add? *What type* of noise?

Randomised response

The basic idea is present in Warner (1965). Say $X_1, \ldots, X_n$ are i.i.d. binary variables and we want to estimate $\pi = \mathbb{P}(X_1 = 1)$.

Figure: Source: Adapted from Pitsch & Emrich (2012)

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Randomised response

Ask individual i to calculate and transmit

$$Z_i = \begin{cases} X_i & \text{with probability } p \\ 1 - X_i & \text{otherwise} \end{cases},$$

where p controls the amount of noise. Typically $p > 1/2$.

Given $Z_1, \ldots, Z_n$, the MLE for θ is

$$\hat{\theta}_n = \frac{1}{n(2p-1)} \sum_{i=1}^{n} \{Z_i - (1-p)\}.$$

We have $\mathbb{E}\hat{\theta}_n = \theta$ and

$$\mathrm{Var}(\hat{\theta}_n) = \frac{\theta(1-\theta)}{n} + \frac{p(1-p)}{n(2p-1)^2},$$

an inflation of the non-private ($p = 1$) variance.

- *How much* noise should we add?
- Beyond binary data: *What type* of noise?

Randomised response

Ask individual i to calculate and transmit

$$Z_i = \begin{cases} X_i & \text{with probability } p \\ 1 - X_i & \text{otherwise} \end{cases},$$

where p controls the amount of noise. Typically $p > 1/2$.

Given $Z_1, \ldots, Z_n$, the MLE for θ is

$$\hat{\theta}_n = \frac{1}{n(2p-1)} \sum_{i=1}^{n} \{Z_i - (1-p)\}.$$

We have $\mathbb{E}\hat{\theta}_n = \theta$ and

$$\mathrm{Var}(\hat{\theta}_n) = \frac{\theta(1-\theta)}{n} + \frac{p(1-p)}{n(2p-1)^2},$$

an inflation of the non-private ($p = 1$) variance.

- *How much* noise should we add?
- Beyond binary data: *What type* of noise?

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Differential privacy

Privacy mechanism Q is called α-*differentially private* (Dwork et al., 2006) if

$$\sup_A \frac{Q(A|\mathbf{x})}{Q(A|\mathbf{x}')} = \sup_A \frac{\mathbb{P}(\mathbf{Z} \in A|\mathbf{X} = \mathbf{x})}{\mathbb{P}(\mathbf{Z} \in A|\mathbf{X} = \mathbf{x}')} \leq e^{\alpha}$$

for all $\mathbf{x}, \mathbf{x}'$ such that $d_{\mathrm{Ham}}(\mathbf{x}, \mathbf{x}') := \sum_{i=1}^n \mathbb{1}_{\{x_i \neq x_i'\}} \leq 1$. Assume $\alpha \leq 1$.

Large scale applications include

- Google Chrome (Erlingsson, Pihur & Korolova, 2014);
- Apple in iOS and macOS (Tang et al., 2017);
- Microsoft (Ding, Kulkarni & Yekhanin, 2017);
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Can also be used to demonstrate GDPR compliance (Cohen & Nissim, 2020).

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A statistical interpretation

Suppose that the data $X_1, \ldots, X_n$ are i.i.d. with distribution P.

If an adversary sees **Z** and, knowing P and Q, tests

$$H_0 : X_1 = x_1 \quad \text{vs.} \quad H_1 : X_1 = x_1'$$

at significance level γ, the power of the test is bounded above by γe^{α}
(Wasserman & Zhou, 2010).

We cannot reliably work out whose data is in the database.

Central model

The least restrictive model is the *central model* (Dwork & Lei, 2009; Wasserman & Zhou, 2010; Cai, Wang & Zhang, 2021).
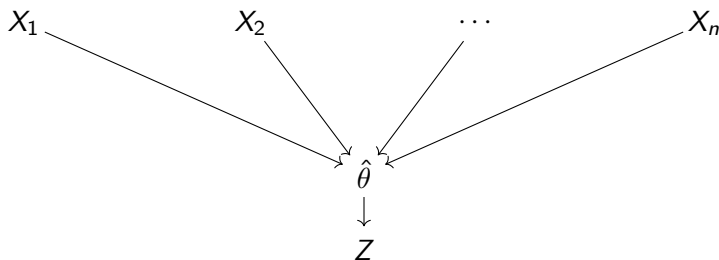
The statistician has access to the raw data, but can only publish randomised statistics. E.g. the census.

E.g., $Z = \frac{1}{n} \sum_{i=1}^{n} X_i + \frac{1}{n\alpha} W$.

Local differential privacy

We consider the *local model* (Kairouz, Oh & Viswanath, 2014; Duchi, Jordan & Wainwright, 2018; Rohde & Steinberger, 2020; Acharya et al., 2024), where data are randomised before collection.

$$\sup_A \sup_{x_i, x_i', z^{(1:i-1)}} \frac{\mathbb{P}(Z_i \in A | X_i = x_i)}{\mathbb{P}(Z_i \in A | X_i = x_i')} \le e^\alpha \text{ for all } i.$$

No trusted third party: analyse $Z = (Z_1, \dots, Z_n)$ with
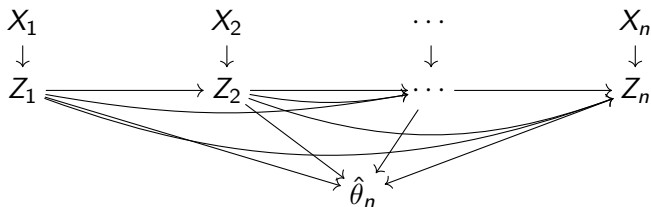
E.g., $Z_i = X_i + \frac{1}{\alpha} W_i$ and $\hat{\theta} = \frac{1}{n} \sum_{i=1}^n Z_i = \frac{1}{n} \sum_{i=1}^n X_i + \frac{1}{n\alpha} \sum_{i=1}^n W_i$.

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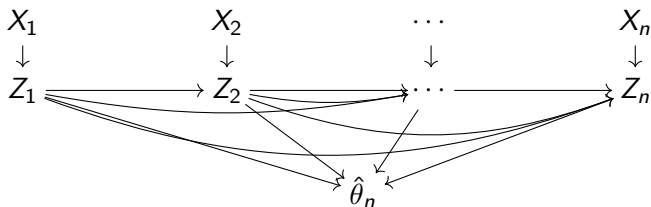
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Simple hypothesis testing – Scheffé test

Let's reconsider the simple hypothesis testing problem

$$H_0 : P = P_0 \quad \text{vs.} \quad H_1 : P = P_0$$

for fixed distributions P_0, P_1, given $X_1, \ldots, X_n \overset{\text{iid}}{\sim} P$.

The classical LR statistic $\prod_{i=1}^n \frac{dP_1}{dP_0}(X_i)$ is difficult to privatise, but we can use ideas from *robust statistics* (e.g. Devroye & Lugosi, 2001; Chen, Gao & Ren, 2016).

In the non-private setting the *Scheffé test* rejects H_0 if and only if

$$\frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{X_i \in A\}} > \frac{1}{2}\{P_0(A) + P_1(A)\},$$

where A is such that $P_1(A) - P_0(A) = \sup_S\{P_1(S) - P_0(S)\}$. We can see that each data point now has a limited effect on the test.

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Simple hypothesis testing

This can be applied to the output of the randomised response mechanism
(Warner, 1965; Gopi et al., 2020)

$$Z_i = \begin{cases} \mathbb{1}_{\{X_i \in A\}}, & \text{w.pr. } e^{\alpha}/(1 + e^{\alpha}), \\ 1 - \mathbb{1}_{\{X_i \in A\}}, & \text{otherwise.} \end{cases}$$

This is a valid α-LDP mechanism since

$$\frac{\mathbb{P}(Z = 1 | X \in A)}{\mathbb{P}(Z = 1 | X \in A^c)} = \frac{\mathbb{P}(Z = 0 | X \in A^c)}{\mathbb{P}(Z = 0 | X \in A)} = e^{\alpha}.$$

Reject H_0 if and only if

$$\frac{e^{\alpha} + 1}{n(e^{\alpha} - 1)} \sum_{i=1}^{n} \left(Z_i - \frac{1}{e^{\alpha} + 1} \right) > \frac{1}{2}\{P_0(A) + P_1(A)\}.$$

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Upper bound

By construction, we have

$$\mathbb{E}\left\{\frac{e^{\alpha}+1}{n(e^{\alpha}-1)}\sum_{i=1}^{n}\left(Z_i-\frac{1}{e^{\alpha}+1}\right)\right\} = \frac{e^{\alpha}+1}{e^{\alpha}-1}\left\{\mathbb{P}(Z=1)-\frac{1}{e^{\alpha}+1}\right\}$$

$$= \frac{e^{\alpha}+1}{e^{\alpha}-1}\left\{\frac{e^{\alpha}}{1+e^{\alpha}}P(A)+\frac{1}{1+e^{\alpha}}P(A^c)-\frac{1}{e^{\alpha}+1}\right\} = P(A).$$

We can then use Hoeffding's inequality:

$$\mathbb{E}_{P_0}(\phi) = \mathbb{P}_{P_0}\left(\frac{e^{\alpha}+1}{n(e^{\alpha}-1)}\sum_{i=1}^{n}(Z_i-\mathbb{E}Z_i) > \frac{1}{2}\{P_0(A)+P_1(A)\}-P_0(A)\right)$$

$$= \mathbb{P}_{P_0}\left(\frac{1}{n}\sum_{i=1}^{n}(Z_i-\mathbb{E}Z_i) > \frac{1}{2}\frac{e^{\alpha}-1}{e^{\alpha}+1}\mathrm{TV}(P_0,P_1)\right)$$

$$\le \exp\left(-\frac{n}{2}\left(\frac{e^{\alpha}-1}{e^{\alpha}+1}\right)^2\mathrm{TV}(P_0,P_1)^2\right),$$

with an almost identical bound for the Type II error.

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$$= \frac{e^\alpha + 1}{e^\alpha - 1}\left\{\frac{e^\alpha}{1 + e^\alpha}P(A) + \frac{1}{1 + e^\alpha}P(A^c) - \frac{1}{e^\alpha + 1}\right\} = P(A).$$

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Private information inequality

To prove a matching lower bound, we require the following result, thought of as a *data processing inequality*.

> **Theorem (Duchi, Jordan & Wainwright 2018)**
>
> *Let Q be any α-LDP privacy mechanism and P_0, P_1 any distributions. Then*
> $$\mathrm{KL}\big(QP_0^{\otimes n}, QP_1^{\otimes n}\big) \leq 4n(e^{\alpha}-1)^2 \mathrm{TV}(P_0, P_1)^2.$$

For small α the RHS scales like $O(\alpha^2)$, so that privacy mechanisms act as contractions obscuring information.

We can bound a private KL divergence by non-private TV distance, the opposite direction to Pinsker!

Lower bound

Let Q be any α-LDP privacy mechanism and ϕ any test based on its output. Then

$$\mathbb{E}_{P_0}\phi + \mathbb{E}_{P_1}(1 - \phi) \geq 1 - \mathrm{TV}(QP_0^{\otimes n}, QP_1^{\otimes n}) \qquad \text{as before}$$

$$\geq \frac{1}{2}\exp\left(-\mathrm{KL}\left(QP_0^{\otimes n}, QP_1^{\otimes n}\right)\right) \qquad \text{Bretagnolle–Huber}$$

$$\geq \frac{1}{2}\exp\left(-4n(e^\alpha - 1)^2\mathrm{TV}(P_0, P_1)^2\right)$$

Duchi, Jordan & Wainwright (2018)

The minimax risk satisfies

$$\frac{1}{2}\exp\left(-4n(e^\alpha - 1)^2\mathrm{TV}(P_0, P_1)^2\right) \leq \inf_Q \inf_\phi \left\{\mathbb{E}_{P_0}\phi + \mathbb{E}_{P_1}(1 - \phi)\right\}$$

$$\leq 2\exp\left(-\frac{n}{2}\left(\frac{e^\alpha - 1}{e^\alpha + 1}\right)^2\mathrm{TV}(P_0, P_1)^2\right).$$

Up to constants, this is tight in the *high-privacy* regime $\alpha \leq 1$.

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Cost of privacy for two-point testing

Let's compare this to classical results and *Huber's contamination model* $X \sim (1 - \epsilon)P + \epsilon G$.

For combined error rate ≤ 0.1 we require:

- Classical model: $\mathrm{H}(P_0, P_1) \gtrsim 1/\sqrt{n}$;

- ε-Huber with $n = \infty$: $\mathrm{TV}(P_0, P_1) > \varepsilon/(1 - \varepsilon)$ (e.g. Chen, Gao & Ren, 2016);

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- α-LDP and ε-Huber: $\mathrm{TV}(P_0, P_1) \gtrsim \varepsilon + 1/\sqrt{n\alpha^2}$ (Li et al., 2023)

when $\alpha \leq 1$.

There are deep connections between robust statistics and (local) differential privacy (e.g. Dwork & Lei, 2009; Avella-Medina, 2021; Li et al., 2023).

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Uniformity testing

Recall that when $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} P$ on $[d]$ we want to test
$$H_0 : P = \text{Unif}([d]) \quad \text{vs.} \quad H_1(\rho) : \text{TV}(P, \text{Unif}([d])) \geq \rho.$$

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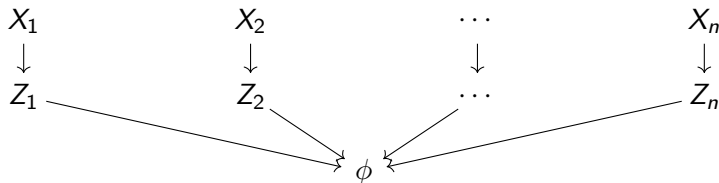In GoF we see that *sequentially interactive* methods can do better.

Non-interactive rates

$$X_1 \qquad X_2 \qquad \cdots \qquad X_n$$

$$Z_1 \qquad Z_2 \qquad \cdots \qquad Z_n$$

Theorem (Acharya et al. 2019)

Among non-interactive α-LDP ($\alpha \leq 1$) procedures, we have

$$\rho_{n,\alpha}^{\mathrm{NI},*} \asymp \frac{d^{3/4}}{\sqrt{n\alpha^2}}.$$

We can compare this to the classical rate of $d^{1/4}/\sqrt{n}$:

- Effective sample size $n \mapsto n\alpha^2$.
- Effect of dimension worsened $d \mapsto d^3$.

Upper bound methodology

We can use the RAPPOR mechanism (Erlingsson, Pihur & Korolova, 2014) to adapt (roughly speaking) the previous non-private test (Acharya et al., 2019).

Independently across individuals and coordinates, calculate

$$Z_{ij} = \begin{cases} \mathbb{1}_{\{X_i = j\}}, & \text{w.pr. } e^{\alpha/2}/(1 + e^{\alpha/2}), \\ 1 - \mathbb{1}_{\{X_i = j\}}, & \text{otherwise}, \end{cases}$$

i.e. one-hot encoding then independent randomised responses.

This is α-LDP: for $z \in \{0, 1\}^d$ and $x \neq x' \in [d]$ we have

$$\frac{\mathbb{P}(Z_1 = z | X_1 = x)}{\mathbb{P}(Z_1 = z | X_1 = x')} = \frac{\mathbb{P}(Z_{1x} = z_x | X_1 = x) \, \mathbb{P}(Z_{1x'} = z_{x'} | X_1 = x)}{\mathbb{P}(Z_{1x} = z_x | X_1 = x') \, \mathbb{P}(Z_{1x'} = z_{x'} | X_1 = x')}$$

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Upper bound

Easy to check that

$$\mathbb{E}(Z_{ij}) = \frac{1}{1 + e^{\alpha/2}} + \frac{e^{\alpha/2} - 1}{e^{\alpha/2} - 1} p(j).$$

We therefore consider the U-statistic

$$T = \frac{1}{\binom{n}{2}} \sum_{i_1 < i_2} \sum_{j=1}^{d} \left(Z_{i_1 j} - \frac{1}{1 + e^{\alpha/2}} \right) \left(Z_{i_2 j} - \frac{1}{1 + e^{\alpha/2}} \right) - \left(\frac{e^{\alpha/2} - 1}{e^{\alpha/2} - 1} \right)^2 \frac{1}{d},$$

which has $\mathbb{E}_P(T) = (\frac{e^{\alpha/2} - 1}{e^{\alpha/2} - 1}) \|p - p_0\|_2^2$. We also have

$$\mathrm{Var}_{P_0}(T) \lesssim \frac{d}{n^2}.$$

Calculations similar to those in the first lecture reveal that we have a powerful test when $\|p - p_0\|_2^2 \gtrsim \sqrt{d}/(n\alpha^2)$, or $\|p - p_0\|_1 \gtrsim d^{3/2}/\sqrt{n\alpha^2}$.

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Public-coin mechanism

Surprisingly, we can do much better with very limited interaction.
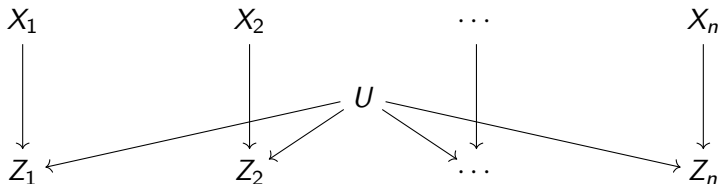
Theorem (Acharya et al. 2019)

Among public-coin α-LDP ($\alpha \leq 1$) procedures, we have

$$\rho_{n,\alpha}^{\mathrm{PC},*} \asymp \frac{\sqrt{d}}{\sqrt{n\alpha^2}}.$$

This rate cannot be improved by allowing more general sequentially interactive mechanisms (Amin et al., 2020).

Public-coin method

Uses the `RAPTOR` mechanism (Acharya et al., 2019): generate $S \subset [d]$ with $|S| = d/2$ uniformly at random, then use randomised response

$$Z_i = \begin{cases} \mathbb{1}_{\{X_i \in S\}}, & \text{w.pr. } e^\alpha/(1 + e^\alpha), \\ 1 - \mathbb{1}_{\{X_i \in S\}}, & \text{otherwise.} \end{cases}$$

Clearly $\mathbb{E}(Z|S) = \frac{1}{1+e^\alpha} + \frac{e^\alpha - 1}{e^\alpha + 1} p(S)$, so we can reject H_0 if

$$\left| \bar{Z}_n - \frac{1}{1 + e^\alpha} - \frac{1}{2} \frac{e^\alpha - 1}{e^\alpha + 1} \right| \geq \frac{C}{\sqrt{n}}$$

some suitable constant C. This has power if $|p(S) - 1/2| \gtrsim (n\alpha^2)^{-1/2}$.

Lemma (Acharya et al. 2019)

We have

$$\mathbb{P}\left(\left| p(S) - \frac{1}{2} \right| > \frac{\mathrm{TV}(P, P_0)}{\sqrt{5d}} \right) > \frac{1}{477}.$$

If $\mathrm{TV}(P, P_0) \gtrsim \frac{\sqrt{d}}{\sqrt{n\alpha^2}}$ there is a constant probability of having power. By boosting, this is enough to get the rate.

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Summary of uniformity testing

Recall that when $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} P$ on $[d]$ we have been testing

$$H_0 : P = \mathrm{Unif}([d]) \quad \text{vs.} \quad H_1(\rho) : \mathrm{TV}(P, \mathrm{Unif}([d])) \geq \rho.$$

We see costs of privacy and lack of interaction. Optimal separation rates are

- $\frac{d^{1/4}}{\sqrt{n}}$ without privacy constraints
- $\frac{\sqrt{d}}{\sqrt{n\alpha^2}}$ with public-coin α-LDP
- $\frac{d^{3/4}}{\sqrt{n\alpha^2}}$ with non-interactive α-LDP.
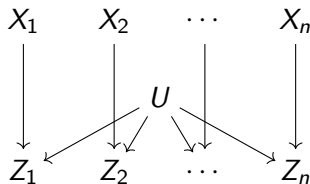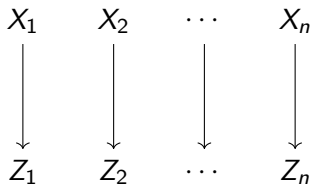
Instance optimality – non-interactive rates

We get similar phenomena with non-uniform nulls. Recall that in the classical model the optimal separation rate is like $\sqrt{\|p_0\|_{2/3}/n}$.

$$X_1 \qquad\qquad X_2 \qquad\qquad \cdots \qquad\qquad X_n$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \cdots \qquad\qquad \downarrow$$

$$Z_1 \qquad\qquad Z_2 \qquad\qquad \cdots \qquad\qquad Z_n$$

Theorem (B. & Butucea 2020)
$$\rho_{n,\alpha}^{\mathrm{NI},*}(p_0) \lesssim j_*^{3/4}/\sqrt{n\alpha^2},$$

with nearly matching lower bound. Here j_* is an 'effective support size'

$$j_* = j_*(n\alpha^2, p_0,) := \min\left\{ j \in \mathbb{N} : \frac{j^{3/4}}{(n\alpha^2)^{1/2}} \geq \sum_{j'=j+1}^{\infty} p_0(j') \right\}.$$

Interactive rates

Going back to the interactive setting:
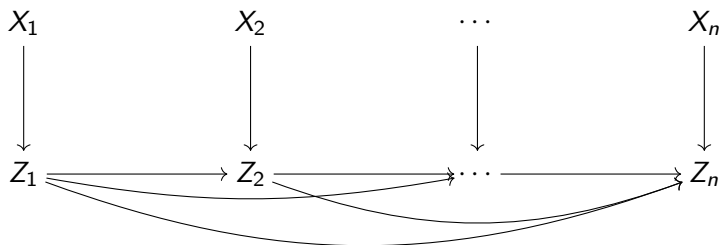
Theorem (B. & Butucea 2020)

$$\rho_{n,\alpha}^{\mathrm{I},*}(p_0) \lesssim \tilde{j}^{1/4}/\sqrt{n\alpha^2},$$

with (nearly) matching lower bound, where $\tilde{j}$ is another 'effective support size'.

Examples

Table: Separation rates (up to log factors) for testing discrete distributions on $\mathbb{N}$.

p_0	Non-interactive	Interactive	Non-private
Unif$[d]$	$\frac{d^{3/4}}{\sqrt{n\alpha^2}}$	$\frac{d^{1/2}}{\sqrt{n\alpha^2}}$	$\frac{d^{1/4}}{\sqrt{n}}$
$\propto j^{-1-\beta}$	$\left(n\alpha^2\right)^{-\frac{2\beta}{4\beta+3}}$	$\left(n\alpha^2\right)^{-\frac{2\beta}{4\beta+2}}$	$n^{-1/2} \vee n^{-\frac{2\beta}{2\beta+1}}$

We can also extend to continuous data (Dubois et al., 2022).

Conclusion

LDP constraints reduce the effective sample size and can change rates of convergence.

By considering simple hypothesis testing we see links between robust statistics and LDP (also in mean/median estimation, density estimation...)

With more complex problems there can be a gap between non-interactive and sequentially interactive rates.

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